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On the cohomology of arrangements of subtori

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Abstract

Given an arrangement of subtori of arbitrary codimension in a complex torus, we compute the cohomology groups of the complement. Then, by using the Leray spectral sequence, we describe the multiplicative structure on the associated graded cohomology. We also provide a differential model for the cohomology ring, by considering a toric wonderful model and its Morgan algebra. Finally, we focus on the divisorial case, proving a new presentation for the cohomology of toric arrangements.

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Contents

INTRODUCTION

The cohomology ring of the complement of an arrangement of affine hyperplanes in a complex vector space admits a renowned combinatorial presentation in terms of the poset of intersections of the arrangement, due to Orlik and Solomon [21]. For a *toric arrangement*, that is, a collection of 1-codimensional subtori in a complex algebraic torus, a similar presentation was recently provided by [5].

A different way of generalizing arrangements of hyperplanes is considering a family of affine subspaces, not necessarily of codimension 1. The complement of such a *subspace arrangement* was studied by several authors (see [10, 13, 14, 17, 25, 26]; see also [2] and the bibliography therein). In particular, Goresky and MacPherson provided the following description of the cohomology groups.

Theorem A [17, III.1.5, Theorem A]. *Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{R}^d$, and let $M_\mathcal{A} = \mathbb{R}^d \setminus \cup\mathcal{A}$ be its complement. The reduced cohomology of the complement is given by*

$$\tilde{H}^k(M_\mathcal{A};\mathbb{Z}) \cong \bigoplus_{W\in\mathcal{L}_{>\hat{0}}} \tilde{H}_{\mathrm{cd}_\mathbb{R} W-k-2}\big(\Delta(\hat{0},W);\mathbb{Z}\big),$$

where $\mathcal{L}_{>\hat{0}}$ is the poset of intersections $\mathcal{L}$ without the minimum $\hat{0}=\mathbb{R}^d$, $\mathrm{cd}_\mathbb{R}(W)$ is the real codimension of W, and $\Delta(\hat{0},W)$ is the order complex of the open interval $(\hat{0},W)$ in $\mathcal{L}$.

Later, De Concini and Procesi constructed in [10] a wonderful model for subspace arrangements, that is, a smooth projective variety Y that contains $M_\mathcal{A}$ as an open subset whose complement is a simple normal crossing divisor. They also applied a result of Morgan [20] to present the rational cohomology ring of $M_\mathcal{A}$ as the cohomology of a differential graded algebra explicitly constructed from the combinatorial data only.

In 1996, Yuzvinsky simplified the differential graded algebra (see [26]) by using the order complex of the poset of intersections. He also showed a connection between the results of [17] and of [10]. A further simplification was obtained in [25] by replacing the order complex with the atomic complex.

Yuzvinsky also conjectured an integral version of this presentation. This conjecture was proven in [13, 14]: Deligne, Goresky, and MacPherson proved their result using diagram of spaces, de Longueville and Schultz by showing that the isomorphism of Theorem A is canonical.

In this paper, we consider arrangements of subtori of arbitrary codimension in a complex algebraic torus. Given the complement of such an arrangement, we determine its cohomology groups in terms of the poset of layers $\mathcal{L}$, that is, the set of connected components of intersections of subtori, ordered by reverse inclusion.

Theorem B (Theorem 2.8). *Let $\mathcal{A}$ be an arrangement of subtori of a torus T and $\mathcal{L}$ be its poset of layers. Then the cohomology groups of the complement $M_\mathcal{A}$ are*

$$H^k(M_\mathcal{A};\mathbb{Z}) \cong \bigoplus_{W\in\mathcal{L}} \bigoplus_{p+q=k} H^p(W;\mathbb{Z}) \otimes_\mathbb{Z} \tilde{H}_{2\,\mathrm{cd}(W)-2-q}(\Delta(T,W)),$$

where $\mathrm{cd}(W)$ is the complex codimension of the layer W.

Our proof is based on a suitable embedding of a d-dimensional complex algebraic torus T in the Alexandroff compactification of $\mathbb{C}^d$, that is, the sphere $\mathbb{S}^{2d}$. The embedding is chosen so that the complement of T in $\mathbb{S}^{2d}$ does not intersect the toric subspaces; hence the arrangement decomposes in a wedge of two simpler ones, given, respectively, by the compactifications of the coordinate hyperplanes and of the subtori in the original arrangement (Proposition 1.4). Then we apply some results on homotopy colimits [24], following a strategy outlined by Deshpande in [12]. In that paper, the same result was announced, but the proof therein does not seem to be correct, as several steps fail if the compactification is not chosen carefully.

Moreover we describe the multiplicative structure on the associated graded of the cohomology, by using the Leray spectral sequence for the inclusion map $j : M_{\mathcal{A}} \to T$. We show, by using results of the previous section, that the second page of the spectral sequence is a finitely generated $\mathbb{Z}$-module isomorphic to the cohomology as a module. It follows that the spectral sequence degenerates at the second page and this gives the isomorphism

$$E_2^{p,q} \cong \mathrm{gr}^{\mathrm{L}}_{p+q} H^{p+q}(M_{\mathcal{A}};\mathbb{Z})$$

(Theorem 3.2).

Furthermore, we provide a model for the cohomology of $M_{\mathcal{A}}$: we use the wonderful model for toric arrangements introduced by De Concini and Gaiffi (see [7, 8]) to construct a differential graded algebra (D,d) (Definition 4.5) whose cohomology is isomorphic to the rational cohomology ring of the complement:

$$H^\bullet(\mathrm{D},\mathrm{d}) \cong H^\bullet(M_{\mathcal{A}};\mathbb{Q})$$

(Theorem 4.9). Since our methods are based on the aforementioned Morgan algebra, this d.g.a. codifies also the rational homotopy type of the complement.

Finally we focus on the case of an arrangement of subtori of codimension 1. Given such a *toric arrangement*, and chosen its maximal *building set*, we find a subalgebra of the Morgan model isomorphic to the cohomology ring. This subalgebra, explicitly presented by generators and relations in Theorem 5.12, yields an analogue of the Orlik–Solomon algebra for toric arrangements. This new presentation depends on the oriented arithmetic matroid only (see [22]) and, compared to the previous result of [5], exhibits more clearly the dependence from the orientation. Furthermore, it seems more suitable to be generalized to arrangement of subtori of arbitrary codimensions. We also conjecture that a similar presentation holds for cohomology with integer coefficients (Conjecture 5.18).

1 | POSITIVE SYSTEMS AND EMBEDDING OF SUBTORI

A d-dimensional *complex torus* T is an algebraic group isomorphic to $(\mathbb{C}^*)^d$. A *character* is a morphism of algebraic groups $T \to \mathbb{C}^*$. The group Λ of all characters is a lattice of rank d, that is, it is isomorphic to $\mathbb{Z}^d$. A *subtorus* of T is a translate of a subgroup isomorphic to $(\mathbb{C}^*)^k$, $0 \leqslant k < d$.

Definition 1.1. An *arrangement of subtori* is a finite collection $\mathcal{A} = \{S_1, \ldots, S_n\}$ of subtori of T such that $S_i \nsubseteq S_j$ for all $i \neq j$.

We denote by $M_{\mathcal{A}}$ the complement of this arrangement, that is,

$$T \setminus (S_1 \cup \cdots \cup S_n).$$

The set of characters that are constant on a subtorus S_i is a subgroup of Λ, that we denote by Λ_{S_i}. Let $\mathbf{B}$ be a basis (over $\mathbb{Z}$) of Λ and, for every $i = 1, \dots, n$, let $\mathbf{B}_i$ be a basis (over $\mathbb{Z}$) of Λ_{S_i}.

Definition 1.2. We say that $(\mathbf{B}, \mathbf{B}_1, \dots, \mathbf{B}_n)$ is a *positive system* if the coordinates of all the elements of every $\mathbf{B}_i$ in the basis $\mathbf{B}$ are non-negative.

The above definition is inspired by similar (and indeed stronger) notions introduced by De Concini and Gaiffi in [7, 8].

Lemma 1.3. *Every arrangement of subtori admits a positive system.*

Proof. For each $S_i \in \mathcal{A}$ choose a basis $\mathbf{B}_i$ of Λ_{S_i} and a basis $\mathbf{B}$ of Λ. Consider the matrix A that represent the elements $b_{i,j} \in \mathbf{B}_i$, for $i = 1, \dots, n$ and $j = 1, \dots, |\mathbf{B}_i|$ in the basis $\mathbf{B}$. Hence, the columns of the matrix are indexed by couples (i, j), with $i = 1, \dots, n$ and $j = 1, \dots, |\mathbf{B}_i|$. By changing $b_{i,j}$ with $-b_{i,j}$ we suppose that the last non-zero entry of the (i, j)th column of A is a positive integer: we call this entry the pivot of the column. We perform a sequence of elementary row operation in order to make A a non-negative matrix. The columns with pivot in the first row are already non-negative. We proceed by induction, suppose that all columns with pivot in the first $k - 1$ rows are non-negative. By adding a suitable multiple of the kth row to the previous rows, we can make all the columns with pivot in the kth row non-negative. This operation does not change the columns with pivot in the first $k - 1$ rows. By repeating the procedure for every $k = 2, \dots, d$, we obtain a non-negative matrix. The elementary row operations correspond to a change of basis from $\mathbf{B}$ to a new basis $\mathbf{B}'$ that form a positive system $(\mathbf{B}', \mathbf{B}_1, \dots, \mathbf{B}_n)$. □

We denote by $\mathbb{S}^d$ the d-dimensional real sphere, and by $\mathcal{B}_d$ the Boolean arrangement, that is, the set of the coordinate hyperplanes in $\mathbb{C}^d$.

Given a topological space X, its *Alexandroff compactification* $\widehat{X}$ is defined as the pointed topological space on the set $X \cup \{\infty\}$ (with base point ∞) whose basis of open sets is given by the open sets of X and the sets $(X \setminus C) \cup \{\infty\}$, where C ranges over all the closed and compact sets of X. For instance, the Alexandroff compactification of $\mathbb{C}^d$ is isomorphic to the sphere $\mathbb{S}^{2d}$. The wedge sum of two pointed topological spaces (X, x), (Y, y) is $X \vee Y$, that is, the disjoint union of X and Y with the base points identified.

Proposition 1.4. *Let $\mathcal{A}$ be a arrangement of subtori in a d-dimensional torus T. Then there exists an embedding $M_{\mathcal{A}} \hookrightarrow \mathbb{S}^{2d}$ such that*

$$\mathbb{S}^{2d} \setminus M_{\mathcal{A}} = \widehat{\cup \mathcal{A}} \vee \widehat{\cup \mathcal{B}_d}.$$

Proof. We choose a positive system $(\mathbf{B}, \mathbf{B}_1, \dots, \mathbf{B}_n)$. The basis $\mathbf{B}$ defines an isomorphism $T \cong (\mathbb{C}^*)^d$, and consider the composition

$$M_{\mathcal{A}} \subset T \cong (\mathbb{C}^*)^d \subset \mathbb{C}^d \subset \widehat{\mathbb{C}^d} \cong \mathbb{S}^{2d}.$$

Note that $\mathbb{C}^d \setminus M_{\mathcal{A}}$ is the disjoint union of $\cup\mathcal{A}$ and $\cup\mathcal{B}_d$ because the system

$$\begin{cases} z_1^{n_1} z_2^{n_2} \cdots z_d^{n_d} = c \\ z_j = 0 \end{cases}$$

for $n_i \in \mathbb{N}_{>0}$, $c \in \mathbb{C}^*$, and $j \leqslant d$, has no solutions. The condition of positive system ensures that each subtorus $S_i \in \mathcal{A}$ is contained in a hypertorus of the form

$$\{\underline{z} \in (\mathbb{C}^*)^d \mid z_1^{n_1} z_2^{n_2} \ldots z_d^{n_d} = c\}$$

for some $c \in \mathbb{C}^*$ and some $n_i \in \mathbb{N}$, not all equal to zero. Now, $\mathbb{S}^{2d} \setminus M_{\mathcal{A}}$ is the Alexandroff compactification of $\mathbb{C}^d \setminus M_{\mathcal{A}}$, hence

$$\mathbb{S}^{2d} \setminus M_{\mathcal{A}} \cong \widehat{\mathbb{C}^d \setminus M_{\mathcal{A}}} \cong \widehat{\cup\mathcal{A} \sqcup \cup\mathcal{B}_d} \cong \widehat{\cup\mathcal{A}} \vee \widehat{\cup\mathcal{B}_d}.$$

□

2 | COHOMOLOGY GROUPS OF ARRANGEMENTS OF SUBTORI

Let $\mathcal{P}$ be a poset. We recall that the *order complex* $\Delta(\mathcal{P})$ is the simplicial complex whose simplices are the totally ordered subsets of $\mathcal{P}$. For any $W, L \in \mathcal{P}$ with $W > L$ we denote $\Delta(L, W)$ the order complex of the sub-poset

$$\{X \in \mathcal{P} \mid W > X > L\}.$$

Definition 2.1. Given two pointed CW-complexes (X, x) and (Y, y), we define:

- the *wedge sum* $X \vee Y$ as $X \sqcup Y / x \sim y$;
- the *smash product* $X \wedge Y$ as the topological quotient $X \times Y / X \vee Y$;
- the *join* $X * Y$ as $X \wedge Y \wedge \mathbb{S}^1$.

Let $\hat{\mathcal{L}}_{>0}$ be the poset obtained from the poset of layers $\mathcal{L}$ by removing the minimum $\hat{0} = T$ and adding a maximum $\hat{1}$. We think of $\hat{\mathcal{L}}_{>0}$ as a category, having one morphism $p \to q$ for every $p, q \in \hat{\mathcal{L}}_{>0}$ such that $p \leqslant q$.

Given a poset $\mathcal{P}$, a $\mathcal{P}$-*diagram* is a functor from the category $\mathcal{P}$ to the category Top_* of pointed topological spaces.

Definition 2.2. We define two $\hat{\mathcal{L}}_{>0}$-diagrams $\mathcal{D}$ and $\mathcal{E}$ as follows.

For every object $W \in \mathcal{L}_{>0}$,

$$\mathcal{D}(W) = \mathcal{E}(W) = \widehat{W} \text{ and } \mathcal{D}(\hat{1}) = \mathcal{E}(\hat{1}) = \{\infty\};$$

for every map $W > L$ $(W \neq \hat{1})$, $\mathcal{D}$ is defined by the natural inclusions:

$$\mathcal{D}(W > L) = \widehat{W} \hookrightarrow \widehat{L} \text{ and } \mathcal{D}(\hat{1} > L) = \{\infty\} \hookrightarrow \widehat{L}$$

while $\mathcal{E}$ by the constant maps at the point ∞:

$$\mathcal{E}(W > L) = \widehat{W} \to \widehat{L} \text{ and } \mathcal{E}(\hat{1} > L) = \{\infty\} \to \widehat{L}.$$

The *colimit* of a $\mathcal{P}$-diagram $\mathcal{F}$ is the union of all topological spaces $\mathcal{F}(p)$ for all $p \in \mathcal{P}$ with the identification given by the maps between them (that is, $x = \mathcal{F}(p \to q)(x)$ for all maps $p \to q$ in $\mathcal{P}$ and all $x \in \mathcal{F}(p)$).

The *homotopy colimit* of $\mathcal{F}$ can be constructed by replacing all the maps $\mathcal{F}(p \to q)$ with homotopy equivalent cofibrations and then taking the colimit of the resulting diagram.

We recall the following results of Welker, Ziegler and Živaljević:

Lemma 2.3 (Projection Lemma [24, Lemma 4.5]). *Let $\mathcal{D}$ be a $\mathcal{P}$-diagram such that all maps are inclusions and closed cofibrations. Then the natural map* $\operatorname{hcolim} \mathcal{D} \to \operatorname{colim} \mathcal{D}$ *from the homotopy colimit to the colimit of $\mathcal{D}$ is a homotopy equivalence.*

Lemma 2.4 (Homotopy Lemma [24, Lemma 4.6]). *Let $\mathcal{D}$ and $\mathcal{E}$ be $\mathcal{P}$-diagrams and $h : \mathcal{D} \to \mathcal{E}$ be a morphism of P-diagrams (that is, a natural transformation between the two functors). Suppose that for all $W \in \mathcal{P}$ the map $h_W : \mathcal{D}(W) \to \mathcal{E}(W)$ is a homotopy equivalence, then the induced map* $\operatorname{hcolim} \mathcal{D} \to \operatorname{hcolim} \mathcal{E}$ *is a homotopy equivalence.*

Lemma 2.5 (Wedge Lemma [24, Lemma 4.9]). *Let $\mathcal{P}$ be a poset with a maximal element and let $\mathcal{E}$ be a $\mathcal{P}$-diagram. Suppose that all maps in $\mathcal{E}$ are constant morphisms of pointed spaces, then*

$$\operatorname{hcolim} \mathcal{E} \simeq \bigvee_{p \in \mathcal{P}} \big(\Delta(\mathcal{P}_{<p}) * \mathcal{E}(p)\big).$$

We now prove the following result on compactifications of subtori.

Lemma 2.6. *Let $\mathcal{A}$ be an arrangement of subtori. For each $W \in \mathcal{L}$ there exists a homotopy equivalence $h_W : \widehat{W} \to \widehat{W}$ such that, for all $L > W$, the following diagram commutes.*

$$\begin{tikzcd} \widehat{L} \arrow[r, hook] \arrow[d] & \widehat{W} \arrow[d, "h_W"] \\ \{\infty\} \arrow[r] & \widehat{W} \end{tikzcd}$$

Proof. Consider a positive system $\mathbf{B}, \mathbf{B}_i$ for the restricted arrangement in W

$$\mathcal{A}^W = \{L \mid L \text{ is a connected component of } S \cap W, S \in \mathcal{A}\},$$

that is, a basis $\mathbf{B}$ of Λ_W and a basis $\mathbf{B}_i$ of Λ_{S_i} for each atom $S_i \in \mathcal{A}^W$. The basis $\mathbf{B}$ gives an isomorphism between W and $(\mathbb{C}^*)^{\dim W}$. Let $\epsilon \in \mathbb{R}^+$ be the minimum of the distance between $0 \in \mathbb{C}^{\dim W}$ and S_i for all atoms $S_i \in \mathcal{A}^W$.

Each layer $L \in \mathcal{L}_{>W}$ of the restricted arrangement $\mathcal{A}^W$ is contained in a hypertorus

$$\{\underline{z} \in \mathbb{C}^{\dim W} \mid z_1^{n_1} \dots z_{\dim W}^{n_{\dim W}} = c\}$$

for some $n_i \in \mathbb{N}$ and some $c \in \mathbb{C}^*$. Hence, ϵ is positive and each layer L is disjoint to the ball $D_\epsilon \subset \mathbb{C}^{\dim W} \subset \mathbb{S}^{2 \dim W}$ of center 0 and radius ϵ.

Choose a continuous, increasing and surjective function $f\colon [0,\epsilon) \to [0,\infty)$ and define $\tilde{h}_W\colon \mathbb{S}^{2\dim W} \to \mathbb{S}^{2\dim W}$ by

$$x \mapsto \begin{cases} f(|x|)x & \text{if } x \in D_\epsilon, \\ \infty & \text{otherwise,} \end{cases}$$

where $|x|$ is the distance of x from 0 and ∞ is the unique point in $\mathbb{S}^{2\dim W} \setminus \mathbb{C}^{2\dim W}$. It easy to see that $\tilde{h}_W$ induces a homotopy equivalence $h_W\colon \widehat{W} \to \widehat{W}$. The commutativity of the diagram above follows from $L \cap D_\epsilon = \emptyset$. □

The previous results now allow us to describe the Alexandroff compactification of the union of the subtori of the arrangement.

Lemma 2.7. *There exists a homotopy equivalence*

$$\widehat{\cup\mathcal{A}} \simeq \bigvee_{w\in\hat{\mathcal{L}}_{>0}} \big(\widehat{W} * \Delta(T,W)\big).$$

Proof. Consider the maps h_W given by Lemma 2.6 and let $h_{\hat{1}}\colon \{\infty\} \to \{\infty\}$ be the only map. These data define a morphism $h\colon \mathcal{D} \to \mathcal{E}$ of $\hat{\mathcal{L}}_{>0}$-diagrams. We have

$$\widehat{\cup\mathcal{A}} \simeq \operatorname{colim}\mathcal{D} \simeq \operatorname{hcolim}\mathcal{D} \simeq \operatorname{hcolim}\mathcal{E} \simeq \bigvee_{w\in\hat{\mathcal{L}}_{>0}} \big(\widehat{W} * \Delta(T,W)\big),$$

where the first isomorphism follow by the definition of colim, the others by the projection Lemma 2.3, the homotopy Lemma 2.4 applied to h, and the wedge Lemma 2.5, respectively. □

Theorem 2.8. *Let $\mathcal{A}$ be an arrangement of subtori of a torus T and $\mathcal{L}$ be its poset of layers. Then the cohomology groups of the complement $M_{\mathcal{A}}$ are*

$$H^k(M_{\mathcal{A}};\mathbb{Z}) \cong \bigoplus_{W\in\mathcal{L}} \bigoplus_{p+q=k} H^p(W;\mathbb{Z}) \otimes_{\mathbb{Z}} \tilde{H}_{2\operatorname{cd} W-2-q}(\Delta(T,W)).$$

Proof. Consider the embedding $M_{\mathcal{A}} \subset S^{2d}$ of such that $S^{2d} \setminus M_{\mathcal{A}} = \widehat{\cup\mathcal{A}} \vee \widehat{B_d}$, provided by Proposition 1.4. We use the Alexander duality (see, for instance, [18, Theorem 3.44]) to obtain

$$\tilde{H}^k(M_{\mathcal{A}}) \cong \tilde{H}_{2d-k-1}(\widehat{\cup\mathcal{A}} \vee \widehat{B_d}) \cong \tilde{H}_{2d-k-1}(\widehat{\cup\mathcal{A}}) \oplus \tilde{H}_{2d-k-1}(\widehat{B_d}).$$

Again Alexander duality for the embedding $\widehat{B_d} \subset S^{2d}$ gives the isomorphism $\tilde{H}_{2d-k-1}(\widehat{B_d}) \cong \tilde{H}_k(T)$. Now, Lemma 2.7 implies

$$\begin{aligned}\tilde{H}_{2d-k-1}(\widehat{\cup\mathcal{A}}) &\cong \tilde{H}_{2d-k-1}\left(\bigvee_{w\in\hat{\mathcal{L}}_{>0}} \big(\widehat{W} * \Delta(T,W)\big)\right) \\ &\cong \bigoplus_{w\in\hat{\mathcal{L}}_{>0}} \tilde{H}_{2d-k-1}\big(\widehat{W} * \Delta(T,W)\big)\end{aligned}$$

$$\cong \bigoplus_{w\in\hat{\mathcal{L}}_{>0}} \tilde{H}_{2d-k-2}\big(\widehat{W}\wedge\Delta(T,W)\big)$$

$$\cong \bigoplus_{w\in\hat{\mathcal{L}}_{>0}} \bigoplus_{p+q=k} \tilde{H}_{2\dim W-p}(\widehat{W})\otimes_{\mathbb{Z}} \tilde{H}_{2\operatorname{cd} W-q-2}(\Delta(T,W))),$$

where the last isomorphism is the Kunneth isomorphism for reduced cohomology applied to the smash product. We conclude the proof by the Poincarè duality on W between Borel–Moore homology and cohomology (see [4, Theorem 9.2]):

$$\tilde{H}_{2\dim W-p}(\widehat{W}) = H^{BM}_{2\dim W-p}(W)\cong H^p(W). \qquad \square$$

3 | GRADED OF THE COHOMOLOGY RING

In this section, we study the Leray spectral sequence for the inclusion map $j : M_{\mathcal{A}} \to T$ to give a description of the graded cohomology ring $\mathrm{gr}^{\mathrm{L}}_{\bullet} H^{\bullet}(M_{\mathcal{A}};\mathbb{Z})$. We refer to [4, section 6] as a general reference on this spectral sequence.

Let $R^q j_*$ be the higher direct image functor of j. In our case, the Leray spectral sequence

$$E_r^{p,q} \Rightarrow H^{p+q}(M_{\mathcal{A}};\mathbb{Z})$$

converges to $H^{p+q}(M_{\mathcal{A}};\mathbb{Z})$; the second page of this spectral sequence is

$$E_2^{p,q} = H^p(R^q j_*\mathbb{Z}_{M_{\mathcal{A}}}),$$

where $\mathbb{Z}_{M_{\mathcal{A}}}$ is the constant sheaf and the Leray filtration $\mathrm{L}^{\bullet}$ on $H^k(M_{\mathcal{A}};\mathbb{Z})$ is defined by

$$\mathrm{L}^q = \operatorname{Im}\big(H^k(T;\tau_{k-q}\mathbb{R}j_*\mathbb{Z}) \to H^k(T;\mathbb{R}j_*\mathbb{Z})\cong H^k(M_{\mathcal{A}};\mathbb{Z})\big),$$

where $\tau_{k-q}\mathbb{R}j_*\mathbb{Z}$ is the truncation at position $k-q$ of the complex $\mathbb{R}j_*\mathbb{Z}$.

For each $W\in\mathcal{L}$, let ϵ^q_W be the sheaf on T defined by

$$\epsilon^q_W = (i_W)_*\mathbb{Z}_W \otimes_Z H_{2\operatorname{cd} W-q-2}(\Delta(T,W))$$

where i_W is the closed embedding of W in T. We set

$$\epsilon^q = \bigoplus_{W\in\mathcal{L}} \epsilon^q_W.$$

The following lemma generalizes [1, Lemma 3.1].

Lemma 3.1. *Let $\mathcal{A}$ be an arrangement of subtori. Then*

$$E_2^{p,q} = H^p(R^q j_*\mathbb{Z}_{M_{\mathcal{A}}}) \cong \bigoplus_{W\in\mathcal{L}} H^p(W;\mathbb{Z})\otimes \tilde{H}_{2\operatorname{cd} W-q-2}(\Delta(T,W);\mathbb{Z}).$$

Proof. First we prove that $\epsilon^q \cong R^q j_*\mathbb{Z}_{M_{\mathcal{A}}}$: for each point $x\in T$ there exists an open set U_x isomorphic to an open subset V_x of the tangent space $\mathrm{T}_x T$ (containing the origin). We also take a neighborhood basis $\mathcal{U}$ given for every $x \in T$ by the inverse image of all open balls in V_x centered in $0 \in \mathrm{T}_x T$. Note that the arrangement of subtori $\mathcal{A}$ defines a central arrangement of subspaces

$$\mathcal{A}[x] = \{\mathrm{T}_x S \mid x \in S \in \mathcal{A}\}$$

in $\mathrm{T}_x T$ for all $x \in T$.

We define a morphism of sheaves $f : \epsilon^q \to R^q j_* \mathbb{Z}_{M_{\mathcal{A}}}$ on the neighborhood basis $\mathcal{U}$ as follow. For all $U \in \mathcal{U}$ centered in x, let $f(U) : \epsilon(U) \to R^q j_* \mathbb{Z}(U)$ be the composition

$$\epsilon(U) = \bigoplus_{W \ni x} \mathbb{Z} \otimes_{\mathbb{Z}} \tilde{H}_{2\,\mathrm{cd}\,W-q-2}(\Delta(T,W)) \cong H^q(M_{\mathcal{A}[x]}) \cong H^q(U \cap M_{\mathcal{A}}) = R^q j_* \mathbb{Z}(U),$$

where the first isomorphism is given by Theorem A and the second one is given by the composition

$$M_{\mathcal{A}[x]} \simeq V_x \setminus \cup\mathcal{A}[x] \cong U \setminus \cup\mathcal{A}.$$

Since $f(U)$ is an isomorphism for all $U \in \mathcal{U}$ then f is an isomorphism of sheaves. Now, the isomorphism

$$H^p(\epsilon^q) \cong \bigoplus_{W \in \mathcal{L}} H^p(W;\mathbb{Z}) \otimes \tilde{H}_{2\,\mathrm{cd}\,W-q-2}(\Delta(T,W);\mathbb{Z})$$

completes the proof. □

The minimum of the poset $\mathcal{L}$ (and of $\mathcal{L}_{\leqslant W}$ for all $W \in \mathcal{L}$) is $\hat{0} = T$. Let $E_W^{p,q} \subset E_2^{p,q}$ be the $\mathbb{Z}$−module $H^p(W;\mathbb{Z}) \otimes \tilde{H}_{2\,\mathrm{cd}\,W-q-2}(\Delta(T,W);\mathbb{Z})$: this module depends only on the cohomology of W and on the poset $\mathcal{L}_{\leqslant W}$.

The multiplication in E_2 is induced by the maps

$$\eta^L_{W,W'} : E_W^{p,q} \otimes E_{W'}^{p',q'} \to E_L^{p+p',q+q'},$$

where $\eta^L_{W,W'} = 0$ if $\mathrm{cd}\,L \neq \mathrm{cd}\,W + \mathrm{cd}\,W'$ or if L is not a connected component of $W \cap W'$, otherwise

$$\eta^L_{W,W'}(a \otimes b \otimes a' \otimes b') = (-1)^{p'q}(a \smile a') \otimes (b \bullet b'),$$

where $\bullet : \tilde{H}_k(\Delta(\hat{0},W)) \otimes \tilde{H}_{k'}(\Delta(\hat{0},W')) \to \tilde{H}_{k+k'+2}(\Delta(\hat{0},L))$ is the map of [26, Theorem 6.6(ii)]. Under the isomorphism of Theorem A the map $\bullet$ corresponds to the cup product in the cohomology of the subspace arrangement $\mathcal{A}[x]$ (for any $x \in L$).

Theorem 3.2. *The Leray spectral sequence $E_r^{p,q}$ for the inclusion $M_{\mathcal{A}} \hookrightarrow T$ degenerates at the second page, that is,*

$$E_2^{p,q} \cong \mathrm{gr}^{\mathrm{L}}_{p+2q} H^{p+q}(M_{\mathcal{A}};\mathbb{Z}).$$

Proof. We know that $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$ and that the last page is $E_\infty^{p,q} \cong \mathrm{gr}^{\mathrm{L}}_{p+2q} H^{p+q}(M_{\mathcal{A}};\mathbb{Z})$. By Theorem 2.8 and Lemma 3.1, $E_\infty^{p,q}$ and $E_2^{p,q}$ are isomorphic and finitely generated; hence $E_2^{p,q} = E_\infty^{p,q}$. □

4 A MODEL FOR THE COMPLEMENT

As in the previous sections, we denote by $\mathcal{A}$ an arrangement of subtori in T and by Λ the character group of T. For any layer $W \in \mathcal{L}$, Λ_W is the set of characters that are constant on W. Let Λ^* be the dual lattice of Λ. We refer to [6] for a general introduction to fans and toric varieties. Let Δ be a smooth and complete fan in Λ^*. Every ray of Δ is generated by a (uniquely determined) primitive vector in Λ^*: we denote by $\mathcal{R}_\Delta \subset \Lambda^*$ the set of primitive vectors corresponding to the rays of Δ. Let $\mathcal{P}(\mathcal{R}_\Delta)$ be power set of $\mathcal{R}_\Delta$; we denote by $\mathcal{C}_\Delta \subseteq \mathcal{P}(\mathcal{R}_\Delta)$ the collection of the sets of primitive vectors that span a cone in Δ. Thus, from now on we identify a cone in Δ with the set of its extremal primitive vectors.

Definition 4.1. A fan Δ in Λ^* describes a *good toric variety* X_Δ (with respect to $\mathcal{A}$) if Δ is complete and smooth and each maximal cone $C \in \mathcal{C}_\Delta$ can be completed to a positive system $(\mathbf{C}, \mathbf{C}_1, \dots, \mathbf{C}_n)$ (where $\mathbf{C}$ is the dual basis of C and $\mathbf{C}_i$ a basis of Λ_{S_i} for each $S_i \in \mathcal{A}$).

The second condition in the above definition can be reformulated as follows: for each $W \in \mathcal{A}$, there exists a basis $\beta_1, \dots, \beta_{\operatorname{cd} W}$ of Λ_W such that for each maximal cone $C \in \mathcal{C}_\Delta$ and each $i = 1, \dots, \operatorname{cd} W$ the natural pairing $\langle \beta_i, c\rangle$ is non-negative (or non-positive) for all $c \in C$. In this case, we say that the basis $\beta_1, \dots, \beta_{\operatorname{cd} W}$ of Λ_W has the *equal sign property* with respect to Δ (see [7, Definition 3.2]).

Let $\mathcal{G} \subseteq \mathcal{L}_{>0}$ be a *well-connected building set* in the sense of [8, Definition 4.1] and Δ a good toric variety. These data define a smooth projective variety $Y(\Delta, \mathcal{G})$ obtained from X_Δ by subsequently blowing up (the strict transform of) W for all $W \in \mathcal{G}$ in any total order refining the partial order given by inclusion (so that smaller layers are blown up first). The variety $Y(\Delta, \mathcal{G})$ is the *wonderful model* for $M_\mathcal{A}$ described in [8], that is, a smooth projective variety containing $M_\mathcal{A}$ such that the complement $Y(\Delta, \mathcal{G}) \setminus M_\mathcal{A}$ is a simple normal crossing divisor. The irreducible components of the divisor $Y(\Delta, \mathcal{G}) \setminus M_\mathcal{A}$ are indexed by $\mathcal{G} \sqcup \mathcal{R}_\Delta$, indeed these components are:

- the exceptional divisor D_W associated to the blowup along W for each $W \in \mathcal{G}$,
- the strict transform D_j of the torus equivariant divisor for each ray $j \in \mathcal{R}_\Delta$.

We want to describe the Morgan algebra (cf. [20]) for the pair $(Y(\Delta, \mathcal{G}), M_\mathcal{A})$. For the convenience of the reader, we will briefly recall here the definition of this algebra. Consider a smooth complete algebraic variety Y and a simple normal crossing divisor $D = \bigcup_{i\in I} D_i$ with complement M. The *Morgan algebra* (see [20]) is the vector space

$$\bigoplus_{A\subseteq I} H^\bullet\left(\bigcap_{i\in A} D_i\right)$$

with $H^p(\bigcap_{i\in A} D_i)$ of bi-degree $(p, |A|)$. The multiplication is given by

$$H^p\left(\bigcap_{i\in A} D_i\right) \otimes H^{p'}\left(\bigcap_{i\in A'} D_i\right) \to H^{p+p'}\left(\bigcap_{i\in A\sqcup A'} D_i\right),$$

that is, the composition of the restriction maps and the cup product. The differential is induced by the Gysin morphisms $H^p(\bigcap_{i\in A} D_i) \to H^{p+2}(\bigcap_{i\in B} D_i)$ for every $B = A \setminus \{a\}$.

Let E be the graded commutative algebra over $\mathbb{Q}$ on generators s_W, t_W, b_j, c_j for $W \in \mathcal{G}$ and $j \in \mathcal{R}_\Delta$, where the bi-degree of s_W and b_j is (0,1) and the bi-degree of t_W and c_j is (2,0). Consider the differential d on E defined on generators by $\mathrm{d}(s_W) = t_W$, by $\mathrm{d}(b_j) = c_j$, and by $\mathrm{d}(t_W) = \mathrm{d}(c_j) = 0$ for all $W \in \mathcal{G}$ and $j \in \mathcal{R}_\Delta$, so that (E, d) is a differential graded commutative algebra.

To understand what relations should put on E, we start by recalling the definition of some polynomials $P_W^V(t)$, which were introduced in [8, section 8] as *good lifting* of the Chern polynomials[†].

For each pair $V \leqslant W$ in $\mathcal{L}$, and for each tuple $\beta_1, \dots, \beta_{\mathrm{cd}\, V} \in \Lambda_V$ with the equal sign property with respect to Δ such that $\beta_{\mathrm{cd}\, W - \mathrm{cd}\, V + 1}, \dots, \beta_{\mathrm{cd}\, V}$ form an integral basis of Λ_W, define

$$P_W^V(t) = \prod_{i=1}^{\mathrm{cd}\, W - \mathrm{cd}\, V} \left(t - \sum_{j \in \mathcal{R}_\Delta} \min(0, \langle \beta_i, j \rangle) c_j \right). \tag{1}$$

The polynomial $P_W^V(t)$ depends on the choice of $\beta_1, \dots, \beta_{\mathrm{cd}\, W - \mathrm{cd}\, V}$.

Definition 4.2. A set $A \subset \mathcal{G}$ is *nested* if the irreducible components of the normal crossing divisor of $Y(\Delta, \mathcal{G})$ that correspond to the elements of A have non-empty intersection. When we want to emphasize the dependence on $\mathcal{G}$ we will say that A is *$\mathcal{G}$-nested*.

The property of being nested does not depend on the choice of Δ, and can be expressed in a purely combinatorial way (see [8, Definition 2.7]). For each $A \sqcup B \subseteq \mathcal{G} \sqcup \mathcal{R}_\Delta$ we denote with $Y_{A \sqcup B}$ the intersections of all divisors associated to A and B. We recall the following result of De Concini and Gaiffi.

Theorem 4.3. *[8, Theorem 9.1] Let $A \subseteq \mathcal{G}$ and $B \subseteq \mathcal{R}_\Delta$. The intersection $Y_{A \sqcup B}$ is non-empty if and only if A is $\mathcal{G}$-nested, B is contained in $\bigcap_{W \in A} \operatorname{Ann} \Lambda_W$, and B is a cone in Δ. In this case the cohomology ring $H^\bullet(Y_{A \sqcup B})$ is the algebra on generators $\{t_W\}_{W \in \mathcal{G}}$ and $\{c_j\}_{j \in \mathcal{R}_\Delta}$ of degree two with relations:*

- (T1) $\prod_{j \in C} c_j$ *if* $C \notin C_\Delta$,
- (T2) $\sum_{j \in \mathcal{R}_\Delta} \langle \beta, j \rangle c_j$ *for every* $\beta \in \Lambda$ *(or equivalently for* β *in a fixed basis of* Λ*),*
- (W1) $\prod_{j \in C} c_j$ *if* $C \cup B$ *is not a cone in* Δ *or* $C \not\subset \bigcap_{W \in A} \operatorname{Ann} \Lambda_W$, [†]
- (W2) $t_W c_j$ *if* $j \notin \operatorname{Ann} \Lambda_W$,
- (W3a) *for all* $W \in \mathcal{G}$ *and all* $C \subseteq \mathcal{G}_{<W}$*, the relations*

$$P_W^V \left(\sum_{L \in \mathcal{G}_{\geqslant W}} -t_L \right) \prod_{L \in C} t_L,$$

where V is the connected component of $\bigcap_{L \in A_{<W} \cup C} L$ containing W,

- (W3b) $\prod_{W \in C} t_W$ *if* $C \cup A$ *is not* $\mathcal{G}$*-nested or* $B \not\subset \bigcap_{W \in C} \operatorname{Ann} \Lambda_W$.

[†] The authors of [8] forgot to specify that, in order to define a good lifting, the basis of Λ_W must have the equal sign property with respect to Δ.

[†] In [8], these relations are stated only for $|C| = 1$; however they hold, before performing blow-ups, for any set C, by the well-known theory of toric varieties. Thus, by adding the relations with $|C| > 1$ to the presentation given in [8], we get a correct presentation.

Although the polynomials $P^V_W(t)$ in (W3a) depend on the choice of a basis, the ideal generated by all the relations is independent from this choice, as shown in [8, Proposition 6.3].

Remark 4.4. Another possible choice of $P^V_W(t)$ consist of taking the polynomials:

$$t^{\operatorname{cd} W - \operatorname{cd} V} + \prod_{i=1}^{\operatorname{cd} W - \operatorname{cd} V} \sum_{j \in \mathcal{R}_\Delta} - \min(0, \langle \beta_i, j \rangle) c_j. \tag{2}$$

Indeed, the difference between the polynomials (1) and (2) evaluated at $t = \sum_{L \in \mathcal{G}_{\geqslant W}} -t_L$ is in the ideal generated by the relations of type (W2).

To simplify the notations, in the definition above we denoted by $\prod_{a \in A}$ the exterior product taken in the order of $A \subseteq \{1, \dots, n\}$. The same notation will be used from now on. Let A be a nested set, W be any element in $\mathcal{G}$, and $B \subseteq \mathcal{G}$ be such that each $L \in B$ is smaller than W ($L \lneq W$ in $\mathcal{L}$). We define the element $F(A, W, B)$ in E by

$$F(A, W, B) = P^V_W\left(\sum_{L \in \mathcal{G}_{\geqslant W}} -t_L\right) \prod_{L \in A} s_L \prod_{L \in B} t_L,$$

where V is the connected component of $\bigcap_{L \in A_{<W} \cup B} L$ containing W (so $V \leqslant W$).

Definition 4.5. Let (D, d) be the differential graded algebra given by E with relations:

(1) $x_W y_j$ if $j \notin \operatorname{Ann} \Lambda_W$, where $x_W = s_W$ or t_W and $y_j = b_j$ or c_j,
(2) $\prod_{W \in A} s_W \prod_{W \in B} t_W$ if $A \cup B$ is not a $\mathcal{G}$-nested set,
(3) $\prod_{j \in A} b_j \prod_{j \in B} c_j$ if $A \cup B$ is not a cone in Δ (that is, $A \cup B \in \mathcal{P}(\mathcal{R}_\Delta) \setminus \mathcal{C}_\Delta$),
(4) $\sum_{j \in \mathcal{R}_\Delta} \langle \chi, j \rangle c_j$ for every $\chi \in \Lambda$ (or equivalently for χ in a fixed basis of Λ),
(5) $F(A, W, B)$ for A $\mathcal{G}$-nested set, $W \in \mathcal{G}$, and $B \subseteq \mathcal{G}$ be such that each $L \in B$ is smaller than W (that is, $B \subseteq \mathcal{G}_{<W}$),

and differential d induced by the one of E, that is, defined on generators by $\mathrm{d}(s_W) = t_W$, $\mathrm{d}(b_j) = c_j$, and by $\mathrm{d}(t_W) = \mathrm{d}(c_j) = 0$.

Lemma 4.6. *The ideal generated by (1)–(5) is stable with respect to* d, *so* (D, d) *is a differential graded algebra.*

Proof. It is obvious that the ideal generated by (1)–(4) is d-stable. The relation

$$\mathrm{d}(F(A, W, B)) = \sum_{L \in A_{<W}} \pm F(A \setminus \{L\}, W, B \cup \{L\}) + \sum_{L \in A_{\not< W}} \pm t_L F(A \setminus \{L\}, W, B)$$

show that the ideal generated by (5) is d-stable. □

Let M be the Morgan algebra associated to the pair $(Y(\Delta, \mathcal{G}), M_{\mathcal{A}})$. The complement $Y(\Delta, \mathcal{G}) \setminus M_{\mathcal{A}}$ is a simple normal crossing divisor $\bigcup_{W \in \mathcal{G}} D_W \cup \bigcup_{j \in \mathcal{R}_\Delta} D_j$, whose irreducible components are indexed by $\mathcal{G} \sqcup \mathcal{R}_\Delta$.

For each $A \subseteq \mathcal{G} \sqcup \mathcal{R}_\Delta$ we denote with Y_A the intersections of all divisors associated to A. The graded differential algebra M is the direct sum of vector spaces

$$\bigoplus_{A \subset \mathcal{G} \sqcup \mathcal{R}_\Delta} H^\bullet(Y_A, \mathbb{Q}),$$

on which:

- the total degree of the elements in $H^p(Y_A)$ is $|A| + p$;
- the multiplication is induced by the restriction maps and the cup product

$$H^p(Y_A) \otimes H^{p'}(Y_B) \to H^{p+p'}(Y_{A \cup B});$$

- the differential is defined from the Gysin map $H^p(Y_A) \to H^{p+2}(Y_{A \setminus \{a\}})$.

The cohomology of each stratum Y_A is computed in [8, Theorem 9.1] in terms of some generators $t_W, s_j \in H^2(Y_A)$.

We define a morphism $\tilde{f} : E \to \mathrm{M}$ on generators by

$$s_W \mapsto 1 \in H^0(D_W), \qquad t_W \mapsto t_W \in H^2(Y(\Delta, \mathcal{G})),$$
$$b_j \mapsto 1 \in H^0(D_j), \qquad c_j \mapsto c_j \in H^2(Y(\Delta, \mathcal{G})).$$

Lemma 4.7. *The map $\tilde{f}$ is a surjective morphism of differential graded algebras.*

Proof. As shown in [8, Theorem 9.1], the restriction maps $H^\bullet(Y_A) \to H^\bullet(Y_B)$ for $A \subset B$ are surjective. Since $\operatorname{Im} \tilde{f}$ contains $H^\bullet(Y(\Delta, \mathcal{G}))$ and the elements $1 \in H^0(D)$ for all divisors D, the morphisms $\tilde{f}$ is surjective. By construction of the cohomology algebra, the elements t_W and c_j of $H^2(Y(\Delta, \mathcal{G}))$ are $t_W = (i_W)_*(1)$ and $c_j = (i_j)_*(1)$, where i_* is the Gysin morphism for the regular embedding $i : D \hookrightarrow Y(\Delta, \mathcal{G})$. Therefore, $\tilde{f}$ is a morphism of differential graded algebras. □

The map $\tilde{f}$ factors through $f : \mathrm{D} \to \mathrm{M}$, indeed we have the following lemma.

Lemma 4.8. *The map f is well-defined and is an isomorphism.*

Proof. We first check that (1)–(5) belong to $\ker \tilde{f}$.

(1) There are four cases to check:
   - $\tilde{f}(s_W b_j)$ is zero since D_W and D_j do not intersect for $j \notin \operatorname{Ann} \Lambda_W$;
   - $\tilde{f}(s_W c_j)$ is zero since $c_j = 0$ in $H^\bullet(D_W)$ by (W1);
   - $\tilde{f}(t_W b_j) = t_W \in H^\bullet(D_j)$ is zero by (W3b);
   - $t_W c_j = 0$ in $H^\bullet(Y(\Delta, \mathcal{G}))$ by (W2).

(2) We have $\tilde{f}(\prod_{W \in A} s_W \prod_{W \in B} t_W) = \prod_{W \in B} t_W = 0$ in $H^\bullet(Y_A)$ by (W3b) since $A \cup B$ is not $\mathcal{G}$-nested set.

(3) The element $\tilde{f}(\prod_{j \in A} b_j \prod_{j \in B} c_j) = \prod_{j \in B} c_j$ is zero in $H^\bullet(Y_A)$ by (W1).

(4) The vanishing of the linear relation follows from (T2).

(5) We have

$$\tilde{f}(F(A,W,B)) = P_W^V\left(\sum_{L\in\mathcal{G}_{\geqslant W}} -t_L\right)\prod_{L\in B} t_L$$

that is zero by (W3a).

We have proven that f is well-defined and surjective, since $\tilde{f}$ is. Let I be the ideal in E generated by (1)–(5) and note that I is a monomial ideal in the variable s_W and b_j for $W \in \mathcal{G}$ and $j \in \mathcal{R}_\Delta$. It is enough to prove that

$$f\left(\prod_{W\in A} s_W \prod_{j\in B} b_j z\right) = 0 \text{ implies } \prod_{W\in A} s_W \prod_{j\in B} b_j z = 0$$

in D for all subsets $A \subseteq \mathcal{G}$, $B \subseteq \mathcal{R}_\Delta$ and all polynomials z in the variables $\{t_W\}_{W\in\mathcal{G}}$ and $\{c_j\}_{j\in\mathcal{R}_\Delta}$.

The monomials $\prod_{W\in A} s_W \prod_{j\in B} b_j$ with A a non-nested set belong to I by (2), the ones with B not a cone belong to I by (3), and the ones with $B \not\subset \bigcap_{W\in A} \operatorname{Ann}\Lambda_W$ are in I by (1).

Now, let A be a $\mathcal{G}$-nested set and $B \in \mathcal{C}_\Delta$ be a cone contained in $\bigcap_{W\in A} \operatorname{Ann}\Lambda_W$. We define a map $H^\bullet(Y_{A\sqcup B}) \to \mathrm{D}$ by using the presentation of Theorem 4.3: the morphism is defined by $z \mapsto \prod_{W\in A} s_W \prod_{j\in B} b_j z$ for all z in the exterior algebra on generators $\{t_W\}_{W\in\mathcal{G}}$ and $\{c_j\}_{j\in\mathcal{R}_\Delta}$. It is well-defined:

(T1) holds by relation (3),
(T2) holds by relation (4),
(W1) holds by relations (1) and (3),
(W3a) holds by relation (5),
(W3b) holds by relations (2) and (1).

The composition

$$H^\bullet(Y_{A\sqcup B}) \to \mathrm{D} \to \mathrm{M} \twoheadrightarrow H^\bullet(Y_{A\sqcup B})$$

is the identity, therefore if $f(\prod_{W\in A} s_W \prod_{j\in B} b_j z) = 0$ then $z = 0$ in $H^\bullet(Y_{A\cup B})$ and $\prod_{W\in A} s_W \prod_{j\in B} b_j z = 0$ in D. □

Lemma 4.8 together with the main result of [20] imply the following result.

Theorem 4.9. *The differential graded algebra* (D, d) *built in Definition 4.5 is a model for the complement* $M_\mathcal{A}$. *Therefore,* $H^\bullet(\mathrm{D}, \mathrm{d}) \cong H^\bullet(M_\mathcal{A}; \mathbb{Q})$. □

Remark 4.10. Since our methods are based on the Morgan algebra, the d.g.a. (D, d) also codifies the rational homotopy type of the complement, which was studied also in [9]. Unlike the d.g.a. introduced therein, our d.g.a. is finite-dimensional.

5 | DIVISORIAL CASE

In this section, we consider arrangements of subtori of codimension 1, usually known in the literature as *toric arrangements*. Given such an arrangement $\mathcal{A} = \{S_1, \ldots, S_n\}$ we consider the toric wonderful model $Y(\Delta, \mathcal{G})$ where $\mathcal{G} = \mathcal{L}_{>\hat{0}}$ is the maximal building set. In this case, the $\mathcal{G}$-nested subsets coincide with the chains in $\mathcal{L}_{>\hat{0}}$.

Inspired by Yuzvinsky [25, 26], we introduce a different set of generators σ_W, τ_W for the differential graded algebra (d.g.a. for short) D and we determine the relations between them (Lemma 5.2). By using these generators we define some elements $\Xi_{W,A}$ of D (Definition 5.3) that belongs to the kernel of d (Lemma 5.4). We study the multiplication between them in Lemma 5.8 and their relation with the cohomology of the ambient torus (Lemma 5.9). The linear relation between $\Xi_{W,A}$ are rather complicated to prove (Lemmas 5.5, 5.10 and 5.11 and Corollary 5.6). In the main result of this section, Theorem 5.12, we introduce a Orlik–Solomon type algebra R and we prove that the composition

$$R \to H(\mathrm{D}, \mathrm{d}) \cong H(M(\mathcal{A}); \mathbb{Q})$$

is an isomorphism. The map $R \to H(\mathrm{D}, \mathrm{d})$ is well-defined by all the lemmas preceding the main theorem, is injective by Lemmas 5.1, 5.14 and 5.15, and surjective by dimensional argument (Lemma 5.14).

Although the d.g.a. D depends on the choice of a good fan Δ, the algebra R and its isomorphic image in D are independent from the choice of the fan.

In this section, we will use basic notions of matroid theory: a set of subtori $I \subseteq \mathcal{A}$ is an *independent set* if the codimension of the intersection $\cap_{S \in I} S$ is equal to the cardinality of I, and a *dependent set* otherwise. A *circuit* is a minimal dependent set. We fix a order of the subtori, that is, a total order $\prec$ on $\mathcal{A}$. We recall that a *broken circuit* is a circuit with the maximum (with respect to the fixed order $\prec$) removed, and that a *no broken circuit* is a set that does not contain any broken circuits.

Define the elements $\sigma_W = \sum_{L \geqslant W} s_L$ and $\tau_W = \sum_{L \geqslant W} t_L$ in D for all $W \in \mathcal{L}_{\geqslant \hat{0}}$. Moreover, for every $\chi \in \Lambda$ define

$$\beta_\chi^- = -\sum_{j \in \mathcal{R}_\Delta} \min(0, \langle \chi, j \rangle) b_j, \qquad \beta_\chi^+ = \sum_{j \in \mathcal{R}_\Delta} \max(0, \langle \chi, j \rangle) b_j$$

and

$$\beta_\chi = \beta_\chi^+ - \beta_\chi^-, \qquad \gamma_\chi^- = -\sum_{j \in \mathcal{R}_\Delta} \min(0, \langle \chi, j \rangle) c_j.$$

As in the previous section, we consider the bi-gradation of D given by $\deg(s_W) = \deg(b_j) = (0, 1)$ and $\deg(t_W) = \deg(c_j) = (2, 0)$, so that the differential d has bi-degree $(2, -1)$.

Lemma 5.1. *The set* $\{\prod_{W \in A} s_W \prod_{j \in C} b_j\}_{A,C}$, *where* A *runs over all the* $\mathcal{G}$*-nested sets and* $C \in C_\Delta$ *over all the cones contained in* $\cap_{W \in A} \operatorname{Ann} \Lambda_W$, *is a linear basis of* $\mathrm{D}^{0,\bullet}$.

Moreover, the set $\{\prod_{W \in A} \sigma_W \prod_{j \in C} b_j\}_{A,C}$ *(where* A *and* C *runs over the range described above) is a linear basis of* $\mathrm{D}^{0,\bullet}$.

Proof. Note that $\mathrm{D}^{0,\bullet}$ is the exterior algebra on generators s_W and b_j with relations:

(1') $s_W b_j$ if $j \notin \operatorname{Ann} \Lambda_W$,
(2') $\prod_{W \in A} s_W$ if A is not $\mathcal{G}$-nested,
(3') $\prod_{j \in C} b_j$ if A is not a cone in Δ.

These relations generate a monomial ideal and so the monomials not divisible by the relations form an basis $\mathcal{B}$ of the vector space $D^{0,\bullet}$. Thus, it can be easily seen that the set $\{\prod_{W\in A} s_W \prod_{j\in C} b_j\}_{A,C}$ is the basis $\mathcal{B}$.

For the second basis, we choose a total order on the set $\mathcal{G}$ that refines the partial order on it. This total order induces a lexicographical order on the set of $\mathcal{G}$-nested sets. The matrix that represents the elements $\{\prod_{W\in A} \sigma_W \prod_{j\in C} b_j\}_{A,C}$ in the basis $\{\prod_{W\in A} s_W \prod_{j\in C} b_j\}_{A,C}$ is upper triangular with ones on the diagonal entries. This proves the claim. □

For a subset A of the poset of layers $\mathcal{L}$, we denote by $\sup(A)$ the set of all minimal upper bounds of N, that is, $\sup(A)$ is the set of connected components of the intersection $\bigcap_{W\in A} W$. To simplify notations, we write $\sup(A_1,\dots,\mathcal{A}_n)$ for $\sup(\{A_1,\dots,\mathcal{A}_n\})$.

Lemma 5.2. *In* D, *we have the following relations:*

(1) $\sigma_W\sigma_L = (\sigma_W-\sigma_L)(\sum_{V\in\sup(W,L)}\sigma_V)$ *for all* $W,L\in\mathcal{L}$,
(2) *if* $\chi\in\Lambda_V$ *then* $x_\chi y_V = 0$ *where* $x=\beta,\beta^-,\beta^+$ *or* γ^- *and* $y=\sigma$ *or* τ,
(3) *if* $W \gtrdot V$ *and* $\chi\in\Lambda_W$ *is an element that generates* Λ_W/Λ_V, *then:*

$$\sigma_V(-\tau_W+\gamma_\chi^-) = -\sigma_W\tau_W, \qquad \tau_V(-\tau_W+\gamma_\chi^-) = -\tau_W^2.$$

Proof.

(1) Let $x_1=\sum_{\substack{V\geqslant W\\ V\ngeqslant L}} s_V$, $x_2=\sum_{\substack{V\geqslant L\\ V\ngeqslant W}} s_V$ and $x_3=\sum_{\substack{V\geqslant W\\ V\geqslant L}} s_V$. The claimed equality can be rewritten as $(x_1+x_3)(x_2+x_3)=(x_1-x_2)x_3$. Since x_3 has degree one we have $x_3^2=0$ and we need to prove that $x_1x_2=0$. This follows from $x_1x_2=\sum s_Vs_U$ where the sum runs over all $V\geqslant W$, $V\ngeqslant L$ and $U\ngeqslant W$, $U\geqslant L$: we have $s_Vs_U=0$ because V and U do not form a chain.
(2) Note that for $\chi\in\Lambda_W$ we have $\min(0,\langle\chi,j\rangle)a_jr_W=0$ for $a=b$ or $a=c$ and $r=s$ or $r=t$, by relation (1) of Definition 4.5. Since $W\geqslant V$ implies $\Lambda_W\supseteq\Lambda_V$, we have

$$x_\chi y_V = -\sum_{\substack{W\geqslant V\\ j\in\mathcal{R}_\Delta}} \min(0,\langle\chi,j\rangle)a_jr_W = 0,$$

that proves the statement for $x=\beta^-,\gamma^-$. Analogously, the relations $\max(0,\langle\chi,j\rangle)a_jr_W=0$ imply the statement for $x=\beta^+$. Finally, we have $\beta_\chi y_V=\beta_\chi^+y_V-\beta_\chi^-y_V=0$.
(3) If $L\geqslant W$, we have that $s_L(-\tau_W+\gamma_\chi^-)=-s_L\tau_W$ since $\chi\in\Lambda_L$. On the other hand, if $L\geqslant V$, $L\ngeqslant W$, we will show that $s_L(\tau_W+\gamma_\chi^-)=0$. Indeed, we have

$$s_L(-\tau_W+\gamma_\chi^-) = -\left(\sum_{U\in\sup(L,W)} s_L\tau_U\right)+s_L\gamma_\chi^-.$$

Since $\operatorname{cd}V=\operatorname{cd}W+1$ and $U>L$, then $\operatorname{cd}L=\operatorname{cd}U+1$. Let $\eta\in\Lambda_U$ be an element that generates Λ_U/Λ_L. By [15, Lemma 3.4], we have $|L\cap W|=|\Lambda_U/(\Lambda_L+\Lambda_W)|$ and we set $a=|L\cap W|$. Moreover, $\Lambda_L+\Lambda_W=\Lambda_L+\mathbb{Z}\chi$, so there exists $\eta'\in\Lambda_L$ such that $a\eta=\eta'+\chi$. Observe that

$$\begin{aligned}\gamma_\chi^-s_L &= \sum_{j\in\mathcal{R}_\Delta} -\min(0,\langle\chi,j\rangle)c_js_L\\ &= \sum_{\substack{j\in\mathcal{R}_\Delta\\ j\in\operatorname{Ann}\Lambda_L}} -\min(0,\langle a\eta-\eta',j\rangle)c_js_L\end{aligned}$$

$$= \sum_{\substack{j \in \mathcal{R}_\Delta \\ j \in \operatorname{Ann} \Lambda_L}} -a \min(0, \langle \eta, j \rangle) c_j s_L$$

$$= \sum_{j \in \mathcal{R}_\Delta} -a \min(0, \langle \eta, j \rangle) c_j s_L = a\gamma_\eta^- s_L$$

and so

$$s_L(-\tau_W + \gamma_\chi^-) = \sum_{U \in \sup(L,W)} s_L(-\tau_U + \gamma_\eta^-) = \sum_{U \in \sup(L,W)} F(\{L\}, U, \emptyset) = 0$$

by relation (5) of Definition 4.5. The proof of $\tau_V(-\tau_W + \gamma_\chi^-) = -\tau_W^2$ is analogous. □

Let $A \subseteq \{1, \dots, n\}$ be an independent set and W a connected component of $\cap_{a\in A} S_a$. Following [19], we denote by $m(A)$ the number of connected components in such intersection. Let $\mathcal{L}_A^W$ be the subposet of $\mathcal{L}_{\leqslant W} = \{L \in \mathcal{L} \mid L \leqslant W\}$ generated by the atoms in A.

A *flag* adapted to A and W is a sequence of layers $T = F_0 \lessdot F_1 \lessdot \cdots \lessdot F_k$ in the lattice $\mathcal{L}_A^W$ such that $F_0 = T$ and each F_i covers F_{i-1}. Since $\mathcal{L}_A^W$ is Boolean, for every $i = 1, \dots, k$ there exists a unique $a_i \in A$ such that $F_i = S_{a_i} \cap F_{i-1}$. Therefore, each flag adapted to A and W is uniquely determined by the sequence $\mathcal{F} = (a_1, a_2, \dots, a_k)$ of k distinct elements of A.

For a flag $\mathcal{F} = (a_1, a_2, \dots, a_k)$ adapted to A and W, we denote by $m(\mathcal{F})$ the number of connected components in the intersection $\cap_{i=1}^k S_{a_i}$.

For every $i \in \{1, \dots, n\}$ we choose a character χ_i that generates Λ_{S_i}. For each flag $\mathcal{F}$ and each $a \in A$, we define the elements $x(\mathcal{F}, a) = -\sigma_{F_i}$ if $a = a_i$ and $x(\mathcal{F}, a) = \beta_{\chi_a}^-$ otherwise. Analogously we set $y(\mathcal{F}, a) = -\tau_{F_i}$ if $a = a_i$ and $y(\mathcal{F}, a) = \gamma_{\chi_a}^-$, otherwise.

Definition 5.3. For each independent set $A \subseteq \{1, \dots, n\}$ and each connected component W of $\cap_{a\in A} S_a$ we define the following element of D:

$$\Xi_{W,A} = \sum_{\mathcal{F}} \frac{m(\mathcal{F})}{m(A)} \prod_{a\in A} x(\mathcal{F}, a),$$

where the sum is taken over all the flags adapted to A and W.

In the previous definition we use the notation defined above: by $\prod_{a\in A}$ we indicate the exterior product taken in the order of $A \subseteq \{1, \dots, n\}$.

Lemma 5.4. *For each independent set $A \subseteq \{1, \dots, n\}$ and each connected component W of $\cap_{a\in A} S_a$ we have* $\mathrm{d}(\Xi_{W,A}) = 0$.

Proof. We have that

$$\mathrm{d}(\Xi_{W,A}) = \sum_{\mathcal{F}} \frac{m(\mathcal{F})}{m(A)} \sum_{b\in A} (-1)^{|A_{<b}|} y(\mathcal{F}, b) \prod_{a \in A\setminus\{b\}} x(\mathcal{F}, a),$$

so define $Z(\mathcal{F}, b) = y(\mathcal{F}, b) \prod_{a\in A\setminus\{b\}} x(\mathcal{F}, a)$.

Let $\mathcal{F} = (a_1, a_2, \dots, a_k)$ be a flag adapted to A and W. For $i > 1$, let us denote by $\tilde{Z}(\mathcal{F}, a_i)$ be the same product defining $Z(\mathcal{F}, a_i)$ but with $-\sigma_{F_{i-1}}$ replaced by $-\sigma_{F_i}$ (in position labeled by a_{i-1}, that is, the $|A_{<a_{i-1}}|$-th position in the product). We analyze the elements $Z(\mathcal{F}, a_i)$ for $i \in \mathcal{F}$ dividing in cases:

$k > 1, i = 1$: We have

$$Z(\mathcal{F}, a_1) = 0 \tag{3}$$

because $\tau_{F_1}\sigma_{F_2} = -(-\tau_{F_1} + \gamma^-_{\chi_{a_1}})\sigma_{F_2} = -F(\emptyset, F_1, \emptyset)\sigma_{F_2} = 0.$

$i \neq 1, k$: We have

$$Z(\mathcal{F}, a_i) = \tilde{Z}(\mathcal{F}, a_i)$$

because $\sigma_{F_{i-1}}\tau_{F_i}\sigma_{F_{i+1}} = -\sigma_{F_{i-1}}(-\tau_{F_i} + \gamma^-_{\chi_{a_i}})\sigma_{F_{i+1}} = \sigma_{F_i}\tau_{F_i}\sigma_{F_{i+1}}$. Consider the flag $\mathcal{F}' = (a_1, \dots, a_{i-2}, a_i, a_{i-1}, a_{i+1}, \dots, a_k)$ and note that

$$\tilde{Z}(\mathcal{F}', a_{i-1}) = (-1)^{|A_{<a_i}| - |A_{<a_{i-1}}| - 1}\tilde{Z}(\mathcal{F}, a_i),$$

because $F_j = F'_j$ for all $j \neq i-1$ and the factor $-\sigma_{F_i}$ appears in the $|A_{<a_{i-1}}|$th position in $\tilde{Z}(\mathcal{F}, a_i)$ and in the $|A_{<a_i}|$th position in $\tilde{Z}(\mathcal{F}', a_{i-1})$. Therefore,

$$(-1)^{|A_{<a_i}|}Z(\mathcal{F}, a_i) + (-1)^{|A_{<a_{i-1}}|}Z(\mathcal{F}', a_{i-1}) = 0. \tag{4}$$

$k = i = 1$: We have

$$Z((a_1), a_1) + \frac{1}{m(a_1)}Z(\emptyset, a_1) = 0 \tag{5}$$

because $\frac{\chi_{a_1}}{m(a_1)}$ generates Λ_{F_1} where F_1 is the connected component of S_{a_1} containing W and so $-\tau_{F_1} + \frac{1}{m(a_1)}\gamma^-_{\chi_{a_1}} = -\tau_{F_1} + \gamma^-_{\frac{\chi_{a_1}}{m(a_1)}} = F(\emptyset, F_1, \emptyset) = 0.$

$i = k > 1$: We have

$$Z(\mathcal{F}, a_k) + \frac{m(\mathcal{F} \setminus a_k)}{m(\mathcal{F})}Z(\mathcal{F} \setminus a_k, a_k) = \tilde{Z}(\mathcal{F}, a_k) \tag{6}$$

because $\sigma_{F_{k-1}}(-\tau_{F_k} + \frac{m(\mathcal{F}\setminus a_k)}{m(\mathcal{F})}\gamma^-_{\chi_{a_k}}) = \sigma_{F_{k-1}}(-\tau_{F_k} + \gamma^-_{\chi}) = -\sigma_{F_k}\tau_{F_k}$, where $\chi \in \Lambda_{F_k}$ is any element such that $m(\mathcal{F})\chi - m(\mathcal{F} \setminus a_k)\chi_{a_k} \in \Lambda_{F_{k-1}}$. Moreover, for $\mathcal{F} = (a_1, a_2, \dots, a_k)$ we have defined $\mathcal{F}' = (a_1, \dots, a_{k-2}, a_k, a_{k-1})$ and we have

$$\tilde{Z}(\mathcal{F}', a_{k-1}) = (-1)^{|A_{<a_k}| - |A_{<a_{k-1}}| - 1}\tilde{Z}(\mathcal{F}, a_k). \tag{7}$$

Finally, we have:

$$\begin{aligned} m(A)\,\mathrm{d}(\Xi_{W,A}) &= \sum_{\mathcal{F}} m(\mathcal{F}) \sum_{b \in A} (-1)^{|A_{<b}|} Z(\mathcal{F}, b) \\ &= \sum_{|\mathcal{F}|>0} \sum_{i=1}^{k} (-1)^{|A_{<a_i}|} m(\mathcal{F}) Z(\mathcal{F}, a_i) + \sum_{\mathcal{F}} \sum_{a \notin \mathcal{F}} (-1)^{|A_{<a}|} m(\mathcal{F}) Z(\mathcal{F}, a). \end{aligned}$$

By Equations (3) and (4), the terms with $i < k$ cancel with each other, hence the sum above is equal to

$$\sum_{|\mathcal{F}|>0} (-1)^{|A_{<a_k}|} m(\mathcal{F}) Z(\mathcal{F}, a_k) + \sum_{\mathcal{F}} \sum_{a \notin \mathcal{F}} (-1)^{|A_{<a}|} m(\mathcal{F}) Z(\mathcal{F}, a).$$

By formula (5), the terms with $k = 0$ vanish, thus we obtain

$$\sum_{|\mathcal{F}|>1} (-1)^{|A_{<a_k}|} m(\mathcal{F}) Z(\mathcal{F}, a_k) + \sum_{\mathcal{F}\neq\emptyset} \sum_{a \notin \mathcal{F}} (-1)^{|A_{<a}|} m(\mathcal{F}) Z(\mathcal{F}, a).$$

Equation (6) allows us to rewrite the above sums using the monomials $\tilde{Z}$:

$$\sum_{|\mathcal{F}|>1} (-1)^{|A_{<a_k}|} m(\mathcal{F}) \tilde{Z}(\mathcal{F}, a_k)$$

and finally we apply formula (7) to obtain

$$\sum_{\substack{|\mathcal{F}|>1 \\ a_k > a_{k-1}}} (1 + (-1)^1) m(\mathcal{F}) \tilde{Z}(\mathcal{F}, a_k) = 0.$$

This completes the proof. □

Let $s \in \{+, -\}$ and $\chi \in \Lambda$, we define the open half space $H^s_\chi \subset \Lambda^*$ as

$$H^s_\chi = \{v \in \Lambda^* \mid \langle v, s\chi \rangle > 0\}.$$

Recall that C_Δ is the collection of all cones in Δ. We denote C^l_Δ the set of all cones in Δ of dimension l.

Lemma 5.5. *Let A be an independent set and $s_a \in \{+, -\}$ for $a \in A$. Let Z be the set $\bigcap_{a\in A} H^{s_a}_{\chi_a}$. Consider the projection $\pi : \Lambda^* \to \Lambda^* / \operatorname{Ann} \Lambda_A$. We have that*

$$\prod_{a\in A} \beta^{s_a}_{\chi_a} = m(A) \sum_{\substack{K \in C^{|A|}_\Delta \\ K \subset Z}} \operatorname{Vol}(\pi(K)) \prod_{c\in K} b_c,$$

where the last product is taken in any order such that the two bases $(s_a\chi_a)_{a\in A}$ and $(\pi(c))_{c\in K}$ are both positive or both negative.

Moreover, if A is dependent then $\prod_{a\in A} \beta^{s_a}_{\chi_a} = 0$.

Proof. Let $K \in C^{|A|}_\Delta$ be a $|A|$-dimensional cone not contained in Z: then there exists $c \in K$ such that $c \notin Z$. So, for some $a \in A$, we have $\min(0, \langle s_a\chi_a, c' \rangle) = 0$ for all $c' \in K$ by using the equal sign property. It easy to see that the monomial $\prod_{c\in K} b_c$ does not appear in $\prod_{a\in A} \beta^{s_a}_{\chi_a}$.

Now suppose that a $K = (k_1, \dots, k_l) \in C^l_\Delta$ is contained in Z, the coefficient of $\prod_{i=1}^l b_{k_i}$ in $\prod_{a\in A} \beta^{s_a}_{\chi_a}$ is

$$\sum_{\sigma\in\mathfrak{S}_l}(-1)^{\operatorname{sgn}\sigma}\prod_{i=1}^{l}\langle s_i\chi_i, k_{\sigma(i)}\rangle.$$

Now note that $\langle s_i\chi_i, k_{\sigma(i)}\rangle = \langle s_i\chi_i, \pi(k_{\sigma(i)})\rangle$ for all i and σ.

The equality

$$\sum_{\sigma\in\mathfrak{S}_l}(-1)^{\operatorname{sgn}\sigma}\prod_{i=1}^{l}\langle s_i\chi_i, \pi(k_{\sigma(i)})\rangle = \det(s_i\chi_i)\det(\pi(k_i))$$

follows from the multilinearity in the entries $s_i\chi_i$ and $\pi(k_i)$.

If A is dependent then $\dim \Lambda^*/\operatorname{Ann}\Lambda_A = \operatorname{rk}(A) < |A|$, so $\det(\pi(k_i)) = 0$ and $\prod_{a\in A}\beta_{\chi_a}^{s_a} = 0$.

Otherwise, the two bases $(s_a\chi_a)_{a\in A}$ and $(\pi(k))_{k\in K}$ are both positive (respectively, negative) then $\det(s_i\chi_i)\det(\pi(k_i))$ is positive and equals to $m(A)\operatorname{Vol}(\pi(K))$. □

The proof of this corollary follows from the proof of Lemma 5.5 by omitting some steps.

Corollary 5.6. *Let A be an independent set, then:*

$$\prod_{a\in A}\beta_{\chi_a} = m(A)\sum_{K\in C_\Delta^{|A|}}\operatorname{Vol}(\pi(K))\prod_{c\in K}b_c,$$

where the last product is taken in any order such that the two bases $(\chi_a)_{a\in A}$ and $(\pi(c))_{c\in K}$ are both positive or both negative. □

Corollary 5.7. *Let $V \in \mathcal{L}$, $A \subseteq E$ and $s_a \in \{+,-\}$ for $a \in A$. Let Z be the set $\bigcap_{a\in A} H_{\chi_a}^{s_a}$ and $\pi : \Lambda^* \to \Lambda^*/\operatorname{Ann}\Lambda_A$. If the vectors χ_a for $a \in A$ are dependent in Λ/Λ_V then $\sigma_V\prod_{a\in A}\beta_{\chi_a}^{s_a} = 0$, otherwise*

$$\sigma_V\prod_{a\in A}\beta_{\chi_a}^{s_a} = \sigma_V m(A)\sum_{\substack{K\in C_\Delta^{|A|}\\ K\subset Z\cap\operatorname{Ann}\Lambda_V}}\operatorname{Vol}(\pi(K))\prod_{c\in K}b_c,$$

where the last product is taken in any order such that the two bases $(s_a\chi_a)_{a\in A}$ and $(\pi(c))_{c\in K}$ are both positive or both negative.

Proof. We use Lemma 5.5 and then we multiply both sides by σ_V. If $K \not\subseteq \operatorname{Ann}\Lambda_V$ then $\sigma_V\prod_{c\in K}b_c = 0$ and the second claim follows. Since we can assume $K \subset \operatorname{Ann}\Lambda_V$ the map π restricts to the canonical projection $\operatorname{Ann}\Lambda_V \to \operatorname{Ann}\Lambda_V/\operatorname{Ann}(\Lambda_V + \Lambda_A)$. As in the proof of Lemma 5.5, it follows that $\det(\pi(k_i)) = 0$ because they are $|A|$ vectors in a vector space of strictly less dimension. □

We recall that, given two positive integers k, h, a (k,h)-*shuffle* is an element p of the symmetric group on the elements $\{1,\dots,k+h\}$ such that $p(i) < p(j)$ for every couple $i < j$ such that either $i, j \in \{1,\dots,k\}$, or $i, j \in \{k+1,\dots,k+h\}$.

Lemma 5.8. *For all independent set A and B, and for all connected components W of $\cap_{a\in A} S_a$ and L of $\cap_{b\in B} S_b$, we have $\Xi_{W,A}\Xi_{L,B} = 0$ if $A \cap B$ is not empty or if $A \sqcup B$ is not independent. Otherwise*

$$\Xi_{W,A}\Xi_{L,B} = (-1)^{l(A,B)} \sum_{V\in \sup(W,L)} \Xi_{V,A\cup B},$$

where $l(A,B)$ is the sign of the permutation reordering (A,B).

Proof. Let $\mathcal{F} = (a_1, \dots, a_k)$ be a flag adapted to A and W, and $\mathcal{H} = (a_{k+1}, \dots, a_{k+h})$ be a flag adapted to B and L. Let $C = \{a_1, \dots, a_{k+h}\}$ and suppose that C is not independent. Then using Equation (1) of Lemma 5.2, we can write the product $\prod_{i=1}^{|\mathcal{F}|} \sigma_{F_i} \prod_{i=1}^{|\mathcal{H}|} \sigma_{H_i}$ as linear combination of monomials $\prod_{i=1}^{k+h} \sigma_{G_i}$ for $G_1 \leqslant G_2 \leqslant \dots \leqslant G_{k+h} \leqslant F_h \vee H_k$. Since C is dependent, $\mathrm{rk}(G_{k+h}) < k+h$ and so there exists i such that $G_i = G_{i+1}$. Since $\sigma_{G_i}^2 = 0$, all such monomial are zero and so the product $\prod_{i=1}^{|\mathcal{F}|} \sigma_{F_i} \prod_{i=1}^{|\mathcal{H}|} \sigma_{H_i}$ vanishes. If C is independent of cardinality $k+h$, then for each (k,h)-shuffle p and each element $V \in \sup(F_k, H_h)$ we have a flag $\mathcal{F} *_p \mathcal{H} := (a_{p(1)}, \dots, a_{p(k+h)})$ adapted to C and V. By using only Equation (1) of Lemma 5.2, we have

$$\prod_{i=1}^{k} \sigma_{F_i} \prod_{j=1}^{h} \sigma_{H_j} = (-1)^{l(A,B)} \sum_{V\in \sup(F_k,H_h)} \sum_{p \text{ shuffle}} \prod_{i=1}^{k+h} \sigma_{(\mathcal{F} *_p \mathcal{H})_i},$$

where the products are taken in increasing order of the corresponding a_i.

Now we prove that $\Xi_{W,A}\Xi_{L,B} = 0$ if $\mathrm{rk}(A\cup B) < |A| + |B|$. It is enough to verify that $\prod_{a\in A} x(\mathcal{F}, a) \prod_{b\in B} x(\mathcal{H}, b) = 0$ for all flags $\mathcal{F}$ and $\mathcal{H}$ as above. If C is dependent then we have already prove that the product is zero, so suppose C to be independent. Let $a_{k+h+1}, \dots, a_{|A|+|B|}$ be the list of the elements in $A \setminus \mathcal{F}$ and in $B \setminus \mathcal{H}$. By Corollary 5.7, we have $\sigma_V \prod_{i=k+h+1}^{|A|+|B|} \beta^-_{\chi_{a_i}} = 0$ for all $V \in \sup(F_k, H_h)$.

It remains to prove the case $A \sqcup B$ an independent set. The number of connected components of $W \cap L$ contained in V is equal to

$$\frac{m(A\cup B)}{m(A)m(B)} \frac{m(\mathcal{F})m(\mathcal{H})}{m(\mathcal{F}\cup\mathcal{H})}.$$

Finally,

$$\begin{aligned}
\Xi_{W,A}\Xi_{L,B} &= \sum_{\mathcal{F},\mathcal{H}} \frac{m(\mathcal{F})m(\mathcal{H})}{m(A)m(B)} \prod_{a\in A} x(\mathcal{F}, a) \prod_{b\in B} x(\mathcal{H}, b) \\
&= (-1)^{l(A,B)} \sum_{\mathcal{F},\mathcal{H}} \frac{m(\mathcal{F})m(\mathcal{H})}{m(A)m(B)} \sum_{V\in \sup(F_k,H_h)} \sum_{p \text{ shuffle}} \prod_{a\in A\sqcup B} x(\mathcal{F} *_p \mathcal{H}, a) \\
&= (-1)^{l(A,B)} \sum_{\mathcal{F},\mathcal{H}} \frac{m(\mathcal{F}\cup\mathcal{H})}{m(A\cup B)} \sum_{V\in \sup(W,L)} \sum_{p \text{ shuffle}} \prod_{a\in A\sqcup B} x(\mathcal{F} *_p \mathcal{H}, a) \\
&= (-1)^{l(A,B)} \sum_{V\in \sup(W,L)} \sum_{\mathcal{F},\mathcal{H}} \frac{m(\mathcal{F}\cup\mathcal{H})}{m(A\cup B)} \sum_{p \text{ shuffle}} \prod_{a\in A\sqcup B} x(\mathcal{F} *_p \mathcal{H}, a)
\end{aligned}$$

$$= (-1)^{l(A,B)} \sum_{V \in \sup(W,L)} \sum_{\mathcal{K}} \frac{m(\mathcal{K})}{m(A \cup B)} \prod_{a \in A \sqcup B} x(\mathcal{K}, a)$$

$$= (-1)^{l(A,B)} \sum_{V \in \sup(W,L)} \Xi_{V, A \cup B},$$

where we used the fact that flags $\mathcal{K}$ adapted to $A \sqcup B$ and V are in bijection with flags $\mathcal{F} *_p \mathcal{H}$ where p runs over all the (k,h)-shuffles, $\mathcal{F}$ over all flags adapted to A and W, and $\mathcal{H}$ over all flags adapted to B and L. So, the claim follows. □

Lemma 5.9. *Let A be an independent set and W a connected component of $\cap_{a \in A} S_a$. If $\chi \in \Lambda_W$, then $\Xi_{W,A} \beta_\chi = 0$.*

Proof. Let $\mathcal{F}$ be a flag adapted to A and W, we show that $\beta_\chi^s \prod_{a \in A} x(\mathcal{F}, a) = 0$ for $s \in \{+,-\}$. The element $\beta_\chi^s \sigma_{F_k} \prod_{a \in A \setminus \mathcal{F}} \beta_{\chi_a}^-$ is zero by Corollary 5.7, because the vectors χ, χ_a for $a \in A \setminus \mathcal{F}$ are linearly dependent in Λ / Λ_{F_k}. The Lemma follows. □

Recall that by definition of a circuit C, there exists a minimal relation $\sum_{i \in C} n_i \chi_i$ with $n_i \neq 0$ for all $i \in C$ and this coefficients n_i are unique up to scalars. Moreover, we can choose n_i as $c_i m(C \setminus \{i\})$, where m is the multiplicity function of the arithmetic matroid and c_i is the orientation of the oriented matroid.

Lemma 5.10. *Let X be a subset such that $|X| = \mathrm{rk}(X) + 1$, $C \subseteq X$ be the unique circuit, $A \subset X$ be an independent set, F be a connected component of $\cap_{a \in A} S_a$. There exists a minimal relation $\sum_{i \in C} c_i m(C \setminus \{i\}) \chi_i = 0$ for some $c_i \in \{+,-\}$. Suppose that $C' := C \setminus A$ has cardinality at least 2, then*

$$\sigma_F \sum_{j \in C'} \frac{(-1)^{|C'_{<j}|}}{m(X \setminus \{j\})} \prod_{i \in C' \setminus \{j\}} \beta_{\chi_i}^{\delta(i,j)} = 0,$$

where $\delta(i,j) = c_i c_j$ if $i < j$ and $\delta(i,j) = -$ if $i > j$.

Proof. For the sake of simplifying the notation, let us suppose $C' = \{0, 1, \dots, l\}$. The first step of the proof is to reduce to the case $c_i = -$ for $i < k$ and $c_i = +$ for $i \geqslant k$ for some $k \in C'$. Let $\mu \in \mathfrak{S}_{|C'|}$ be the unique shuffle that reorders C' in such a way that $c_i = -$ for $i < k$ and $c_i = +$ for $i \geqslant k$. We have

$$\sum_{j \in C'} \frac{(-1)^j}{m(X \setminus \{j\})} \prod_{i \in C' \setminus \{j\}} \beta_{\chi_i}^{\delta(i,j)} = \mathrm{sgn}(\mu) \sum_{j \in C'} \frac{(-1)^{\mu(j)}}{m(X \setminus \{j\})} \prod_{i \in \mu(C' \setminus \{j\})} \beta_{\chi_i}^{\delta(i,\mu(j))},$$

where we use $\mathrm{sgn}(\mu) = (-1)^{j - \mu(j)} \mathrm{sgn}(\mu_{|C' \setminus \{j\}})$. Moreover, note that $\delta(i,j) = \delta(\mu(i), \mu(j))$ since (i,j) is an inversion of μ only if $c_i c_j = -$. Thus, from now on we assume $c_i = -$ for $i < k$ and $c_i = +$ for $i \geqslant k$.

Define $Z_j = (\cap_{i<j} H_{\chi_i}^{c_i c_j} \cap_{i>j} H_{\chi_i}^-) \cap \mathrm{Ann}\, \Lambda_F$, $X_j = Z_j \cap H_{\chi_j}^+$ and $Y_j = Z_j \cap H_{\chi_j}^-$. The following properties follows easily from the definition:

$$Z_j = X_j \cup Y_j \qquad \dim(X_j \cap Y_j) < l$$
$$X_l = \emptyset \qquad Y_{k-1} = \emptyset$$
$$X_k = Y_0 \qquad X_{j-1} = Y_j \text{ for all } j \neq k.$$

Note that $\sigma_F b_c = 0$ if $c \notin \operatorname{Ann} \Lambda_F$ by Definition 4.5. By Lemma 5.5, we have

$$\begin{aligned}\frac{\sigma_F}{m(X\setminus\{j\})}\prod_{i\in C'\setminus\{j\}}\beta_i^{\delta(i,j)} &= \sigma_F\frac{m(C'\setminus\{j\})}{m(X\setminus\{j\})}\sum_{\substack{K\in C^l_\Delta\\ K\subset Z_j}}\operatorname{Vol}(\pi_j(K))\prod_{c\in K}b_c\\ &= \sigma_F\frac{m(C')}{m(X)}\sum_{\substack{K\in C^l_\Delta\\ K\subset Z_j}}\operatorname{Vol}(\pi_j(K))\prod_{c\in K}b_c,\end{aligned}$$

where in the last equality we used the property (P) of *arithmetic matroids* (see [3]).

The map π_j restricted to $\operatorname{Ann}\Lambda_F$ does not depend on $j \in C'$. Indeed the restriction of $\pi_j : \Lambda^* \to \Lambda^*/\operatorname{Ann}\Lambda_{C'\setminus\{j\}}$ is the canonical projection $\operatorname{Ann}\Lambda_F \to \operatorname{Ann}\Lambda_F/\operatorname{Ann}\Lambda_{X\setminus\{j\}}$ and it does not depend on j because $\Lambda_X = \Lambda_{X\setminus\{j\}}$ for all $j \in C$.

For $j \neq k$ the bases $(\delta(i,j)\chi_i)_{i\neq j}$ and $(\delta(i,j-1)\chi_i)_{i\neq j-1}$ have the same orientation. The bases $(-\chi_i)_{i>0}$ and $(\delta(i,k)\chi_i)_{i\neq k}$ have the same orientation if and only if $(-1)^{k-1} = 1$.

Since

$$\sigma_F\sum_{j\in C'}\frac{(-1)^{|C'_{<j}|}}{m(X\setminus\{j\})}\prod_{i\in C'\setminus\{j\}}\beta_i^{\delta(i,j)} = \frac{m(C')}{m(X)}\sigma_F\sum_{j\in C'}\sum_{\substack{K\in C^l_\Delta\\ K\subset Z_j}}(-1)^j\operatorname{Vol}(\pi(K))\prod_{c\in K}b_c,$$

it is enough to consider the following:

$$\begin{aligned}\sum_{j\in C'}\sum_{K\subset Z_j}(-1)^j\operatorname{Vol}(\pi(K))\prod_{c\in K}b_c &= \sum_{j\in C'}\sum_{K\subset X_j}(-1)^j\operatorname{Vol}(\pi(K))\prod_{c\in K}b_c+\\ &\quad+\sum_{j\in C'}\sum_{K\subset Y_j}(-1)^j\operatorname{Vol}(\pi(K))\prod_{c\in K}b_c\\ &= \sum_{K\subset X_k}(-1)^k\operatorname{Vol}(\pi(K))\prod_{c\in K}b_c + \sum_{K\subset Y_0}\operatorname{Vol}(\pi(K))\prod_{c\in K}b_c = 0,\end{aligned}$$

so the claim follows. □

Let $C \subseteq \{1,\dots,n\}$ be a circuit oriented by the signs $(c_i)_{i\in C}$. We recall the following definition, which was introduced by Postnikov in [23]. For each $A \subseteq \{1,\dots,n\}$, we say that C/A is a *positroid* if $c_i = c_j$ for all $i,j \in C\setminus A$. Since the orientation is defined up to a global sign, we can assume c_i is positive for all $i \in C\setminus A$.

Lemma 5.11. *Consider $X \subseteq \{1,\dots,n\}$ such that $|X| = \operatorname{rk}(X)+1$, let $C \subseteq X$ be the unique circuit and L be a connected component of $\cap_{i\in X}S_i$. There exists a minimal relation $\sum_{i\in C}c_i m(C\setminus\{i\})\chi_i = 0$*

for some $c_i \in \{+,-\}$. *Then, we have*

$$\sum_{\substack{X\setminus C\subseteq A\subsetneq X\\ C/A \text{ positroid}}} (-1)^{|X_{<j}|+l(A,B)} \frac{m(A)}{m(X\setminus\{j\})}\Xi_{W,A}\beta_B = 0,$$

where $j = \max(X\setminus A)$, $B = C\setminus(A\cup\{j\})$, W *is the connected component of* $\cap_{a\in A} S_a$ *containing* L *and* $l(A,B)$ *is the sign of the permutation that reorders* (A,B).

Proof. We may assume that $X = \{0,1,\dots,\mathrm{rk}(X)\}$ and $C = \{0,1,\dots,\mathrm{rk}(C)\}$. Let $R = X\setminus C$, we can rewrite the left-hand side as follows:

$$\sum_{\substack{R\subseteq A\subsetneq X\\ C/A \text{ pos.}}} (-1)^{j+l(A,B)} \frac{m(A)}{m(X\setminus\{j\})}\Xi_{W,A}\beta_B =$$

$$= \sum_{\substack{R\subseteq A\subsetneq X\\ C/A \text{ pos.}}} \sum_{\mathcal{F}\subseteq A} (-1)^{j+|\mathcal{F}|+l(A,B)+l(\mathcal{F},A\setminus\mathcal{F})} \frac{m(\mathcal{F})}{m(X\setminus\{j\})} \prod_{i=1}^{|\mathcal{F}|}\sigma_{F_i} \prod_{a\in A\setminus\mathcal{F}}\beta_a^- \prod_{b\in B}\beta_b$$

$$= \sum_{\mathcal{F}\subsetneq X} \sum_{\substack{\mathcal{F}\cup R\subseteq A\subsetneq X\\ C/A \text{ pos.}}} (-1)^{j+|\mathcal{F}|+l(A,B)+l(\mathcal{F},A\setminus\mathcal{F})} \frac{m(\mathcal{F})}{m(X\setminus\{j\})} \prod_{i=1}^{|\mathcal{F}|}\sigma_{F_i} \prod_{a\in A\setminus\mathcal{F}}\beta_a^- \prod_{b\in B}\beta_b,$$

where we applied the definition of $\Xi_{W,A}$ and then we exchanged the two sums. By setting $D = A\setminus(\mathcal{F}\cup R)$ so that $A = D\sqcup\mathcal{F}\sqcup(R\setminus\mathcal{F})$, we rewrite the above equation as

$$\sum_{\substack{R\subseteq A\subsetneq X\\ C/A \text{ pos.}}} (-1)^{j+l(A,B)} \frac{m(A)}{m(X\setminus\{j\})}\Xi_{W,A}\beta_B =$$

$$= \sum_{\mathcal{F}\subsetneq X} \frac{m(\mathcal{F})}{m(X\setminus\{j\})} \sum_{\substack{D\subsetneq C\setminus\mathcal{F}\\ C/(D\cup\mathcal{F}) \text{ pos.}}} (-1)^{j+|\mathcal{F}|+l(A,B)+l(\mathcal{F},A\setminus\mathcal{F})} \prod_{i=1}^{|\mathcal{F}|}\sigma_{F_i} \prod_{a\in D\cup(R\setminus\mathcal{F})}\beta_a^- \prod_{b\in B}\beta_b. \tag{8}$$

Let $C' = C\setminus\mathcal{F}$, $j = \max(C\setminus A)$ and $C(j) = \{i\in C_{<j}\setminus\mathcal{F} \mid c_i = c_j\}$, we need the following equality:

$$\sum_{\substack{B\subseteq C_{<j}\setminus\mathcal{F}\\ B\cup\{j\} \text{ pos.}}} (-1)^{l(C'\setminus(B\cup\{j\}),B)} \prod_{a\in C'\setminus(B\cup\{j\})}\beta_a^- \prod_{b\in B}\beta_b =$$

$$\overset{(1)}{=} \sum_{B\subseteq C(j)} \sum_{D\subseteq B} (-1)^{|B\setminus D|+l(C'\setminus(D\cup\{j\}),D)} \prod_{a\in C'\setminus(D\cup\{j\})}\beta_a^- \prod_{b\in D}\beta_b^+$$

$$\overset{(2)}{=} \sum_{D\subseteq C(j)} (-1)^{l(C'\setminus(D\cup\{j\}),D)} \prod_{a\in C'\setminus(D\cup\{j\})}\beta_a^- \prod_{b\in D}\beta_b^+ \sum_{E\subseteq C(j)\setminus D}(-1)^{|E|} \tag{9}$$

$$\stackrel{(3)}{=} (-1)^{l(C'\setminus(C(j)\cup\{j\}),C(j))} \prod_{a\in C'\setminus(C(j)\cup\{j\})} \beta_a^- \prod_{b\in C(j)} \beta_b^+$$

$$\stackrel{(4)}{=} \prod_{\substack{b=0,\dots,j-1\\ b\notin\mathcal{F}}} \beta_b^{c_b c_j} \prod_{\substack{a=j+1,\dots,\mathrm{rk}(C)\\ a\notin\mathcal{F}}} \beta_a^-.$$

In equality (1), we used $\beta_b = \beta_b^+ - \beta_b^-$ and expanded the product. In equality (2) we set $E = B \setminus D$ and we exchanged the two sums. Equality (3) follows from the fact that $\sum_{E\subseteq C(j)\setminus D}(-1)^{|E|} = 0$ if $C(j) \setminus D \neq \emptyset$. Equality (4) follows from the fact that, for $b < j$, $c_b c_j = -$ if $b \in C' \setminus (C(j) \cup \{j\})$ and $c_b c_j = +$ if $b \in C(j)$.

We also need, for $|C'| > 1$ the following:

$$\sigma_{F_k} \sum_{j\in C'} \frac{(-1)^{|C'_{<j}|}}{m(X\setminus\{j\})} \sum_{\substack{B\subseteq C'_{<j}\\ B\cup\{j\}\text{ pos.}}} (-1)^{l(C'\setminus(B\cup\{j\}),B)} \prod_{a\in C'\setminus(B\cup\{j\})} \beta_a^- \prod_{b\in B} \beta_b =$$

$$= \sigma_{F_k} \sum_{j\in C'} \frac{(-1)^{|C'_{<j}|}}{m(X\setminus\{j\})} \prod_{b\in C'_{<j}} \beta_b^{c_b c_j} \prod_{a\in C'_{>j}} \beta_a^-$$

$$= 0,$$

by Equation (9) and Lemma 5.10. This proves that all summands in eq. (8) such that $|C \setminus \mathcal{F}| > 1$ cancel each other. Therefore,

$$\sum_{\substack{R\subseteq A\subsetneq X\\ C/A\text{ pos.}}} (-1)^{j+l(A,B)} \frac{m(A)}{m(X\setminus\{j\})} \Xi_{W,A}\beta_B =$$

$$= \sum_{j\in C} \sum_{C\setminus\{j\}\subseteq\mathcal{F}\subsetneq X} (-1)^{j+|\mathcal{F}|+l(\mathcal{F},X\setminus(\mathcal{F}\cup\{j\})} \frac{m(\mathcal{F})}{m(X\setminus\{j\})} \prod_{i=1}^{|\mathcal{F}|} \sigma_{F_i} \prod_{a\in X\setminus(\mathcal{F}\cup\{j\})} \beta_a^-.$$

Recall also that $\frac{m(\mathcal{F})}{m(X\setminus\{j\})} = \frac{m(\mathcal{F}\cup\{j\})}{m(X)}$ by property (P) of arithmetic matroids. Thus, we have:

$$\sum_{\substack{R\subseteq A\subsetneq X\\ C/A\text{ pos.}}} (-1)^{j+l(A,B)} \frac{m(A)}{m(X\setminus\{j\})} \Xi_{W,A}\beta_B =$$

$$= \sum_{j\in C} \sum_{C\setminus\{j\}\subseteq\mathcal{F}\subsetneq X} (-1)^{j+|\mathcal{F}|+l(\mathcal{F},X\setminus(\mathcal{F}\cup\{j\}))} \frac{m(\mathcal{F}\cup\{j\})}{m(X)} \prod_{i=1}^{|\mathcal{F}|} \sigma_{F_i} \prod_{a\in X\setminus(\mathcal{F}\cup\{j\})} \beta_a^-.$$

We prove that the terms in the above sum cancels in pairs: consider $j \in C$, a flag $\mathcal{F}$ with last element k and define $\tilde{\mathcal{F}}$ the flag obtained from $\mathcal{F}$ substituting k with j. This gives a pairing between $(j, \mathcal{F})$ and $(k, \tilde{\mathcal{F}})$. We prove that the summands associated with $(j, \mathcal{F})$ and $(k, \tilde{\mathcal{F}})$ cancel each other. Note that $\mathcal{F} \cup \{j\} = \tilde{\mathcal{F}} \cup \{k\}$ and that $(-1)^{l(\mathcal{F},X\setminus(\mathcal{F}\cup\{j\}))} \prod_{i=1}^{|\mathcal{F}|} \sigma_{F_i} \prod_{a\in X\setminus(\mathcal{F}\cup\{j\})} \beta_a^-$ differ from $(-1)^{l(\tilde{\mathcal{F}},X\setminus(\tilde{\mathcal{F}}\cup\{k\}))}\prod_{i=1}^{|\tilde{\mathcal{F}}|}\sigma_{F_i}\prod_{a\in X\setminus(\tilde{\mathcal{F}}\cup\{k\})}\beta_a^-$ by $(-1)^{k-j-1}$, because the element $\sigma_{F_{\mathcal{F}}}$ appears in position k in the first element and in position j in the second one. Therefore, the two monomials associated with $(j,\mathcal{F})$ and $(k,\tilde{\mathcal{F}})$ are the same but with opposite sign. □

Let ω be the generator of $H^1(\mathbb{C}^*;\mathbb{Z})$.

Theorem 5.12. *Let $\mathcal{A}$ be a toric arrangement. The rational cohomology algebra $H^*(M(\mathcal{A});\mathbb{Q})$ is isomorphic to the graded commutative algebra*

$$H^\bullet(T;\mathbb{Q})[e_{W,A}]\big/_{I},$$

where A ranges over all the independent subsets of $\{1,\dots,n\}$ and W ranges over all connected components of $\cap_{a\in A}S_a$. The degree of the generator $e_{W,A}$ is $|A|$. The ideal I is generated by the following elements.

- *For any two generators $e_{W,A}$, $e_{W',A'}$,*

$$e_{W,A}e_{W',A'}$$

if $A\cap A'\neq\emptyset$ or $A\sqcup A'$ is a dependent set, and otherwise

$$e_{W,A}e_{W',A'}-(-1)^{l(A,A')}\sum_{L\in\pi_0(W\cap W')}e_{L,A\cup A'}. \tag{10}$$

- *For any $\psi\in H^\bullet(T;\mathbb{Q})$ such that $\psi_{|W}=0$,*

$$e_{W,A}\psi. \tag{11}$$

- *For every $X\subseteq\{1,\dots,n\}$ such that $\mathrm{rk}(X)=|X|-1$ write $X=C\sqcup F$ with C the unique circuit in X. Consider the associated linear dependency $\sum_{i\in C}n_i\chi_i=0$ with $n_i\in\mathbb{Z}$, and for every connected component L of $\cap_{i\in X}S_i$ a relation*

$$\sum_{\substack{X\setminus C\subseteq A\subsetneq X\\ C/A\text{ positroid}}}(-1)^{|X_{<j}|+l(A,B)}\frac{m(A)}{m(X\setminus\{j\})}e_{W,A}\psi_B, \tag{12}$$

where $j=\max(C\setminus A)$, $B=C\setminus(A\cup\{j\})$, $\psi_B=\prod_{b\in B}\chi_b^*(\omega)$ an element in $H^\bullet(T)$, and W is the connected component of $\cap_{i\in A}S_i$ containing L.

Remark 5.13. Compared to the previous result of [5], this new presentation exhibits more clearly its dependence on the orientation. Different choices of the orientation give rise to different presentations of the same algebra. Furthermore, our presentation depends only on the *oriented arithmetic matroid* associated with the set of characters defining the toric arrangement. The notion of of oriented arithmetic matroid was defined in [22], by refining the notion of arithmetic matroid introduced in [3, 15]. Since, by [22, Theorem 6.1], all the orientations are equivalent, the isomorphism class of the cohomology algebra only depends on the arithmetic matroid.

We also remark that the presentation provided by Theorem 5.12 seems more suitable to be generalized to arrangement of subtori of arbitrary codimensions.

Before proving the above theorem, we need a couple of lemmas. We denote the ring $H^\bullet(T;\mathbb{Q})[e_{W,A}]/I$ by R.

Lemma 5.14. *There exists a filtration* $\mathrm{F}_\bullet$ *of* $H^\bullet(T;\mathbb{Q})[e_{W,A}]/I$ *such that*

$$\operatorname{gr}^\bullet_{\mathrm{F}} R \cong \bigoplus_{W\in\mathcal{L}} H^\bullet(W;\mathbb{Q}) \otimes \tilde{H}_{\operatorname{cd} W-2}(\Delta(T,W)).$$

In particular, the set $e_{W,A}$ *with* A *a no broken circuit set in* $\mathcal{L}_{\leqslant W}$ *generates* R *as* $H(T;\mathbb{Q})$*-module. Moreover,* R *and* $H^\bullet(M(\mathcal{A});\mathbb{Q})$ *have the same dimension.*

Proof. Let $\mathrm{F}_\bullet$ be the filtration defined by

$$\mathrm{F}_h R = \sum_{\substack{\operatorname{cd}(W)\leqslant h \\ A}} e_{W,A} H^\bullet(T;\mathbb{Q}).$$

The graded ring $\operatorname{gr}_{\mathrm{F}} R$ is isomorphic to $H^\bullet(T;\mathbb{Q})[e_{W,A}]/I'$, where I' is the ideal generated by Equations (10), (11) and

$$\sum_{j\in C} (-1)^{|X_{<j}|} e_{L,X\setminus\{j\}} \qquad (12')$$

for all X such that $\operatorname{rk}(X) = |X| - 1$ and all L connected components of $\cap_{a\in X} S_a$. Note that $\operatorname{gr}_{\mathrm{F}} R$ is $\mathcal{L}$-graded and isomorphic to

$$\bigoplus_{W\in\mathcal{L}} H(W;\mathbb{Q})[e_{W,A}]_A \big/ I_W$$

as $H(T)$-module, where I_W is the ideal generated by Equation (12') for all X such that $\operatorname{rk}(X) = |X| - 1$ and W is a connected component of $\cap_{a\in X} S_a$. Finally, we have

$$\begin{aligned} H^\bullet(T;\mathbb{Q})[e_{W,A}] \big/ I &\cong \operatorname{gr}_{\mathrm{F}} R \\ &\cong \bigoplus_{W\in\mathcal{L}} H(W;\mathbb{Q})[e_{W,A}]_A \big/ I_W \\ &\cong \bigoplus_{W\in\mathcal{L}} H^\bullet(W;\mathbb{Q}) \otimes \tilde{H}_{\operatorname{rk} W-2}(\Delta(T,W)), \end{aligned}$$

where we use the Brieskorn isomorphism for the Orlik–Solomon algebra associated to the geometric lattice $\mathcal{L}_{\leqslant W}$.

From Theorem 2.8, we deduce that $R \cong H^\bullet(M(\mathcal{A});\mathbb{Q})$ as $\mathbb{Q}$-vector space and so they have the same dimension. ☐

We want to construct a bijection for any geometric lattice $\mathcal{L}_{\leqslant W}$ between *no broken circuit sets* and certain maximal flags. For any maximal flag of layers $\mathcal{F} = (T = F_0 \lessdot F_1 \lessdot \cdots \lessdot F_k = W)$ we define the *edge labeling* $\epsilon(\mathcal{F})$ as the list $(b_1, \dots, b_k)$ where $b_k = \max\{i \in \{1, \dots, n\} \mid F_k \in \sup(F_{k-1}, S_i)\}$. We say that $\mathcal{F}$ is *increasing* if $b_i < b_j$ for all $i < j$ (where $\epsilon(\mathcal{F}) = (b_1, \dots, b_k)$).

Note that if $\mathcal{F}$ is a maximal flag adapted to A and W, $\epsilon(\mathcal{F})$ may not be a subset of A.

Lemma 5.15. *We fix a layer W of rank k and consider the geometric lattice $\mathcal{L}_{\leqslant W}$. If $A = \{a_1 < a_2 < \cdots < a_k\}$ is a no broken circuit set, then a maximal flag $\mathcal{F}$ adapted to A and W is increasing in $\mathcal{L}_{\leqslant W}$ if and only if $\mathcal{F} = (a_1, a_2, \dots, a_k)$.*

Proof. The key observation is the following: if $b > a_k$ then $A \cup \{b\}$ is an independent set (since A is a no broken circuit set). We prove that every maximal, increasing flag adapted to A and W is $\mathcal{F} = (a_1, a_2, \dots, a_k) = \epsilon(\mathcal{F})$ by induction on k; the base case is trivial. Let $\mathcal{F} = (a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(k)})$ be a maximal increasing flag adapted to A and W, by inductive step we assume that the flag $\mathcal{F}' = (a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(k-1)}) = (a_1, \dots, \widehat{a_{\sigma(k)}}, \dots, a_k)$ has labeling $\epsilon(\mathcal{F}') = (a_1, \dots, \widehat{a_{\sigma(k)}}, \dots, a_k)$. The labeling $\epsilon(\mathcal{F}) = (a_1, \dots, \widehat{a_{\sigma(k)}}, \dots, a_k, b)$ for some $b \in \{1, \dots, n\}$ is increasing but from the key observation we have $b \leqslant a_k$. By definition of the labeling $b \geqslant a_k$, so $b = a_k$ and $\sigma(k) = k$.

Again by induction, we prove that the flag $\mathcal{F} = (a_1, a_2, \dots, a_k)$ has labeling $\epsilon(\mathcal{F}) = (a_1, a_2, \dots, a_k)$ and so is increasing. By inductive step $\epsilon((a_1, a_2, \dots, a_{k-1})) = (a_1, a_2, \dots, a_{k-1})$, so $\mathcal{F}$ has labeling $\epsilon(\mathcal{F}) = (a_1, \dots, a_{k-1}, b)$ with $b \geqslant a_k$ by definition and with $b \leqslant a_k$ by the key observation. We have proven that the flag $(a_1, a_2, \dots, a_k)$ is increasing. □

Proof of Theorem 5.12. Let $g : H(T; \mathbb{Q})[e_{W,A}] \to \mathrm{D}$ be the map defined by $g(\chi^*(\omega)) = \beta_\chi$ for all $\chi \in \Lambda$ and by $g(e_{W,A}) = \Xi_{W,A}$. It is well-defined since $\beta_{a\chi + b\eta} = a\beta_\chi + b\beta_\eta$ for all $a, b \in \mathbb{Z}$. The ideal I is contained in $\ker g$ by Lemmas 5.8, 5.9 and 5.11, so $g : H(T; \mathbb{Q})[e_{W,A}]/I \to \mathrm{D}$ is well-defined.

We will show the injectivity of g considering it as morphism of $H^\bullet(T)$-module. Consider the monomial base of $\mathrm{D}^{0,\bullet}$ provided in the second part of Lemma 5.1. Note that in the expansion of $g(e_{W,A}) = \Xi_{W,A}$, for A no broken circuit set in $\mathcal{L}_{\leqslant W}$, appears only one monomial $\prod_{L \in \mathcal{F}} \sigma_L$ with $\mathcal{F}$ increasing chain (in $\mathcal{L}_{\leqslant W}$) by Lemma 5.15. For each $W \in \mathcal{L}$ and A no broken circuit set in $\mathcal{L}_{\leqslant W}$, we choose a set $B(A)$ such that $A \sqcup B(A)$ is a basis and a cone $C(A) \in \Delta$ contained in $\operatorname{Ann} \Lambda_A$ of maximal dimension. Let us suppose that

$$g\left(\sum_{W \in \mathcal{L}} \sum_{A \text{ n.b.c. in } \mathcal{L}_{\leqslant W}} \alpha_{W,A} e_{W,A} \psi_{W,A}\right) = 0$$

for some $\psi_{W,A} \in H^\bullet(W; \mathbb{Q})$ and some $\alpha_{W,A} \in \mathbb{Q}$ with at least one $\alpha_{W,A}$ different from zero. Let $(\overline{W}, \overline{A})$ such that $|\overline{A}|$ is maximal among all (W, A) with $\alpha_{W,A} \neq 0$. Let $\psi \in H(T)$ such that $\psi_{\overline{W},\overline{A}} \psi|_{\overline{W}} = \psi_{B(\overline{A})}$. Let $\overline{\mathcal{F}}$ be the list of all elements in $\overline{A}$ ordered increasingly. By Lemma 5.15, in

$$g\left(\sum_{W \in \mathcal{L}} \sum_{A \text{ n.b.c. in } \mathcal{L}_{\leqslant W}} \alpha_{W,A} e_{W,A} \psi_{W,A} \psi\right) = \sum_{W \in \mathcal{L}} \sum_{A \text{ n.b.c.}} \alpha_{W,A} \Xi_{W,A} g(\psi_{W,A} \psi)$$

$$= \sum_{W \in \mathcal{L}} \sum_{A \text{ n.b.c.}} \sum_{\mathcal{F} \text{ adap. } A,W} \alpha_{W,A} \frac{m(\mathcal{F})}{m(A)} \left(\prod_{a \in A} x(\mathcal{F}, a)\right) g(\psi_{W,A} \psi)$$

the monomial $z = \prod_{L\in\overline{F}} -\sigma_L \prod_{j\in C(\overline{A})} b_j$ associated to the increasing flag $\overline{\mathcal{F}}$, can appear only in the addendum $\alpha_{\overline{W},\overline{A}}\Xi_{\overline{W},\overline{A}}g(\psi_{\overline{W},\overline{A}}\psi)$. In particular, z appears only in the expansion of

$$\begin{aligned}\left(\prod_{L\in\overline{F}} -\sigma_L\right) g(\psi_{\overline{W},\overline{A}}\psi) &= \left(\prod_{L\in\overline{F}} -\sigma_L\right) g(\psi_{B(\overline{A})})\\ &= \prod_{L\in\overline{F}} -\sigma_L \prod_{b\in B(\overline{A})} \beta_b\\ &= m(B(\overline{A}))\prod_{L\in\overline{F}} -\sigma_L \sum_{K\in C_\Delta^{|B(\overline{A})|}} \mathrm{Vol}(\pi(K))\prod_{c\in K} b_c.\end{aligned}$$

The coefficient of z in $(\prod_{L\in\overline{F}} -\sigma_L)g(\psi_{\overline{W},\overline{A}}\psi)$ must be zero, but it is (up to a sign) equal to $\alpha_{\overline{W},\overline{A}} m(B(\overline{A}))\,\mathrm{Vol}(\pi(C(\overline{A})))$ (cf. Corollary 5.6). The volume $\mathrm{Vol}(\pi(C(\overline{A})))$ is different from zero because $\Lambda_{\overline{A}} \otimes \mathbb{Q} \oplus \Lambda_{B(\overline{A})} \otimes \mathbb{Q} = \Lambda \otimes \mathbb{Q}$. We have $\alpha_{\overline{W},\overline{A}} = 0$ contradicting the assumption, hence g is injective.

Note that the range of g is contained in $\ker \mathrm{d}$ by Lemma 5.4 and in the subalgebra $\mathrm{D}^{0,\bullet}$. The map g induces an injective map

$$g : {H^\bullet(T;\mathbb{Q})[e_{W,A}]}/{I} \to H^\bullet(\mathrm{D},\mathrm{d}) \cong H^\bullet(M(\mathcal{A});\mathbb{Q})$$

since d is of bi-degree $(2,-1)$. It is also surjective because $H^\bullet(T;\mathbb{Q})[e_{W,A}]/I$ and $H^\bullet(M(\mathcal{A});\mathbb{Q})$ have the same dimension (see Lemma 5.14). We have proven the theorem. □

Remark 5.16. Theorem 5.12 is a generalization of [11, Theorem 5.2] and analogous to [5, Theorem 6.13]. Indeed, if $\mathcal{A}$ is totally unimodular and the circuit $C = \{0,1,\dots,n\}$ is oriented with $c_0 = -$, $c_i = +$ for $i > 0$, we obtain [11, eq. 20].

We have chosen the generator associated with an hypertorus S_a as $\Xi_{S_a,\{a\}} = -\sigma_{S_a} + \beta^-_{\chi_a}$ that depends on the choice of one between χ_a and $-\chi_a$. Another possible choice of generators were $\Xi_{S_a,\{a\}} = -2\sigma_{S_a} + \beta^-_{\chi_a} + \beta^+_{\chi_a}$, this would be lead to the same presentation of [5, Theorem 6.13].

Remark 5.17. Theorem 5.12 gives another proof of the rational formality of toric arrangements, previously proven in [5, 16].

Conjecture 5.18. *Substituting in Equation (12)* $\frac{m(A)}{m(X\setminus\{j\})}\psi_B$ *with* $\prod_{i=1}^{|B|}\psi_{\chi_i}$, *where* $(\overline{\chi_i})_i$ *form a basis of* Λ_C/Λ_A *with the same orientation of* $(\overline{\chi}_b)_{b\in B}$, *the cohomology ring with integer coefficients have a presentation analogous to the one in Theorem 5.12.*

Furthermore, our approach to the computation of cohomology ring for toric arrangements seems suitable to be extended to the non-divisorial case. We hope to develop this line of research in a future paper.

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