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OLIGOPOLY DYNAMICS WITH ISOELASTIC DEMAND: THE JOINT EFFECTS OF MARKET SATURATION AND STRATEGIC DELEGATION

Lorenzo Cerboni Baiardi*,[1] and Fabio Lamantia*,+

* University of Calabria, Via Bucci 0-1C, Rende (CS), Italy

+ Faculty of Economics, VŠB–Technical University of Ostrava, Ostrava, Czech Republic

Abstract. In the framework of a Cournot oligopoly game with isoelastic demand, we examine the simultaneous presence of both market saturation and strategic delegation. Although these two (realistic) aspects have already been considered in the literature each on its own, we aim at deepening their joint interactions when matched together in oligopolistic competition. In addition, we admit the possibility that delegation activities actuated by firms to weaken or even exclude competitors from the market may cease if undertaken by successful players, which thus regain their pure profit maximizing behavior. In this context, a limited market saturation level (positively) influences the effectiveness of delegation strategies and, at the same time, can sustain equilibrium configurations for the winning (monopolistic) firm even under the isoelastic market structure. Through local stability analysis, we show how the combination of strategic delegation with market saturation contributes to determine the equilibrium number of active players and the local asymptotic stability of the (economically relevant) equilibrium. Moreover, non-equilibrium dynamics reveal the presence of periodic cycles along which a firm is active while its competitors alternatively exits and enters the market. We show why these interesting scenarios are due to the joint interplay between strategic delegation and market saturation.

Keywords: strategic delegation; market saturation; Cournot oligopoly; isoelastic demand; non-linear dynamics.

1. Introduction

The seminal paper [10] presents chaotic dynamics in a micro-founded duopoly with isoelastic demand. [10] has been a fundamental source of inspiration for

[1]Corresponding author: lorenzo.cerboni@unical.it

other works on mathematical economics and industrial organization, see [3] for a broad overview.

In this paper, we extend the model in [10] by considering the joint effects of two further elements, namely market saturation and strategic delegation. The first addition that we consider, the market saturation of the demand proposed in [14], represents an important and realistic element of the model, linking inter-temporally the market demand at a certain time with the quantity placed on that market at previous times, which is mostly the case for durable goods. A further important point related to market saturation, often overlooked in the literature, concerns the economic significance of the model with isoelastic demand even in the case of a single active firm, consult [15] for a discussion about the monopoly model with isoelastic demand.

The second addition we address is related to strategic delegation, which entails firms setting their choices according to the maximization of a performance criterion different from profit. The concept of strategic delegation has been prevalent in the industrial organization literature starting from the seminal contributions in [16] and [5]. In particular, here we assume that firms maximize a convex combination of profits and revenues, see for the isoelastic case [4]. When the company management is committed to maximizing (partially) sales, firms behave more aggressively. This behavior, if not reciprocated, can severely damage competitors.

The joint presence of market saturation and strategic delegation in the same model leads to interesting dynamic scenarios. This is indeed more evident when more than two players operate with different costs of production, see also [11] and [12] on this point. More precisely, in the context considered, the presence of delegation accentuates the possible outcomes with asymmetries in firms' costs. As highlighted in [6], oligopoly models with asymmetric costs have been relatively under-studied, with their equilibrium configurations in which some firms may be inactive. This is particularly interesting also for its dynamic consequences, as shown in this paper.

For instance, starting from an oligopoly it is possible that companies that compete harder will push less aggressive companies out of the market, even to the case of having only a monopolist in the market, which is economically meaningful under isoelastic demand due to the presence of market saturation.

To introduce a further element of interest in the model, we assume the possibility that the degree of *aggressiveness* of the delegating firms depends on the number of active firms in the market. For example, if only a monopolist operates in the market, it is likely that, having no competitors, it maximizes standard profit. The degree of delegation, therefore, can dynamically depend on the competitive pressure of the market according to a trigger-like strategy.

Taking into account market saturation and strategic delegation as a function of competitive market pressure, we can mathematically express the model as a discontinuous $(N+1)$-dimensional map, where N is the number of potential incumbents in the market. We study the possible equilibrium configurations under asymmetric costs and asymmetric degrees of delegations, in particular highlighting the number of active firms of the different configurations. The global dynamics analysis of the model shows the possible endogenous entry/exit scenarios by firms that can adapt their level of delegation, that is aggressiveness, according to actual market pressure.

More precisely, we first discuss the presence of such periodic entry/exit patterns in the duopoly: the more aggressive firm manages to send the competitor out of the market thus becoming monopolist and, after some latent period, a new incumbent firm succeeds in re-entering that market once the active firm reverts to standard profit maximizing behavior as market saturation reduces over time. We then detect similar cases with more than two firms and non-equilibrium productivity cycles whenever no steady state exists or it is unstable. To the best of our knowledge, this is the first time that this kind of entry/exit dynamics arises endogenously in repeated oligopoly games. Notice that the model we propose does not address directly firms' entry/exit decisions, which are rather a possible dynamical outcome of myopic interaction between boundedly rational players which adjust their competitiveness level according to the pressure in the market.[2]

The paper is organized as follows. Section 2 describes the model, characterizes the equilibrium, and provides interpretations of the system with saturation only and with saturation and delegation. Here in particular we consider the case of the monopolist with saturation. Section 3 describes the typical behavior of the system when the equilibrium is locally unstable or does not exist and

[2]In the recent literature, several works model directly firms' entry/exit decisions within the framework of deterministic or stochastic dynamic games, see for recent contributions [2], [9] and [1]

provides an economic interpretation of such dynamics. Section 4 concludes. All technical proofs are contained in the Appendix.

2. Formulation of the model

We consider a Cournot oligopoly with $N \geq 2$ firms in a market characterized by an isoelastic demand with unitary elasticity (see [10]). Player i sets its output q_i at the (constant) unit cost c_i, where $i = 1, \cdots, N$. In the following, $Q = \sum_{j=1}^{N} q_j$ denotes the aggregate production of the industry and we set $Q_{-i} = Q - q_i$. Along the line marked by [14] (see also [8]), the market's saturation $\mathcal{Q}$ is the sum between the aggregate current production and the unsold production from previous periods. Assuming a constant absorption rate, the saturation level is recursively defined as

$$\mathcal{Q}' = \delta \mathcal{Q} + Q' \tag{1}$$

where parameter $\delta \in (0, 1)$ is the *persistence* rate while its complement $1 - \delta$ is the *absorption* rate and where " $'$ " denotes the unit time advancement operator. Assuming the market price to be determined as $p = \mathcal{Q}^{-1}$, profits of firm i can be expressed as $\pi_i = q_i \mathcal{Q}^{-1} - c_i q_i$.

Remark 1. *A possible economic motivation for the market saturation model proposed in* [14] *can be given assuming the aggregate production Q to be entirely delivered by manufacturers to the market and integrated with the part $\delta\mathcal{Q}$ of goods unsold in the previous period and kept as inventories by retailers. In this setting, retailers are willing to manage the entire quantity $\mathcal{Q}$, corresponding to the demand for the product, so that having price $p = \mathcal{Q}^{-1}$ is consistent with the isoelastic demand hypothesis. Under this interpretation, p is the price recognized to manufacturers, which depends not only on the current total quantity placed on the market but also on the warehouse stocks present at the retailers. It is worth noting that retailers can buy from manufacturers at lower prices in the next period by withholding some of the supply from direct sales in the present and allocating it as inventory to be used later. The term $\delta\mathcal{Q}$ is inversely related with the next period price indeed, being $p' = (\delta\mathcal{Q} + Q')^{-1}$. We also refer the reader to the literature on inventories and supply chain management strategies as well as on retailers' (secondary) market operating modes (see e.g.* [7] *or* [17]*).*

Implementing a delegation scheme as in [5], we assume player's i objective to be the following convex combinations of profits and sales:

$$W_i = \alpha_i \pi_i + (1 - \alpha_i) q_i p = \frac{q_i}{Q} - \alpha_i c_i q_i \tag{2}$$

where coefficient α_i and its complement $1 - \alpha_i$ represent the incentive weights firm's i management poses on maximizing profits and sales respectively. With this performance criterion, objective maximising players are interested both in raising profits and the volume of sales. As it is clear from the second equality in (2), such players behave as profits maximizing, treating production costs as if they were lower than they actually are. The reason for this attitude is to allow players to set non-optimal over-productions, which, in turn, results in aggressive behavior to harm competitors. Indeed, the very existence of a management incentive based on (2) makes this non-optimal behavior credible for opponents, that is (2) introduces a commitment device for such behaviors. As already remarked, according to (2) manager i views $\alpha_i c_i$ as its marginal cost of production. Thus, the lower the *adjusted* cost $\alpha_i c_i$ considered by the manager of firm i, the higher the quantity that firm i places on the market, thus resulting in a greater level of competitiveness (or aggressiveness) of firm i.[3] Accordingly, in the following we will refer to the quantity $\alpha_i c_i$ as the level of *competitiveness* of player i.

Similarly, a monopolist could put more incentives on sales as well, in order to maintain a sufficiently high level of competitive pressure, aiming at discouraging other firms to enter the market. Assuming that the relevance given by a player to sales in their objectives is straightened by the presence of active competitors, we consider endogenous weights $\alpha_i : \mathbb{R}_{\geq 0} \to [0, 1]$ defined by

$$\alpha_i(Q_{-i}) = \begin{cases} \alpha_i^0 & \text{if } Q_{-i} > 0 \\ \alpha_i^1 & \text{otherwise} \end{cases} \tag{3}$$

where $0 < \alpha_i^0 \leq \alpha_i^1 \leq 1$, for all $i = 1, \cdots, N$.[4]

The dynamic choice of quantities by the oligopolists occurs through the well-known best reply dynamics described in [10], although here firms try to

[3]If firm i maximizes W_i with respect to q_i, so that $q_i^* = \arg\max_{q_i} W_i$, this claim easily follows by observing that $\frac{\partial q_i^*}{\partial(\alpha_i c_i)} < 0$.

[4]Clearly, the assumption of a stepwise dependence of α_i on the aggregate current production of the rest of the industry Q_{-i} is a simplification, but it generalizes the delegation scheme in [5] and introduces a dependence of the level of aggressiveness of a firm on the presence of active competitors.

maximize convex combinations of profits and sales as performance criteria, having expectations on the next period saturation level.

According to equation (1), at period $t+1$ firm i forecasts a saturation level

$$\mathcal{Q}^{i,e}(t+1) = \delta\mathcal{Q}(t) + Q_{-i}^{i,e}(t+1) + q_i(t+1) \tag{4}$$

where $Q_{-i}^{i,e}(t+1) = \sum_{j\neq i} q_j^{i,e}(t+1)$ is the aggregate output that i expects from the rest of the industry. At decision stage $t+1$, firm i sets its output at its (non negative) best reply level

$$q_i(t+1) := \max\left(0,\ R_i(t+1)\right) \tag{5}$$

where $R_i(t+1)$ is

$$R_i(t+1) = \arg\max_q W_i^e(t+1) = \sqrt{\frac{\delta Q(t) + Q_{-i}^{i,e}(t+1)}{\alpha_i(Q_{-i}^{i,e}(t+1))c_i}} - \left(\delta Q(t) + Q_{-i}^{i,e}(t+1)\right)$$

In the following, we will consider static expectations for all players. Hence, $q_j^{i,e}(t+1) = q_j(t)$ holds for all i and $j \neq i$. As a consequence, the equality $Q_{-i}^{i,e}(t+1) = Q_{-i}(t)$ follows. The oligopoly dynamics is determined by the $(N+1)$-dimensional map $T : (q_1, \cdots, q_N, \mathcal{Q}) \to (q_1', \cdots, q_N', \mathcal{Q}')$ given by recurrences (5) and (4):

$$T : \begin{cases} q_i' &= \max\left(0,\ \sqrt{\dfrac{\delta Q + \sum_{j\neq i} q_j}{\alpha_i\left(\sum_{j\neq i} q_j\right) c_i}} - \left(\delta Q + \sum_{j\neq i} q_j\right)\right), \quad i = 1,\cdots,N \\[2em] \mathcal{Q}' &= \delta\mathcal{Q} + \sum_{i=1}^N q_i' \end{cases}$$

Remark 2. *In the special case in which $\alpha_i^0 = \alpha_0^1 = 1$ for all $i = 1,\cdots,N$, players maximize own profits and map T differs from the one considered in Section 3 of* [14] *for the case of static expectations because of the* max *operator in the first N components of map T.*

Without loss of generality and to present the results in a more concise way, we consider firms' labels to be assigned as specified in the following assumption.

Assumption 1. *We set $\alpha_i^0 c_i \leq \alpha_j^0 c_j$, for all $i, j = 1,\cdots,N$ and $j \geq i$.*

Under Assumption 1, firms are ordered with decreasing levels of competitiveness. The following proposition provides analytic expressions of the steady states of map T. Differently from [14], we characterize not only the inner equilibrium in which all firms operate, but also boundary equilibria in which some but not all firms deliver their output in the market. In the following, we distinguish between N, the total number of potential firms in the market, and n the number of active firms, that are firms that deliver positive quantity in the market.

Proposition 1. *Under Assumption 1, a steady state $E = (\bar{q}_1, \cdots, \bar{q}_N, \bar{Q})$, at which n active firms are present, exists in the following alternatives:*

i) *if $\alpha_1^1 c_1 \leq \delta \alpha_2^0 c_2$ holds; in this case $n = 1$, $\bar{q}_1 = \delta(1-\delta)/(\alpha_1^1 c_1)$, $q_i = 0$ for all $1 < i \leq N$ and $\bar{Q} = \delta/(\alpha_1^1 c_1)$;*

ii) *if $\alpha_1^0 c_1 > \delta \alpha_2^0 c_2$ holds; in this case $n > 1$ is determined as*

$$n = \max\left\{k : \frac{k+\delta-1}{\sum_{i=1}^k \alpha_i^0 c_i} - \frac{1}{\alpha_k^0 c_k} < 0\right\}$$

Moreover

$$\bar{q}_i = \frac{n+\delta-1}{\sum_{i=1}^n \alpha_i^0 c_i}\left(1 - \alpha_i^0 c_i \frac{n+\delta-1}{\sum_{i=1}^n \alpha_i^0 c_i}\right)$$

if $i \leq n$ and $q_i = 0$ otherwise and

$$\bar{Q} = \frac{n+\delta-1}{\sum_{i=1}^n \alpha_i^0 c_i}$$

Proof. See Appendix A. □

In the remaining part of this section we will discuss some economic consequences of Proposition 1.

2.1. **Economic insights for the scenario with saturation and without delegation.** In this case, players are standard profit maximizers, namely $\alpha_i^0 = \alpha_i^1 = 1$ for all i. As observed in Remark 2, map T corresponds to the dynamical system studied in [14] in the case of static expectations, with the inclusion of the "max" operator to account for possible inactive firms. It follows that steady states of T may be internal, i.e. characterised by strictly positive components, describing equilibrium configurations of the game characterized by active players only. Such internal points have already been studied in [14]. Moreover, boundary steady states of T may also be present, which are characterised by some (but not all) null components, describing equilibrium configurations in which some players decide not to produce. These boundary steady states will play an important role in the dynamics of the system, as described below.

2.1.1. *Market activity and saturation.* Saturation discourages firms to enter the market with only the most efficient ones surviving at equilibrium. Indeed, if n is the equilibrium number of active firms for a given value of δ, then for any other value δ' such that $\delta' > \delta$ no more than n firms can be active at equilibrium. In other words, n is non-increasing in δ. This is an immediate consequence of the definition of the number of active firms provided in Proposition 1. In addition, we note that active firms are the most efficient ones. Indeed, under Assumption 1, firms are ordered (i.e. indexed) with non decreasing marginal costs, so that n corresponds to the index of the less efficient player among those that are active. Clearly, all subsequent indexes denote players with higher unit costs.

It is worth noticing that the higher the value of parameter δ is (or equivalently the less the absorption rate is), the more past productions persist in the market and, hence, the higher values of equilibrium supply $\bar{\mathcal{Q}}$ are reached (see Figure 1 left panel, showing the equilibrium saturation level $\bar{\mathcal{Q}}$ at increasing values of δ).[5] On one hand, this is coherent with the inverse relationship between the number of active firms n and the persistence rate δ: market's saturation causes the reduction of prices with less efficient firms being forced to be inactive in order not to earn negative profits (see Figure 1 right panel, showing the number of active firms at increasing values of δ). On the other hand, in the presence of saturation, active firms will set restrained productions in order to maintain sufficiently high prices, so that their profits remain positive. This attests for an inverse relationship between the average individual equilibrium productions and the persistence rate δ. To show this, let us pick δ_1 and δ_2 such that $\delta_1 < \delta_2$. If $n \geq 2$ firms are active, the average individual production is lower when $\delta = \delta_2$. If $\tilde{q}_{\delta_1}$ and $\tilde{q}_{\delta_2}$ denote such average productions, the relation $\tilde{q}_{\delta_2} < \tilde{q}_{\delta_1}$ always holds, which follows from the relation

$$\frac{\bar{q}_{\delta_1}}{\bar{q}_{\delta_2}} = \frac{1-\delta_1}{1-\delta_2} \cdot \frac{n+\delta_1-1}{n+\delta_2-1} > 1$$

that is always satisfied whenever $n \geq 2$.

[5]Saturation $\bar{\mathcal{Q}}$ is indeed strictly increasing in δ. Indeed, $\bar{\mathcal{Q}}$ is continuous with respect to δ. Moreover, $\bar{\mathcal{Q}}$ is piecewise differentiable and, when the derivative is defined, it results $\partial\bar{\mathcal{Q}}/\partial\delta = 1/\left(\sum_{i=1}^{n} c_i\right) > 0$.

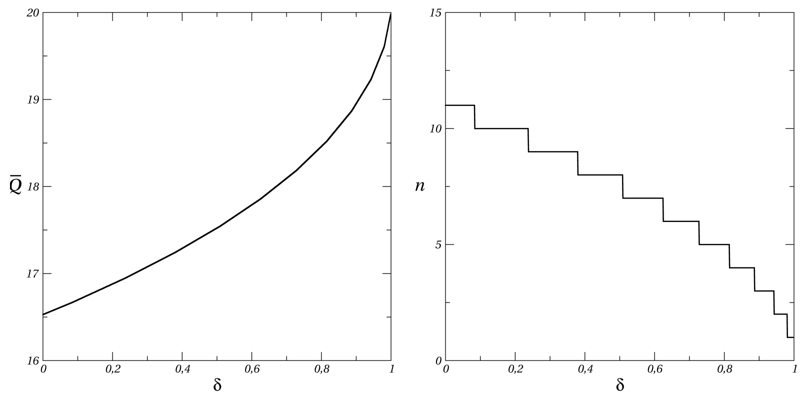

FIGURE 1. $\bar{\mathcal{Q}}$ (left panel) and n (right panel) at increasing δ. Common parameters are $N = 50$, $c_i = 0.05(1 + i/N)$ with $0 \leq i < N$. Moreover, $\alpha_i^0 = \alpha_i^1 = 1$, for all $i = 1, \cdots, N$.

2.1.2. *Monopolistic market.* The non-existence of an equilibrium in the monopoly case with isoelastic demand is clearly described in [15]: "*one of the best known results in economic theory is that a monopolist's price setting problem does not possess a solution in the case of isoelastic demand curves with own-price elasticity (in absolute value) smaller than or equal to one. However, many estimates of elasticities yield values not exceeding one. Moreover, the non-existence of a solution is somewhat embarrassing for economic modelling since for example Cobb-Douglas utility functions, which are probably the most basic and the most frequently adopted type of utility functions used by economists, yield unit elastic demand functions and thus cannot be employed in general equilibrium models with a monopolist*".

Indeed, in absence of saturation, the monopolistic profit function $1 - C(q_1)$ does not attain the maximum over the set of positive quantities (for any increasing cost function). Then, a monopolistic firm has the incentive to set infinitesimal amounts of output at unbounded prices. However, the presence of market's saturation allows to solve the monopolist's quantity setting problem in the case of isoelastic demand with unitary elasticity. This situation is described by the monopolistic equilibrium of map T, to which steady state E reduces when only one active firm is present. The monopolistic equilibrium is feasible because of the presence of unsold products, which implies a minimum level of market supply and, in turn, bounded prices (even if the monopolist sets null output). From the monopolist's viewpoint, the presence of unsold products is equivalent to the presence of a (virtual) competitor (i.e. the *saturated* market behaves as the *zero player* or *nature*). Saturation causes a change in the structure of monopolist's profits, so that optimal productions belong to a set of feasible (i.e. non-negative) outputs.[6] In the present model, monopolistic equilibrium arises when only player 1, the most efficient, is active while all its competitors have no convenience in entering the market because of their higher production costs (recall that firms are labeled in descending order of marginal costs). By Proposition 1, this occurs when condition $c_1 \leq \delta c_2$ holds, ensuring positiveness of firm 1's output and null productions of all its competitors.

2.2. **Economic insights for the scenario with both saturation and delegation.** Incentives on sales are now included in the discussion. The outcomes under unitary isoelastic demand are analogous to those occurring in the linear case, examined in the seminal works [5] and [13]: such incentives on sales lead firms to set non-optimal over-productions in order to harm competitors. Indeed, if we apply comments given in subsection 2.1.1 to this case (where the role of efficiency of firms is replaced by competitiveness) it can be stated that saturation discourages firms to join the market with only the most *competitive* ones surviving at equilibrium. Moreover, the presence of saturation straightens the conflicting power of delegation. Indeed, suppose that $n \geq 2$ active firms are present at equilibrium. Hence, the inequality

$$\frac{n+\delta-1}{\sum_{i=1}^{n} \alpha_i^0 c_i} - \frac{1}{\alpha_n^0 c_n} < 0$$

holds. This is equivalent to $\alpha_n^0 c_n < \dfrac{n}{n+\delta-1}\left(\dfrac{1}{n}\sum_{i=1}^{n} \alpha_i^0 c_i\right)$, where the average level of competitiveness is highlighted in the brackets. Taking into account Assumption 1, it follows that

$$\alpha_1^0 c_1 \leq \alpha_2^0 c_2 \leq \cdots \leq \alpha_n^0 c_n < \frac{n}{n+\delta-1}\left(\frac{1}{n}\sum_{i=1}^{n} \alpha_i^0 c_i\right) \tag{6}$$

Note that the value $n/(n+\delta-1)$ is decreasing in δ and belongs to the interval $(1,2)$. The sequence of inequalities in (6) is satisfied whenever competitiveness levels are not exceeding too much their average value. In detail, the value of δ fixes an upper threshold that bound competitiveness near their average. Because of the multiplicative factor $n/(n+\delta-1)$, which is decreasing in δ, such upper threshold moves towards the average level of competitiveness the more

[6] which indeed becomes $q_1/(\delta\bar{Q}+q_1) - c_1 q_1$ that is concave in $q_1 \in [0,+\infty)$

saturated the market is. Then, saturation reduces the displacement between firms' competitiveness and the competitiveness average value that is allowed for them to remain active. As a result, if n firms survive at equilibrium at a given value of δ, the same firms may not be all active for another value $\delta' > \delta$. A further consequence of the sequence of inequalities in (6) is the requirement of having similar competitiveness degrees among active players at equilibrium. In other words, active firms are characterized by sufficiently similar competitiveness level. This entails that saturation favors, or rather pushes, equilibrium configurations at which active firms are almost homogeneous with respect to their competitiveness level.

This result is not surprising recalling the literature on delegation, which highlights how the deformation of the objective function of the firms implies the existence of suboptimal equilibrium configurations with respect to the standard Cournot-Nash ones. Despite their sub-optimality, these equilibria are indeed achieved if the delegating behavior is reciprocated in the market. In other words, the final outcome in the oligopoly resembles a prisoner's dilemma, in which all agents delegate but they get less than they would have gotten by maximizing standard profit, see again [13] on this point. What we add in the picture is how the presence of saturation reinforces the need for homogeneous behavior to survive in equilibrium, as clearly implied by the inequalities in (6). In other words, saturation results in a greater need to conform to the most aggressive agents, that is, to reciprocate the delegating behavior to survive in the market. Indeed, the higher the saturation level is (which corresponds to low absorption rates) the more the competitiveness levels must approach the same value (their average) to maintain the number of active firms unchanged.

To highlight a second relevant interplay between saturation and delegation, let us consider the extreme consequence to which delegation may lead, namely the persistence of a unique dominant firm and the exclusion of any other competitor in the market. This scenario corresponds to the monopolistic equilibrium described in Proposition 1, where the dominant firm is the most competitive one. Remarkably, the possibility for the delegation mechanisms to bring the industry towards a monopolistic configuration is possible in the present framework because of saturation, see the discussion in Subsection 2.1.2. Indeed, the condition for the emergence of the monopolistic equilibrium requires both sufficiently high competitiveness level of firm 1, even when it operates as a monopolist, and sufficiently high values of δ (sufficiently low absorption rates). Conversely, monopolistic configurations are possible when δ is small (high absorption rates) only in the residual occurrence of extremely high competitiveness level of firm 1, even in absence of competitors. We stress again that no monopolistic equilibrium is defined with no saturation (i.e. when $\delta = 0$).

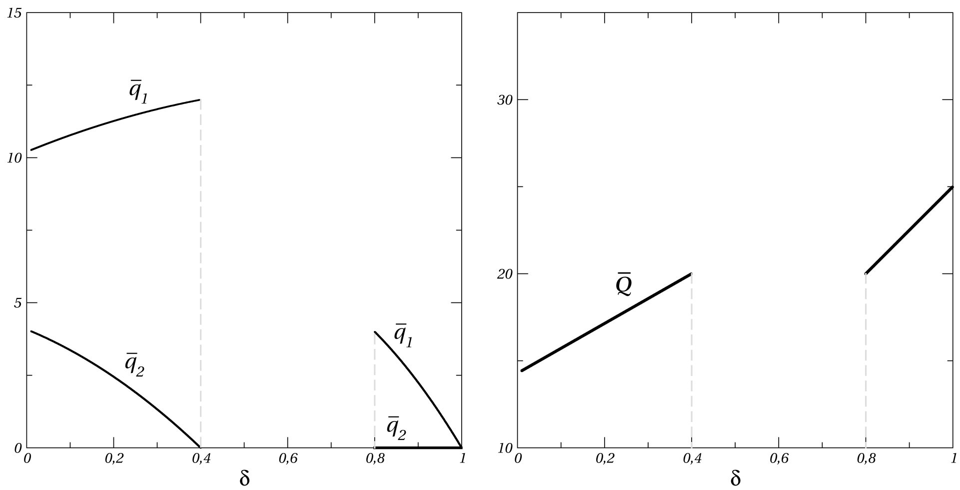

FIGURE 2. Equilibrium outputs $\bar{q}_1$ and $\bar{q}_2$ (left panel) and saturation level $\bar{Q}$ (right panel) when varying δ. $N = 2$, $c_1 = c_2 = 0.05$, $\alpha_1^0 = 0.4$, $\alpha_1^1 = 0.8$, $\alpha_2^0 = \alpha_2^1 = 1$.

We show in Figure 2 simulations of equilibrium outputs (left panel) and saturation level (right panel) in a duopoly as parameter δ increases. In this example, $N = 2$ and both firms are active when δ is below the threshold $\alpha_1^0 c_1/\alpha_2^0 c_2$ (0.4 in that simulation). In this occurrence, competitiveness parameters allow both firms to set positive outputs. Differently, values of δ beyond that threshold makes non sustainable the simultaneous presence of two active firms, so either the equilibrium does not exist or monopolistic configurations are observed for sufficiently low absorption values (high δ). Scenarios in which equilibria do not exist describe cases in which the more competitive firm 1 forces firm 2 not to produce in order not to incur in negative values of its own objective. Once firm 1 becomes dominant, it tunes its incentive according to (3), that is by putting the higher weight α_1^1 on profits.

In this situation, some other competitor (starting with firm 2) has incentive to enter the market, which fact, however, implies the adoption of higher incentive α_1^0 by firm 1 in the next period. This mechanism prevents the formation of equilibrium configurations. Differently, as it will be discussed in Section 3, this makes possible the appearance of periodic cycles along which some less competitive firms may enter and exit the market. Sufficiently high levels of

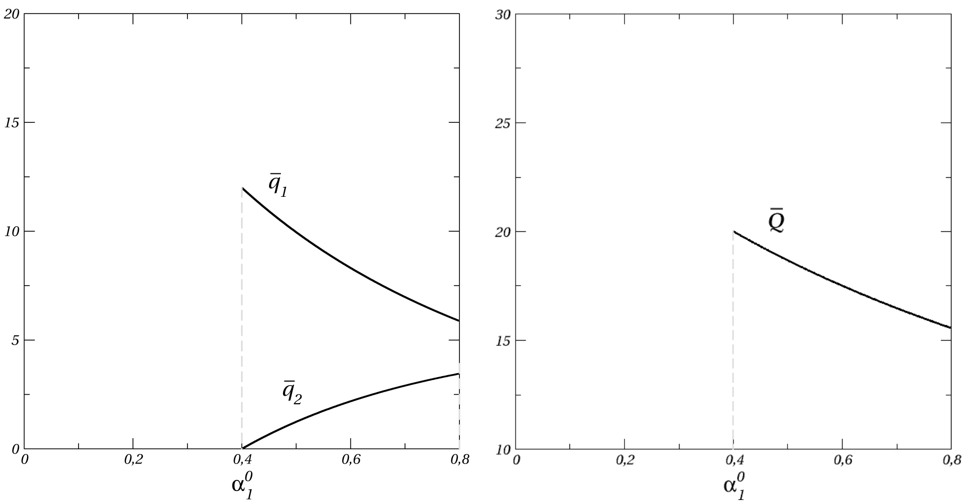

Figure 3. Equilibrium productions $\bar{q}_1$ and $\bar{q}_2$ (left panel) and saturation $\bar{Q}$ (right panel) varying α_1^0 up to α_1^1. $N = 2$, $c_1 = c_2 = 0.05$, $\alpha_1^1 = 0.8$, $\alpha_2^0 = \alpha_2^1 = 1$, $\delta = 0.4$.

heterogeneity in competitiveness at a given δ imply the non-existence of steady states as well. In Figure 3, equilibrium outputs $\bar{q}_1$ and $\bar{q}_2$ (left panel) and $\bar{Q}$ (right panel) are shown in a duopoly, with α_1^0 varying from 0 to α_1^1. When α_1^0 is close to zero, competitiveness of firms 1 and 2 are so much different to exclude the existence of a steady state. Instead, as α_1^0 increases, heterogeneity between players reduces, steady state E emerges and the outputs of the two players approach each other.

We end this section with Figure 4, which shows equilibrium outputs (left panel) and saturation (right panel) in the case of an oligopoly with 5 competitors as parameter δ increases. The increment of δ entails increasing levels of saturation, which, in turn, explain the step-by-step exit of less competitive firms. The qualitative behavior of equilibrium production as δ increases can be explained as in the duopoly case once only the most competitive firms 1 and 2 remain active.

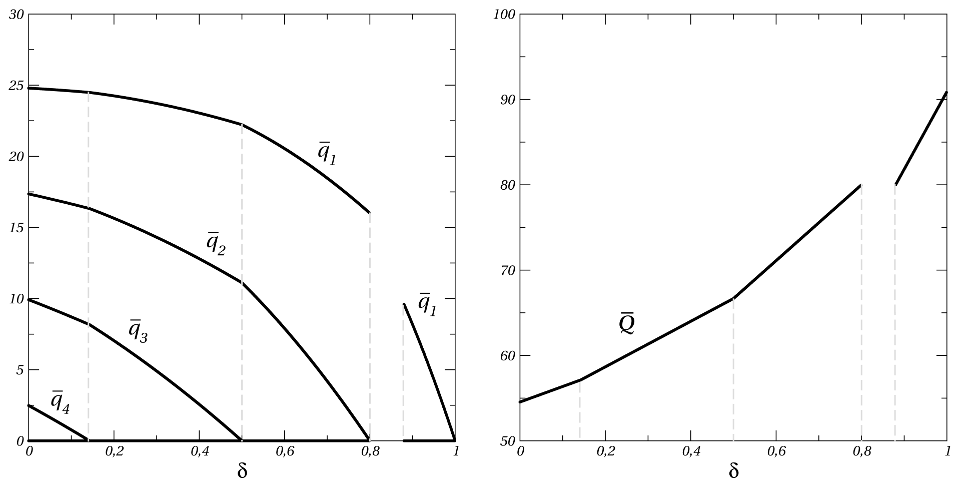

FIGURE 4. Equilibrium productions $\bar{q}_i$, $i = 1, \cdots 5$ (left panel) and saturation $\bar{\mathcal{Q}}$ (right panel) varying δ. $N = 5$, $c_i = 0.05$, $\{\alpha_i^0\}_{i=1}^5 = \{0.2, 0.25, 0.3, 0.35, 0.4\}$, $\{\alpha_i^1\}_{i=1}^5 = \{0.22, 0.9, 1, 1, 1\}$.

3. Non-equilibrium dynamics

Let us focus on dynamic patterns described by map T. We limit our attention to those trajectories that are iterations of points in the set

$$F = \{\boldsymbol{x} \in \mathbb{R}^{N+1} : \ x_k \geq 0, \text{ all } k = 1, \cdots, N \text{ and } x_{N+1} > 0\}. \tag{7}$$

Such points are indeed characterised by having the first N components non-negative and a strictly positive value of the last component, as they are the only points that represent feasible configurations of this Cournot game. In the following, we will call *feasible* those trajectories that are bounded and made up by elements of F only. Feasible trajectories represent non-exploding production paths along which market prices are defined. The presence of feasible trajectories of T is considered in the following Proposition, which shows that trajectories generated by T with feasible initial conditions are feasible as well. As a consequence, T always describes meaningful economic dynamics, whichever economically relevant initial configuration is given.

Proposition 2. *Let the point $(q_1, \cdots, q_N, \mathcal{Q})$ be feasible. Then, there exists a bounded set $A \subset F$ of feasible points such that $(q_1(t), \cdots, q_N(t), \mathcal{Q}(t)) \subseteq A$ for all $t \in \mathbb{N}$, where $(q_1(t), \cdots, q_N(t), \mathcal{Q}(t)) := T^t((q_1, \cdots, q_N, \mathcal{Q}))$.*

Proof. See Appendix B. □

Remarkably, non-equilibrium dynamics emerge both when no steady state of T exists (see to this purpose Proposition 1) as well as when point E is locally unstable. We refer the reader to [14] for sufficient local stability conditions of equilibrium E involving $n \leq N$ active firms. Such conditions, however, should be joined up at those parameters' values at which one player turns from being active to being inactive. Indeed, at those points, map T is not differentiable and stability conditions cannot by obtained through the study of eigenvalues of the Jacobian matrix of T. In the following, we detect trajectories starting with several active firms which eventually lead to a monopoly, that is with only one active firm. Such scenarios are not encompassed in [14] but are a remarkable feature of our setting. Therefore, in the following proposition we highlight that the monopolistic equilibrium is always locally asymptotically stable.

Proposition 3. *Under Assumption 1, let condition $\alpha_1^1 c_1 < \delta\alpha_2^0 c_2$ be fulfilled. Then, the monopolistic equilibrium*

$$E = \left(\frac{\delta(1-\delta)}{\alpha_1^1 c_1}, 0, \cdots, 0, \frac{\delta}{\alpha_1^1 c_1}\right)$$

is locally asymptotically stable.

Proof. See Appendix D. □

3.1. **Economic insights from numerical simulations.** Proposition 2 claims the existence of relevant economic dynamics even when steady state E is either unstable or does not exist. Numerical simulations allow to further investigate such non-equilibrium paths. In detail, as observed in equilibrium scenarios, both saturation and heterogeneity among players disincentive market participation, forcing firms with lower competitiveness to marginally contribute to overall production. This fact extends to non-equilibrium paths through emergence of endogenous competition dynamics, where the activity of the most competitive firm prevails at the expense of its competitors' productions and cyclical entry/exit patterns are observed.

Figure 5 shows the long-run dynamics of both individual productions (left panel) and market saturation (right panel) in the duopoly case varying δ. Those diagrams reveal a locally asymptotically stable steady state E for low values of δ. The increment of δ causes the loss of stability of E and the emergence of a stable cycle of period 4. Moreover, the further increase of δ beyond

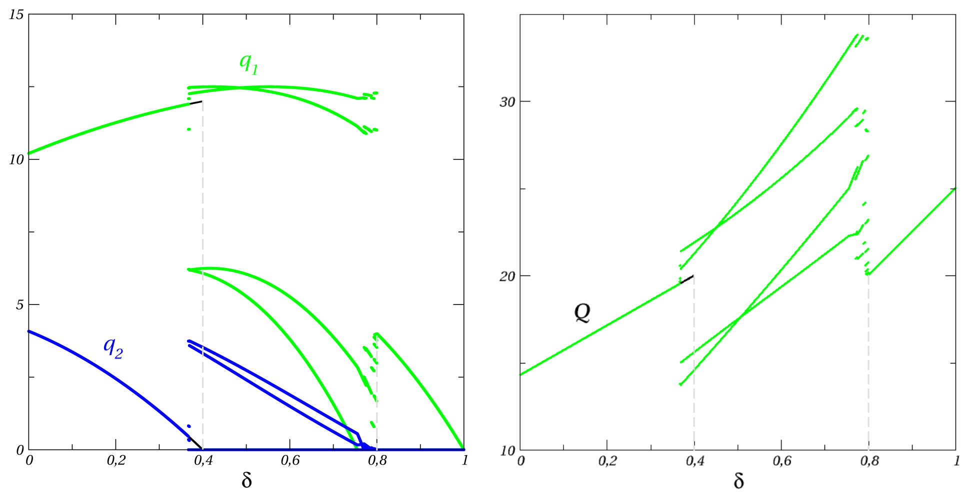

FIGURE 5. Long run productions q_1 (green) and q_2 (blue) (left panel) and saturation $\mathcal{Q}$ (green) (right panel) varying δ. $N = 2$, $c_1 = c_2 = 0.05$, $\alpha_1^0 = 0.4$, $\alpha_1^1 = 0.8$, $\alpha_2^0 = \alpha_2^1 = 1$.

the threshold $\alpha_1^0 c_1/\alpha_2^0 c_2$ causes the disappearance of E, preserving, however, the qualitative structure and stability property of the remaining attractor. We stress that the presence of stable periodic cycles is found to be a robust configuration, being observed in simulations whenever no steady state is present.

In Figure 6 the asymptotic behaviors of three trajectories are shown for different values of δ when no steady state of T exists. Such trajectories are represented through time series of variables q_1 and q_2 (left panel) and by means of the corresponding periodic points (right panel) in the phase plan (q_1, q_2). The time series show non-equilibrium regimes where the activity of the less competitive firm 2 becomes more and more marginal as δ increases and, at the same time, the most competitive firm 1 produces more than its opponent. This translates into a decreasing player's 2 willingness to be active in the market as saturation increases. We note that the same effect can be observed as firm's 1 competitiveness increases, which occurs for example at decreasing values of parameter α_1^0 (see Figure 7).

Moreover, time series show player's 2 alternating activity and inactivity periods along cyclic production scenarios. The basic (endogenous) mechanism underlying this observed cyclic behavior can be explained in the simplest case of a period 4 cycle. Suppose that, at period t, player 1 sets a positive output

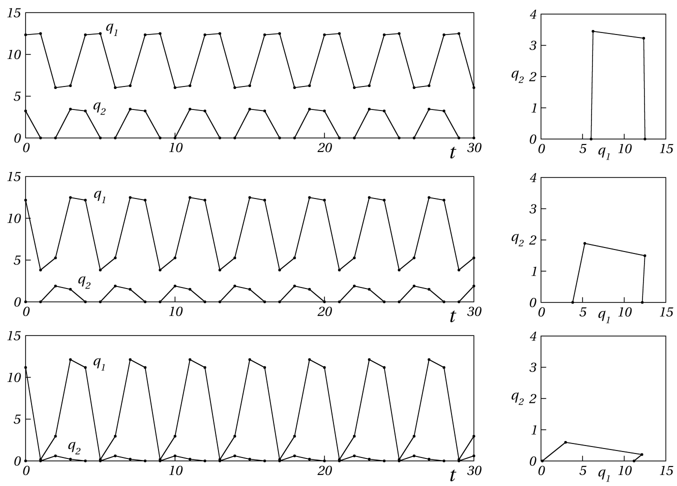

FIGURE 6. Time series of productions q_1 and q_2 in the long run (left panel) and corresponding periodic cycles in the phase plan (q_1, q_2) (right panel). Parameters are $N = 2$, $c_1 = c_2 = 0.05$, $\alpha_1^0 = 0.4$, $\alpha_1^1 = 0.8$, $\alpha_2^0 = \alpha_2^1 = 1$ with, from top to bottom, $\delta = 0.41$, 0.6, 0.75.

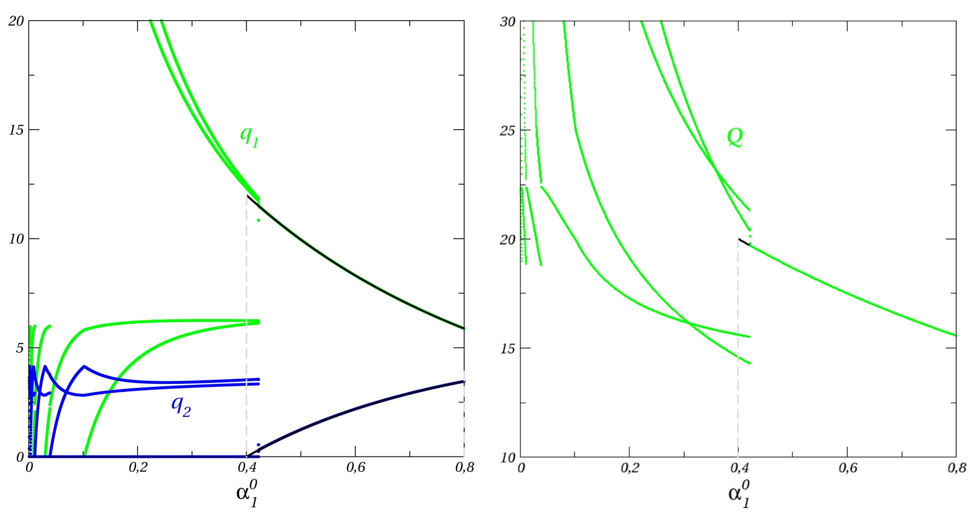

FIGURE 7. Long run productions q_1 (green) and q_2 (blue) (left panel) and saturation $\mathcal{Q}$ (green) (right panel) varying α_1^0 up to α_1^1. $N = 2$, $c_1 = c_2 = 0.05$, $\alpha_1^1 = 0.8$, $\alpha_2^0 = \alpha_2^1 = 1$, $\delta = 0.4$.

while player 2 is inactive. In this configuration, firm 1 is the monopolist and adopts a less aggressive behavior in deciding its own next-period production, expecting its opponent to remain inactive (due to the assumption of static expectations). Then, firm 1 will produce at a sufficiently low level at period

$t+1$. This allows player 2 to enter the market in period $t+2$, expecting the same non-aggressive behavior by firm 1 at that time (due, again, to firm's 2 static expectations). However, player 1 observes an active opponent at period $t+2$, which led it to become more aggressive to harm the opponent. This implies a high production of firm 1 in period $t+3$. This forces player 2 to set zero output in period $t+4$ not to achieve negative values of its own objective, thus exiting the market and becoming inactive.

Figure 8 shows non-equilibrium productivity cycles when no steady state is present, even with more than two firms. Along such cycles, the most competitive firm forces its competitors to alternate activity and inactivity periods. As for the duopoly case, such a behavior is supported by the attitude of firm 1 in changing its competitiveness from a low level - when it is the monopolist - to a higher level in the presence of competitors. We remark that entry/exit patterns of less competitive firms as well as the rise of a stable monopolistic equilibrium at sufficiently high values of δ are possible due to the presence of the saturation mechanism, which allows to define the best reply of the monopolist even in the presence of isoelastic demand (see Subsection 2.1.2). Furthermore, the same example shows non-equilibrium dynamics arising when the steady state E is unstable, which occurs due to supercritical flip bifurcations occurring as parameter δ varies. In detail, at a given number of active firms, increasing values of δ implies higher levels of market saturation and, in turn, decreasing individual productions at equilibrium. Then, competitive pressure reaches important levels, which are paired with the rise of instability of industry's outputs. However, the max operator limits the amplitude of oscillations, so the further increase of δ leads the less competitive firm to be less and less productive, until it becomes inactive. This fact relieves competitive pressure and steady state E retrieves local stability. For the sake of completeness, we mention that the periodic cycle that emerged from the second flip bifurcation loses its stability as well and trajectories are trapped by a third attractor, coexisting with the other unstable ones.

4. Conclusions

In this paper, inspired by the works of T. Puu in [10] and [11], we have proposed and studied an oligopoly model with isoelastic demand and saturation, where firms may employ different delegation schemes. In particular,

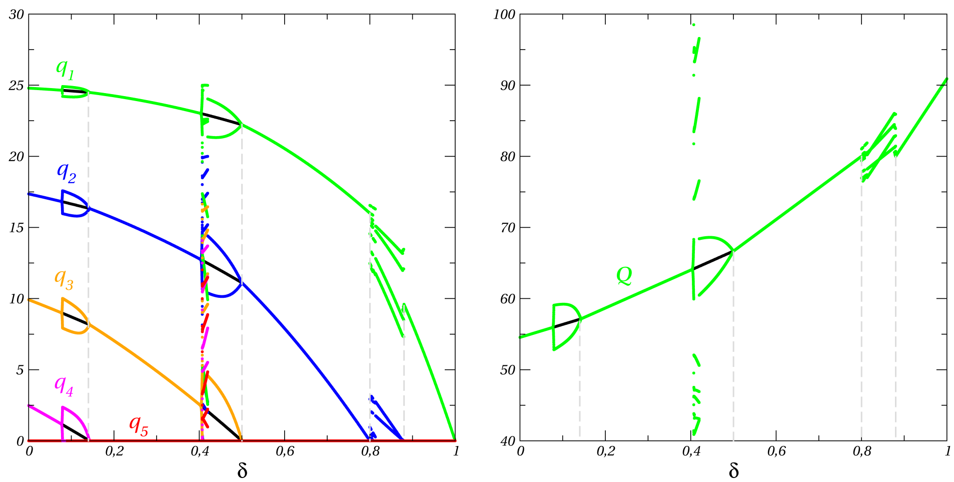

FIGURE 8. Bifurcation diagrams varying δ of variables q_i, with $i = 1, \cdots, 5$ (left panel), and $\mathcal{Q}$ (right panel). Parameters are $N = 5$, $c_i = 0.05$, $\{\alpha_i^0\}_{i=1}^5 = \{0.2, 0.25, 0.3, 0.35, 0.4\}$, $\{\alpha_i^1\}_{i=1}^5 = \{0.22, 0.9, 1, 1, 1\}$.

the level of delegation of each firm is heterogeneous and can be based on the current competitive pressure. In fact, we assumed that the same firm may behave differently if it is a monopolist or if other competitors are active on the market. From a dynamic point of view, we find the scenarios of interest when the internal equilibrium does not exist or is unstable with endogenous dynamics of firms entering and exiting the market. We believe that the results obtained are a starting point for studying similar models with different demand functions and with incentive schemes that depend more generally on the actual level of competitive pressure that each firm faces on the market.

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Appendix A. Proof of Proposition 1

Proof. Steady states of map T are the solutions of the following system of $N+1$ equations

$$\begin{cases} q_i = \max\left(0, \sqrt{\dfrac{\delta\mathcal{Q}+Q_{-i}}{\alpha_i(Q_{-i})c_i}} - (\delta\mathcal{Q}+Q_{-i})\right), \quad i = 1, \cdots, N \\ \\ \mathcal{Q} = \delta\mathcal{Q} + Q \end{cases} \tag{A-8}$$

where $Q_{-i} = Q - q_i$ and $Q = \sum_{j=1}^{N} q_j$. It is immediate that the origin E_0 is a steady state of T. Let $E = (q_1, \cdots, q_N, \mathcal{Q}) \neq E_0$ be a solution of system (A-8). It is easily verified that $q_i > 0$ if and only if $0 < \mathcal{Q} < 1/(\alpha_i(Q_{-i})c_i)$. Moreover, $q_i > 0$ implies $q_i = \mathcal{Q}(1 - \alpha_i(Q_i)c_i\mathcal{Q})$. We now we distinguish the following cases.

Case i)

Suppose E be such that $q_i > 0$ for a given i and $q_j = 0$ for all $j \neq i$. From the i-th and the last equations in system (A-8) and taking into account that $Q_{-i} = 0$ holds (so that $Q = q_i$), the equations $\mathcal{Q} = \delta/(\alpha_i^1 c_i)$ and $q_i = \delta(1-\delta)/(\alpha_i^1 c_i)$ follow. Moreover, the j-th equation in system (A-8) together with $q_j = 0$ imply

$$\sqrt{\frac{\delta\mathcal{Q}+Q_{-j}}{\alpha_j^0 c_j}} - (\delta\mathcal{Q}+Q_{-j}) \leq 0$$

for all $j \neq i$. The previous inequalities are satisfied whenever $\alpha_i^1 c_i \leq \delta\alpha_j^0 c_j$ for all $j \neq i$. However, we claim that $i = 1$ must hold. Indeed, suppose $i > 1$. Then, for any $j < i$, the relation $\alpha_j^0 c_j \leq \alpha_i^0 c_i$ follows from Assumption 1. Since, by construction $\alpha_i^0 \leq \alpha_i^1$, it results

$$\alpha_j^0 c_j < \alpha_i^0 c_i \leq \alpha_i^1 c_i \leq \delta\alpha_j^0 c_j < \alpha_j^0 c_j$$

which is impossible. To conclude, E is a steady state whenever $\alpha_1^1 c_1 \leq \delta\alpha_j^0 c_j$ for all $j > 1$. Under Assumption 1, this is equivalent to $\alpha_1^1 c_1 \leq \delta\alpha_2^0 c_2$.

Case ii)

Let E be such that the set of active firms $\{k : \ q_k > 0\}$ includes at least two players. Since, in this case, $Q_{-i} > 0$ holds for all i, then $\alpha_i = \alpha_i^0$. We claim that $q_i > 0$ holds for a given i if and only if $q_j > 0$ for all $j \leq i$. Indeed, if

$q_i > 0$, then $\mathcal{Q} < 1/(\alpha_i^0 c_i)$. By Assumption 1, it follows $\mathcal{Q} < 1/(\alpha_j^0 c_j)$ for all $j \leq i$, which, in turn, implies $q_j > 0$. Similarly, it can be shown that $q_i = 0$ implies $q_j = 0$ for all $j \geq i$. Hence, the number of active firms can be expressed as $n = \max\{k : \ q_k > 0\}$ or, equivalently, as

$$n = \max\{k : \ \mathcal{Q} < 1/(\alpha_k^0 c_k)\}$$

As a consequence, the aggregate production can be expressed as $Q = \sum_{i=1}^N q_i = \sum_{i=1}^n \mathcal{Q}(1 - \alpha_i^0 c_i \mathcal{Q})$ and the last equation of system (A-8) fixes $\mathcal{Q}$ at the level

$$\mathcal{Q} = \frac{n + \delta - 1}{\sum_{i=1}^n \alpha_i^0 c_i}$$

The number of active firms n can be determined as $n = \max\{k : \ f(k) < 0\}$, where

$$f(k) = \frac{k + \delta - 1}{\sum_{i=1}^k \alpha_i^0 c_i} - \frac{1}{\alpha_k^0 c_k}$$

To conclude, E is a steady state of map T whenever $n \geq 2$ or, equivalently, $f(2) < 0$.[7] This is satisfied provided that condition $\alpha_1^0 c_1 > \delta \alpha_2^0 c_2$ holds. $\square$

Appendix B. Proof of Proposition 2

Proof. Suppose by contradiction that for all bounded sets A of feasible points, it results

$$\{T^t((q_1, \cdots, q_N, \mathcal{Q})), \ t \in \mathbb{N}\} \not\subseteq A$$

Since all the components of vector $(q_1(t), \cdots, q_N(t), \mathcal{Q}(t))$ are non negative for all $t \in \mathbb{N}$, then either **(1)** $\mathcal{Q}(t) \to 0$ holds in the limit $t \to +\infty$ or **(2)** at least one component of vector $(q_1(t), \cdots, q_N(t), \mathcal{Q}(t))$ tends to $+\infty$ as $t \to +\infty$. If **(1)** holds, then $q_i(t) \to 0$ for all $i = 1, \cdots N$, which can be deduced from the last component of map T. Hence, a number $\hat{t} \geq 0$ can be found such that for all $i = 1, \cdots, N$ the relation $q_i(t) < \hat{q} := \max\{q_1(\hat{t}), \cdots, q_N(\hat{t})\}$ holds for all $t > \hat{t}$. However, when $\hat{q}$ is sufficiently small, a component q_i always exists for which it is

$$q_i(\hat{t} + 1) \geq \sqrt{\frac{\delta \mathcal{Q}(\hat{t}) + \hat{q}}{\alpha_i^0 c_i}} - \left(\delta \mathcal{Q}(\hat{t}) + \hat{q}\right) > \delta \mathcal{Q}(\hat{t}) + \hat{q} \geq \hat{q}$$

[7] Clearly, $f(k) < 0$ implies $f(k') < 0$ for all $k' \leq k$. Also $f(k) \geq 0$ implies $f(k') \geq 0$ for all $k' \geq k$.

leading to a contradiction. If **(2)** holds, then as $t \to +\infty$ either $q_i(t) \to +\infty$ or $\mathcal{Q} \to +\infty$ is satisfied. If $q_i(t) \to +\infty$, then for all $j \neq i$ it results

$$R_j(t) = \sqrt{\frac{\delta\mathcal{Q}(t) + \sum_{k\neq j} q_k(t)}{\alpha_j^0 c_j}} - \left(\delta\mathcal{Q}(t) + \sum_{k\neq j} q_k(t)\right) \to -\infty$$

Hence, for all $j \neq i$, it is $q_j(t) = \max(0, R_j(t)) \to 0$ as $t \to +\infty$. In addition, $q_i(t+1) \to +\infty$ implies $\mathcal{Q}(t+1) = \delta\mathcal{Q}(t) + Q(t+1) \to +\infty$ as well. From this, it follows

$$R_i(t) = \sqrt{\frac{\delta\mathcal{Q}(t)}{\alpha_i^1 c_i}} - \delta\mathcal{Q}(t) \to -\infty$$

implying $q_i(t) = \max(0, R_i(t)) \to 0$, a contradiction. Similarly, if $\mathcal{Q}(t) \to +\infty$, then $q_i(t) \to 0$ for all i and the evolution of variable $\mathcal{Q}$ approaches a contraction (with fixed point 0) in the long run. Hence $\mathcal{Q}(t) \to 0$ as $t \to +\infty$, a contradiction. □

Appendix C. Existence of a positively invariant set

We highlight the presence of a positively invariant set M such that $T(M) \subseteq M$, which is included in the plane of $\mathbb{R}^{N+1}$ identified by points of the form $(x_1, 0, \cdots, 0, x_{N+1})$ and representing monopolistic scenarios. We consider the set

$$M = \left\{(x_1, 0, \cdots, 0, x_{N+1}) \in \mathbb{R}^{N+1} :\ x_1 \geq 0,\ x_{N+1} \geq f(x_1)\right\}$$

where

$$f(x_1) = \max\left(\frac{1}{\delta}\left(\frac{1}{\alpha_2^0 c_2} - x_1\right), \mathcal{Q}_-\right)$$

with

$$\mathcal{Q}_- = \frac{1}{\delta\alpha_1^1 c_1}\left(\frac{(1+\delta)^2}{2}\left(1 - \sqrt{1 - \frac{4}{(1+\delta)^2}\frac{\alpha_1^1 c_1}{\alpha_2^0 c_2}}\right) - \frac{\alpha_1^1 c_1}{\alpha_2^0 c_2}\right)$$

Note that $0 < \mathcal{Q}_- < 1/(\delta\alpha_1^1 c_1)$. In the following, with $\mathrm{int}M$ we will denote the interior of M. We omit the proof of the following lemma.

Lemma C.1. *Under Assumption 1, let the condition $\alpha_1^1 c_1 < \delta\alpha_2^0 c_2$ be fulfilled. Then, $E \in \mathrm{int}M$ and $T(M) \subseteq M$ hold.*

Since the set M is positively invariant, the restriction of map T to M can defined, which fact will be used in the proof of Proposition 3.

Appendix D. Proof of Proposition3

Proof of Proposition 3. The restriction T_M of map T to the invariant set M (see Lemma C.1 in Appendix C) is the two dimensional map

$$T_M : \begin{cases} q_1' = \max\left(0, \sqrt{\dfrac{\delta \mathcal{Q}}{\alpha_1^1 c_1}} - \delta \mathcal{Q}\right) \\ \\ \mathcal{Q}' = \delta \mathcal{Q} + q_1' \end{cases} \tag{A-9}$$

with fixed point $E_M = \big(\delta(1-\delta)/(\alpha_1^1 c_1), \delta/(\alpha_1^1 c_1)\big)$. The Jacobian matrix of T_M evaluated at E_M results

$$J_M(\bar{q}_1, \bar{\mathcal{Q}}) = \begin{pmatrix} 0 & 1/2 - \delta \\ \\ 0 & 1/2 \end{pmatrix}$$

whose eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1/2$ are included in the unit circle of the complex plane. Hence, trajectories with initial conditions in M and placed sufficiently close to E converges to E as $t \to +\infty$. Now, let us pick an initial condition in a sufficiently small neighborhood of E, namely a point

$$P = \left(\frac{\delta(1-\delta)}{\alpha_1^1 c_1} + \epsilon_1, \epsilon_2, \cdots, \epsilon_N, \frac{\delta}{\alpha_1^1 c_1} + \epsilon_{\mathcal{Q}}\right)$$

where $\epsilon_1, \epsilon_{\mathcal{Q}} \neq 0$ and $\epsilon_j > 0$ for all $j \neq 1$. For all $j \neq 1$ it results

$$R_j' = \frac{1}{\alpha_j^0 c_j}\left(\sqrt{\frac{\delta \alpha_j^0 c_j}{\alpha_1^1 c_1} + \alpha_j^0 c_j\left(\epsilon_1 + \epsilon_{\mathcal{Q}} + \sum_{k \neq 1, j} \epsilon_k\right)} + \right.$$
$$\left. - \left(\frac{\delta \alpha_j^0 c_j}{\alpha_1^1 c_1} + \alpha_j^0 c_j\left(\epsilon_1 + \epsilon_{\mathcal{Q}} + \sum_{k \neq 1, j} \epsilon_k\right)\right)\right)$$

Since, by hypotheses, $1 < \dfrac{\delta \alpha_2^0 c_2}{\alpha_1^1 c_1} \leq \dfrac{\delta \alpha_j^0 c_j}{\alpha_1^1 c_1}$, then $R_j' < 0$ holds for all $j \neq 1$ provided sufficiently small $|\epsilon_1|$ and $|\epsilon_{\mathcal{Q}}|$ are given. Hence, after the first iteration of P through map T, it results $q_j' = 0$, for all $j \neq 1$. Moreover, it results $q_1' \geq 0$ and, after some analytic manipulations, the relation $\mathcal{Q}' \geq f(q_1')$ can be verified. Then, $T(P) \in M$ and, taking into account the previous argument, it follows $T^t(P) \to E$ as $t \to +\infty$. □

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