



ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

ARCHIVIO ISTITUZIONALE
DELLA RICERCA

Alma Mater Studiorum Università di Bologna
Archivio istituzionale della ricerca

A remark on Kohn's theorem on sums of squares of complex vector fields

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

Published Version:

Parmeggiani, A. (2022). A remark on Kohn's theorem on sums of squares of complex vector fields. *ADVANCES IN MATHEMATICS*, 404(A), 1-24 [10.1016/j.aim.2022.108378].

Availability:

This version is available at: <https://hdl.handle.net/11585/893780> since: 2022-09-09

Published:

DOI: <http://doi.org/10.1016/j.aim.2022.108378>

Terms of use:

Some rights reserved. The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (<https://cris.unibo.it/>).
When citing, please refer to the published version.

(Article begins on next page)

This is the final peer-reviewed accepted manuscript of:

Parmeggiani, A. (2022). A remark on kohn's theorem on sums of squares of complex vector fields. *Advances in Mathematics*, 404

The final published version is available online at
<https://dx.doi.org/10.1016/j.aim.2022.108378>

Terms of use:

Some rights reserved. The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (<https://cris.unibo.it/>)

When citing, please refer to the published version.

A REMARK ON KOHN'S THEOREM ON SUMS OF SQUARES OF COMPLEX VECTOR FIELDS

ALBERTO PARMEGGIANI

Dedicated to Gerardo Mendoza

ABSTRACT. The plan of this paper is to give an alternate proof of Kohn's subelliptic estimate for systems of N smooth complex vector fields on an open set of \mathbb{R}^n , and to improve it in extending the result to perturbations by a first-order term. A pseudodifferential generalization will also be given.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be an open set, and let X_1, \dots, X_N be first order homogeneous differential operators with *real* coefficients of the form $X_j = X_j(x, D)$, $D_j = -i\partial_j$. Hence the iX_j are smooth *real vector fields* on Ω . Let

$$P = \sum_{j=1}^N X_j^* X_j$$

be the sum-of-squares operator associated with our system of vector fields. Let

$$\mathcal{L}_X(x) := \text{Span}_{\mathbb{R}} \{iX_1, \dots, iX_N, [iX_{j_1}, [iX_{j_2}, [\dots, [iX_{j_{h-1}}, iX_{j_h}] \dots]], 1 \leq j_h \leq N, h \geq 2\}(x)$$

be the real vector space spanned by the given vector fields and their repeated commutators, all frozen at $x \in \Omega$.

Recall that P is said to be C^∞ -hypoelliptic if for every $u \in \mathcal{D}'(\Omega)$ and for every open $V \subset \Omega$, having $Pu \in C^\infty(V)$ yields $u \in C^\infty(V)$. (Analytic hypoellipticity is defined similarly by replacing C^∞ with C^ω .) Hörmander's celebrated and fundamental hypoellipticity theorem (see [9], for instance) states the following.

Theorem 1.1. *Suppose that at any given $x \in \Omega$ one has $\mathcal{L}_X(x) = T_x\Omega$. Then P is C^∞ hypoelliptic.*

Subsequently, L. Rothschild and E. Stein [18] made the theorem more precise by proving that if $\mathcal{L}_X(x)$ is spanned by repeated commutators of length at most k at all x (the X_j s have length 1, a commutator has length 2, and so on) then one has the following *subelliptic* estimate: *For all compact $K \subset \Omega$ there is a constant $c_K > 0$ such that, with $\varepsilon = 1/k$,*

$$(1.1) \quad c_K \|u\|_\varepsilon^2 \leq \text{Re}(Pu, u) + \|u\|_0^2, \quad \forall u \in C_c^\infty(K).$$

Here $\|\cdot\|_s$ denotes the norm of the H^s Sobolev space, and (\cdot, \cdot) the L^2 -scalar product. The result for polynomials in the operators X_j is due to J. Nourrigat and B. Helffer [7], and the

2020 Mathematics Subject Classification. Primary 35H20; Secondary 35A23, 35H10.

Key words and phrases: Subelliptic estimate; hypoellipticity; sums of squares of complex vector fields; Melin's inequality .

microlocal generalization to pseudodifferential operators to P. Bolley, J. Camus and J. Nourrigat [3].

It is important to observe that the Lie-algebra condition is *not* necessary: V. S. Fedii gave in [6] an example of operator (with infinite degeneracy of the coefficients) for which the Lie algebra is strictly smaller than the tangent space at certain points and yet the operator is C^∞ -hypoelliptic, and Y. Morimoto [12] obtained a general theorem for the C^∞ -hypoellipticity of that kind of operators. However, in the case of *analytic* coefficients the Lie algebra condition is also *necessary* for the C^∞ -hypoellipticity, as proved by M. Derridj in [5]. In the case of analytic coefficients and the C^ω -hypoellipticity, the Lie algebra condition is not sufficient for the analytic hypoellipticity. In fact, the Baouendi-Goulaouic operator gives a counterexample and is the starting point for all the subsequent work related to the so-called *Treves' conjecture* regarding C^ω -hypoellipticity (see Treves [19] and [20]).

Note that when an inequality such as (1.1) holds with $0 < \varepsilon \leq 1$, one also talks about *hypoellipticity with a loss of $2 - 2\varepsilon$ derivatives*. In general, the operator P of order $m > 0$ is said to be (C^∞ -) *hypoelliptic with a loss of $r \geq 0$ derivatives at a point $x_0 \in \Omega$* if for all $u \in \mathcal{D}'(\Omega)$ and all $s \in \mathbb{R}$

$$Pu \in H^s(x_0) \implies u \in H^{s+m-r}(x_0),$$

where $H^s(x_0)$ denotes the H^s Sobolev space localized at x_0 (see [9] or [16]).

J. J. Kohn, motivated by Y. T. Siu's program to use multipliers for the $\bar{\partial}$ -Neumann problem to get an explicit construction of critical varieties that control the D'Angelo type, considered a system of N *complex* first order homogeneous operators $Z_1(x, D), \dots, Z_N(x, D)$ with C^∞ coefficients in $\Omega \subset \mathbb{R}^n$ (no zeroth-order terms), so that the iZ_j are *complex* vector fields on Ω , and proved in [10] the following remarkable two theorems for the associated sum-of-squares operator

$$(1.2) \quad P = \sum_{j=1}^N Z_j^* Z_j.$$

Theorem 1.2. *Suppose that*

$$(1.3) \quad \text{Span}_{\mathbb{C}}\{iZ_j, [iZ_j, iZ_k]; 1 \leq j, k \leq N\}(x) = \mathbb{C}T_x\Omega, \quad \forall x \in \Omega,$$

where $\mathbb{C}T_x\Omega$ denotes the complexification of $T_x\Omega$. Then the following subelliptic estimate holds: For any given compact $K \subset \Omega$ there is a constant $c_K > 0$ such that

$$(1.4) \quad c_K \|u\|_{1/2}^2 \leq \text{Re}(Pu, u) + \|u\|_0^2, \quad \forall u \in C_c^\infty(K).$$

Remark 1.3. *It is important to observe that there is a relevant difference between sums of squares of complex vector fields and sums of squares of real vector fields, in that in the former case at a characteristic point a subprincipal part (see (2.9) and (2.10) below) is present, whereas in the latter at a characteristic point a subprincipal part is always absent. The presence of such a subprincipal part in general spoils the subelliptic estimate.*

Note also that Kohn's result is stronger than the part of Nourrigat's result in [13] concerned with the (maximal) hypoellipticity of a sum of squares of complex vector fields that is transversally elliptic with respect to its characteristic set.

Therefore, if for any given $x \in \Omega$ the *complex* vector space generated by the vectors fields iZ_j and their commutators, evaluated at x , is the complexified tangent space to Ω at x , then one has the same kind of subelliptic estimate as in the real case (and hence also has the

C^∞ -hypoellipticity of P). However, this is no longer true if one has to consider repeated commutators of length > 2 . In fact, Kohn proved the following further result.

Theorem 1.4. *For any given $k \in \mathbb{Z}_+$ there exist complex vector fields Z_1, Z_{2k} near $0 \in \mathbb{R}^3$ such that Z_1, Z_{2k} and their repeated commutators of length $k+1$ span $\mathbb{C}T_0\Omega$, and when $k \geq 2$ the subelliptic estimate (1.4) does not hold anymore. Moreover, the operator P associated with Z_1 and Z_{2k} is hypoelliptic with a loss of $k+1$ derivatives.*

Kohn in fact takes in $\mathbb{R}_{x_1, x_2, x_3}^3$ the following version of the Lewy operator

$$\bar{L} = \frac{\partial}{\partial \bar{z}_1} - iz_1 \frac{\partial}{\partial x_3}, \quad z_1 = x_1 + ix_2,$$

and considers

$$iZ_1 = \bar{L}, \quad iZ_{2k} = \bar{z}_1^k L, \quad P = Z_1^* Z_1 + Z_{2k}^* Z_{2k} = -(L\bar{L} + \bar{L}|z_1|^{2k}L).$$

At the same time Theorem 1.4 appeared, Parenti and Parmeggiani [14] came out with a technique, based on the Boutet de Monvel-Grigis-Helffer approach to the study of hypoellipticity with a loss of 1 derivative, to study the hypoellipticity with a large loss of derivatives of classes of transversally elliptic operators. That approach allowed them to give an explanation of E. Stein's example $\square_b^{(0,q)} + c$, $c \neq 0$ a complex number, $q = 0$ or n , where $\square_b^{(0,q)}$ is the Kohn-Laplacian acting on $(0, q)$ -forms on the Heisenberg group $\mathbb{C}^n \times \mathbb{R}$, which is hypoelliptic with a loss of exactly 2 derivatives (and also analytic hypoelliptic), while $\square_b^{(0,q)}$ for $q = 0, n$, cannot be hypoelliptic. In this respect, Stein's example is the first one (to my knowledge) exhibiting such a behavior. I note in passing that this example shows that the study of hypoelliptic operators which remain hypoelliptic after a perturbation by lower order terms is interesting and open (see [17] and [16]). However, Stein's example concerns addition by a zeroth order term and therefore cannot be represented as a "mere" sum of squares, whereas Kohn's example concerns indeed the latter case.

At any rate, a short time after Theorem 1.4 appeared, M. Christ [4] wrote down an example exhibiting the same behavior of Kohn's sum of squares, and Parenti and Parmeggiani [15], wrote down a class of examples generalizing Christ's one in the perspective of the theory developed in [14].

Out of curiosity, Parenti and Parmeggiani's approach in [14] allows to give very simple examples of hypoelliptic operators that lose derivatives. Let $d \geq 1$ be an integer, let $\mu > 0$ and let $S = \{\pm(2\ell + 1); \ell \in \mathbb{Z}_+\}$. Let $\gamma \in \mathbb{R}$, and consider the operator in \mathbb{R}_{x_1, x_2}^2

$$P_\gamma := (1 + x_1^{2d})(D_{x_1}^2 + \mu^2 x_1^2 D_{x_2}^2) + (\gamma + \mu x_1^{2d})D_{x_2} - 2ix_1^{2d-1}(D_{x_1} + i\mu x_1 D_{x_2}).$$

Then, P_γ is C^∞ -hypoelliptic with a loss of exactly $d+1$ derivatives when $\gamma \in S$. When $\gamma \notin S$ the operator P_γ is hypoelliptic with a loss of 1 derivative by Boutet de Monvel-Grigis-Helffer's result.

I wish also to mention the very interesting paper by Altomani, Hill, Nacinovich and Porter [1], in which, in the context of distributions of complex vector fields on a real manifold, the authors prove a subelliptic estimate under certain assumptions, that generalize the essential pseudoconcavity for CR manifolds, and the Hörmander bracket condition for real vector fields.

In this paper, I will be concerned with giving a different proof of Kohn's Theorem 1.2, improve it to deal also with the stability of the subelliptic estimate under perturbations by lower order terms, and give an extension to a pseudodifferential setting. My point here is

to exploit the strong form of A. Melin's inequality (see [11]) as given by Hörmander in [9] (Thm. 22.3.3, page 364). So, the main issue will be to use the symplectic geometry associated with the principal part of P at characteristic points, namely the linearization of the Hamilton field (i.e. the Hamilton map F) of the principal symbol of P , the subprincipal symbol p_1^s of P , and their relative sizes.

The main result here is the following theorem, that improves in this case Kohn's result (and hence also the transversally elliptic case of Nourrigat's one) to deal also with first order perturbations (see also the more detailed statement of Theorem 4.1 below).

Theorem 1.5. *Suppose Kohn's condition (1.3) holds. Then P given in (1.2) satisfies the subelliptic estimate (1.4). Moreover, (when P is non-elliptic and) denoting by Σ the characteristic set of P , there exist continuous functions $\Lambda_{\pm}: \Sigma \rightarrow \mathbb{R}$ (positively homogeneous of degree 1 in the fibers), explicitly given in terms of the subprincipal symbol and the Poisson brackets of the real and imaginary parts of the Z_j (see (4.20) and (4.21) below), such that if Q is a first-order classical properly supported pseudodifferential operator whose principal symbol has real part q_1 that satisfies*

$$(1.5) \quad \Lambda_-(\rho) < q_1(\rho) < \Lambda_+(\rho), \quad \forall \rho \in \Sigma,$$

then $P + Q$ keeps satisfying (1.4). In particular, (1.5) holds if

$$(1.6) \quad |q_1(\rho)| < \min\{-\Lambda_-(\rho), \Lambda_+(\rho)\}, \quad \forall \rho \in \Sigma.$$

So, the result is quite precise, in that the quantity on the right-hand side of inequality (1.6) is a precise measure of the first-order perturbations Q allowed by Kohn's condition (1.3) in order for $P + Q$ still to satisfy the subelliptic estimate (1.4). (See Section 4 for more details on that.) Note in addition that no smoothness assumption on Σ is required. I will also extend Theorem 1.5 to the natural pseudodifferential analog (see Theorem 5.2 below).

In the next section, I will recall what is needed about the Melin inequality. Then, I will study the spectrum of the Hamilton map which gives the fundamental symplectic quantity to have Melin's inequality in the strong form, and finally, in the last two sections, prove both the subelliptic estimate (1.4), the improvement to perturbations of P by lower order terms and the extension to a pseudodifferential case. The perturbations considered here are indeed meaningful in the study of complexes of operators associated with structures that are not necessarily involutive throughout Ω (see [2] for an introduction to involutive structures).

My plan then will be to further develop such an approach for Kohn's sums of squares of *complex* vector fields to study (among other things) unique-continuation phenomena and propagation of smoothness (following the lines indicated by Hörmander in his study of solvability of operators satisfying condition (P)).

I wish to thank the anonymous referee for the very useful comments and suggestions.

2. THE STRONG MELIN INEQUALITY

The strong form of Melin's inequality is concerned with the existence of subelliptic estimates of the kind of (1.1). Suppose $p_m: T^*\Omega \setminus 0 \rightarrow \mathbb{R}$ is a homogeneous m th-order pseudodifferential symbol, and that $p_m \geq 0$. Let Σ denote the zero-set of p_m in $T^*\Omega \setminus 0$. On Σ the Hessian of p_m is therefore invariantly defined. Using the symplectic form $\sigma = \sum_j d\xi_j \wedge dx_j$ on $T^*\Omega$ we may hence invariantly define the *Hamilton map* (or *fundamental matrix*) $F(\rho)$ of p_m at $\rho \in \Sigma$ by

$$\sigma(w, F(\rho)w') = \frac{1}{2} \langle \text{Hess}(p_m)(\rho)w, w' \rangle, \quad \forall w, w' \in T_\rho T^*\Omega,$$

which is then a linear map $F(\rho): T_\rho T^*\Omega \longrightarrow T_\rho T^*\Omega$. One readily has that $F(\rho)$ is *skew-symmetric* with respect to σ , and it turns out that the spectral structure of $F(\rho)$, at any given $\rho \in \Sigma$, is the following:

- $\text{Ker } F(\rho) \subset \text{Ker}(F(\rho)^2) = \text{Ker}(F(\rho)^3)$;
- 0 is the only generalized eigenvalue, and the other eigenvalues (if present) are semisimple, so that (with repetitions according to multiplicity)

$$\text{Spec}(F(\rho)) = \{0\} \cup \{\pm i\mu_j; \mu_j > 0, 1 \leq j \leq r\} \quad (\text{some } r \in \mathbb{Z}_+);$$

- one has the following σ -orthogonal decomposition

$$T_\rho T^*\Omega = \text{Ker}(F(\rho)^2) \oplus \text{Range}(F(\rho)^2).$$

One defines for any given $\rho \in \Sigma$ the *positive trace* of $F(\rho)$ by

$$\text{Tr}^+ F(\rho) = \sum_{\substack{\mu > 0 \\ i\mu \in \text{Spec}(F(\rho))}} \mu.$$

The positive trace is a *symplectic invariant* (positively homogeneous of degree $m - 1$ in the fibers). Dynamically speaking, the map $F(\rho)$ is the linearization of the bicharacteristic flow $t \mapsto \exp(tH_{p_m})(\rho)$ at $\rho \in \Sigma$, where

$$H_{p_m} = \sum_{j=1}^n \left(\frac{\partial p_m}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p_m}{\partial x_j} \frac{\partial}{\partial \xi_j} \right)$$

is the *Hamilton vector field* associated with p_m .

In [9], Hörmander proved the following strong version of Melin's inequality.

Theorem 2.1. *Let $P = P^*$ be a (formally self-adjoint) classical properly supported pseudo-differential operator of order $m > 0$ on an open set $\Omega \subset \mathbb{R}^n$. Let $p \sim p_m + p_{m-1} + \dots$ be its symbol. Suppose that*

$$p_m(x, \xi) \geq 0, \quad \forall (x, \xi) \in T^*\Omega \setminus 0,$$

and that

$$(2.7) \quad p_m(x, \xi) = 0 \implies p_{m-1}^s(x, \xi) + \text{Tr}^+ F(x, \xi) > 0.$$

Then for any given compact $K \subset \Omega$ there exist $c_K, C_K > 0$ such that

$$(2.8) \quad (Pu, u) \geq c_K \|u\|_{(m-1)/2}^2 - C_K \|u\|_{(m-2)/2}^2, \quad \forall u \in C_c^\infty(K).$$

Here

$$(2.9) \quad p_{m-1}^s(x, \xi) = p_{m-1}(x, \xi) + \frac{i}{2} \sum_{j=1}^n \partial_{x_j} \partial_{\xi_j} p_m(x, \xi)$$

is the *subprincipal symbol* of P , which is *invariantly* defined on the second-order zeros of the principal symbol (in this case, the whole characteristic set $p_m^{-1}(0) \subset T^*\Omega \setminus 0$, because $p_m \geq 0$, whence its vanishing to 2nd order). In particular, in the case of the operator P given in (1.2), denoting by $Z_j(x, \xi)$ the symbol of $Z_j(x, D)$ and putting $d_j(x) = \sum_{k=1}^n \partial_{x_k} \partial_{\xi_k} Z_j(x, \xi)$ (a sort of complex-valued divergence of Z_j) one computes

$$(2.10) \quad p_1^s(x, \xi) = - \sum_{j=1}^N \text{Im} \left(d_j(x) \overline{Z_j(x, \xi)} \right) - \frac{i}{2} \sum_{j=1}^N \{ \bar{Z}_j, Z_j \}(x, \xi),$$

so that at a characteristic point ρ we have $p_1^s(\rho) = -\frac{i}{2} \sum_{j=1}^N \{\bar{Z}_j, Z_j\}(\rho)$.

Note that when P is a *differential operator* of even order (as in the case of operator P in (1.2)), then, because of the symmetry $(x, \xi) \mapsto (x, -\xi)$ in Σ and the fact that

$$\mathrm{Tr}^+ F(x, -\xi) = \mathrm{Tr}^+ F(x, \xi), \quad \forall (x, \xi) \in \Sigma,$$

one has that condition (2.7) is equivalent to

$$(2.11) \quad |p_{m-1}^s(\rho)| < \mathrm{Tr}^+ F(\rho), \quad \forall \rho \in \Sigma.$$

Note also that when $m = 2$, Melin's inequality (2.8) is identical to (1.1) with $\varepsilon = 1/2$. Our aim is to show that indeed Kohn's condition (1.3) implies (2.7), which in the end gives an alternative proof of his result along with a precise control of the stability of the subelliptic estimate under certain lower order perturbations.

In the next section I will reformulate Kohn's condition and study the spectrum of the Hamilton map of Kohn's sum of squares (1.2), showing that $\mathrm{Tr}^+ F$ can be controlled from below in terms of the absolute value of the subprincipal symbol and the moduli of the Poisson brackets of the symbols of Z_1, \dots, Z_N .

3. SETTING OF THE PROBLEM AND SPECTRUM OF THE HAMILTON MAP F

I write for $1 \leq j \leq N$,

$$Z_j(x, \xi) = \langle \zeta_j(x), \xi \rangle, \quad \zeta_j = \alpha_{2j-1} + i\alpha_{2j} \in C^\infty(\Omega; \mathbb{C}T\Omega),$$

so that, with

$$X_{2j-1}(x, \xi) = \langle \alpha_{2j-1}(x), \xi \rangle, \quad X_{2j}(x, \xi) = \langle \alpha_{2j}(x), \xi \rangle,$$

I may also write

$$Z_j(x, \xi) = X_{2j-1}(x, \xi) + iX_{2j}(x, \xi), \quad (x, \xi) \in T^*\Omega \setminus 0.$$

I will write $[w_1, w_2]$ for the commutator of two vector fields w_1 and w_2 , and

$$\{f, g\} = \sum_{k=1}^n \left(\frac{\partial f}{\partial \xi_k} \frac{\partial g}{\partial x_k} - \frac{\partial f}{\partial x_k} \frac{\partial g}{\partial \xi_k} \right)$$

for the Poisson bracket of the functions f and g .

Since the principal symbol of P in (1.2) is

$$p_2(x, \xi) = \sum_{j=1}^N |Z_j(x, \xi)|^2,$$

we have that the characteristic set Σ of P (we shall be considering the case P nonelliptic, i.e. $\Sigma \neq \emptyset$ since when P is elliptic the estimate and perturbation results are well-known) is given by

$$\Sigma = \bigcap_{j=1}^N \{(x, \xi) \in T^*\Omega \setminus 0; Z_j(x, \xi) = 0\}.$$

Hence, on defining for $x \in \Omega$

$$W(x) = \mathrm{Span}_{\mathbb{R}} \{ \alpha_{2j-1}(x), \alpha_{2j}(x); 1 \leq j \leq N \},$$

we have

$$\Sigma = \{(x, \xi) \in T^*\Omega; 0 \neq \xi \in W(x)^\perp\}.$$

Of course, in $(T^*\Omega \setminus 0) \setminus \Sigma$ we have that P is elliptic, and if $\pi: T^*\Omega \rightarrow \Omega$ is the canonical projection, we have that $\pi(\Sigma) = \{x \in \Omega; \exists \xi \neq 0 \text{ with } \xi \in W(x)^\perp\}$.

In this context, Kohn's condition (1.3) is written as

$$(3.12) \quad \text{Span}_{\mathbb{C}}\{\zeta_j(x), [\zeta_j, \zeta_k](x); 1 \leq j, k \leq N\} = \mathbb{C}T_x\Omega, \quad \forall x \in \Omega.$$

Proposition 3.1. *Suppose that Kohn's condition (3.12) holds at $x_0 \in \pi(\Sigma)$. Then for all $0 \neq \xi \in W(x_0)^\perp$ there exist $j, k \in \{1, \dots, N\}$ such that $\{Z_j, Z_k\}(x_0, \xi) \neq 0$, that is,*

$$(3.13) \quad \begin{cases} \text{Re}\{Z_j, Z_k\}(x_0, \xi) = \{X_{2j-1}, X_{2k-1}\}(x_0, \xi) - \{X_{2j}, X_{2k}\}(x_0, \xi) \neq 0, \\ \text{or } \text{Im}\{Z_j, Z_k\}(x_0, \xi) = \{X_{2j-1}, X_{2k}\}(x_0, \xi) + \{X_{2j}, X_{2k-1}\}(x_0, \xi) \neq 0. \end{cases}$$

Proof. Suppose (3.12) holds at $x_0 \in \pi(\Sigma)$. By contradiction, if there is $0 \neq \xi \in W(x_0)^\perp$ such that $\{Z_j, Z_k\}(x_0, \xi) = 0$ for all $j, k \in \{1, \dots, N\}$, then ξ is a real nonzero solution to the system

$$\begin{cases} \langle \zeta_j(x_0), \xi \rangle = 0, & 1 \leq j \leq N, \\ \langle [\zeta_j, \zeta_k](x_0), \xi \rangle = 0, & 1 \leq j < k \leq N. \end{cases}$$

Therefore $\xi \in \mathbb{R}^n \setminus \{0\}$ is orthogonal to $\mathbb{C}T_{x_0}\Omega = \mathbb{C}^n$, which is a contradiction. \square

Remark 3.2. *Since P is microlocally elliptic outside Σ , when the compact K does not intersect $\pi(\Sigma)$ one has the Gårding inequality and hence an inequality of the kind (1.1) with $\varepsilon = 1$. Therefore, assuming (3.12) becomes crucial for obtaining the subelliptic estimate only when $K \cap \pi(\Sigma) \neq \emptyset$ and hence only for any given $x \in \pi(\Sigma)$.*

Furthermore, note that condition (3.13) is only necessary for Kohn's condition (3.12) to hold since the nonzero covector ξ is only supposed to be real. This contrasts with the strong Melin condition (2.7), which is necessary and sufficient for (2.8) to hold (as is seen by testing the inequality on suitable families of wave-packets). Hence Kohn's condition is not necessary for the subelliptic estimate to hold.

As for the subprincipal symbol p_1^s of P at $\rho = (x, \xi) \in \Sigma$ we have (see (2.10) above)

$$p_1^s(\rho) = -\frac{i}{2} \sum_{j=1}^N \{\bar{Z}_j, Z_j\}(\rho) = \sum_{j=1}^N \{X_{2j-1}, X_{2j}\}(\rho).$$

Now, if $H_j(\rho)$ is the Hamilton vector field of X_j at $\rho \in \Sigma$, a computation gives (using the fact that $df(\rho)w = \sigma(w, H_f(\rho))$)

$$F(\rho)w = \sum_{j=1}^N \left(\sigma(w, H_{2j-1}(\rho))H_{2j-1}(\rho) + \sigma(w, H_{2j}(\rho))H_{2j}(\rho) \right), \quad w \in T_\rho T^*\Omega.$$

To lighten notation, I shall at times drop the dependence on $\rho \in \Sigma$, retaining it when necessary.

Let, for $\rho \in \Sigma$,

$$V = V(\rho) := \text{Span}_{\mathbb{R}}\{H_{2j-1}, H_{2j}; 1 \leq j \leq N\}.$$

Then

$$\text{Range}(F) \subset V$$

and $F(\rho): T_\rho T^* \Omega \longrightarrow V(\rho) \subset T_\rho T^* \Omega$. By considering the obvious \mathbb{C} -linear extension of σ to $\mathbb{C}T_\rho T^* \Omega$, we think of F as mapping $\mathbb{C}T_\rho T^* \Omega$ into itself. Let then

$$T = T(\rho): \mathbb{C}^{2N} \ni z := \begin{bmatrix} z'_j \\ z''_j \end{bmatrix}_{1 \leq j \leq N} \longmapsto \sum_{j=1}^N (z'_j H_{2j-1} + z''_j H_{2j}) \in \mathbb{C}V(\rho),$$

and

$$L = L(\rho): \mathbb{C}T_\rho T^* \Omega \ni w \longmapsto \begin{bmatrix} \sigma(w, H_{2j-1}) \\ \sigma(w, H_{2j}) \end{bmatrix}_{1 \leq j \leq N} \in \mathbb{C}^{2N}.$$

For short, I will write $\begin{bmatrix} z' \\ z'' \end{bmatrix}$ in place of $\begin{bmatrix} z'_j \\ z''_j \end{bmatrix}_{1 \leq j \leq N}$. We have

$$Fw = (T \circ L)w, \quad \forall w \in \mathbb{C}T_\rho T^* \Omega.$$

Since the eigenvectors of F belonging to the nonzero eigenvalues are necessarily lying in $\text{Range}(F) \subset \mathbb{C}V$, to understand $\text{Spec}(F) \setminus \{0\}$ I will therefore work in $\mathbb{C}V$.

Now, for $w = \sum_{j=1}^N (z'_j H_{2j-1} + z''_j H_{2j}) \in \mathbb{C}V$ one has

$$\begin{aligned} Fw &= \sum_{j=1}^N (z'_j F H_{2j-1} + z''_j F H_{2j}) = \sum_{j,k=1}^N \left(z'_j \left(\sigma(H_{2j-1}, H_{2k-1}) H_{2k-1} + \sigma(H_{2j-1}, H_{2k}) H_{2k} \right) \right. \\ &\quad \left. + z''_j \left(\sigma(H_{2j}, H_{2k-1}) H_{2k-1} + \sigma(H_{2j}, H_{2k}) H_{2k} \right) \right) \\ &= \sum_{k=1}^N \left(\sum_{j=1}^N \left(z'_j \sigma(H_{2j-1}, H_{2k-1}) + z''_j \sigma(H_{2j}, H_{2k-1}) \right) \right) H_{2k-1} \\ &\quad + \sum_{k=1}^N \left(\sum_{j=1}^N \left(z'_j \sigma(H_{2j-1}, H_{2k}) + z''_j \sigma(H_{2j}, H_{2k}) \right) \right) H_{2k}. \end{aligned}$$

It is thus natural to define the $2N \times 2N$ real matrix

(3.14)

$$M = M(\rho) = \left[\begin{array}{c|c} \sigma(H_{2j-1}, H_{2k-1})_{1 \leq j, k \leq N} & \sigma(H_{2j-1}, H_{2k})_{1 \leq j, k \leq N} \\ \hline \sigma(H_{2j}, H_{2k-1})_{1 \leq j, k \leq N} & \sigma(H_{2j}, H_{2k})_{1 \leq j, k \leq N} \end{array} \right] =: \left[\begin{array}{c|c} A_1 & B \\ \hline -{}^t B & A_2 \end{array} \right],$$

where $A_j = -{}^t A_j$, $j = 1, 2$. Hence $M: \mathbb{C}^{2N} \longrightarrow \mathbb{C}^{2N}$ is a *skew-symmetric* matrix with real entries and

$$M \begin{bmatrix} z' \\ z'' \end{bmatrix} = \left[\begin{array}{c} \sum_{k=1}^N \left(\sigma(H_{2j-1}, H_{2k-1}) z'_k + \sigma(H_{2j-1}, H_{2k}) z''_k \right)_{1 \leq j \leq N} \\ \sum_{k=1}^N \left(\sigma(H_{2j}, H_{2k-1}) z'_k + \sigma(H_{2j}, H_{2k}) z''_k \right)_{1 \leq j \leq N} \end{array} \right].$$

Therefore, if $w = \sum_{j=1}^N (z'_j H_{2j-1} + z''_j H_{2j})$, then

$$(3.15) \quad Fw = \sum_{k=1}^N \left(\left(-M \begin{bmatrix} z' \\ z'' \end{bmatrix} \right)'_k H_{2k-1} + \left(-M \begin{bmatrix} z' \\ z'' \end{bmatrix} \right)''_k H_{2k} \right).$$

Since

$$LH_{2j-1} = \left[\frac{\sigma(H_{2j-1}, H_{2k-1})}{\sigma(H_{2j-1}, H_{2k})} \right]_{1 \leq k \leq N}, \quad LH_{2j} = \left[\frac{\sigma(H_{2j}, H_{2k-1})}{\sigma(H_{2j}, H_{2k})} \right]_{1 \leq k \leq N},$$

we also have

$$(3.16) \quad [(LH_{2j-1})_{1 \leq j \leq N} \mid (LH_{2j})_{1 \leq j \leq N}] = -M.$$

Notice therefore that

$$(3.17) \quad L \circ T = -M.$$

Hence

$$(3.18) \quad \text{Ker } T \subset \text{Ker } M.$$

By interpreting F as a complex linear map, we have the following.

Proposition 3.3. *The eigenvalue problem*

$$\begin{cases} Fw = \lambda w, \\ \lambda \neq 0, \\ 0 \neq w \in \text{Range}(F) \subset \mathbb{C}V, \end{cases}$$

is equivalent to the problem

$$\begin{cases} (M + \lambda) \begin{bmatrix} z' \\ z'' \end{bmatrix} \in \text{Ker } T, \\ \lambda \neq 0, \\ w = \sum_{j=1}^N (z'_j H_{2j-1} + z''_j H_{2j}) = T \begin{bmatrix} z' \\ z'' \end{bmatrix} \neq 0. \end{cases}$$

Proof. Since

$$w \in \text{Range}(F) \Leftrightarrow \exists w' \text{ such that } w = \sum_{j=1}^N (\sigma(w', H_{2j-1}) H_{2j-1} + \sigma(w', H_{2j}) H_{2j}),$$

we have that the eigenvalue equation $Fw = \lambda w$, with $w = \sum_{j=1}^N (z'_j H_{2j-1} + z''_j H_{2j}) \neq 0$ belonging to $\mathbb{C}V$, may be rewritten as

$$\sum_{j=1}^N \left(\left((-M - \lambda) \begin{bmatrix} z' \\ z'' \end{bmatrix} \right)'_j H_{2j-1} + \left((-M - \lambda) \begin{bmatrix} z' \\ z'' \end{bmatrix} \right)''_j H_{2j} \right) = 0,$$

that is,

$$(M + \lambda) \begin{bmatrix} z' \\ z'' \end{bmatrix} \in \text{Ker } T,$$

where $T \begin{bmatrix} z' \\ z'' \end{bmatrix} \neq 0$.

Conversely, if $\lambda \neq 0$ and $(M + \lambda) \begin{bmatrix} z' \\ z'' \end{bmatrix} \in \text{Ker } T$, with $T \begin{bmatrix} z' \\ z'' \end{bmatrix} \neq 0$, then

$$F \circ T \begin{bmatrix} z' \\ z'' \end{bmatrix} = (T \circ L) \circ T \begin{bmatrix} z' \\ z'' \end{bmatrix} = T \circ (L \circ T) \begin{bmatrix} z' \\ z'' \end{bmatrix} = -T \circ M \begin{bmatrix} z' \\ z'' \end{bmatrix} = \lambda T \begin{bmatrix} z' \\ z'' \end{bmatrix}.$$

This proves the proposition. \square

Recall that, since M is real and skew-symmetric, its spectrum is given by 0 and by $\pm i\mu_j$, $1 \leq j \leq r_0$ (possibly repeated according to the multiplicity), where the μ_j are > 0 and $r_0 = \text{rk} M/2$ (in case $r_0 = N$ then 0 is not in the spectrum of M). Using (3.18) and Proposition 3.3 we have the following crucial result about the spectrum of F .

Theorem 3.4. *One has*

$$\text{Spec}(F) \setminus \{0\} = \text{Spec}(M) \setminus \{0\},$$

with the same multiplicities. In particular, $\lambda \in \text{Spec}(F) \setminus \{0\}$ iff $-\lambda \in \text{Spec}(M) \setminus \{0\}$.

Proof. Let $0 \neq w = T \begin{bmatrix} z' \\ z'' \end{bmatrix}$, be an eigenvector of F belonging to $\lambda \neq 0$. By (3.18) and Proposition 3.3 we then have, equivalently, that

$$(M + \lambda) \begin{bmatrix} z' \\ z'' \end{bmatrix} \in \text{Ker} T.$$

Hence

$$M(M + \lambda) \begin{bmatrix} z' \\ z'' \end{bmatrix} = (M + \lambda)M \begin{bmatrix} z' \\ z'' \end{bmatrix} = 0.$$

We now have that

- (i) *either $0 \neq -\lambda \in \text{Spec}(M)$, hence $(M + \lambda) \begin{bmatrix} z' \\ z'' \end{bmatrix} = 0$,*
- (ii) *or $0 \neq -\lambda \notin \text{Spec}(M)$, hence $(M + \lambda) \begin{bmatrix} z' \\ z'' \end{bmatrix} \neq 0$.*

In the former case the nonzero vector $\begin{bmatrix} z' \\ z'' \end{bmatrix}$ is an eigenvector of M belonging to $-\lambda \neq 0$.

In the latter case we have

$$(3.19) \quad 0 \neq (M + \lambda) \begin{bmatrix} z' \\ z'' \end{bmatrix} =: t_\lambda \in \text{Ker} T,$$

and

$$\begin{bmatrix} z' \\ z'' \end{bmatrix} = (M + \lambda)^{-1} t_\lambda.$$

Since M commutes with $(M + \lambda)^{-1}$ we obtain

$$M \begin{bmatrix} z' \\ z'' \end{bmatrix} = (M + \lambda)^{-1} M t_\lambda = 0,$$

by (3.18). Therefore from (3.19) we have $t_\lambda = \lambda \begin{bmatrix} z' \\ z'' \end{bmatrix}$, $\lambda \neq 0$, which finally yields $w = T \begin{bmatrix} z' \\ z'' \end{bmatrix} = 0$. Hence w cannot be an eigenvector of F belonging to λ . Since for both F and M one has that λ is a nonzero eigenvalue iff $-\lambda$ is a nonzero eigenvalue, this shows that

$$\text{Spec}(F) \setminus \{0\} \subset \text{Spec}(M) \setminus \{0\}.$$

To show the converse, suppose that $0 \neq -\lambda \in \text{Spec}(M)$, and that $\begin{bmatrix} z' \\ z'' \end{bmatrix} \neq 0$ is an eigenvector of M belonging to $-\lambda$. Then it cannot be $\begin{bmatrix} z' \\ z'' \end{bmatrix} \in \text{Ker} T$ because by (3.18) we would then

have $(M + \lambda) \begin{bmatrix} z' \\ z'' \end{bmatrix} = \lambda \begin{bmatrix} z' \\ z'' \end{bmatrix} = 0$ which is impossible. Hence $w := T \begin{bmatrix} z' \\ z'' \end{bmatrix} \neq 0$ is an eigenvector of F belonging to λ , which shows that

$$\text{Spec}(M) \setminus \{0\} \subset \text{Spec}(F) \setminus \{0\}.$$

As for the multiplicities, let w_1, \dots, w_k be a basis of $\text{Ker}(F - \lambda)$. Then, by (i) above, we have that there are $z_j = \begin{bmatrix} z'_j \\ z''_j \end{bmatrix}$, $1 \leq j \leq k$, such that $Tz_j = w_j$ and $(M + \lambda)z_j = 0$. I claim that z_1, \dots, z_k are linearly independent. In fact, suppose $\sum_{j=1}^k \alpha_j z_j = 0$, $\alpha_j \in \mathbb{C}$. Then $\sum_{j=1}^k \alpha_j Tz_j = \sum_{j=1}^k \alpha_j w_j = 0$, which yields $\alpha_1 = \dots = \alpha_k = 0$. Therefore

$$\dim \text{Ker}(F - \lambda) \leq \dim \text{Ker}(M + \lambda).$$

If we had a strict inequality, we could find $z_{k+1} \in \text{Ker}(M + \lambda)$ such that z_1, \dots, z_k, z_{k+1} are still linearly independent. However, we would have $Tz_{k+1} = \sum_{j=1}^k \beta_j w_j = \sum_{j=1}^k \beta_j Tz_j$, for some $\beta_j \in \mathbb{C}$ not all zero, whence

$$z_{k+1} - \sum_{j=1}^k \beta_j z_j \in \text{Ker} T \subset \text{Ker} M.$$

Therefore

$$0 = M \left(z_{k+1} - \sum_{j=1}^k \beta_j z_j \right) = -\lambda \left(z_{k+1} - \sum_{j=1}^k \beta_j z_j \right) \Rightarrow z_{k+1} - \sum_{j=1}^k \beta_j z_j = 0,$$

since $\lambda \neq 0$, which is a contradiction. This concludes the proof. \square

Corollary 3.5. *One therefore has*

$$\text{Tr}^+ F = \text{Tr}^+ M.$$

4. ANOTHER PROOF OF THEOREM 1.2, AND ITS IMPROVEMENT

In this section I give the proof of Theorem 1.5, that I restate in a more detailed version. To do that, I next introduce the relevant functions involved in its statement.

Recall that we are given the complex operators Z_1, \dots, Z_N in n dimensions and are considering the corresponding sum of squares P as given in (1.2), and that we are denoting by $\Sigma \subset T^*\Omega \setminus 0$ the characteristic set of P . We will always be considering only the case $\Sigma \neq \emptyset$, i.e. P is *not elliptic*, for otherwise the estimate and the perturbation results are well-known. Recall also that we write $Z_j = X_{2j-1} + iX_{2j}$. Define the continuous functions on Σ , $\kappa: \Sigma \rightarrow [0, +\infty)$ and $\Lambda_{\pm}: \Sigma \rightarrow \mathbb{R}$ by

$$(4.20) \quad \kappa(\rho) = \left(p_1^s(\rho)^2 + \max_{1 \leq j < k \leq N} |\{Z_j, Z_k\}(\rho)|^2 \right)^{1/2}, \quad \rho \in \Sigma,$$

and

$$(4.21) \quad \Lambda_{\pm}(\rho) = -p_1^s(\rho) \pm \kappa(\rho), \quad \rho \in \Sigma.$$

It is important to observe that the functions κ and Λ_{\pm} are positively homogeneous of degree 1 in the fibers and that:

(i) $\kappa(\rho)^2$ is expressed through Poisson brackets of the X_{2j-1}, X_{2j} at $\rho \in \Sigma$. In fact,

$$p_1^s(\rho) = \sum_{j=1}^N \{X_{2j-1}, X_{2j}\}(\rho),$$

$$\operatorname{Re}\{Z_j, Z_k\}(\rho) = \{X_{2j-1}, X_{2k-1}\}(\rho) - \{X_{2j}, X_{2k}\}(\rho),$$

$$\operatorname{Im}\{Z_j, Z_k\}(\rho) = \{X_{2j-1}, X_{2k}\}(\rho) - \{X_{2j}, X_{2k-1}\}(\rho);$$

(ii) The functions Λ_{\pm} are the solutions of the quadratic equation in λ

$$(\lambda + p_1^s(\rho))^2 = \kappa(\rho)^2;$$

(iii) Since $\kappa(\rho) \geq |p_1^s(\rho)|$ throughout Σ (with strict inequality when Kohn's condition is satisfied) we have

$$\Lambda_-(\rho) \leq 0 \leq \Lambda_+(\rho)$$

(again, with strict inequality when Kohn's condition is satisfied).

I am now in a position to state and prove the main result of this paper.

Theorem 4.1. *Suppose Kohn's condition (1.3) holds. Then P given in (1.2) satisfies the subelliptic estimate (1.4). Moreover, if Q is a first order classical properly supported pseudodifferential operator whose principal symbol has real part q_1 that satisfies*

$$(4.22) \quad \Lambda_-(\rho) < q_1(\rho) < \Lambda_+(\rho), \quad \forall \rho \in \Sigma,$$

then $P + Q$ keeps satisfying (1.4). In particular, condition (4.22) holds if

$$(4.23) \quad |q_1(\rho)| < \min\{-\Lambda_-(\rho), \Lambda_+(\rho)\} = \kappa(\rho) - |p_1^s(\rho)|, \quad \forall \rho \in \Sigma.$$

Remark 4.2. *Hence, the quantity $\kappa - |p_1^s|_{\Sigma}$ appearing on the right-hand side of (4.23) may be thought of as a precise measure of the allowed "width", determined by Kohn's condition (1.3), of an admissible first-order perturbation Q of P .*

Note also that Σ is not supposed to be a smooth manifold.

The proof of Theorem 4.1 follows from the following result.

Theorem 4.3. *One always has the estimate*

$$(4.24) \quad \left(\operatorname{Tr}^+ F(\rho)\right)^2 = \left(\operatorname{Tr}^+ M(\rho)\right)^2 \geq p_1^s(\rho)^2 + \max_{1 \leq h < r \leq N} |\{Z_h, Z_r\}(\rho)|^2, \quad \forall \rho \in \Sigma.$$

Therefore, by Proposition 3.1, Kohn's condition (1.3) implies

$$\operatorname{Tr}^+ F(\rho) > |p_1^s(\rho)|, \quad \forall \rho \in \Sigma,$$

whence Melin's condition (2.7) holds.

Proof of Thm. 4.3. The proof of estimate (4.24) depends on the variational characterization of the singular values of a given matrix. In the first place I will recall the basic facts that will be needed in the proof.

Given a complex $N \times N$ matrix A , its singular values $s_1(A) \geq s_2(A) \geq \dots \geq s_N(A) \geq 0$ are the eigenvalues (repeated according to multiplicity) of A^*A (equivalently AA^*). Denoting by $U(N)$ the unitary group, one has (see Horn and Johnson [8]) that

$$\sum_{j=1}^N s_j(A) = \max_{U \in U(N)} |\operatorname{Tr}(AU)|.$$

Denote by $B(N)$ the set of $N \times N$ contractions, that is the set of the complex matrices C such that $s_j(C) \in [0, 1]$, $1 \leq j \leq N$. One then has (see [8]) that $B(N)$ is compact and that

$$U(N) = \text{ext}(B(N)) \subset B(N),$$

that is, $U(N)$ is the set of extreme points of $B(N)$ and, moreover, for any given $C \in B(N)$ there exist $U_1, \dots, U_r \in U(N)$ such that

$$C = \sum_{j=1}^r \alpha_j U_j, \quad \alpha_j \geq 0, \quad \sum_{j=1}^r \alpha_j = 1.$$

It hence follows, by convexity, that

$$|\text{Tr}(AC)|^2 \leq \max_{U \in U(N)} |\text{Tr}(AU)|^2, \quad \forall C \in B(N).$$

Now, in the present case we are considering $M = -{}^tM$, real $2N \times 2N$ matrix, so that we readily have that

$$2\text{Tr}^+M = \max_{U \in U(2N)} |\text{Tr}(MU)|,$$

and that for any given contraction $C \in B(2N)$,

$$|\text{Tr}(MC)|^2 \leq 4(\text{Tr}^+M)^2.$$

For simplicity, I will consider

$$f_M: B(2N) \ni C \longmapsto \frac{1}{4} |\text{Tr}(MC)|^2 \in [0, (\text{Tr}^+M)^2].$$

Now, since $U(2N)$ is compact and arcwise connected, and since M is skew-symmetric and hence traceless, one has $f_M(I_{2N}) = 0$ (as I_{2N} is a unitary matrix and $\text{Tr}(MI_{2N}) = 0$), whence

$$f_M(U(2N)) = f_M(B(2N)) = [0, (\text{Tr}^+M)^2].$$

So, the point is to understand the values of f_M . Remark that, since we consider the nontrivial case $M \neq 0$ at each $\rho \in \Sigma$ (by virtue of the Kohn condition and the fact that we consider the nontrivial case of P nonelliptic), we must have $\text{Tr}^+M > 0$.

For that purpose it will be convenient to use the Weyl basis E_{jk} , $1 \leq j, k \leq N$, of the $N \times N$ complex matrices, whose entries are all 0 except for the jk -th entry, which is 1. I also consider, for $1 \leq h < r \leq N$, the $N \times N$ skew-symmetric matrix

$$J_{hr} = E_{hr} - E_{rh}.$$

Therefore

$$(4.25) \quad E_{jk}E_{hr} = \delta_{kh}E_{jr}, \quad J_{hr}^2 = -E_{hh} - E_{rr}, \quad J_{hr} + J_{hr}^* = 0,$$

with δ_{kh} the Kronecker δ .

Next we write down the matrix M introduced in (3.14). It is useful to use the following blockwise expression of M : for $1 \leq j < k \leq N$ we put $x_{jk} = \{X_j, X_k\}$, then

$$M = \left[\begin{array}{c|c} A_1 & B \\ \hline -{}^tB & A_2 \end{array} \right],$$

where A_1, A_2 , and B are $N \times N$ real matrices with

$$A_1 = -{}^tA_1 = \sum_{1 \leq j < k \leq N} x_{2j-1, 2k-1} J_{jk},$$

$$A_2 = -{}^t A_2 = \sum_{1 \leq j < k \leq N} x_{2j,2k} J_{jk},$$

and

$$B = \sum_{\substack{1 \leq j, k \leq N \\ 2j-1 < 2k}} x_{2j-1,2k} E_{jk} - \sum_{\substack{1 \leq j, k \leq N \\ 2j-1 > 2k}} x_{2k,2j-1} E_{jk}.$$

Note that

$$\text{Tr}(B) = p_1^s(\rho).$$

Using (4.25), one computes for $j < k$, $h < r$,

$$J_{jk} J_{hr} = \delta_{kh} E_{jr} - \delta_{kr} E_{jh} - \delta_{jh} E_{kr} + \delta_{jr} E_{kh} \Rightarrow \text{Tr}(J_{jk} J_{hr}) = 2\delta_{kh} \delta_{jr} - 2\delta_{kr} \delta_{jh}.$$

It then follows that for $1 \leq h < r \leq N$

$$\begin{aligned} \text{Tr}(A_1 J_{hr}) &= \sum_{1 \leq j < k \leq N} x_{2j-1,2k-1} \text{Tr}(J_{jk} J_{hr}) \\ &= 2 \sum_{1 \leq j < k \leq N} x_{2j-1,2k-1} (\delta_{hk} \delta_{jr} - \delta_{kr} \delta_{jh}) = -2x_{2h-1,2r-1}, \end{aligned}$$

and in the same way

$$\text{Tr}(A_2 J_{hr}) = -2x_{2h,2r}.$$

Therefore

$$(4.26) \quad \text{Tr}(A_1 J_{hr}) - \text{Tr}(A_2 J_{hr}) = -2(x_{2h-1,2r-1} - x_{2h,2r}) = -2 \text{Re}\{Z_h, Z_r\}(\rho).$$

As for B , one has, using

$$\text{Tr}(E_{jk} J_{hr}) = \delta_{kh} \delta_{jr} - \delta_{kr} \delta_{jh},$$

that

$$\text{Tr}(B J_{hr}) = \sum_{j,k; 2j-1 < 2k} x_{2j-1,2k} (\delta_{kh} \delta_{jr} - \delta_{kr} \delta_{jh}) - \sum_{j,k; 2j-1 > 2k} x_{2k,2j-1} (\delta_{kh} \delta_{jr} - \delta_{kr} \delta_{jh}),$$

whence, with $h < r$,

$$\text{Tr}(B J_{hr}) = -(x_{2h-1,2r} + x_{2h,2r-1}) = -\text{Im}\{Z_h, Z_r\}(\rho),$$

and, using the properties of Tr , also that

$$\text{Tr}(-{}^t B J_{hr}) = \text{Tr}(B J_{hr}) = -(x_{2h-1,2r} + x_{2h,2r-1}).$$

Hence

$$(4.27) \quad \text{Tr}(B J_{hr}) + \text{Tr}(-{}^t B J_{hr}) = -2 \text{Im}\{Z_h, Z_r\}(\rho)$$

Now, consider the unitary matrix $U_s := \begin{bmatrix} 0 & -I_N \\ I_N & 0 \end{bmatrix} \in \text{U}(2N)$, for which it is readily seen that $f_M(U_s) = p_1^s(\rho)^2$. Therefore we obtain a first remarkable relation.

Lemma 4.4. *For any given sum-of-squares of complex vector fields operator P as in (1.2) one always has (in the nonelliptic case, i.e. $\Sigma \neq \emptyset$, which is the case we are interested in)*

$$(4.28) \quad |p_1^s(\rho)| \leq \text{Tr}^+ M(\rho) = \text{Tr}^+ F(\rho), \quad \forall \rho \in \Sigma.$$

But the above estimate must be strict in view of Kohn's condition. In fact, suppose that $|p_1^s(\rho)| = \text{Tr}^+ M(\rho)$ for some $\rho \in \Sigma$ (hence $|p_1^s(\rho)| > 0$). Let $1 \leq h < r \leq N$ be such that $|\{Z_h, Z_r\}(\rho)| \neq 0$. Say that $\text{Re}\{Z_h, Z_r\}(\rho) \neq 0$. Then, if one defines

$$\psi_0: \mathbb{R} \rightarrow [0, +\infty), \quad \psi_0(t) = f_M(U_s \exp\left(t \begin{bmatrix} 0 & J_{hr} \\ J_{hr} & 0 \end{bmatrix}\right)),$$

then $0 < p_1^s(\rho)^2 = \psi_0(0) = f_M(U_s)$ is a maximum, whence (by (4.26))

$$\psi_0'(0) = \text{Tr}(B) \text{Tr}\left(M \begin{bmatrix} -J_{hr} & 0 \\ 0 & J_{hr} \end{bmatrix}\right) = 2p_1^s(\rho) \text{Re}\{Z_h, Z_r\}(\rho) = 0,$$

which is impossible. The same happens in case $\text{Im}\{Z_h, Z_r\}(\rho) \neq 0$. One in fact considers

$$\psi_1: \mathbb{R} \rightarrow [0, +\infty), \quad \psi_1(t) = f_M(U_s \exp\left(t \begin{bmatrix} -J_{hr} & 0 \\ 0 & J_{hr} \end{bmatrix}\right)),$$

for which (by (4.27))

$$\psi_1(0) = p_1^s(\rho)^2 > 0, \quad \psi_1'(0) = p_1^s(\rho) \text{Tr}\left(M \begin{bmatrix} 0 & -J_{hr} \\ -J_{hr} & 0 \end{bmatrix}\right) = 2p_1^s(\rho) \text{Im}\{Z_h, Z_r\}(\rho).$$

Therefore we obtain a second remarkable relation.

Lemma 4.5. *For any given sum-of-squares of complex vector fields operator P as in (1.2) which satisfies Kohn's condition one always has (in the nonelliptic case)*

$$(4.29) \quad |p_1^s(\rho)| < \text{Tr}^+ M(\rho) = \text{Tr}^+ F(\rho), \quad \forall \rho \in \Sigma.$$

Therefore Melin's strong inequality (2.8) (for $m = 2$) holds, and hence also the subelliptic estimate (1.4).

To complete the proof of Theorem 4.3, we have to show inequality (4.24). To prove it, for any given $\rho \in \Sigma$ I consider all the h, r with $1 \leq h < r \leq N$ such that $|\{Z_h, Z_r\}(\rho)| \neq 0$. Take an h and r with that property. We construct the following contraction belonging to $B(2N)$. For $t_1, t_2, t_3, t_4 \in \mathbb{R}$ such that $t_1^2 + t_2^2 = t_3^2 + t_4^2 = 1$ consider the matrix

$$C_{hr} = C_{hr}(t_1, t_2; t_3, t_4) = \left[\begin{array}{c|c} -t_1 J_{hr} & t_2(t_3 I_N - t_4 J_{hr}) \\ \hline -t_2(t_3 I_N + t_4 J_{hr}) & t_1 J_{hr} \end{array} \right].$$

One computes

$$C_{hr}^* C_{hr} = \left[\begin{array}{c|c} E_{hh} + E_{rr} + t_2^2 t_3^2 \sum_{j \neq h, r} E_{jj} & 0 \\ \hline 0 & E_{hh} + E_{rr} + t_2^2 t_3^2 \sum_{j \neq h, r} E_{jj} \end{array} \right],$$

so that the singular values of C_{hr} are 1 and $t_2^2 t_3^2 \in [0, 1]$, whence $C_{hr} \in B(2N)$.

I next compute $f_M(C_{hr})$ and maximize it for $t_1^2 + t_2^2 = 1 = t_3^2 + t_4^2$. One has

$$\text{Tr}(M C_{hr}) = 2t_1 \text{Re}\{Z_h, Z_r\}(\rho) - 2t_2 \left(t_3 p_1^s(\rho) + t_4 \text{Im}\{Z_h, Z_r\}(\rho) \right).$$

Since C_{hr} is a contraction for all $t_1^2 + t_2^2 = 1 = t_3^2 + t_4^2$ we have

$$f_M(C_{hr}) \leq (\text{Tr}^+ M)^2,$$

and hence

$$(4.30) \quad \max_{t_3^2 + t_4^2 = 1} \max_{t_1^2 + t_2^2 = 1} f_M(C_{hr}) = p_1^s(\rho)^2 + |\{Z_h, Z_r\}(\rho)|^2 \leq (\text{Tr}^+ M)^2.$$

The equality on the left-hand side of inequality (4.30) is obtained by using the fact that since C_{hr} is a real matrix then f_M is the square of a real-valued linear form. To see (4.30), in the first place one has

$$\max_{t_1^2+t_2^2=1} f_M(C_{hr}) = \operatorname{Re}\{Z_h, Z_r\}(\rho)^2 + \left(t_3 p_1^s(\rho) + t_4 \operatorname{Im}\{Z_h, Z_r\}(\rho) \right)^2,$$

and then

$$\max_{t_3^2+t_4^2=1} \left(\operatorname{Re}\{Z_h, Z_r\}(\rho)^2 + \left(t_3 p_1^s(\rho) + t_4 \operatorname{Im}\{Z_h, Z_r\}(\rho) \right)^2 \right) = p_1^s(\rho)^2 + |\{Z_h, Z_r\}(\rho)|^2.$$

Therefore estimate (4.24) and, in turn, Melin's condition (2.7) hold and the theorem is proved. \square

Proof of Thm. 4.1. The fact that P satisfies the subelliptic estimate follows from Melin's theorem. We need only to show the perturbation part of the result. Recall that, with the present notation,

$$\kappa(\rho) = \left(p_1^s(\rho)^2 + \max_{1 \leq h < r \leq N} |\{Z_h, Z_r\}(\rho)|^2 \right)^{1/2}, \quad \rho \in \Sigma.$$

The condition for $P + Q$ to satisfy (2.7) at $\rho \in \Sigma$ is that

$$p_1^s(\rho) + q_1(\rho) + \operatorname{Tr}^+ F(\rho) > 0,$$

which, in view of the above computations, is implied by requiring that

$$(4.31) \quad |p_1^s(\rho) + q_1(\rho)| < \operatorname{Tr}^+ F(\rho).$$

Since $\kappa(\rho) \leq \operatorname{Tr}^+ F(\rho)$, (4.31) holds if

$$(4.32) \quad |p_1^s(\rho) + q_1(\rho)| < \kappa(\rho).$$

Taking the squares of both sides and recalling the definition of $\Lambda_{\pm}(\rho)$, we have that (4.32) holds iff

$$\Lambda_-(\rho) < q_1(\rho) < \Lambda_+(\rho), \quad \forall \rho \in \Sigma,$$

which is condition (4.22) in the statement of the theorem.

Finally, since

$$|p_1^s(\rho)| + |q_1(\rho)| < \kappa(\rho)$$

(and $|p_1^s(\rho)| < \kappa(\rho)$) implies (4.32), the last statement also follows, and this concludes the proof. \square

5. A PSEUDODIFFERENTIAL GENERALIZATION

In this final section I generalize Theorem 4.1 to a pseudodifferential setting.

Definition 5.1. Let $P_1, \dots, P_N \in \Psi_{\text{cl}}^m(\Omega; \mathbb{C})$ be m th-order complex classical properly supported pseudodifferential operators on an open set $\Omega \subset \mathbb{R}^n$. I will denote by $\Sigma \subset T^*\Omega \setminus 0$ the characteristic set (supposed to be nonempty) of the operator $P = \sum_{j=1}^N P_j^* P_j$, and by $\mathbb{S}^*\Sigma$ the set

$$\mathbb{S}^*\Sigma = \{(x, \xi) \in \Sigma; |\xi| = 1\}.$$

Denoting by $p_j(x, \xi)$ the principal symbol of the operator P_j , $1 \leq j \leq N$, I say that the system (P_1, \dots, P_N) satisfies condition (K_{Σ}) if

$$(5.33) \quad \sum_{1 \leq j < k \leq N} |\{p_j, p_k\}(\rho)| > 0, \quad \forall \rho \in \mathbb{S}^*\Sigma.$$

Let

$$\kappa(\rho) = \left(p_{2m-1}^s(\rho)^2 + \max_{1 \leq j < k \leq N} |\{p_j, p_k\}(\rho)|^2 \right)^{1/2}, \quad \rho \in \Sigma.$$

Notice that $\kappa: \Sigma \rightarrow [0, +\infty)$ is continuous and positively homogeneous of degree $2m - 1$ in the fibers.

Exactly the same proof of Theorem 4.1 above gives the following pseudodifferential generalization.

Theorem 5.2. *Suppose that $P_1, \dots, P_N \in \Psi_{cl}^m(\Omega; \mathbb{C})$ are properly supported and satisfy condition (K_Σ) . Then for $P = \sum_{j=1}^N P_j^* P_j$ (now a $2m$ th-order classical pseudodifferential operator) we have*

$$p_{2m-1}^s(\rho)^2 < p_{2m-1}^s(\rho)^2 + \max_{1 \leq j < k \leq N} |\{p_j, p_k\}(\rho)|^2 \leq \left(\text{Tr}^+ F(\rho) \right)^2, \quad \forall \rho \in \Sigma.$$

Hence P satisfies the following subelliptic estimate: For every compact $K \subset \Omega$ there exists $c_K > 0$ such that

$$(5.34) \quad c_K \|u\|_{m-1/2}^2 \leq \text{Re}(Pu, u) + \|u\|_{m-1}^2, \quad \forall u \in C_c^\infty(K)$$

(hence (1.4) is a particular case of (5.34) for $m = 1$). Moreover, if Q is a $(2m - 1)$ st-order classical properly supported pseudodifferential operator whose principal symbol has real part q_{2m-1} that satisfies

$$(5.35) \quad -\kappa(\rho) - p_{2m-1}^s(\rho) < q_{2m-1}(\rho) < \kappa(\rho) - p_{2m-1}^s(\rho), \quad \forall \rho \in \Sigma,$$

then $P + Q$ keeps satisfying (5.34). In particular, condition (5.35) holds if

$$(5.36) \quad |q_{2m-1}(\rho)| < \kappa(\rho) - |p_{2m-1}^s(\rho)|, \quad \forall \rho \in \Sigma.$$

Once more, note that *no smoothness assumption* on Σ is required.

REFERENCES

- [1] A. Altomani, C. D. Hill, M. Nacinovich and E. Porten. Complex vector fields and hypoelliptic partial differential operators. *Ann. Inst. Fourier* **60** (2010), 987–1034.
- [2] S. Berhanu, P. D. Cordaro and J. Hounie. *An introduction to involutive structures*. New Mathematical Monographs, **6**. Cambridge University Press, Cambridge, 2008. xii+392 pp.
- [3] P. Bolley, J. Camus and J. Nourrigat. La condition de Hörmander-Kohn pour les opérateurs pseudo-différentiels. *Comm. Partial Differential Equations* **7** (1982), 197–221.
- [4] M. Christ. A remark on sums of squares of complex vector fields. Preprint (March 2005).
- [5] M. Derridj. Sur une classe d'opérateurs différentiels hypoelliptiques à coefficients analytiques. Séminaire Goulaouic-Schwartz 1970–1971: Équations aux dérivées partielles et analyse fonctionnelle, Exp. No. 12, 6 pp. Centre de Math., École Polytech., Paris, 1971.
- [6] V. S. Fedii. A certain criterion for hypoellipticity. *Mat. Sb. (N.S.)* **85** (1971), 18–48.
- [7] B. Helffer and J. Nourrigat. *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs*. Progress in Mathematics, **58**. Birkhäuser Boston, Inc., Boston, MA, 1985. x+278 pp.
- [8] R. A. Horn and C. R. Johnson. *Topics in matrix analysis*. Cambridge University Press, Cambridge, 1991. viii+607 pp.
- [9] L. Hörmander. *The analysis of linear partial differential operators. III. Pseudo-differential operators*. (Corrected second printing.) Grundlehren der Mathematischen Wissenschaften, **274**. Springer-Verlag, Berlin, 1994. viii+525 pp.
- [10] J. J. Kohn. Hypoellipticity and loss of derivatives. (With an appendix by Makhlof Derridj and David S. Tartakoff.) *Ann. of Math.* **162** (2005), 943–986.
- [11] A. Melin. Lower bounds for pseudo-differential operators. *Ark. Mat.* **9** (1971), 117–140, doi:10.1007/BF02383640

- [12] Y. Morimoto. On the hypoellipticity for infinitely degenerate semi-elliptic operators. *J. Math. Soc. Japan* **30** (1978), 327–358, doi: 10.2969/jmsj/03020327
- [13] J. F. Nourrigat. Systèmes sous-elliptiques. II. *Invent. Math.* **104** (1991), 377–400.
- [14] C. Parenti and A. Parmeggiani. On the hypoellipticity with a big loss of derivatives. *Kyushu J. Math.* **59** (2005), 155–230, doi: 10.2206/kyushujm.59.155
- [15] C. Parenti and A. Parmeggiani. A note on Kohn’s and Christ’s examples. *Hyperbolic problems and regularity questions*, 151–158, Trends Math., Birkhäuser, Basel, 2007, doi: https://doi.org/10.1007/978-3-7643-7451-8_15
- [16] C. Parenti and A. Parmeggiani. On the stability of the C^∞ -hypoellipticity under perturbations. *Proc. Amer. Math. Soc.* **146** (2018), 1097–1104, doi:10.1090/proc/13800
- [17] A. Parmeggiani. A remark on the stability of C^∞ -hypoellipticity under lower-order perturbations. *J. Pseudo-Differ. Oper. Appl.* **6** (2015), 227–235, doi: 10.1007/s11868-015-0118-8
- [18] L. Rothschild and E. M. Stein. Hypoelliptic differential operators and nilpotent groups. *Acta Math.* **137** (1976), 247–320, doi: 10.1007/BF02392419
- [19] F. Trèves. Symplectic geometry and analytic hypo-ellipticity. *Differential equations: La Pietra 1996. Conference on differential equations marking the 70th birthdays of Peter Lax and Louis Nirenberg*, 201–219, AMS Proc. Symp. Pure Math. **65**, 1999.
- [20] F. Trèves. On the analyticity of solutions of sums of squares of vector fields. *Phase space analysis of partial differential equations*, 315–329. Birkhäuser. *Progress in Nonlinear Differential Equations and Their Applications* **69**, 2006.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BOLOGNA, PIAZZA DI PORTA S.DONATO 5, 40126 BOLOGNA, ITALY

Email address: alberto.parmeggiani@unibo.it