Classification of finite irreducible conformal modules for K'_4

Published Version:
Bagnoli, L., Caselli, F. (2022). Classification of finite irreducible conformal modules for K'_4. JOURNAL OF MATHEMATICAL PHYSICS, 63(9), 1-41 [10.1063/5.0098441].

Published:
DOI: http://doi.org/10.1063/5.0098441

CLASSIFICATION OF FINITE IRREDUCIBLE CONFORMAL MODULES FOR K_4'

LUCIA BAGNOLI AND FABRIZIO CASELLI

Abstract. We classify the finite irreducible modules over the conformal superalgebra K_4' by their correspondence with finite conformal modules over the associated annihilation superalgebra $\mathcal{A}(K_4')$. This is achieved by a complete classification of singular vectors in generalized Verma modules for $\mathcal{A}(K_4')$. We also show that morphisms between generalized Verma modules can be arranged in infinitely many bilateral complexes.

1. Introduction

Finite simple conformal superalgebras were completely classified in [13] and consist of the following list: $\mathrm{Cur}\,\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite−dimensional Lie superalgebra, $W_n(n \geq 0)$, $S_{n,b}$, $\tilde{S}_n$ $(n \geq 2,\, b \in \mathbb{C})$, $K_n(n \geq 0,\, n \neq 4)$, K_4', CK_6. The finite irreducible modules over the conformal superalgebras $\mathrm{Cur}\,\mathfrak{g}$, K_0, K_1 were studied in [8]. Boyallian, Kac, Liberati and Rudakov classified all finite irreducible modules over the conformal superalgebras of type W and S in [3]; Boyallian, Kac and Liberati classified all finite irreducible modules over the conformal superalgebras of type K_n in [1]. The classification of all finite irreducible modules over the conformal superalgebras of type K_n, for $n \leq 4$, had been previously studied also by Cheng and Lam in [11]. Finally, a classification of all finite irreducible modules over the conformal superalgebra CK_6 was obtained in [2] and [23] with different approaches. For $n = 4$ the conformal superalgebra K_4 is not simple and its the derived subalgebra K_4' is instead a simple conformal superalgebra.

A possible strategy for studying modules over conformal superalgebras is the following. If R is a conformal superalgebra one considers the Lie superalgebra $\mathfrak{g} = \mathcal{A}(R)$, called the annihilation superalgebra of R. The annihilation superalgebra has a fundamental role since the study of the finite modules over R is equivalent to the study of *finite conformal* modules over $\mathfrak{g}$. Furthermore, if R is $\mathbb{Z}$-graded then $\mathfrak{g}$ is also $\mathbb{Z}$-graded and one can reduce the problem to finite Verma modules of $\mathfrak{g}$, i.e. induced modules $\mathrm{Ind}(F) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\geq 0})} F$, where F is a finite dimensional $\mathfrak{g}_{\geq 0}$-module [18, 11].

This is the case for the simple conformal superalgebra K_4', and its annihilation superalgebra $\mathcal{A}(K_4')$. The main goal of this paper is therefore to classify irreducible conformal modules for K_4' through the classification of all degenerate (i.e., non irreducible) finite Verma modules for $\mathcal{A}(K_4')$; in turn, this is equivalent to the classification of (highest weight) singular vectors in these modules, i.e. vectors which are annihilated by $\mathcal{A}(K_4')_{>0}$. The final result is much richer than in the "standard" conformal contact superalgebras K_n where, up to duality, there is only one family of singular vectors, all of degree 1: we show that for $\mathcal{A}(K_4')$ there are four

2010 *Mathematics Subject Classification.* 08A05, 17B05 (primary), 17B65, 17B70 (secondary).
Key words and phrases. conformal superalgebras, linearly compact Lie superalgebras, finite Verma modules, singular vectors.

families of singular vectors of degree 1, four families of singular vectors of degree 2 and two "exceptional" singular vectors of degree 3.

Since the classification of singular vectors in finite Verma modules is equivalent to the classification of morphisms between such modules, we show that these morphisms can be arranged in an infinite number of bilateral complexes in a picture (see Figure 1) which is similar to those obtained for the exceptional linearly compact Lie superalgebras $E(1,6)$, $E(3,6)$, $E(3,8)$ and $E(5,10)$ (see [17, 18, 19, 20, 4, 6]). In a subsequent publication we will compute the homology of these complexes and provide an explicit construction of all irreducible quotients.

The paper is organized as follows. In section 2 we collect all preliminaries on conformal superalgebras which are needed, in section 3 we describe the conformal superalgebra K_4' and in section 4 its annihilation superalgebra $\mathcal{A}(K_4')$. In section 5 we show explicitly how the conformal superalgebra K_4' acts on a finite Verma module. In section 6 we deduce the crucial conditions that must be satisfied by a singular vector and we show that singular vectors have degree at most 3. Finally, section 7, 8, 9 contain the classification of singular vectors of degree 2, 3, 1 respectively.

2. Preliminaries on conformal superalgebras

In this section we introduce some notions on conformal superalgebras. For further details see [15, Chapter 2], [12], [3], [1].

Let $\mathfrak{g}$ be a Lie superalgebra; a formal distribution with coefficients in $\mathfrak{g}$, or equivalently a $\mathfrak{g}-$valued formal distribution, in the indeterminate z is an expression of the following form:

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^n,$$

with $a_n \in \mathfrak{g}$ for every $n \in \mathbb{Z}$. We denote the vector space of formal distributions with coefficients in $\mathfrak{g}$ in the indeterminate z by $\mathfrak{g}[[z,z^{-1}]]$. We denote by $\mathrm{Res}(a(z)) = a_{-1}$ the coefficient of z^{-1} of $a(z)$. The vector space $\mathfrak{g}[[z,z^{-1}]]$ has a natural structure of $\mathbb{C}[\partial_z]-$module. We define for all $a(z) \in \mathfrak{g}[[z,z^{-1}]]$ its derivative:

$$\partial_z a(z) = \sum_{n\in\mathbb{Z}} n a_n z^{n-1}.$$

A formal distribution with coefficients in $\mathfrak{g}$ in the indeterminates z and w is an expression of the following form:

$$a(z,w) = \sum_{m,n\in\mathbb{Z}} a_{m,n} z^m w^n,$$

with $a_{m,n} \in \mathfrak{g}$ for every $m,n \in \mathbb{Z}$. We denote the vector space of formal distributions with coefficients in $\mathfrak{g}$ in the indeterminates z and w by $\mathfrak{g}[[z,z^{-1},w,w^{-1}]]$. Given two formal distributions $a(z) \in \mathfrak{g}[[z,z^{-1}]]$ and $b(w) \in \mathfrak{g}[[w,w^{-1}]]$, we define the commutator $[a(z),b(w)]$:

$$[a(z),b(w)] = \left[\sum_{n\in\mathbb{Z}} a_n z^n, \sum_{m\in\mathbb{Z}} b_m w^m\right] = \sum_{m,n\in\mathbb{Z}} [a_n,b_m] z^n w^m.$$

Definition 2.1. Two formal distributions $a(z), b(z) \in \mathfrak{g}[[z,z^{-1}]]$ are called *local* if

$$(z-w)^N [a(z),b(w)] = 0 \text{ for } N \gg 0.$$

We call $\delta-$function the following formal distribution in the indeterminates z and w:

$$\delta(z-w)=\sum_{m,n:\,m+n=-1} z^m w^n.$$

See Corollary 2.2 in [15] for the following equivalent condition of locality.

Proposition 2.2. *Two formal distributions $a(z), b(z) \in \mathfrak{g}[[z,z^{-1}]]$ are local if and only if $[a(z),b(w)]$ can be expressed as a finite sum of the form:*

$$[a(z),b(w)]=\sum_j (a(w)_{(j)}b(w))\frac{\partial_w^j}{j!}\delta(z-w),$$

where the coefficients $(a(w)_{(j)}b(w))$ are formal distributions in the indeterminate w. Moreover, if $a(z)$ and $b(z)$ are local then necessarily $(a(w)_{(j)}b(w)) = \mathrm{Res}_z(z-w)^j[a(z),b(w)]$.

Definition 2.3 (Formal Distribution Superalgebra)**.** Let $\mathfrak{g}$ be a Lie superalgebra and $\mathcal{F}$ a family of mutually local $\mathfrak{g}-$valued formal distributions in the indeterminate z. The pair $(\mathfrak{g},\mathcal{F})$ is called a *formal distribution superalgebra* if the coefficients of all formal distributions in $\mathcal{F}$ span $\mathfrak{g}$.

We define the $\lambda-$bracket between two local formal distributions $a(z), b(z) \in \mathfrak{g}[[z,z^{-1}]]$ as the generating series of the $(a(z)_{(j)}b(z))$'s:

$$[a(z)_\lambda b(z)]=\sum_{j\geq 0}\frac{\lambda^j}{j!}(a(z)_{(j)}b(z)). \tag{1}$$

The λ-bracket of formal distributions satisfies some algebraic properties which are the motivation of the following definition. If V is any $\mathbb{Z}_2$-graded vector space we denote by p its parity function. As customary, whenever we write $p(v)$ for some $v \in V$ we always implicitly assume that v is a homogeneous element of V.

Definition 2.4 (Conformal superalgebra)**.** A *conformal superalgebra* R is a left $\mathbb{Z}_2-$graded $\mathbb{C}[\partial]-$module endowed with a $\mathbb{C}-$linear map, called $\lambda-$bracket, $R\otimes R \to \mathbb{C}[\lambda]\otimes R$, $a\otimes b \mapsto [a_\lambda b]$, that satisfies the following properties for all $a,b,c \in R$:

(i) $p(\partial a)=p(a)$;
(ii) $[\partial a_\lambda b]=-\lambda[a_\lambda b], \quad [a_\lambda \partial b]=(\lambda+\partial)[a_\lambda b]$;
(iii) $[a_\lambda b]=-(-1)^{p(a)p(b)}[b_{-\lambda-\partial}a]$;
(iv) $[a_\lambda[b_\mu c]]=[[a_\lambda b]_{\lambda+\mu}c]+(-1)^{p(a)p(b)}[b_\mu[a_\lambda c]]$.

We refer to properties (ii), (iii), (iv) in Definition 2.4 as the conformal linearity, conformal symmetry and conformal Jacobi identity respectively. We call $n-$products the coefficients $(a_{(n)}b)$ that appear in $[a_\lambda b]=\sum_{n\geq 0}\frac{\lambda^n}{n!}(a_{(n)}b)$ and give an equivalent definition of conformal superalgebra.

Definition 2.5 (Conformal superalgebra)**.** A *conformal superalgebra* R is a left $\mathbb{Z}_2-$graded $\mathbb{C}[\partial]-$module endowed with a $\mathbb{C}-$bilinear product $(a_{(n)}b): R\otimes R \to R$, defined for every $n\geq 0$, that satisfies the following properties for all $a,b,c\in R$:

(i) $p(\partial a)=p(a)$;
(ii) $(a_{(n)}b)=0,\ for\ n\gg 0$;
(iii) $(\partial a_{(0)}b)=0$ and $(\partial a_{(n)}b)=-n(a_{(n-1)}b)$ for all $n\geq 1$;

(iv) $(a_{(n)}b) = -(-1)^{p(a)p(b)} \sum_{j\geq 0}(-1)^{j+n}\frac{\partial^j}{j!}(b_{(n+j)}a)$ for all $n \geq 0$;
(v) $(a_{(m)}(b_{(n)}c)) = \sum_{j=0}^{m}\binom{m}{j}((a_{(j)}b)_{(m+n-j)}c) + (-1)^{p(a)p(b)}(b_{(n)}(a_{(m)}c))$ for all $m, n \geq 0$.

Using conditions (iii) and (iv) in Definition 2.5 it is easy to show that for all $a, b \in R$, $n \geq 0$:

$$(a_{(n)}\partial b) = \partial(a_{(n)}b) + n(a_{(n-1)}b). \tag{2}$$

In particular, by the first part of (iii) in Definition 2.5, the map $\partial : R \to R$, $a \mapsto \partial a$ is a derivation with respect to the $0-$product.

Remark 2.6. Let $\mathcal{F}$ be a formal distribution superalgebra in the indeterminate z which is a vector subspace of $\mathbb{C}[[z]]$ and is invariant under the operator ∂_z. Then the formal distribution algebra $\mathcal{F}$, endowed with $\lambda-$bracket (1) and operator $\partial = \partial_z$ is a conformal superalgebra (for a proof see [15, Proposition 2.3]).

We say that a conformal superalgebra R is *finite* if it is finitely generated as a $\mathbb{C}[\partial]-$module. An *ideal* I of R is a $\mathbb{C}[\partial]-$submodule of R such that $a_{(n)}b \in I$ for every $a \in R$, $b \in I$, $n \geq 0$. A conformal superalgebra R is *simple* if it has no non-trivial ideals and the $\lambda-$bracket is not identically zero. We denote by R' the *derived subalgebra* of R, i.e. the $\mathbb{C}-$span of all $n-$products.

Definition 2.7. A module M over a conformal superalgebra R is a $\mathbb{Z}_2-$graded $\mathbb{C}[\partial]-$module endowed with $\mathbb{C}-$linear maps $R \to \mathrm{End}_{\mathbb{C}} M$, $a \mapsto a_{(n)}$, defined for every $n \geq 0$, that satisfy the following properties for all $a, b \in R$, $v \in M$:

(i) $a_{(n)}v = 0$ *for* $n \gg 0$;
(ii) $(\partial a)_{(n)}v = [\partial, a_{(n)}]v = -na_{(n-1)}v$ for all $n \geq 0$;
(iii) $[a_{(m)}, b_{(n)}]v = \sum_j \binom{m}{j}(a_{(j)}b)_{(m+n-j)}v$ for all $m, n \geq 0$.

A module M is called *finite* if it is a finitely generated $\mathbb{C}[\partial]-$module.
We can construct a conformal superalgebra starting from a formal distribution superalgebra $(\mathfrak{g}, \mathcal{F})$. Let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}$ under all the $n-$products, ∂_z and linear combinations. By Dong's Lemma, $\overline{\mathcal{F}}$ is still a family of mutually local formal distributions (see [15]) and it turns out that $\overline{\mathcal{F}}$ is a conformal superalgebra. We will refer to $\overline{\mathcal{F}}$ as the conformal superalgebra *associated* with $(\mathfrak{g}, \mathcal{F})$.
Let us recall the construction of the annihilation superalgebra associated with a conformal superalgebra R. Let $\widetilde{R} = R[y, y^{-1}]$, set $p(y) = 0$ and $\widetilde{\partial} = \partial + \partial_y$. We define the following $k-$products on $\widetilde{R}$, for all $a, b \in R$, $f, g \in \mathbb{C}[y, y^{-1}]$, $k \geq 0$:

$$(af_{(k)}bg) = \sum_{j\in\mathbb{Z}_+}(a_{(k+j)}b)\Big(\frac{\partial_y^j}{j!}f\Big)g.$$

In particular if $f = y^m$ and $g = y^n$ we have for all $k \geq 0$:

$$(ay^m{}_{(k)}by^n) = \sum_{j\in\mathbb{Z}_+}\binom{m}{j}(a_{(k+j)}b)y^{m+n-j}. \tag{3}$$

We observe that $\widetilde{\partial}\widetilde{R}$ is a two sided ideal of $\widetilde{R}$ with respect to the $0-$product. The quotient $\mathrm{Lie}\, R := \widetilde{R}/\widetilde{\partial}\widetilde{R}$ has a structure of Lie superalgebra with the bracket induced by the

0−product, i.e. for all $a, b \in R$, $f, g \in \mathbb{C}[y, y^{-1}]$,

$$[af, bg] = \sum_{j \in \mathbb{Z}_+} (a_{(j)}b)\Big(\frac{\partial_y^j}{j!} f\Big) g. \tag{4}$$

Definition 2.8. The *annihilation superalgebra* $\mathcal{A}(R)$ of a conformal superalgebra R is the subalgebra of Lie R spanned by all elements ay^n with $n \geq 0$ and $a \in R$.
The extended annihilation superalgebra $\mathcal{A}(R)^e$ of a conformal superalgebra R is the Lie superalgebra $\mathbb{C}\partial \ltimes \mathcal{A}(R)$. The semidirect sum $\mathbb{C}\partial \ltimes \mathcal{A}(R)$ is the vector space $\mathbb{C}\partial \oplus \mathcal{A}(R)$ endowed with the structure of Lie superalgebra uniquely determined by the bracket

$$[\partial, ay^m] = -\partial_y(ay^m) = -may^{m-1},$$

for all $a \in R$, and the fact that $\mathcal{A}(R)$ and $\mathbb{C}\partial$ are Lie subalgebras.

For all $a \in R$ we consider the following formal power series in $\mathcal{A}(R)[[\lambda]]$:

$$a_\lambda = \sum_{n \geq 0} \frac{\lambda^n}{n!} ay^n.$$

For all $a, b \in R$, we have: $[a_\lambda, b_\mu] = [a_\lambda b]_{\lambda+\mu}$ and $(\partial a)_\lambda = -\lambda a_\lambda$ (for a proof see [5]). This notation is coherent with the definition of conformal modules in the following sense.

Proposition 2.9 ([8])**.** *Let R be a conformal superalgebra. If M is a finite conformal R-module then M has a natural structure of $\mathcal{A}(R)^e$-module, where the action of ay^n on M is uniquely determined by $a_\lambda v = \sum_{n \geq 0} \frac{\lambda^n}{n!} ay^n.v$ for all $v \in V$. Viceversa if M is a $\mathcal{A}(R)^e$-module such that for all $a \in R$, $v \in M$ we have $ay^n.v = 0$ for $n \gg 0$ then M is also a finite conformal module by letting $a_\lambda v = \sum_n \frac{\lambda^n}{n!} ay^n.v$.*

One usually refers to Proposition 2.9 by saying that a module over a conformal superalgebra R is the *same* as a *continuous* module over the Lie superalgebra $\mathcal{A}(R)^e$. Proposition 2.9 reduces the study of modules over a conformal superalgebra R to the study of a class of modules over its (extended) annihilation superalgebra.

In some cases one can even avoid to use the extended annihilation algebra and simply consider the annihilation algebra. Recall that a Lie superalgebra $\mathfrak{g}$ is $\mathbb{Z}$-graded if $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ with $[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m}$ for all $n, m \in \mathbb{Z}$. We say in this case that $\mathfrak{g}$ has finite depth $d \geq 0$ if $\mathfrak{g}_n = 0$ for all $n < -d$ and $\mathfrak{g}_{-d} \neq 0$.

Proposition 2.10 ([1])**.** *Let $\mathfrak{g}$ be the annihilation superalgebra of a conformal superalgebra R. Assume that $\mathfrak{g}$ satisfies the following conditions:*

- **L1:** *$\mathfrak{g}$ is $\mathbb{Z}-$graded with finite depth d;*
- **L2:** *there exists $t \in \mathfrak{g}$ such that the centralizer of t is contained in $\mathfrak{g}_0$;*
- **L3:** *there exists $\Theta \in \mathfrak{g}_{-d}$ such that $\mathfrak{g}_{i-d} = [\Theta, \mathfrak{g}_i]$, for all $i \geq 0$.*

Finite modules over R are the same as modules V over $\mathfrak{g}$, called finite conformal, that satisfy the following properties:

1. *for every $v \in V$, we have $\mathfrak{g}_n.v = 0$ for $n \gg 0$;*
2. *V is finitely generated as a $\mathbb{C}[\Theta]-$module.*

Remark 2.11. We point out that condition **L2** is automatically satisfied when $\mathfrak{g}$ contains a grading element, i.e. an element $t \in \mathfrak{g}$ such that $[t, b] = \deg(b)b$ for all $b \in \mathfrak{g}$.

Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}-$graded Lie superalgebra. We will use the notation $\mathfrak{g}_{>0} = \bigoplus_{i>0} \mathfrak{g}_i$, $\mathfrak{g}_{<0} = \bigoplus_{i<0} \mathfrak{g}_i$ and $\mathfrak{g}_{\geq 0} = \bigoplus_{i\geq 0} \mathfrak{g}_i$. We denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

Definition 2.12. Let F be a $\mathfrak{g}_{\geq 0}-$module. The *generalized Verma module* associated with F is the $\mathfrak{g}-$module $\mathrm{Ind}(F)$ defined by,

$$\mathrm{Ind}(F) := \mathrm{Ind}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}}(F) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\geq 0})} F.$$

If F is a finite dimensional irreducible $\mathfrak{g}_{\geq 0}-$module we will simply say that $\mathrm{Ind}(F)$ is a finite Verma module. We will identify $\mathrm{Ind}(F)$ with $U(\mathfrak{g}_{<0}) \otimes F$ as vector spaces via the Poincaré−Birkhoff−Witt Theorem. The $\mathbb{Z}-$grading of $\mathfrak{g}$ induces a $\mathbb{Z}-$grading on $U(\mathfrak{g}_{<0})$ and $\mathrm{Ind}(F)$. We will invert the sign of the degree, so that we have a $\mathbb{Z}_{\geq 0}-$grading on $U(\mathfrak{g}_{<0})$ and $\mathrm{Ind}(F)$. We will say that an element $v \in U(\mathfrak{g}_{<0})_k$ is homogeneous of degree k. Analogously an element $m \in U(\mathfrak{g}_{<0})_k \otimes F$ is homogeneous of degree k.

Proposition 2.13. *Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}-$graded Lie superalgebra. If F is an irreducible finite−dimensional $\mathfrak{g}_{\geq 0}-$module, then* $\mathrm{Ind}(F)$ *has a unique maximal submodule. We denote by* $\mathrm{I}(F)$ *the quotient of* $\mathrm{Ind}(F)$ *by the unique maximal submodule.*

Proof. First we point out that a submodule $V \neq \{0\}$ of $\mathrm{Ind}(F)$ is proper if and only if it does not contain nontrivial elements of degree 0. Indeed, if V contains an element $v_0 \neq 0$ of degree 0, then it contains $1 \otimes F = \mathfrak{g}_{\geq 0}.v_0$, due to irreducibility of F. Therefore $\mathfrak{g}_{<0}.F = \mathrm{Ind}(F) \subseteq V$. The union S of all proper submodules is still a proper submodule of $\mathrm{Ind}(F)$, since S does not contain nontrivial elements of degree 0, thus S is the unique maximal proper submodule. □

Definition 2.14. Given a $\mathfrak{g}-$module V, we call *singular vectors* the elements of:

$$\mathrm{Sing}(V) = \{v \in V \mid \mathfrak{g}_{>0}.v = 0\}.$$

Homogeneous components of singular vectors are still singular vectors so we often assume that singular vectors are homogeneous without loss of generality. In the case $V = \mathrm{Ind}(F)$ for a $\mathfrak{g}_{\geq 0}-$module F, we will call *trivial singular vectors* the elements of $\mathrm{Sing}(V)$ of degree 0 and *nontrivial singular vectors* the nonzero elements of $\mathrm{Sing}(V)$ of positive degree.

Theorem 2.15 ([18],[11])**.** *Let $\mathfrak{g}$ be a Lie superalgebra that satisfies L1, L2, L3 in Proposition 2.10; then*

(1) *if F is an irreducible finite−dimensional $\mathfrak{g}_{\geq 0}-$module, then the action of $\mathfrak{g}_{>0}$ on F is trivial;*
(2) *the map $F \mapsto \mathrm{I}(F)$ is a bijective map between irreducible finite−dimensional $\mathfrak{g}_0-$modules and irreducible finite conformal $\mathfrak{g}-$modules;*
(3) *the $\mathfrak{g}-$module* $\mathrm{Ind}(F)$ *is irreducible if and only if the $\mathfrak{g}_0-$module F is irreducible and* $\mathrm{Ind}(F)$ *has no nontrivial singular vectors.*

We recall the notion of duality for conformal modules (see for further details [3], [5]). Let R be a conformal superalgebra and M a conformal module over R.

Definition 2.16. The conformal dual M^* of M is defined by

$$M^* = \{f_\lambda : M \to \mathbb{C}[\lambda] \mid f_\lambda(\partial m) = \lambda f_\lambda(m),\ \forall m \in M\}.$$

The structure of $\mathbb{C}[\partial]-$module is given by $(\partial f)_\lambda(m) = -\lambda f_\lambda(m)$, for all $f \in M^*$, $m \in M$. The $\lambda-$action of R is given, for all $a \in R$, $m \in M$, $f \in M^*$, by:

$$(a_\lambda f)_\mu(m) = -(-1)^{p(a)p(f)} f_{\mu-\lambda}(a_\lambda m).$$

Definition 2.17. Let $T : M \to N$ be a morphism of $R-$modules, i.e. a linear map such that for all $a \in R$ and $m \in M$:

i: $T(\partial m) = \partial T(m)$,

ii: $T(a_\lambda m) = a_\lambda T(m)$.

The dual morphism $T^* : N^* \to M^*$ is defined, for all $f \in N^*$ and $m \in M$, by:

$$[T^*(f)]_\lambda (m) = -f_\lambda (T(m)).$$

3. The conformal superalgebra K_4'

In this section we introduce and study the contact Lie superalgebras and related conformal superalgebras. Let $\bigwedge(N)$ be the Grassmann superalgebra in the N odd indeterminates $\xi_1, ..., \xi_N$. Let t be an even indeterminate and $\bigwedge(1, N) = \mathbb{C}[t, t^{-1}] \otimes \bigwedge(N)$ which we consider as an associative algebra in the natural way omitting the symbol $\wedge$ between the indeterminates ξ_i's. We also consider the Lie superalgebra of derivations of $\bigwedge(1, N)$:

$$W(1, N) = \left\{ D = a\partial_t + \sum_{i=1}^{N} a_i \partial_i \mid a, a_i \in \bigwedge(1, N) \right\},$$

where $\partial_t = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial \xi_i}$ for every $i \in \{1, ..., N\}$.

Let us consider the contact form $\omega = dt - \sum_{i=1}^{N} \xi_i d\xi_i$. The contact Lie superalgebra $K(1, N)$ is defined by:

$$K(1, N) = \left\{ D \in W(1, N) \mid D\omega = f_D \omega \text{ for some } f_D \in \bigwedge(1, N) \right\}.$$

We denote by $K'(1, N)$ the derived algebra $[K(1, N), K(1, N)]$ of $K(1, N)$. Analogously, let $\bigwedge(1, N)_+ = \mathbb{C}[t] \otimes \bigwedge(N)$. We consider the Lie superalgebra of derivations of $\bigwedge(1, N)_+$:

$$W(1, N)_+ = \left\{ D = a\partial_t + \sum_{i=1}^{N} a_i \partial_i \mid a, a_i \in \bigwedge(1, N)_+ \right\}.$$

The Lie superalgebra $K(1, N)_+$ is defined by:

$$K(1, N)_+ = \left\{ D \in W(1, N)_+ \mid D\omega = f_D \omega \ for\ some\ f_D \in \bigwedge(1, N)_+ \right\}.$$

One can define on $\bigwedge(1, N)$ a Lie superalgebra structure as follows: for all $f, g \in \bigwedge(1, N)$ we let:

$$[f, g] = \Big(2f - \sum_{i=1}^{N} \xi_i \partial_i f\Big) \partial_t g - \partial_t f \Big(2g - \sum_{i=1}^{N} \xi_i \partial_i g\Big) + (-1)^{p(f)} \Big(\sum_{i=1}^{N} \partial_i f \partial_i g\Big). \tag{5}$$

It is useful to restate (5) in a more explicit way. We adopt the following notation: we let $\mathcal{I}$ be the set of (finite) sequences of elements in $\{1, \ldots, N\}$; for notational convenience we usually write $I = i_1 \cdots i_r$ instead of $I = (i_1, \ldots, i_r)$ and we think of $\mathcal{I}$ as a monoid by juxtaposition (i.e. if $I = i_1 \cdots i_r$ and $J = j_1 \cdots j_s$ we let $IJ = i_1 \cdots i_r j_1 \cdots j_s$); if $I = i_1 \cdots i_r \in \mathcal{I}$ we let $\xi_I = \xi_{i_1} \cdots \xi_{i_r}$ and $|I| = r$. For $m, n \in \mathbb{Z}$ and $I, J \in \mathcal{I}$ we have

$$[t^m \xi_I, t^n \xi_J] = (2n - 2m - n|I| + m|J|) t^{m+n-1} \xi_{IJ} + (-1)^{|I|} t^{m+n} \sum_i \partial_i \xi_I \, \partial_i \xi_J. \tag{6}$$

We recall that $K(1, N) \cong \bigwedge(1, N)$ as Lie superalgebras via the following map (see [10]):

$$\bigwedge(1, N) \longrightarrow K(1, N)$$

$$f \longmapsto 2f\partial_t + (-1)^{p(f)} \sum_{i=1}^{N} (\xi_i \partial_t f + \partial_i f)(\xi_i \partial_t + \partial_i).$$

From now on we will always identify elements of $K(1,N)$ with elements of $\bigwedge(1,N)$. We consider on $K(1,N)$ the standard grading, i.e. for every $t^m \xi_{i_1} \cdots \xi_{i_s} \in K(1,N)$ we have $\deg(t^m \xi_{i_1} \cdots \xi_{i_s}) = 2m + s - 2$.
Next target is to realize $K(1,N)_+$ as the annihilation superalgebra of a conformal superalgebra. In order to do this, we construct a formal distribution superalgebra using the following family of formal distributions:

$$\mathcal{F} = \left\{ A(z) := \sum_{m\in\mathbb{Z}} (t^m A) z^{-m-1} = A\delta(t-z), \ \forall A \in \bigwedge(N) \right\}.$$

Note that the set of all the coefficients of formal distributions in $\mathcal{F}$ spans $\bigwedge(1,N)$.

Proposition 3.1. *The pair $(\bigwedge(1,N), \mathcal{F})$ is a formal distribution superalgebra. More precisely, for all $I, J \in \mathcal{I}$ we have*

$$(\xi_I(z)_{(0)}\xi_J(z)) = (|I|-2)\partial_z \xi_{IJ}(z) + (-1)^{|I|} \sum_{i=1}^{N} (\partial_i \xi_I \, \partial_i \xi_J)(z); \tag{7}$$

$$(\xi_I(z)_{(1)}\xi_J(z)) = (|I|+|J|-4)\xi_{IJ}(z);$$

$$(\xi_I(z)_{(n)}\xi_J(z)) = 0 \ for \ n > 1.$$

In particular the conformal superalgebra associated with $(\bigwedge(1,N), \mathcal{F})$ is $\bar{\mathcal{F}} = \mathbb{C}[\partial_z]\mathcal{F}$.

Proof. Let's show that $\xi_I(z)$ and $\xi_J(z)$ are local. We have:

$$[\xi_I(z), \xi_J(w)] = \sum_{m,n\in\mathbb{Z}} [t^m \xi_I, t^n \xi_J] z^{-m-1} w^{-n-1}$$

$$= \sum_{m,n\in\mathbb{Z}} \Big((n\,(2-|I|) - m\,(2-|J|))\, t^{m+n-1} \xi_{IJ} \ + (-1)^{|I|} t^{m+n} \sum_{i=1}^{N} \partial_i \xi_I \, \partial_i \xi_J \Big) z^{-m-1} w^{-n-1}$$

We let $h = m+n-1$ in the former sum and $l = m+n$ in the latter and we obtain

$$\begin{aligned}
&[\xi_I(z), \xi_J(w)] \\
&= \sum_{h,m\in\mathbb{Z}} ((h-m+1)\,(2-|I|) - m\,(2-|J|))\, t^h \xi_{IJ} \frac{z^{-m-1}}{w^{-(m-h-2)}} \\
&\quad + \sum_{l,m\in\mathbb{Z}} (-1)^r \sum_{i=1}^{N} t^l \partial_i \xi_I \partial_i \xi_J \frac{z^{-m-1}}{w^{-(m-l-1)}} \\
&= \sum_{h,m\in\mathbb{Z}} (h+1)(2-|I|) t^h \xi_{IJ} w^{-h-2} z^{-m-1} w^m + \sum_{h,m\in\mathbb{Z}} m(|I|+|J|-4) t^h \xi_{IJ} w^{-h-1} z^{-m-1} w^{m-1} \\
&\quad + \sum_{l,m\in\mathbb{Z}} (-1)^{|I|} \sum_{i=1}^{N} t^l \partial_i \xi_I \partial_i \xi_J w^{-l-1} z^{-m-1} w^m
\end{aligned}$$

$$= (|I|-2)\partial_w(\xi_{IJ}(w))\delta(z-w) + (|I|+|J|-4)\xi_{IJ}(w)\partial_w\delta(z-w)$$
$$+ (-1)^{|I|}\sum_{i=1}^{N}(\partial_i\xi_I\,\partial_i\xi_J)(w)\delta(z-w).$$

All results follow. □

We can say something more about the conformal superalgebra $\bar{\mathcal{F}}$ associated with the formal distribution superalgebra $(K(1,N),\mathcal{F})$.

Proposition 3.2. *The conformal superalgebra $\bar{\mathcal{F}} = \mathbb{C}[\partial_z]\mathcal{F}$ is a free $\mathbb{C}[\partial_z]-$module.*

Proof. If $A_1, A_2, \dots, A_s$ is a basis of $\bigwedge(N)$ then $A_1\delta(t-z), A_2\delta(t-z), \dots, A_s\delta(t-z)$ is a basis of $\mathcal{F}$. Let us consider a finite linear combination, with coefficients in $\mathbb{C}[\partial_z]$, of elements of this basis:

$$\sum_{i=1}^{s} P_i(\partial_z)A_i\delta(t-z) = 0,$$

where $P_i(\partial_z) \in \mathbb{C}[\partial_z]$ for every $1 \leq i \leq s$. From linear independence of the A_i's, we obtain for every $1 \leq i \leq s$:

$$P_i(\partial_z)\delta(t-z) = 0.$$

Therefore every coefficient P_i must be 0. □

We will identify $\bar{\mathcal{F}} = \mathbb{C}[\partial_z] \otimes \mathcal{F}$ with $K_N := \mathbb{C}[\partial] \otimes \bigwedge(N)$. We also identify ∂_z with ∂ and every $A(z) \in \mathcal{F}$ with $A \in \bigwedge(N)$. We will refer to K_N as the conformal superalgebra associated with $K(1,N)$. For all $I, J \in \mathcal{I}$ the $\lambda-$bracket is given by

$$[\xi_I{}_\lambda \xi_J] = (|I|-2)\partial\xi_{IJ} + (-1)^{|I|}\sum_{i=1}^{N}\partial_i\xi_I\partial_i\xi_J + \lambda(|I|+|J|-4)\xi_{IJ}, \tag{8}$$

by Proposition 3.1. In [1] it is shown that the annihilation superalgebra of K_N is $\mathcal{A}(K_N) = K(1,N)_+$ and that it satisfies conditions L1, L2, L3. Thus, the study of finite irreducible modules over the conformal superalgebra K_N is reduced to the study of singular vectors of Verma modules on $K(1,N)_+$.
Now we concentrate in the special case $N = 4$, because the conformal superalgebra K_4 is not simple. The derived superalgebra K_4' is one of the exceptional cases appearing in the classification of finite simple conformal superalgebras in [13]. Our main target is to study all finite irreducible modules over the conformal superalgebra K_4'.

In order to describe K_4' explicitly we need to introduce the following terminology. Let V be a vector space and $B = \{b_i\}_{i\in \mathrm{I}}$ be a basis of V. An element $v \in V$ can be uniquely expressed as $v = \sum_i c_i b_i$. The support of v with respect to B is $\mathrm{Supp}_B\, v = \{b_i : c_i \neq 0\}$. We will usually drop the index B if there is no risk of confusion.

Recall that we denote by $\mathcal{I}$ the set of all sequences with entries in $\{1,2,3,4\}$. We also denote by $\mathcal{I}_{\neq}$ the set of sequences in $\mathcal{I}$ with distinct entries and by $\mathcal{I}_{<}$ the set of sequences in $\mathcal{I}$ with strictly increasing entries. For typographical reasons we simply denote by $i_1 \cdots i_r$ the sequence $(i_1, \dots, i_r)$.

Proposition 3.3. *The element $\xi_{1234} \notin K_4'$. More precisely:*

$$K_4' = \langle\{\partial^k \xi_I,\, \partial^l \xi_{1234} : I \in \mathcal{I}_<,\, I \neq 1234,\, k \geq 0,\, l > 0\}\rangle.$$

Proof. By Proposition 3.2, we know that $\left\{\partial^k \xi_I : k \geq 0,\, I \in \mathcal{I}_<\right\}$ is a basis for K_4. We first show that $\xi_{1234} \notin K_4'$. Since the $j-$products are bilinear maps, it is sufficient to show that ξ_{1234} does not belong to $\mathrm{Supp}(\xi_{I\,(j)}\xi_J)$, for all $I, J \in \mathcal{I}_<$. This is an immediate consequence of (8).

Now we show that every element $\partial^k \xi_I$ with $k > 0$ or $I \neq 1234$ lies in K_4':

(1) if $k > 0$, then $\partial^k \xi_I = \left(-\frac{1}{2}{}_{(0)}\partial^{k-1}\xi_I\right)$ by (8);
(2) if $k = 0$ and $I \neq 1234$ let $i \in \{1234\}$ be such that $\xi_{iI} \neq 0$. Then we have $\xi_I = -(\xi_{i\,(0)}\xi_{iI})$ by (8).

$\square$

Proposition 3.4. *The element $t^{-1}\xi_{1234} \notin K'(1,4)$. More precisely:*

$$K'(1,4) = \langle\{t^k \xi_I,\, t^l \xi_{1234} : I \in \mathcal{I}_<,\, I \neq 1234,\, k, l \in \mathbb{Z},\, l \neq -1\}\rangle.$$

Proof. We know that $\left\{t^k \xi_I : k \in \mathbb{Z}, I \in \mathcal{I}_<\right\}$ is a basis for $K(1,4)$. Let us first show that $t^{-1}\xi_{1234} \notin K'(1,4)$. Since the bracket (5) is bilinear, it is sufficient to prove that $t^{-1}\xi_{1234}$ does not belong to $\mathrm{Supp}[t^m\xi_I, t^n\xi_J]$ for all $m, n \in \mathbb{Z}$ and $I, J \in \mathcal{I}_<$. Indeed, if $t^{-1}\xi_{1234} \in \mathrm{Supp}[t^m\xi_I, t^n\xi_J]$ then necessarily $m+n = 0$ and $|I|+|J| = 4$, but these conditions imply that the coefficient $2n - 2m - n|I| + m|J|$ in (6) vanishes, leading to a contradiction.

Next we show that every monomial $t^n\xi_I$ with $n \neq -1$ or $I \neq 1234$ belongs to $K'(1,4)$:

(1) recall that $[t, t^n\xi_I] = \deg(t^n\xi_I)t^n\xi_I$. In particular, if $\deg(t^n\xi_I) \neq 0$ the result follows.
(2) if $\deg(t^n\xi_I) = 0$, then either $n = 0$ and $I = ij$, or $n = 1$ and $I = \emptyset$. The result follows since $\xi_{ij} = -[\xi_{kij}, \xi_{ij}]$ (for any $k \neq i, j$) and $t = -[t\xi_1, \xi_1]$.

$\square$

4. The annihilation superalgebra of K_4'

Motivated by Proposition 2.10 and Theorem 2.15, we want to understand the structure of $\mathcal{A}(K_4')$.

Let us recall some notions on central extensions of Lie superalgebras.

Definition 4.1. Let $\mathfrak{g}$ be a Lie superalgebra. A $2-$*cocycle* on $\mathfrak{g}$ is a bilinear map $\psi : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ that satisfies the following conditions:

(1) $\psi(a,b) = -(-1)^{p(a)p(b)}\psi(b,a)$,
(2) $(-1)^{p(a)p(c)}\psi(a,[b,c]) + (-1)^{p(a)p(b)}\psi(b,[c,a]) + (-1)^{p(a)p(c)}\psi(c,[a,b]) = 0$,

for all $a, b, c \in \mathfrak{g}$. The set of all $2-$cocycles on $\mathfrak{g}$ is denoted by $Z^2(\mathfrak{g}, \mathbb{C})$.

Remark 4.2. We denote the set of linear maps $\mathfrak{g} \to \mathbb{C}$ by $C^1(\mathfrak{g}, \mathbb{C})$ and we call its elements $1-$*cochains*. For every $1-$cochain $f \in C^1(\mathfrak{g}, \mathbb{C})$, it is possible to construct a $2-$cocycle δf on $\mathfrak{g}$. For all $a, b \in \mathfrak{g}$ we define:

$$\delta f(a,b) = f([a,b]).$$

It is a straightforward verification that δf is a $2-$cocycle on $\mathfrak{g}$. The map $\delta : C^1(\mathfrak{g}, \mathbb{C}) \to Z^2(\mathfrak{g}, \mathbb{C})$, $f \to \delta f$, is called *coboundary operator*.

Definition 4.3. We denote by $B^2(\mathfrak{g},\mathbb{C})$ the image of $\delta : C^1(\mathfrak{g},\mathbb{C}) \to Z^2(\mathfrak{g},\mathbb{C})$. Two 2−cocycles $\psi_1, \psi_2 \in Z^2(\mathfrak{g},\mathbb{C})$ are *cohomologous* when $\psi_1 - \psi_2 \in B^2(\mathfrak{g},\mathbb{C})$. We denote by $H^2(\mathfrak{g},\mathbb{C})$ the quotient $\frac{Z^2(\mathfrak{g},\mathbb{C})}{B^2(\mathfrak{g},\mathbb{C})}$.

Definition 4.4. A Lie superalgebra $\hat{\mathfrak{g}}$ is a *central extension* of $\mathfrak{g}$ by a one−dimensional center $\mathbb{C}C$ if there exist two (Lie superalgebras) homomorphisms $i : \mathbb{C}C \to \hat{\mathfrak{g}}$ and $s : \hat{\mathfrak{g}} \to \mathfrak{g}$ such that the following sequence is exact:

$$0 \to \mathbb{C}C \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{s} \mathfrak{g} \to 0,$$

and $\mathrm{Ker}(s)$ lies in the center of $\hat{\mathfrak{g}}$.

Definition 4.5. Two central extentions $\hat{\mathfrak{g}}_1$ and $\hat{\mathfrak{g}}_2$ of $\mathfrak{g}$ by a one−dimensional center $\mathbb{C}C$ are isomorphic if there exists an isomorphism of Lie superalgebras $\Phi : \hat{\mathfrak{g}}_1 \to \hat{\mathfrak{g}}_2$ such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}C \arrow[r, "i_1"] \arrow[d, "\mathrm{Id}"] & \hat{\mathfrak{g}}_1 \arrow[r, "s_1"] \arrow[d, "\Phi"] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & \mathbb{C}C \arrow[r, "i_2"] & \hat{\mathfrak{g}}_2 \arrow[r, "s_2"] & \mathfrak{g} \arrow[r, "d"] & 0.
\end{tikzcd}$$

Next result is certainly well-known but we include a sketch of the proof for completeness and for the reader's convenience.

Proposition 4.6. *There is a bijection between (isomorphism classes of) central extensions of $\mathfrak{g}$ by a one−dimensional center and elements of $H^2(\mathfrak{g},\mathbb{C})$. If $\psi \in Z^2(\mathfrak{g},\mathbb{C})$ the corresponding central extension is, up to isomorphism, $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}C$ where:*

$$[C, a] = 0 \quad \textit{and} \quad [a, b]_{\hat{\mathfrak{g}}} = [a, b]_{\mathfrak{g}} + \psi(a, b)C,$$

for all $a, b \in \mathfrak{g}$.

Proof. Let

$$0 \to \mathbb{C}C \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{s} \mathfrak{g} \to 0,$$

be a central extension of $\mathfrak{g}$. In particular, $\hat{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathbb{C}i(C)$ as vector spaces and we have the following relation between the bracket $[\cdot,\cdot]_{\hat{\mathfrak{g}}}$ in $\hat{\mathfrak{g}}$ and the bracket $[\cdot,\cdot]_{\mathfrak{g}}$ in $\mathfrak{g}$ for all $a, b \in \mathfrak{g}$, $\alpha, \beta \in \mathbb{C}$:

$$[a + \alpha i(C), b + \beta i(C)]_{\hat{\mathfrak{g}}} = [a, b]_{\mathfrak{g}} + \psi(a, b)i(C),$$

where $\psi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a 2−cocycle.
Conversely, given $\psi \in C^2(\mathfrak{g},\mathbb{C})$, we can construct a central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$. We define $\hat{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{C}C$. For all $a, b \in \mathfrak{g}$, $\alpha, \beta \in \mathbb{C}$, we set $i(\alpha C) := \alpha C$, $s(a + \alpha C) := a$ and $[a + \alpha C, b + \beta C]_{\hat{\mathfrak{g}}} := [a, b]_{\mathfrak{g}} + \psi(a, b)C$. It follows directly from the definition of 2−cocycles that it is a central extension.
Finally we show that two isomorphic central extensions $\hat{\mathfrak{g}}_1 \cong \mathfrak{g} \oplus \mathbb{C}C$ and $\hat{\mathfrak{g}}_2 \cong \mathfrak{g} \oplus \mathbb{C}C$ correspond to cohomologous 2−cocycles. Since $\hat{\mathfrak{g}}_1$ and $\hat{\mathfrak{g}}_2$ are isomorphic, we have an isomorphism

$\Phi : \hat{\mathfrak{g}}_1 \to \hat{\mathfrak{g}}_2$ such that the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}C \arrow[r, "i_1"] \arrow[d, "\mathrm{Id}"] & \hat{\mathfrak{g}}_1 \arrow[r, "s_1"] \arrow[d, "\Phi"] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & \mathbb{C}C \arrow[r, "i_2"] & \hat{\mathfrak{g}}_2 \arrow[r, "s_2"] & \mathfrak{g} \arrow[r, "d"] & 0. \end{tikzcd}$$

Thus for all $a \in \mathfrak{g}$, $\alpha \in \mathbb{C}$:

$$\Phi(a + \alpha C) = a + \rho(a)C + \alpha C, \tag{9}$$

where $\rho \in C^1(\mathfrak{g}, \mathbb{C})$.
We call ψ_1(resp. ψ_2) the 2−cocycle that corresponds to $\hat{\mathfrak{g}}_1$(resp. $\hat{\mathfrak{g}}_2$). We have for all $a, b \in \mathfrak{g}$:

$$\begin{aligned} \Phi([a,b]_{\hat{\mathfrak{g}}_1}) &= \Phi([a,b]_{\mathfrak{g}} + \psi_1(a,b)C) \\ &= [a,b]_{\mathfrak{g}} + (\rho([a,b]_{\mathfrak{g}}) + \psi_1(a,b))\, C. \end{aligned}$$

But from the fact that Φ is an isomorphism we also have:

$$\begin{aligned} \Phi([a,b]_{\hat{\mathfrak{g}}_1}) &= [\Phi(a), \Phi(b)]_{\hat{\mathfrak{g}}_2} \\ &= [a + \rho(a)C, b + \rho(b)C]_{\hat{\mathfrak{g}}_2} \\ &= [a,b]_{\mathfrak{g}} + \psi_2(a,b)C. \end{aligned}$$

Therefore, $\delta\rho + \psi_1 = \psi_2$.
Analogously, if $\psi_1, \psi_2 \in Z^2(\mathfrak{g}, \mathbb{C})$ are cohomologous, i.e. $\psi_1 - \psi_2 = \delta\eta \in B^2(\mathfrak{g}, \mathbb{C})$, then we can construct an isomorphism between the central extensions defined by ψ_1 and ψ_2 as in (9) letting $\rho := \eta$. □

The following proposition is the main result of this section.

Proposition 4.7. *There exists a (unique) surjective morphism of Lie superalgebras* ϕ : $\mathrm{Lie}\, K'_4 \to K'(1,4)$ *such that for all* $m \in \mathbb{Z}$

$$\begin{aligned} &\phi(\xi_I y^m) = t^m \xi_I, \textit{ for all } I \in \mathcal{I}_<,\ I \neq 1234 \\ &\phi(\partial \xi_{1234} y^n) = -m \xi_{1234} t^{m-1} \end{aligned}$$

and $\mathrm{Ker}(\phi) = \mathbb{C}\, \partial\xi_{1234}$. *The annihilation superalgebra of* K'_4 *is a central extension of* $K(1,4)_+$ *by a one−dimensional center* $\mathbb{C}C$*:*

$$\mathcal{A}(K'_4) = K(1,4)_+ \oplus \mathbb{C}C.$$

The extension is given by the 2−cocycle $\psi \in Z^2(K(1,4)_+, \mathbb{C})$ *which computed on basis elements returns non−zero values in the following cases only (up to skew-symmetry of* ψ*):*

$$\begin{aligned} &\psi(1, \xi_{1234}) = -2, \\ &\psi(\xi_i, \partial_i \xi_{1234}) = -1. \end{aligned}$$

We need a lemma in order to prove Proposition 4.7.

Lemma 4.8. *The element* $\partial\xi_{1234} y^0 \in \mathrm{Lie}\, K'_4$ *is central.*

Proof. By (3) and (4) we have, for all $ay^l \in \mathrm{Lie}\, K_4'$, with $a \in K_4'$:

$$\left[\partial\xi_{1234}y^0, ay^l\right] = \left(\partial\xi_{1234\,(0)}a\right) y^l = 0.$$

In the last equality we used the fact that $\left(\partial\xi_{1234\,(0)}a\right)$ is computed as the restriction of the $0-$product in K_4 and (iii) in Definition 2.5. □

We adopt the following notation. Given a proposition P, we let

$$\chi_P = \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{if } P \text{ is false.} \end{cases}$$

Remark 4.9. Recall that by the definition of $\mathrm{Lie}\, K_4'$, for all $a \in K_4'$ and $m \in \mathbb{Z}$, we have $\partial ay^m = -may^{m-1}$ and that $\xi_{1234} \notin K_4'$ by Proposition 3.3. Hence, every monomial $\partial^k P(\xi)y^n$ can be represented in $\mathrm{Lie}\, K_4'$ as a scalar multiple of a monomial $\xi_I y^n$ for some $I \neq 1234$ or of a monomial $\partial\xi_{1234}y^n$.

More precisely we have that the set $\{\xi_I y^m,\ \partial\xi_{1234}y^m : m \in \mathbb{Z},\ I \in \mathcal{I}_<,\ I \neq 1234\}$ is a basis of $\mathrm{Lie}\, K_4'$.

Proof of Proposition 4.7. By Remark 4.9 we know that there exists a unique linear map ϕ satisfying the prescribed conditions. It is clear from its definition that ϕ is surjective and so we only need to prove that ϕ is a morphism of Lie superalgebras. We have to distinguish four cases:

(1) Let $I, J \in \mathcal{I}_<$ with $I, J \neq 1234$ and $\xi_{IJ} \neq \pm\xi_{1234}$, and $m, n \in \mathbb{Z}$. In $\mathrm{Lie}\, K_4'$ we have, by (3) and (4) and $j-$products (7):

$$\begin{aligned}
[\xi_I y^m, \xi_J y^n] &= \sum_{j\in\mathbb{Z}_+} \binom{m}{j} (\xi_{I\,(j)}\xi_J) y^{m+n-j} \\
&= (\xi_{I\,(0)}\xi_J) y^{m+n} + m(\xi_{I\,(1)}\xi_J) y^{m+n-1} \\
&= (|I|-2)\partial\xi_{IJ}y^{m+n} + (-1)^{|I|}\sum_{i=1}^{4} \partial_i\xi_I\,\partial_i\xi_J\, y^{m+n} + m(|I|+|J|-4)\xi_{IJ}y^{m+n-1} \\
&= (2n-2m-n|I|+m|J|)\xi_{IJ}y^{m+n-1} + (-1)^{|I|}\sum_{i=1}^{4} \partial_i\xi_I\,\partial_i\xi_J\, y^{m+n}.
\end{aligned}$$

Therefore, by (6), we have:

$$\begin{aligned}
[\phi(\xi_I y^m), \phi(\xi_J y^n)] &= \left[t^m\xi_I, t^n\xi_J\right] \\
&= (2n-2m-n|I|+m|J|)t^{m+n-1}\xi_{IJ} + (-1)^{|I|}t^{m+n}\sum_{i=1}^{4} \partial_i\xi_I\,\partial_i\xi_J \\
&= \phi([\xi_I y^m, \xi_J y^n]).
\end{aligned}$$

(2) Let $I, J \in \mathcal{I}_<$ with $|I|, |J| \neq 4$ and $\xi_{IJ} = \xi_{1234}$, and $m, n \in \mathbb{Z}$. We proceed like in the previous case and we have

$$\begin{aligned}
[\xi_I y^m, \xi_J y^n] &= (\xi_{I\,(0)}\xi_J) y^{m+n} + m(\xi_{I\,(1)}\xi_J) y^{m+n-1} \\
&= (|I|-2)\partial\xi_{1234}y^{m+n}.
\end{aligned}$$

and so in $K'(1,4)$ we have

$$\begin{aligned}[\phi(\xi_I y^m), \phi(\xi_J y^n)] &= \left[t^m\xi_I, t^n\xi_J\right] \\ &= (2-|I|)(m+n)t^{m+n-1}\xi_{1234} \\ &= \phi([\xi_I y^m, \xi_J y^n]).\end{aligned}$$

(3) Let $m, n \in \mathbb{Z}$. We have $f = \partial\xi_1\xi_2\xi_3\xi_4 y^m$ and $g = \partial\xi_1\xi_2\xi_3\xi_4 y^n$ in Lie K'_4, with $m, l \in \mathbb{Z}$. In Lie K'_4 we have, using bracket (4) and $n-$products (7):

$$[\partial\xi_{1234}y^m, \partial\xi_{1234}y^n] = \sum_{j\in\mathbb{Z}_+}\binom{h}{j}(\partial\xi_{1234\,(j)}\partial\xi_{1234})y^{m+n-j} = 0.$$

by (iii) of Definition 2.5, (2) and (6). On the other hand

$$[\phi(\partial\xi_{1234}y^m), \phi(\partial\xi_{1234}y^n)] = \left[-m\xi_{1234}t^{m-1}, -n\xi_{1234}t^{n-1}\right] = 0,$$

by (6).

(4) Finally, let $J \in \mathcal{I}_<$, $J \neq 1234$ and $m, n \in \mathbb{Z}$. First, we point out that $(\partial\xi_{1234\,(j)}\xi_J) = -j(\xi_{1234\,(j-1)}\xi_J) = 0$ for all $j \geq 2$ by (7). Therefore in Lie K'_4 we have

$$\begin{aligned}\left[\partial\xi_{1234}y^m, \xi_J y^n\right] &= (\partial\xi_{1234(0)}\xi_J)y^{m+n} + m(\partial\xi_{1234\,(1)}\xi_J)y^{m+n-1} \\ &= -m(\xi_{1234\,(0)}\xi_J)y^{m+n-1} \\ &= -2m\chi_{J=\emptyset}\,\partial\xi_{1234}\;y^{m+n-1} - m\sum_{i=1}^{4}\partial_i\xi_{1234}\partial_i\xi_J\,y^{m+n-1}.\end{aligned}$$

In $K'(1,4)$ we have, using bracket (6):

$$\begin{aligned}[\phi(\partial\xi_{1234}y^m), \phi(\xi_J y^n)] &= \left[-m\xi_{1234}t^{m-1}, \xi_J t^n\right] \\ &= -\chi_{J=\emptyset}m(-2n-2m+2)\,t^{m+n-2}\xi_{1234} - m\sum_{i=1}^{4}\partial_i\xi_{1234}\,\partial_i\xi_J\,t^{m+n-1} \\ &= \phi([\partial\xi_{1234}y^m, \xi_J y^n]).\end{aligned}$$

The previous computations imply that the kernel of the map ϕ : Lie $K'_4 \longrightarrow K'(1,4)$ is Ker $\phi = \langle\partial\xi_1\xi_2\xi_3\xi_4\rangle$ and so the following sequence is exact:

$$0 \to \langle\partial\xi_1\xi_2\xi_3\xi_4\rangle \xrightarrow{i} \mathrm{Lie}\,K'_4 \xrightarrow{\phi} K'(1,4) \to 0.$$

By Lemma 4.8 the Lie superalgebra Lie K'_4 is therefore a central extension of $K'(1,4)$ by the one$-$dimensional center $\langle\partial\xi_1\xi_2\xi_3\xi_4\rangle$.
In particular, we point out that ϕ : Lie $K'_4/\mathbb{C}\partial\xi_1\xi_2\xi_3\xi_4 \to K'(1,4)$ is an isomorphism. In the previous computations we computed all the possible brackets between monomials in Lie K'_4, therefore in particular all the possible brackets between monomials in $\mathcal{A}(K'_4)$ and we can observe that the central element $\partial\xi_{1234}$ lies in the support of the bracket of two basis elements only in the case (2) of the previous computations. The other parts of the statement follow. $\square$

5. Verma modules

In this section we study the action of $\mathfrak{g} := \mathcal{A}(K_4') = K(1,4)_+ \oplus \mathbb{C}C$ on a finite Verma module $\mathrm{Ind}(F)$, where F is a finite−dimensional irreducible $\mathfrak{g}_{\geq 0}$−module, on which $\mathfrak{g}_{>0}$ acts trivially. The grading on $\mathfrak{g}$ is the standard grading of $K(1,4)_+$ and C has degree 0. We have:

$$\mathfrak{g}_{-2} = \langle 1 \rangle ,$$
$$\mathfrak{g}_{-1} = \langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle ,$$
$$\mathfrak{g}_0 = \langle \{C, t, \xi_{ij} : 1 \leq i < j \leq 4\} \rangle .$$

Remark 5.1. The annihilation superalgebra $\mathfrak{g}$ satisfies conditions L1, L2, L3 of Proposition 2.10. Indeed:

L1. This is obvious.

L2. The element t is a grading element, i.e. $[t,a] = \deg(a)a$ for all $a \in \mathfrak{g}$. Hence, by Remark 2.11, t satisfies condition L2.

L3. The element Θ is chosen as $-\frac{1}{2}\xi_{\emptyset} = -\frac{1}{2} \in \mathfrak{g}_{-2}$. Indeed for all $m \geq 0$ and $I \in \mathcal{I}_{<}$ we have $t^m \xi_I \in \mathfrak{g}_{2m+|I|-2}$ and

$$t^m \xi_I = -\frac{1}{m+1}[\Theta, t^{m+1}\xi_I]$$

and $C = [\Theta, \xi_{1234}]$.

Remark 5.2. Since $\mathrm{Ind}(F) \cong U(\mathfrak{g}_{<0}) \otimes F$, it follows that $\mathrm{Ind}(F) \cong \mathbb{C}[\Theta] \otimes \bigwedge(4) \otimes F$. Indeed, let us denote by η_i the image in $U(\mathfrak{g})$ of $\xi_i \in \bigwedge(4)$, for all $i \in \{1,2,3,4\}$. In $U(\mathfrak{g})$ we have that $\eta_i^2 = \Theta$, for all $i \in \{1,2,3,4\}$: since $[\xi_i, \xi_i] = -1$ in $\mathfrak{g}$, we have $\eta_i \eta_i = -\eta_i \eta_i - 1$ in $U(\mathfrak{g})$.

From now on it is always assumed that F is a finite−dimensional irreducible $\mathfrak{g}_{\geq 0}$−module.
We will study the action of $\mathfrak{g}$ on $\mathrm{Ind}(F)$ using the λ−action notation by Proposition 2.9:

$$\xi_{I\ \lambda}(g \otimes v) = \sum_{j \geq 0} \frac{\lambda^j}{j!} t^j \xi_I.(g \otimes v),$$

for $I \in \mathcal{I}$, $g \in U(\mathfrak{g}_{<0})$ and $v \in F$. In order to find an explicit formula for $\xi_{I\ \lambda}(g \otimes v)$ we need some preliminary lemmas.

We will make the following slight abuse of notation: if $I, J \in \mathcal{I}_{\neq}$ we will denote by $I \cap J$ (resp. $I \setminus J$) the increasingly ordered sequence whose elements are the elements of the intersection of the underlying sets of I and J (resp. the elements of the difference of the underlying sets of I and J). We will say $I \subseteq J$ when the underlying set of I is contained in the underlying set of J. Analogously we will denote by I^c the increasingly ordered sequence whose elements are the elements of the complement of the underlying set of I. Given $I = (i_1, i_2, \cdots i_k) \in \mathcal{I}_{\neq}$, we will use the notation η_I to denote the element $\eta_{i_1}\eta_{i_2}\cdots\eta_{i_k} \in U(\mathfrak{g}_{<0})$ and we will denote $|\eta_I| = |I| = k$. We will denote $\xi_* = \xi_{1234}$ (resp. $\eta_* = \eta_{1234}$). Given $I = (i_1, i_2, \cdots i_k)$ and $I^c = (j_{k+1}, j_{k+2}, \cdots j_4)$, we will denote by ε_I the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \cdots & 4 \\ i_1 & i_2 & \cdots & i_k & j_{k+1} & \cdots & j_4 \end{pmatrix}.$$

We will also use the following notation: if $(i_1, \ldots, i_k) \in \mathcal{I}_{\neq}$ and $i \in \{1,2,3,4\}$ we let

$$\partial_i \eta_{i_1,\dots,i_k} = \begin{cases} (-1)^{j+1}\eta_{i_1,\dots,\hat{i_j},\dots,i_k} & \text{if } i = i_j \text{ for some (necessarily unique) } j \\ 0 & \text{otherwise.} \end{cases}$$

and for $a \in \mathbb{C}$, $I = (i_1, i_2, \cdots i_k)$, $J \in \mathcal{I}_{\neq}$:

$$\begin{aligned} \partial_I \eta_J &= \partial_{i_1}\partial_{i_2}\dots\partial_{i_k}\eta_J & \partial_I \xi_J &= \partial_{i_1}\partial_{i_2}\dots\partial_{i_k}\xi_J; \\ \partial_{a\xi_I}\eta_J &= a\partial_I\eta_J & \partial_{a\xi_I}\xi_J &= a\partial_I\xi_J; \\ \partial_\emptyset \eta_S &= \eta_S & \partial_\emptyset \xi_S &= \xi_S. \end{aligned}$$

Given $I, J \in \mathcal{I}_{\neq}$ we let

$$\xi_I \star \eta_J = \chi_{I\cap J=\emptyset}\eta_{IJ},$$

$$\eta_J \star \xi_I = \chi_{I\cap J=\emptyset}\eta_{JI}.$$

and we extend this notation by linearity in both arguments.

We observe that in $\mathfrak{g}$, by (6) and Proposition 4.7 we have

$$[t^m\xi_I, \xi_r] = -mt^{m-1}\xi_{Ir} + (-1)^{|I|}t^m\partial_r\xi_I + \psi(t^m\xi_I, \xi_r)C.$$

and in particular

$$[t^m\xi_I, \xi_r] = -mt^{m-1}\xi_{Ir} + (-1)^{|I|}t^m\partial_r\xi_I + \chi_{m=0}\chi_{r=I^c}\varepsilon_I C \tag{10}$$

for all $I \in \mathcal{I}_{\neq}$, $m \geq 0$ and $r \in \{1, 2, 3, 4\}$.

Lemma 5.3. *Let $I, L \in \mathcal{I}_{\neq}$, $v \in F$ and $m \geq 3$. We have:*

$$t^m\xi_I(\eta_L \otimes v) = \begin{cases} -6\varepsilon_L \otimes Cv & \text{if } m = 3,\ |I| = 0 \text{ and } |L| = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We can always assume, without loss of generality, that $\eta_L = \eta_J\eta_K$ with $I \cap J = \emptyset$, $K \subseteq I$.

We first point out that $t^m\xi_I(\eta_L \otimes v) = 0$ when $m > 3$ because $\deg(t^m\xi_I) = 2m + |I| - 2 > 4 \geq \deg(\eta_L)$.

If $m = 3$, $|I| > 0$, $t^3\xi_I(\eta_L \otimes v) = 0$ because $\deg(t^3\xi_I) = 2m + |I| - 2 > 4 \geq \deg(\eta_L)$.

If $m = 3$, $|I| = 0$ and $|L| \neq 4$, $t^3(\eta_L \otimes v) = 0$ because $\deg(t^3) = 4 > \deg(\eta_L)$.

If $m = 3$, $|I| = 0$ and $|L| = 4$ we can assume $L = 1234$ without loss of generality and by (10), we have

$$\begin{aligned} t^3(\eta_{1234} \otimes v) &= -3(t^2\xi_1)\eta_{234} \otimes v - 3\eta_1(t^2\xi_2)\eta_{34} \otimes v - 3\eta_{12}(t^2\xi_3)\eta_4 \otimes v - 3\eta_{123}(t^2\xi_4) \otimes v \\ &= 6(t\xi_{12})\eta_{34} \otimes v \\ &= -6(\xi_{123})\eta_4 \otimes v \\ &= -6 \otimes Cv. \end{aligned}$$

□

Lemma 5.3 describes the terms of degree at least 3 in the variable λ in the λ-action of K'_4 on a Verma module. Next target is to study the terms of degree 0, 1 and 2: this will be accomplished in Lemmas 5.5, 5.6 and 5.7 respectively. But we first state a technical lemma.

Lemma 5.4. *Let $I, J, K \in \mathcal{I}_{\neq}$ with $I \cap J = \emptyset$, $K \subseteq I$. We have:*

$$\xi_I(\eta_{JK} \otimes v) = \sum_{L \subseteq K} (-1)^{|I|(|J|+|K|)+|L|(|L|-1)/2-|L|(|K|-|L|)} \eta_J \partial_L \eta_K \otimes \partial_L \xi_I.v$$
$$+ \chi_{|I|=3} \chi_{J=I^c}\, \varepsilon_I\, \eta_K \otimes Cv.$$

Proof. From repeated applications of (10) we have

$$\xi_I(\eta_J \eta_K) \otimes v = (-1)^{|I||J|} \eta_J \xi_I \eta_K \otimes v + \chi_{|I|=3} \chi_{J=I^c}\, \varepsilon_I \eta_K \otimes Cv. \tag{11}$$

Indeed, from (10), if $|I| = 1, 2$, or $|I| = 3$ with $J \neq I^c$, then $[\xi_I, \xi_r] = 0$ for all $r \in J$ and formula (11) is straightforward. In the case $|I| = 3$ and $J = I^c$, using (10), we have:

$$\xi_I(\eta_{I^c} \eta_K) \otimes v = -\eta_{I^c} \xi_I \eta_K \otimes v + \chi_{|I|=3} \chi_{J=I^c} \varepsilon_I \eta_K \otimes Cv.$$

The rest of the proof is the same as the proof of Lemma A.2 in [1] and it is done by induction on $|K|$ using formula (11) and is therefore omitted. ☐

Lemma 5.5. *Let $I, L \in \mathcal{I}_{\neq}$. We have:*

$$\xi_I(\eta_L \otimes v) =(-1)^{|I|}(|I|-2)\Theta \partial_I \eta_L \otimes v + \sum_{i=1}^{4} \partial_{(\partial_i \xi_I)}(\xi_i \star \eta_L) \otimes v + (-1)^{|I|} \sum_{i<j} \partial_{(\partial_{ij}\xi_I)} \eta_L \otimes \xi_{j,i}.v$$
$$+ \chi_{|I|=3}\, \varepsilon_I\, \partial_{I^c} \eta_L \otimes Cv.$$

Proof. The proof is analogous to the proof in [1] of Lemma A.3, and it is based on Lemma 5.4. The extra term in C is due to the additional term of Lemma 5.4, which is not present in Lemma A.2 of [1]. ☐

Now we study the term of degree 1 in λ of the $\lambda-$action.

Lemma 5.6. *Let $I, L \in \mathcal{I}_{\neq}$. We have:*

$$t\xi_I(\eta_L \otimes v) =(-1)^{|I|} \partial_I \eta_L \otimes t.v + (-1)^{|I|+|L|} \sum_{i=1}^{4} (\partial_{Ii} \eta_L \star \xi_i) \otimes v$$
$$- \sum_{i \neq j} \partial_{\partial_i \xi_I}(\partial_j \eta_L) \otimes \xi_{i,j}.v + \chi_{|I|=2}\, \varepsilon_I\, \partial_{I^c} \eta_L \otimes Cv.$$

Proof. Without loss of generality we can suppose that $\eta_L = \eta_J \eta_K$ with $I \cap J = \emptyset$, $K \subseteq I$. Let us prove that:

$$t\xi_I(\eta_J \eta_K \otimes v) =(-1)^{|I||J|} \eta_J (t\xi_I) \eta_K \otimes v + \sum_{j=1}^{4} (-1)^{|I||J|-|I|+|J|} (\partial_j \eta_J) \xi_{Ij} \eta_K \otimes v \tag{12}$$
$$+ \chi_{|I|=2}\, \varepsilon_I \partial_{I^c} \eta_{JK} \otimes Cv.$$

The formula is the same as the relation proved for $K(1, N)_+$ in the proof of Lemma A.4 of [1], except for an additional term in C. We point out that a term with C is involved only if $|I| = 2$ and $|J| = 2$. Let us prove (12) by induction on $|J|$. If $|J| = 0$, (12) is straightforward.

Let $\tilde{J}$ be such that $|\tilde{J}|>0$ and $\widetilde{J}\cap I=\emptyset$. Let J,s be such that $\eta_{\widetilde{J}}=\eta_J\eta_s$. We have, using (12) for η_J, that:

$$\begin{aligned} t\xi_I(\eta_J\eta_s\eta_K\otimes v) =&(-1)^{|I||J|}\eta_J(t\xi_I)\eta_s\eta_K\otimes v+\sum_{j=1}^{4}(-1)^{|I||J|-|I|+|J|}(\partial_j\eta_J)\xi_{Ij}\eta_s\eta_K\otimes v\\ &+\chi_{|I|=2}\,\varepsilon_I(\partial_{I^c}\eta_J)\eta_s\eta_K\otimes Cv. \end{aligned}$$

Notice that, since we are supposing $\eta_{\widetilde{J}}=\eta_J\eta_s$ with $\widetilde{J}\cap I=\emptyset$ and $s\notin J$, the term $\chi_{|I|=2}\,\varepsilon_I(\partial_{I^c}\eta_J)\eta_s\eta_K\otimes Cv$ is 0 because if $|I|=2$, then $|J|<2$. We have, using (10), that:

$$\begin{aligned} t\xi_I(\eta_J\eta_s\eta_K\otimes v) =&(-1)^{|I|(|J|+1)}\eta_J\eta_s(t\xi_I)\eta_K\otimes v-(-1)^{|I||J|}\eta_J\xi_{Is}\eta_K\otimes v\\ &+\sum_{j=1}^{4}(-1)^{|I||J|-|I|+|J|+|I|+1}(\partial_j\eta_J)\eta_s\,\xi_{Ij}\,\eta_K\otimes v\\ &-(-1)^{|J|}\chi_{|I|=2}\chi_{|J|=1}\varepsilon_I\partial_{I^c}\eta_{JsK}\otimes Cv. \end{aligned}$$

We observe that:

$$\begin{aligned} -(-1)^{|I||J|}\eta_J\xi_{Is}\eta_K\otimes v &= (-1)^{|I||J|+1+|J|}(\partial_s\eta_{\widetilde{J}})\xi_{Is}\eta_K\otimes v\\ &= (-1)^{|I||\widetilde{J}|-|I|+|\widetilde{J}|}(\partial_s\eta_{\widetilde{J}})\xi_{Is}\eta_K\otimes v. \end{aligned}$$

Therefore:

$$\begin{aligned} t\xi_I(\eta_J\eta_s\eta_K\otimes v) =&(-1)^{|I|(|\widetilde{J}|)}\eta_{\widetilde{J}}(t\xi_I)\eta_K\otimes v+\sum_{j=1}^{4}(-1)^{|I||\widetilde{J}|-|I|+|\widetilde{J}|}(\partial_j\eta_{\widetilde{J}})\,\xi_{Ij}\,\eta_K\otimes v\\ &+\chi_{|I|=2}\varepsilon_I\partial_{I^c}\eta_{\widetilde{J}K}\otimes Cv. \end{aligned}$$

Hence, formula (12) is proved. The rest of the proof is analogous to the proof of Lemma A.4 in [1] and it is based on (12). □

Now we study the term of degree 2 in λ of the $\lambda-$action.

Lemma 5.7. *Let $I,L\in\mathcal{I}_{\neq}$ and $v\in F$. We have*

$$\frac{1}{2}t^2\xi_I(\eta_L\otimes v) =(-1)^{|I|}\sum_{i<j}\partial_{Iij}\eta_L\otimes\xi_{j,i}.v-\chi_{|I|=1}\,\varepsilon_I\,\partial_{I^c}\eta_L\otimes Cv.$$

Proof. As before, without loss of generality, we can suppose that $\eta_L=\eta_J\eta_K$ with $I\cap J=\emptyset$, $K\subseteq I$. Let us prove that:

$$\begin{aligned} &\frac{1}{2}t^2\xi_I(\eta_J\eta_K\otimes v) \qquad\qquad (13)\\ &=-\chi_{I=K}\sum_{i<j}(-1)^{|I||J|+|I|(|I|+1)/2}\partial_{ij}\eta_J\,\partial_I\eta_K\otimes\xi_{ij}.v-\chi_{|I|=1}\,\varepsilon_I\,\partial_{I^c}\eta_{JK}\otimes Cv. \end{aligned}$$

In order to establish (13), we need to prove the following:

$$\Big(\frac{1}{2}t^2\xi_I\Big)\,\eta_J=\sum_{S\in\mathcal{I}_<:\,|S|\leq 2}\pm\frac{1}{(2-|S|)!}\partial_S\eta_J\,(t^{2-|S|}\xi_{IS})-\chi_{|I|=1}\,\varepsilon_I\,\partial_{I^c}\eta_JC. \qquad (14)$$

We prove (14) by induction on $|J|$. If $|J| = 0$, the result is straightforward.

Let us consider $\eta_{\widetilde{J}} = \eta_J \eta_r$ with $\widetilde{J} \cap I = \emptyset$ and $r \notin J$. We have, using (14) for η_J, that:

$$\left(\frac{1}{2}t^2\xi_I\right)\eta_J\eta_r = \sum_{S\in\mathcal{I}_<:\,|S|\leq 2} \frac{1}{(2-|S|)!}\partial_S\eta_J\,(t^{2-|S|}\xi_{IS})\eta_r - \chi_{|I|=1}\,\varepsilon_I\,(\partial_{I^c}\eta_J)\eta_r C.$$

Notice that, since we are supposing $\eta_{\widetilde{J}} = \eta_J\eta_r$ with $\widetilde{J}\cap I = \emptyset$ and $r \notin J$, the term $-\chi_{|I|=1}\,\varepsilon_I\,(\partial_{I^c}\eta_J)\eta_r C$ is 0 because if $|I| = 1$, then $|J| < 3$. Now, by (10), we have

$$\begin{aligned}
\left(\frac{1}{2}t^2\xi_I\right)\eta_J\,\eta_r &= \sum_{S:\,|S|\leq 2} \pm\frac{1}{(2-|S|)!}(\partial_S\eta_J)\eta_r(t^{2-|S|}\xi_{IS}) + \sum_{|S|\leq 2}\pm\frac{2-|S|}{(2-|S|)!}\partial_S\eta_J\,(t^{1-|S|}\xi_{ISr})\\
&\quad \pm \chi_{|I|=1}\chi_{|J|=2}\varepsilon_{\tilde{J}}C\\
&= \sum_{S:\,|S|\leq 2,\, r\notin S}\pm\frac{1}{(2-|S|)!}\partial_S\eta_{Jr}\,(t^{2-|S|}\xi_{IS}) \pm \sum_{S:\,|S|\leq 1}\partial_{Sr}\eta_{\tilde{J}}(t^{1-|S|}\xi_{ISr})\\
&\quad \pm \chi_{|I|=1}\chi_{|J|=2}\varepsilon_{\tilde{J}}C\\
&= \sum_{S:\,|S|\leq 2}\pm\frac{1}{(2-|S|)!}\partial_S\eta_{\tilde{J}}(t^{2-|S|}\xi_{IS})\eta_K\otimes v \pm \chi_{|I|=1}\chi_{|J|=2}\varepsilon_{\tilde{J}}C
\end{aligned}$$

Now we compute explicitly the sign of the last summand above. Hence we consider I with $|I| = 1$ and $|\widetilde{J}| = 3$. For $I = (i)$ and $\widetilde{J} = (j,k,l) = I^c$, by repeated applications of (10), we have:

$$\begin{aligned}
\left(\frac{1}{2}t^2\xi_i\right)\eta_j\eta_k\eta_l &= -(t\xi_{ij})\eta_k\eta_l + \eta_j(t\xi_{ik})\eta_l - \eta_j\eta_k(t\xi_{il}) - \eta_j\eta_k\eta_l\left(\frac{1}{2}t^2\xi_i\right)\\
&= \xi_{ijk}\eta_l + \eta_k(\xi_{ijl}) - \eta_k\eta_l(t\xi_{ij}) - \eta_j\xi_{ikl} + \eta_j\eta_l(t\xi_{ik}) - \eta_j\eta_k(t\xi_{il}) - \eta_j\eta_k\eta_l\left(\frac{1}{2}t^2\xi_i\right)\\
&= -\eta_l\xi_{ijk} + \varepsilon_I C + \eta_k\xi_{ijl} - \eta_{kl}(t\xi_{ij}) - \eta_j\xi_{ikl}) + \eta_{jl}(t\xi_{ik}) - \eta_{jk}(t\xi_{il}) - \eta_{jkl}\left(\frac{1}{2}t^2\xi_i\right)\\
&= \sum_{S:|S|\leq 2}\pm\frac{1}{(2-|S|)!}\partial_S\eta_{\tilde{J}}(t^{2-|S|}\xi_{IS}) - \varepsilon_I\,\partial_{I^c}\eta_{\tilde{J}}C,
\end{aligned}$$

The result follows with the simple verification that the coefficient of C above agrees with the coefficient of C in (14).

Now we observe that if $|S| = 0, 1$ then $\deg(t^2\xi_I\xi_S) > \deg(\eta_K)$ and $\deg(t\xi_I\xi_S) > \deg(\eta_K)$ and therefore, by (14), we have

$$\frac{1}{2}t^2\xi_I(\eta_J\eta_K\otimes v) = \sum_{S\,:|S|=2}\pm(\partial_S\eta_J)\xi_{IS}\eta_K\otimes v - \chi_{|I|=1}\,\varepsilon_I\,\partial_{I^c}\eta_J\otimes Cv.$$

Proceeding as in the proof of Lemma A.5 in [1], one can show that the signs in this sum do not depend on S and are all equal to $-(-1)^{|I||J|}$. It follows that this relation reduces to:

$$\frac{1}{2}t^2\xi_I(\eta_J\eta_K\otimes v) = -(-1)^{|I||J|}\sum_{i<j}\partial_{ij}\eta_J(\xi_{Iij})\eta_K\otimes v - \chi_{|I|=1}\varepsilon_I\partial_{I^c}\eta_J\eta_K\otimes Cv. \qquad (15)$$

Formula (13) can be proved using (15), (10) and induction on $|K|$. The proof is similar to the proof of (15). Finally, the rest of the proof is analogous to the proof of Lemma A.5 in [1] and it is based on (13). $\square$

It is convenient to summarize the previous lemmas in the following result.

Proposition 5.8. *Let $I, L \in \mathcal{I}_{\neq}$. The $\lambda-$action has the following expression:*

$$\begin{aligned}
\xi_{I\lambda}(\eta_L \otimes v) =& (-1)^{|I|}(|I|-2)\Theta\partial_I\eta_L \otimes v + \sum_{i=1}^4 \partial_{(\partial_i\xi_I)}(\xi_i \star \eta_L) \otimes v \\
&+ (-1)^{|I|}\sum_{i<j} \partial_{(\partial_{ij}\xi_I)}\eta_L \otimes \xi_{j,i}.v + \chi_{|I|=3}\,\varepsilon_I\,\partial_{I^c}\eta_L \otimes Cv \\
&+ \lambda\Big((-1)^{|I|}\partial_I\eta_L \otimes t.v + (-1)^{|I|+|L|}\sum_{i=1}^4(\partial_{Ii}\eta_L \star \xi_i) \otimes v \\
&+ \sum_{i\neq j}\partial_{\partial_i\xi_I}(\partial_j\eta_L) \otimes \xi_{j,i}.v + \chi_{|I|=2}\,\varepsilon_I\,\partial_{I^c}\eta_L \otimes Cv\Big) \\
&+ \lambda^2\Big((-1)^{|I|}\sum_{i<j}\partial_{Iij}\eta_L \otimes \xi_{j,i}v - \chi_{|I|=1}\,\varepsilon_I\,\partial_{I^c}\eta_L \otimes Cv\Big) \\
&+ \lambda^3\left(-\chi_{|I|=0}\,\partial_{I^c}\eta_L \otimes Cv\right).
\end{aligned}$$

For $\eta_I \in \bigwedge(4)$ we indicate with $\overline{\eta_I}$ its Hodge dual in $U(\mathfrak{g}_{<0})$, i.e. the unique monomial such that $\overline{\eta_I} \star \xi_I = \eta_{1234}$. Then we extend by linearity the definition of Hodge dual to elements $\sum_I \alpha_I\eta_I \in U(\mathfrak{g}_{<0})$ and we set $\overline{\Theta^k\eta_I} = \Theta^k\overline{\eta_I}$.
We recall Lemma 4.2 from [1].

Lemma 5.9. *For $f \in \bigwedge(4)$, $L \in \mathcal{I}_{\neq}$, $i \in \{1,2,3,4\}$, we have:*

$$\overline{\partial_i\eta_L} = \overline{\eta_L} \star \xi_i = (-1)^{|L|}\xi_i \star \overline{\eta_L}, \tag{16}$$

$$\overline{\partial_f\eta_L} = (-1)^{(|f|(|f|-1)/2)+|f||L|} f \star \overline{\eta_L}, \tag{17}$$

$$\overline{\xi_i \star \eta_L} = -(-1)^{|L|}\partial_i\overline{\eta_L}, \tag{18}$$

$$\overline{\eta_L \star \xi_i} = -\partial_i\overline{\eta_L}. \tag{19}$$

Next result is a consequence of Proposition 5.8.

Proposition 5.10. *Let $I, L \in \mathcal{I}_{\neq}$. Let T be the vector space isomorphism $T : \mathrm{Ind}(F) \to \mathrm{Ind}(F)$ defined by $T(g \otimes v) = \overline{g} \otimes v$, for all $g \in U(\mathfrak{g}_{<0})$, $v \in F$. Then:*

$$\begin{aligned}
&(T \circ \xi_{I\lambda} \circ T^{-1})(\eta_L \otimes v) \\
&= (-1)^{(|I|(|I|+1)/2)+|I||L|}\Big\{(|I|-2)\Theta(\xi_I \star \eta_L) \otimes v - (-1)^{|I|}\sum_{i=1}^4(\partial_i\xi_I \star \partial_i\eta_L) \otimes v \\
&\quad - \sum_{r<s}(\partial_{rs}\xi_I \star \eta_L) \otimes \xi_{s,r}.v + \chi_{|I|=3}\,\varepsilon_I\,(\xi_{I^c} \star \eta_L) \otimes Cv
\end{aligned}$$

$$+\lambda\Bigg[(\xi_I\star\eta_L)\otimes t.v-(-1)^{|I|}\sum_{i=1}^{4}\partial_i(\xi_{Ii}\star\eta_L)\otimes v+(-1)^{|I|}\sum_{i\neq j}(\partial_i\xi_{Ij}\star\eta_L)\otimes\xi_{j,i}.v$$

$$+\chi_{|I|=2}\,\varepsilon_I\,(\xi_{I^c}\star\eta_L)\otimes Cv\Bigg]$$

$$+\lambda^2\Bigg[-\sum_{i<j}(\xi_{Iij}\star\eta_L)\otimes\xi_{j,i}.v-\chi_{|I|=1}\,\varepsilon_I\,(\xi_{I^c}\star\eta_L)\otimes Cv\Bigg]+\lambda^3\Big[-\chi_{|I|=0}(\xi_*\star\eta_L)\otimes Cv\Big]\Bigg\}.$$

Proof. The proof follows by Proposition 5.8 with a straightforward application of Lemma 5.9. □

In the following lemma we give a recursive formula in order to compute $\xi_{I\lambda}(\Theta^k g\otimes v)$ for $I\in\mathcal{I}_{\neq}$ and $g\in U(g_{<0})$.

Lemma 5.11. *Let $I\in\mathcal{I}_{\neq}$, $g\in U(\mathfrak{g}_{<0})$ and $k\in\mathbb{Z}_{>0}$. We have:*

$$\xi_{I\lambda}(\Theta^k g\otimes v)=(\Theta+\lambda)(\xi_{I\lambda}\Theta^{k-1}g\otimes v)-\chi_{|I|=4}\varepsilon_I\Theta^{k-1}g\otimes Cv.$$

Proof. We have by (6) and Proposition 4.7:

$$\begin{aligned}\xi_{I\lambda}(\Theta^k g\otimes v)&=\sum_{j\geq 0}\frac{\lambda^j}{j!}(t^j\xi_I)(\Theta^k g\otimes v)\\&=\sum_{j\geq 0}\frac{\lambda^j}{j!}\Theta(t^j\xi_I)(\Theta^{k-1}g\otimes v)+\sum_{j\geq 0}\frac{\lambda^j}{j!}(jt^{j-1}\xi_I)(\Theta^{k-1}g\otimes v)-\chi_{|I|=4}\varepsilon_I\Theta^{k-1}g\otimes Cv\\&=(\Theta+\lambda)(f_\lambda\Theta^{k-1}g\otimes v)-\chi_{|I|=4}\varepsilon_I\Theta^{k-1}g\otimes Cv.\end{aligned}$$

□

6. Singular vectors

In this section we deduce some necessary conditions that singular vectors must satisfy. These conditions are obtained generalizing some ideas developed in [1].

We first give a more explicit description of $\mathfrak{g}_0$: we have $\mathfrak{g}_0=\langle\{C,t,\xi_{ij}:1\leq i<j\leq 4\}\rangle\cong\mathfrak{so}(4)\oplus\mathbb{C}t\oplus\mathbb{C}C$, where $\mathfrak{so}(4)$ is the Lie algebra of 4×4 skew−symmetric matrices. In the above homomorphism the element ξ_{ij} corresponds to the skew-symmetric matrix $-E_{i,j}+E_{j,i}\in\mathfrak{so}(4)$. We consider the following basis of a Cartan subalgebra $\mathfrak{h}$:

$$h_x:=-i\xi_{12}+i\xi_{34},\;\;h_y:=-i\xi_{12}-i\xi_{34}. \tag{20}$$

Let $\alpha_x,\alpha_y\in\mathfrak{h}^*$ be such that $\alpha_x(h_x)=\alpha_y(h_y)=2$ and $\alpha_x(h_y)=\alpha_y(h_x)=0$. The set of roots is $\Delta=\{\alpha_x,-\alpha_x,\alpha_y,-\alpha_y\}$ and we have the following root decomposition:

$$\mathfrak{so}(4)=\mathfrak{h}\oplus(\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)\;\text{ with }\mathfrak{g}_{\alpha_x}=\mathbb{C}e_x,\,\mathfrak{g}_{-\alpha_x}=\mathbb{C}f_x,\,\mathfrak{g}_{\alpha_y}=\mathbb{C}e_y,\,\mathfrak{g}_{-\alpha_y}=\mathbb{C}f_y$$

where

$$e_x=\frac{1}{2}(-\xi_{1,3}-\xi_{2,4}-i\xi_{1,4}+i\xi_{2,3}),$$
$$e_y=\frac{1}{2}(-\xi_{1,3}+\xi_{2,4}+i\xi_{1,4}+i\xi_{2,3}),$$

$$f_x = \frac{1}{2}(\xi_{1,3} + \xi_{2,4} - i\xi_{1,4} + i\xi_{2,3}),$$
$$f_y = \frac{1}{2}(\xi_{1,3} - \xi_{2,4} + i\xi_{1,4} + i\xi_{2,3}).$$

It will be convenient to use the following notation:

$$e_1 = e_x + e_y = -\xi_{13} + i\xi_{23}, \tag{21}$$
$$e_2 = e_x - e_y = -\xi_{24} - i\xi_{14}. \tag{22}$$

The set $\{e_1, e_2\}$ is a basis of the nilpotent subalgebra $\mathfrak{g}_{\alpha_x} \oplus \mathfrak{g}_{\alpha_y}$.
We will write the weights $\mu = (m, n, \mu_t, \mu_C)$ of weight vectors of $\mathfrak{g}_0-$modules with respect to action of the vectors h_x, h_y, t and C.

Remark 6.1. Since C is central, by Schur's lemma, C acts as a scalar on F.

Remark 6.2. The sets $\{e_x, f_x, h_x\}$ and $\{e_y, f_y, h_y\}$ span two copies of $\mathfrak{sl}_2$ and we think of $\mathfrak{g}_0^{ss}$ in the standard way as a Lie algebra of derivations We have that:

$$\mathfrak{g}_0^{ss} = \langle e_x, f_x, h_x\rangle \oplus \langle e_y, f_y, h_y\rangle \cong \langle x_1\partial_{x_2}, x_2\partial_{x_1}, x_1\partial_{x_1} - x_2\partial_{x_2}\rangle \oplus \langle y_1\partial_{y_2}, y_2\partial_{y_1}, y_1\partial_{y_1} - y_2\partial_{y_2}\rangle.$$

Thanks to Remark 6.2 we will identify the irreducible $\mathfrak{g}_0^{ss}-$module of highest weight (m, n) with respect to h_x, h_y with the space of bihomogeneous polynomials in the four variables x_1, x_2, y_1, y_2 of degree m in the variables x_1, x_2, and of degree n in the variables y_1, y_2.
By direct computations, we obtain the following results.

Lemma 6.3. *The subalgebra $\mathfrak{g}_{>0}$ is generated by $\mathfrak{g}_1$, i.e. $\mathfrak{g}_i = \mathfrak{g}_1^i$ for all $i \geq 2$ and as $\mathfrak{g}_0-$modules:*

$$\mathfrak{g}_1 \cong \langle t\xi_i : 1 \leq i \leq 4\rangle \oplus \langle \xi_I : I \in \mathcal{I}_<, |I| = 3\rangle.$$

The $\mathfrak{g}_0-$modules $\langle t\xi_i : 1 \leq i \leq 4\rangle$ and $\langle \xi_I : I \in \mathcal{I}_<, |I| = 3\rangle$ are irreducible and the corresponding lowest weight vectors are $t(\xi_1 + i\xi_2)$ and $(\xi_1 + i\xi_2)\xi_3\xi_4$.

Lemma 6.4. *As $\mathfrak{g}_0^{ss}-$modules:*

$$\mathfrak{g}_{-1} \cong \langle x_1y_1, x_1y_2, x_2y_1, x_2y_2\rangle.$$

The isomorphism is given by:

$$\xi_2 + i\xi_1 \leftrightarrow x_1y_1,\ \ \xi_2 - i\xi_1 \leftrightarrow x_2y_2,\ \ -\xi_4 + i\xi_3 \leftrightarrow x_1y_2,\ \ \xi_4 + i\xi_3 \leftrightarrow x_2y_1.$$

Motivated by the previous lemma, we will use the notation

$$w_{11} = \eta_2 + i\eta_1,\ w_{22} = \eta_2 - i\eta_1,\ w_{12} = -\eta_4 + i\eta_3,\ w_{21} = \eta_4 + i\eta_3. \tag{23}$$

We point out that $[w_{11}, w_{22}] = 4\Theta$, $[w_{12}, w_{21}] = -4\Theta$ and all other brackets between the w's are 0.

By Lemma 6.3 to check whether a vector $\vec{m}$ in a $\mathfrak{g}$-module is a highest weight singular vector it is enough to show that it is annihilated by e_1, e_2, $t(\xi_1 + i\xi_2)$ and $(\xi_1 + i\xi_2)\xi_3\xi_4$. Nevertheless in the determination of all possible highest weight singular vectors it is convenient to consider the action of all elements in $\mathfrak{g}_{>0}$ and for this it is extremely convenient to use the λ-action.

Remark 6.5. From the definition of the $\lambda-$action we deduce that $\vec{m} \in \mathrm{Ind}(F)$ is a highest weight singular vector if and only if the following hold:

S0: $e_1.\vec{m} = e_2.\vec{m} = 0$;
S1: $\frac{d^2}{d\lambda^2}(\xi_{I\lambda}\vec{m}) = 0$ for all $I \in \mathcal{I}_{\neq}$;
S2: $\frac{d}{d\lambda}(\xi_{I\,\lambda}\vec{m})_{|\lambda=0} = 0$ for all $I \in \mathcal{I}_{\neq}$ such that $|I| \geq 1$;
S3: $(\xi_{I\lambda}\vec{m})_{|\lambda=0} = 0$ for all $I \in \mathcal{I}_{\neq}$ such that $|I| \geq 3$.

Indeed condition **S0** implies that $\vec{m}$ is a highest weight vector. Condition **S1** is equivalent to

$$\sum_{j\geq 2} j(j-1)\frac{\lambda^{j-2}}{j!}(t^j\xi_I)\vec{m} = 0,$$

which implies $(t^j\xi_I)\vec{m} = 0$ for all $I \in \mathcal{I}_{\neq}$ and $j \geq 2$.
Condition **S2** is equivalent to $(t\xi_I)\vec{m} = 0$ for all for all $I \in \mathcal{I}_{\neq}$ such that $|I| \geq 1$.
Condition **S3** is equivalent to $\xi_I\vec{m} = 0$ for all $I \in \mathcal{I}_{\neq}$ such that $|I| \geq 3$.

The aim of this section is to solve equations **S0**–**S3** in order to obtain the following classification of singular vectors. We recall that the highest weight of F is always written with respect to the elements h_x, h_y, t and C. Let us call $M(m, n, \mu_t, \mu_C)$ the Verma module $\mathrm{Ind}(F(m, n, \mu_t, \mu_C))$, where $F(m, n, \mu_t, \mu_C)$ is the irreducible $\mathfrak{g}_0-$module with highest weight (m, n, μ_t, μ_C).

Theorem 6.6. *Let F be an irreducible finite−dimensional $\mathfrak{g}_0-$module, with highest weight μ. A vector in $\mathrm{Ind}(F)$ is a non trivial highest weight singular vector of degree 1 if and only if $\vec{m}$ is (up to a scalar) one of the following vectors:*

a: $\mu = (m, n, -\frac{m+n}{2}, \frac{m-n}{2})$ *with* $m, n \in \mathbb{Z}_{\geq 0}$,
$$\vec{m}_{1a} = w_{11} \otimes x_1^m y_1^n;$$
b: $\mu = (m, n, 1 + \frac{m-n}{2}, -1 - \frac{m+n}{2})$, *with* $m \in \mathbb{Z}_{>0}$, $n \in \mathbb{Z}_{\geq 0}$,
$$\vec{m}_{1b} = w_{21} \otimes x_1^m y_1^n - w_{11} \otimes x_1^{m-1} x_2 y_1^n;$$
c: $\mu = (m, n, 2 + \frac{m+n}{2}, \frac{n-m}{2})$, *with* $m, n \in \mathbb{Z}_{>0}$,
$$\vec{m}_{1c} = w_{22} \otimes x_1^m y_1^n - w_{12} \otimes x_1^{m-1} x_2 y_1^n - w_{21} \otimes x_1^m y_1^{n-1} y_2 + w_{11} \otimes x_1^{m-1} x_2 y_1^{n-1} y_2;$$
d: $\mu = (m, n, 1 + \frac{n-m}{2}, 1 + \frac{m+n}{2})$, *with* $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{>0}$,
$$\vec{m}_{1d} = w_{12} \otimes x_1^m y_1^n - w_{11} \otimes x_1^m y_1^{n-1} y_2.$$

Theorem 6.7. *Let F be an irreducible finite−dimensional $\mathfrak{g}_0-$module, with highest weight μ. A vector $\vec{m} \in \mathrm{Ind}(F)$ is a non trivial highest weight singular vector of degree 2 if and only if $\vec{m}$ is (up to a scalar) one of the following vectors:*

a: $\mu = (0, n, 1 - \frac{n}{2}, -1 - \frac{n}{2})$ *with* $n \in \mathbb{Z}_{\geq 0}$,
$$\vec{m}_{2a} = w_{11} w_{21} \otimes y_1^n;$$
b: $\mu = (m, 0, 1 - \frac{m}{2}, 1 + \frac{m}{2})$ *with* $m \in \mathbb{Z}_{\geq 0}$,
$$\vec{m}_{2b} = w_{11} w_{12} \otimes x_1^m;$$
c: $\mu = (m, 0, 2 + \frac{m}{2}, -\frac{m}{2})$ *with* $m \in \mathbb{Z}_{>1}$,
$$\vec{m}_{2c} = w_{22} w_{21} \otimes x_1^m + (w_{11} w_{22} + w_{21} w_{12}) \otimes x_1^{m-1} x_2 - w_{11} w_{12} \otimes x_1^{m-2} x_2^2;$$

d: $\mu = (0, n, 2 + \frac{n}{2}, \frac{n}{2})$ *with* $n \in \mathbb{Z}_{>1}$,

$$\vec{m}_{2d} = w_{22}w_{12} \otimes y_1^n - (w_{22}w_{11} + w_{21}w_{12}) \otimes y_1^{n-1}y_2 - w_{11}w_{21} \otimes y_1^{n-2}y_2^2.$$

Theorem 6.8. *Let F be an irreducible finite−dimensional $\mathfrak{g}_0$−module, with highest weight μ. A vector $\vec{m} \in \operatorname{Ind}(F)$ is a non trivial highest weight singular vector of degree 3 if and only if $\vec{m}$ is (up to a scalar) one of the following vectors:*

a: $\mu = (1, 0, \frac{5}{2}, -\frac{1}{2})$,

$$\vec{m}_{3a} = w_{11}w_{22}w_{21} \otimes x_1 + w_{21}w_{12}w_{11} \otimes x_2;$$

b: $\mu = (0, 1, \frac{5}{2}, \frac{1}{2})$,

$$\vec{m}_{3b} = w_{11}w_{22}w_{12} \otimes y_1 + w_{12}w_{21}w_{11} \otimes y_2.$$

Theorem 6.9. *There are no singular vectors of degree greater than 3.*

Remark 6.10. We call a Verma module *degenerate* if it is not irreducible. We point out that, given $M(m, n, \mu_t, \mu_C)$ and $M(\widetilde{m}, \widetilde{n}, \widetilde{\mu}_t, \widetilde{\mu}_C)$ Verma modules, we can construct a non trivial morphism of $\mathfrak{g}$−modules from the former to the latter if and only if there exists a highest weight singular vector $\vec{m}$ in $M(\widetilde{m}, \widetilde{n}, \widetilde{\mu}_t, \widetilde{\mu}_C)$ of highest weight (m, n, μ_t, μ_C). The map is uniquely determined by:

$$\begin{aligned} \nabla : M(m, n, \mu_t, \mu_C) &\longrightarrow M(\widetilde{m}, \widetilde{n}, \widetilde{\mu}_t, \widetilde{\mu}_C) \\ v_\mu &\longmapsto \vec{m}, \end{aligned}$$

where v_μ is a highest weight vector of $F(m, n, \mu_t, \mu_C)$. If $\vec{m}$ is a singular vector of degree d, we say that ∇ is a morphism of degree d.

We use Remark 6.10 to construct the maps in Figure 1 of all possible morphisms in the case of K_4'. We also observe that the symmetry of this picture is coherent with conformal duality. Indeed, by the main result in [5] the conformal dual of a Verma module $M(m, n, \mu_t, \mu_C)$ is $M(m, n, -\mu_t + a, -\mu_C + b)$, with

$$a = \operatorname{str}(\operatorname{ad}(t)_{|\mathfrak{g}_{<0}}) = 2$$

and

$$b = \operatorname{str}(\operatorname{ad}(C)_{|\mathfrak{g}_{<0}}) = 0,$$

where $\mathfrak{g} = \mathcal{A}(K_4')$, "str" denotes supertrace, and "ad" denotes the adjoint representation. In particular the duality is obtained with the rotation by 180 degrees of the whole picture. Note also that all compositions of two morphisms in Figure 1 must vanish by the classification of singular vectors, and hence we obtain an infinite number of bilateral complexes of morphisms.

From Theorems 6.6, 6.7 and 6.8 it follows that the module $M(0, 0, 2, 0)$ does not contain non trivial singular vectors, hence it is irreducible due to Theorem 2.15. This is also confirmed by the following result which can be skipped by the reader who is interested in the classification of singular vectors only.

Proposition 6.11. *The module $M(0, 0, 2, 0)$ is irreducible and it is isomorphic to the coadjoint representation of $K(1, 4)_+$ on the restricted dual, i.e. $K(1, 4)_+^* = \bigoplus_{j \in \mathbb{Z}} (K(1, 4)_{+j})^*$.*

Proof. Observe that we consider $M(0,0,2,0)$ as a $K(1,4)_+$-module since the action of C is trivial.

We first show that $K(1,4)_+^*$ is an irreducible $K(1,4)_+-$module.
We recall that the action on the restricted dual is given, for every $x, y \in K(1,4)_+$ and $f \in K(1,4)_+^*$, by:

$$(x.f)(y) = -(-1)^{p(x)p(f)} f([x,y]), \tag{24}$$

where $p(x)$ (resp. $p(f)$) denotes the parity of x (resp. f) and the bracket is given by (5).
Since we are considering the restricted dual, a basis of $K(1,4)_+^*$ is given by the dual basis elements $(t^n\xi_I)^*$ with $n \geq 0$ and $I \in \mathcal{I}_<$. We will also denote $\Theta^* = -2\xi_\emptyset^*$, so that $\Theta^*(\Theta) = 1$.

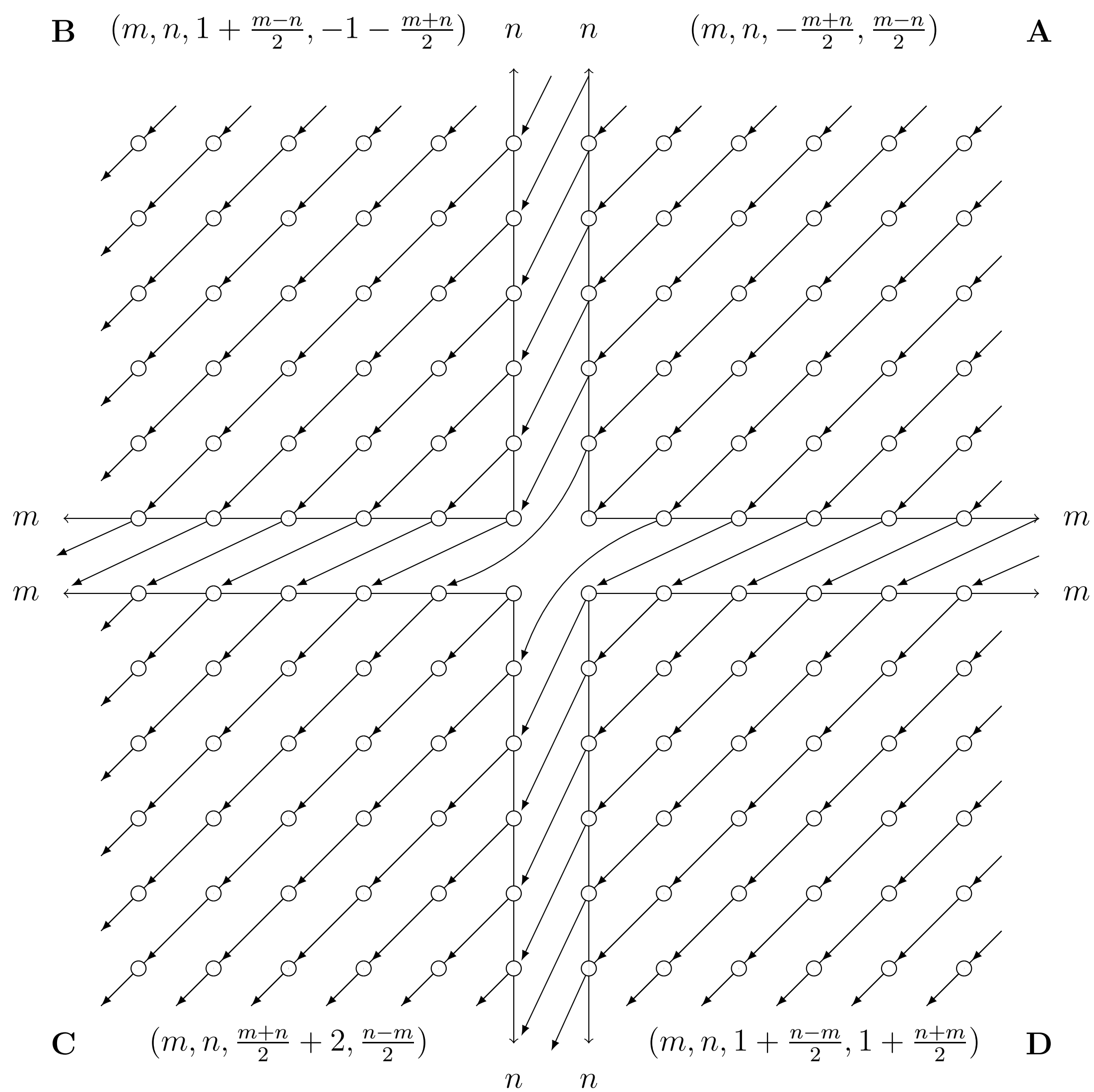

FIGURE 1. Morphisms between finite Verma modules

We first show by induction on $s+p$ that

$$(\underbrace{\Theta.\cdots(\Theta}_{s-times}.(\xi_{i_1}.(\cdots(\xi_{i_p}.\Theta^*))))\cdots)=\gamma_{s,p}(t^s\xi_{i_1\cdots i_p})^*, \tag{25}$$

for some $\gamma_{s,p}\in\mathbb{C}\setminus\{0\}$. If $s+p=0$ the result is trivial, so we assume $s+p>0$.
If $s>0$, by induction hypothesis, for every $n\geq 0$ and $J\in\mathcal{I}_<$ we have

$$\begin{aligned}(\underbrace{\Theta.\cdots(\Theta}_{s-times}.(\xi_{i_1}.(\cdots(\xi_{i_p}.\Theta^*))))(t^n\xi_J)&=-\gamma_{s-1,p}(t^{s-1}\xi_{i_1\cdots i_p})^*([\Theta,t^n\xi_J])\\&=\begin{cases}\gamma_{s-1,p}s & \text{if } n=s \text{ and } J=i_1\cdots i_p,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

and the claim follows in this case.
If $s=0$ (and $p>0$) for every $n\geq 0$ and $J\in\mathcal{I}_<$ we have

$$\begin{aligned}(\xi_{i_1}.\cdots(\xi_{i_p}.(\Theta^*)))(t^n\xi_J)=\gamma_{0,p-1}(\xi_{i_1}.(\xi_{i_2\cdots i_p})^*)(t^n\xi_J)&=(-1)^p\gamma_{0,p-1}(\xi_{i_2\cdots i_p})^*([\xi_{i_1},t^n\xi_J])\\&=\begin{cases}(-1)^{p+1}\gamma_{0,p-1} & \text{if } n=0 \text{ and } J=i_1\cdots i_p,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

and the proof of (25) is complete.

Now we need the following observation: let $m,s\geq 0$ and $I,K\in\mathcal{I}_<$ be such that $\deg(t^m\xi_I)\geq\deg(t^s\xi_K)$, i.e. $2m+|I|-2\geq 2s+|K|-2$. Then

$$t^{m+1}\xi_I.(t^s\xi_K)^*=\begin{cases}\beta_{m,I}\Theta^* & \text{if } K=I \text{ and } s=m\\ 0 & \text{otherwise.}\end{cases}, \tag{26}$$

for suitable $\beta_{m,I}\in\mathbb{C}\setminus\{0\}$. By (26) we deduce that if $f\in K(1,4)_+$ with $f=\sum\alpha_{I,m}(t^m\xi_I)^*\neq 0$ and we choose the pair I_0,m_0 among all pairs (I,m) with $\alpha_{I,m}\neq 0$ such that $2m+|I|-2$ is maximum, then $t^{m_0+1}\xi_{I_0}.f$ is a nonzero scalar multiple of Θ^*. From this observation and (25) we deduce that $K(1,4)^*_+$ is irreducible.

Now consider $M(0,0,2,0)=\mathrm{Ind}(F)$, where $F=\langle v\rangle$ is the 1-dimensional $\mathfrak{g}_0$-module of highest weight $(0,0,2,0)$. Since Θ^* is a highest weight singular vector in $K(1,4)^*_+$ of weight $(0,0,2,0)$ we deduce that there exists a (unique) morphism

$$\varphi: M(0,0,2,0)\to K(1,4)^*_+$$

such that $\varphi(v)=\Theta^*$. The morphism φ is surjective by the irreducibility of $K(1,4)^*_+$. The morphism φ is also injective since it preserves the degree, and homogeneous components of the same degree of $M(0,0,2,0)$ and $K(1,4)^*_+$ have also the same dimension. □

In order to prove Theorems 6.6, 6.7 and 6.8, we need some lemmas.

Remark 6.12. We point out that, by Remark 6.5, a vector $\vec{m}\in\mathrm{Ind}(F)$ is a highest weight singular vector if and only if it satisfies **S0**–**S3**. Since T, defined as in Proposition 5.10, is an isomorphism and $\vec{m}=T^{-1}T(\vec{m})$, the fact that $\vec{m}\in\mathrm{Ind}(F)$ satisfies **S0**–**S3** is equivalent to impose conditions **S0**–**S3** for $(T\circ f_\lambda\circ T^{-1})T(\vec{m})$, using the expression given by Proposition 5.10.
Therefore in the following lemmas we will consider a vector $T(\vec{m})\in\mathrm{Ind}(F)$ and we will

impose that the expression for $(T\circ f_\lambda\circ T^{-1})T(\vec{m}) = T(f_\lambda\vec{m})$, given by Proposition 5.10, satisfies conditions **S0**–**S3**. We will have that $\vec{m}$ is a highest weight singular vector.

Motivated by Remark 6.12, we look for a singular vector $\vec{m}$ such that

$$T(\vec{m}) = \sum_{k=0}^{N}\sum_{L\in\mathcal{I}_<}\Theta^k\eta_L\otimes v_{L,k}, \tag{27}$$

with $v_{L,k}\in F$ for all k. For all k, we will denote $v_{*,k} = v_{1234,k}$.

In order to make clearer how the $\lambda-$action of Proposition 5.10 works for a vector as in (27), let us see the following example.

Example 6.13. Let $T(\vec{m}) = \Theta^2\eta_{13}\otimes v_{13,2} + \eta_2\otimes v_{2,0}$. Using Proposition 5.10 and Lemma 5.11, we have:

$$\begin{aligned}
&T(\xi_{2\lambda}\vec{m}) =\\
&= -(\lambda+\Theta)^2\Big\{-\Theta(\xi_2\star\eta_{13})\otimes v_{13,2} + \sum_{i=1}^{4}(\partial_i\xi_2\star\partial_i\eta_{13})\otimes v_{13,2}\\
&\quad + \lambda\Big[(\xi_2\star\eta_{13})\otimes t.v_{13,2} + \sum_{i=1}^{4}\partial_i(\xi_{2\,i}\star\eta_{13})\otimes v_{13,2} - \sum_{i\neq j}(\partial_i\xi_{2j}\star\eta_{13})\otimes\xi_{j,i}.v_{13,2}\Big]\\
&\quad + \lambda^2\Big[-\sum_{i<j}(\xi_{2\,ij}\star\eta_{13})\otimes\xi_{j,i}.v_{13,2} - \varepsilon_2\,(\xi_{(2)^c}\star\eta_{13})\otimes Cv_{13,2}\Big]\Big\}\\
&\quad - \Theta(\xi_2\star\eta_2)\otimes v_{2,0} + \sum_{i=1}^{4}(\partial_i\xi_2\star\partial_i\eta_2)\otimes v_{2,0} + \lambda\Big[(\xi_2\star\eta_2)\otimes t.v_{2,0} + \sum_{i=1}^{4}\partial_i(\xi_{2\,i}\star\eta_2)\otimes v_{2,0}\\
&\quad - \sum_{i\neq j}(\partial_i\xi_{2j}\star\eta_2)\otimes\xi_{j,i}.v_{2,0}\Big] + \lambda^2\Big[-\sum_{i<j}(\xi_{2ij}\star\eta_2)\otimes\xi_{j,i}.v_{2,0} - \varepsilon_{(2)}\,(\xi_{(2)^c}\star\eta_2)\otimes Cv_{2,0}\Big]\\
&= -\,(\lambda+\Theta)^2\Big\{\Theta\eta_{123}\otimes v_{13,2} + \lambda\Big[-\eta_{123}\otimes t.v_{13,2} + \partial_4(\xi_{24}\star\eta_{13})\otimes v_{13,2}\\
&\quad - (\partial_2\xi_{24}\star\eta_{13})\otimes\xi_{4,2}.v_{13,2}\Big]\Big\} + 1\otimes v_{2,0} + \lambda\Big[-\sum_{j\neq 2}(\xi_j\star\eta_2)\otimes\xi_{j,2}.v_{2,0}\Big] + \lambda^2\eta_{1234}\otimes Cv_{2,0}\\
&= -\,(\lambda+\Theta)^2\Big\{\Theta\eta_{123}\otimes v_{13,2} + \lambda\Big[-\eta_{123}\otimes t.v_{13,2} + \eta_{123}\otimes v_{13,2} + \eta_{134}\otimes\xi_{2,4}.v_{13,2}\Big]\Big\}\\
&\quad + 1\otimes v_{2,0} + \lambda\Big[-\eta_{12}\otimes\xi_{1,2}.v_{2,0} + \eta_{23}\otimes\xi_{2,3}.v_{2,0} + \eta_{24}\otimes\xi_{24}.v_{2,0}\Big] + \lambda^2\eta_{1234}\otimes Cv_{2,0}.
\end{aligned}$$

Lemma 6.14. *Let $\vec{m}\in\operatorname{Ind}(F)$ be a singular vector, such that $T(\vec{m})$ is written as in (27). The degree of $T(\vec{m})$ in Θ is at most 3.*

Proof. Using Proposition 5.10, Lemma 5.11 and Remark 6.12, condition **S1** for $I=\emptyset$ reduces to:

$$\begin{aligned}
0 =& \frac{d^2}{d\lambda^2}(T(1_\lambda\vec{m})) = \sum_{k=2}^{N}\sum_{L}k(k-1)(\lambda+\Theta)^{k-2}\Big[-2\Theta\eta_L\otimes v_{L,k} + \lambda\left(\eta_L\otimes t.v_{L,k} - (4-|L|)\eta_L\otimes v_{L,k}\right)\\
&+ \lambda^2\sum_{i<j}(\xi_{ij}\star\eta_L)\otimes\xi_{ij}.v_{L,k} - \lambda^3\chi_{|L|=0}\eta_{1234}\otimes Cv_{L,k}\Big]
\end{aligned}$$

$$+2\sum_{k=1}^{N}\sum_{L} k(\lambda+\Theta)^{k-1}\Big[\eta_L\otimes t.v_{L,k}-(4-|L|)\eta_L\otimes v_{L,k}$$
$$+2\lambda\sum_{i<j}(\xi_{ij}\star\eta_L)\otimes\xi_{ij}.v_{L,k}-3\lambda^2\chi_{|L|=0}\eta_{1234}\otimes Cv_{L,k}\Big]$$
$$+\sum_{k=0}^{N}\sum_{L}(\lambda+\Theta)^k\Big[2\sum_{i<j}(\xi_{ij}\star\eta_L)\otimes\xi_{ij}.v_{I,k}-6\lambda\chi_{|L|=0}\eta_{1234}\otimes Cv_{L,k}\Big].$$

If we expand this expression with respect to the variables λ and $\lambda+\Theta$, the coefficients of $(\lambda+\Theta)^s\lambda^3$, with $s\geq 0$, are:

$$(s+2)(s+1)\eta_{1234}\otimes Cv_{\emptyset,s+2}=0.$$

and therefore

$$v_{\emptyset,k}=0 \tag{28}$$

for all $k\geq 2$. If we consider the coefficients of $(\lambda+\Theta)^s\lambda^2$ with $s\geq 1$ we obtain:

$$\sum_L\sum_{i<j}(s+2)(s+1)(\xi_{ij}\star\eta_L)\otimes\xi_{ij}.v_{L,s+2}-6(s+1)\eta_{1234}\otimes Cv_{\emptyset,s+1}=0.$$

Therefore we obtain that for $s\geq 1$:

$$\sum_L\sum_{i<j}(\xi_{ij}\star\eta_L)\otimes\xi_{ij}.v_{L,s+2}=0. \tag{29}$$

Now we look at the coefficients of $(\lambda+\Theta)^s\lambda$ with $s\geq 2$ and obtain:

$$\sum_L(s+2)(s+1)(2\eta_L\otimes v_{L,s+2}+\eta_L\otimes t.v_{L,s+2}-(4-|L|)\eta_L\otimes v_{L,s+2})$$
$$+4\sum_L\sum_{i<j}(s+1)(\xi_{ij}\star\eta_L)\otimes\xi_{ij}.v_{L,s+1}-6\eta_{1234}\otimes Cv_{\emptyset,s}=0.$$

Therefore, using (28) and (29), we obtain that for $s\geq 2$:

$$\sum_L((|L|-2)\eta_L\otimes v_{L,s+2}+\eta_L\otimes t.v_{L,s+2})=0. \tag{30}$$

Finally we look at the coefficients of $(\lambda+\Theta)^s$ with $s\geq 3$ and obtain:

$$\sum_L(s+1)s(-2\eta_L\otimes v_{L,s+1})+2(s+1)(\eta_L\otimes t.v_{L,s+1}-(4-|L|)\eta_L\otimes v_{L,s+1})$$
$$+2\sum_L\sum_{i<j}(\xi_{ij}\star\eta_L)\otimes\xi_{ij}.v_{L,s}=0.$$

This equation together with (29) and (30) immediately implies $v_{L,k}=0$ for all $k\geq 4$. □

By Lemma 6.14, for a singular vector $\vec{m}\in\mathrm{Ind}(F)$, $T(\vec{m})$ has the following form:

$$T(\vec{m})=\Theta^3\sum_L\eta_L\otimes v_{L,3}+\Theta^2\sum_L\eta_L\otimes v_{L,2}+\Theta\sum_L\eta_L\otimes v_{L,1}+\sum_L\eta_L\otimes v_{L,0}. \tag{31}$$

We write the $\lambda-$action in the following way, using Proposition 5.10 and Lemma 5.11

$$T(\xi_{I\,\lambda}\vec{m}) \tag{32}$$

$$
\begin{aligned}
&= b_0(I) + G_1(I) + \lambda\big[B_0(I) - a_0(I) - G_2(I)\big] + \lambda^2\big[C_0(I) + G_3(I)\big] + \lambda^3 D_0(I) \\
&+ (\lambda+\Theta)\big(\big[a_0(I) + b_1(I) + 2G_2(I)\big] + \lambda[B_1(I) - a_1(I) - 3G_3(I)\big] + \lambda^2 C_1(I) + \lambda^3 D_1(I)\big) \\
&+ (\lambda+\Theta)^2\big(\big[a_1(I) + b_2(I) + 3G_3(I)\big] + \lambda\big[B_2(I) - a_2(I)\big] + \lambda^2 C_2(I) + \lambda^3 D_2(C)\big) \\
&+ (\lambda+\Theta)^3\big(\big[a_2(I) + b_3(I)\big] + \lambda\big[B_3(I) - a_3(I)\big] + \lambda^2 C_3(I) + \lambda^3 D_3(I)\big) \\
&+ (\lambda+\Theta)^4 a_3(I)
\end{aligned}
$$

where the coefficients $a_p(I)$, $b_p(I)$, $B_p(I)$, $C_p(I)$, $D_p(I)$, $G_p(I)$ depend on I for every $0 \leq p \leq 3$. Here is their explicit expression:

$$
\begin{aligned}
a_p(I) &= \sum_L (-1)^{(|I|(|I|+1)/2)+|I||L|}(|I|-2)(\xi_I \star \eta_L) \otimes v_{L,p}; \\
b_p(I) &= \sum_L (-1)^{(|I|(|I|+1)/2)+|I||L|}\Big[-(-1)^{|I|}\sum_{i=1}^{4}(\partial_i \xi_I \star \partial_i \eta_L) \otimes v_{L,p} + \sum_{r<s}(\partial_{rs}\xi_I \star \eta_L) \otimes \xi_{r,s}.v_{L,p}) \\
&\quad + \chi_{|I|=3}\,\varepsilon_I(\xi_{I^c} \star \eta_L) \otimes C v_{L,p}\Big]; \\
B_p(I) &= \sum_L (-1)^{(|I|(|I|+1)/2)+|I||L|}\Big[(\xi_I \star \eta_L) \otimes t.v_{L,p} - (-1)^{|I|}\sum_{i=1}^{4}\partial_i(\xi_{Ii} \star \eta_L) \otimes v_{L,p} \\
&\quad + (-1)^{|I|}\sum_{i\neq j}(\partial_i \xi_{Ij} \star \eta_L) \otimes \xi_{ji}.v_{L,p} + \chi_{|I|=2}\varepsilon_I(\xi_{I^c} \star \eta_L) \otimes C v_{L,p})\Big]; \\
C_p(I) &= \sum_L (-1)^{(|I|(|I|+1)/2)+|I||L|}\Big[\sum_{i<j}(\xi_{Iij} \star \eta_L) \otimes \xi_{ij}.v_{L,p} - \chi_{|I|=1}\varepsilon_I(\xi_{I^c} \star \eta_L) \otimes C v_{L,p})\Big]; \\
D_p(I) &= \sum_L (-1)^{(|I|(|I|+1)/2)+|I||L|}\Big[-\chi_{|I|=0}\,(\xi_{1234} \star \eta_L) \otimes C v_{L,p}\Big]; \\
G_p(I) &= -\sum_L \chi_{|I|=4}\varepsilon_I \eta_L \otimes C v_{L,p}.
\end{aligned}
$$

We will write a_p instead of $a_p(I)$ if there is no risk of confusion, and similarly for the others.

Proposition 6.15. *Let $\vec{m} \in \mathrm{Ind}(F)$ be such that $T(\vec{m})$ is written as in formula* (31)*. Using notation* (32)*, we have that:*

(1) *condition **S1** implies that for all $I \in \mathcal{I}_<$ we have*

$$
\begin{aligned}
D_3 &= D_2 = C_3 = D_1 + a_3 = C_2 - 3a_3 = B_3 + 2a_3 \\
&= C_1 + 2B_2 + a_2 + 3b_3 = D_0 + C_1 + B_2 + b_3 = C_0 + B_1 + b_2 + G_3 = 0;
\end{aligned}
$$

(2) *condition **S2** implies that for all $I \in \mathcal{I}_<$ such that $|I| \geq 1$ we have*

$$B_0 + b_1 + G_2 = B_1 + a_1 + 2b_2 + 3G_3 = 2a_2 + B_2 + 3b_3 = 3a_3 + B_3 = 0;$$

(3) *condition **S3** implies that for all $I \in \mathcal{I}_<$ such that $|I| \geq 3$ we have*

$$b_0 + G_1 = a_0 + b_1 + 2G_2 = a_1 + b_2 + 3G_3 = a_2 + b_3 = a_3 = 0.$$

Proof. We compute $\frac{d^2}{d\lambda^2}(T(\xi_{I_\lambda}\vec{m}))$ and $\frac{d}{d\lambda}(T(\xi_{I_\lambda}\vec{m}))$ using (32). We have

$$\begin{aligned}\frac{d}{d\lambda}(T(\xi_{I_\lambda}\vec{m})) &= B_0 + b_1 + G_2 + \lambda\big[2C_0 + B_1 - a_1 - G_3\big] + \lambda^2\big[3D_0 + C_1\big] + \lambda^3 D_1\\ &+ (\lambda+\Theta)\big(\big[B_1 + a_1 + 2b_2 + 3G_3\big] + \lambda\big[2C_1 + 2B_2 - 2a_2\big] + \lambda^2\big[3D_1 + 2C_2\big] + 2\lambda^3 D_2\big)\\ &+ (\lambda+\Theta)^2\big(\big[2a_2 + B_2 + 3b_3\big] + \lambda\big[3B_3 - 3a_3 + 2C_2\big] + \lambda^2\big[3D_2 + 3C_3\big] + 3\lambda^3 D_3\big)\\ &+ (\lambda+\Theta)^3\big(\big[3a_3 + B_3\big] + 2\lambda C_3 + 3\lambda^2 D_3\big),\end{aligned}$$

and

$$\begin{aligned}&\frac{d^2}{d\lambda^2}(T(\xi_{I_\lambda}\vec{m}))\\ &= 2C_0 + 2B_1 + 2b_2 + 2G_3 + \lambda\big[6D_0 + 4C_1 + 2B_2 - 2a_2\big] + \lambda^2\big[6D_1 + 2C_2\big] + 2\lambda^3 D_2\\ &\quad + (\lambda+\Theta)\big(\big[2C_1 + 4B_2 + 2a_2 + 6b_3\big] + \lambda\big[6D_1 + 8C_2 + 6B_3 - 6a_3\big] + \lambda^2\big[12D_2 + 6C_3\big] + 6\lambda^3 D_3\big)\\ &\quad + (\lambda+\Theta)^2\big(\big[2C_2 + 6a_3 + 6B_3\big] + \lambda\big[12C_3 + 6D_2\big] + 18\lambda^2 D_3\big)\\ &\quad + (\lambda+\Theta)^3\big(2C_3 + 6\lambda D_3\big).\end{aligned}$$

The result follows. □

Let us show some other reductions on singular vectors.

Lemma 6.16. *Let $\vec{m} \in \mathrm{Ind}(F)$ be a singular vector, such that $T(\vec{m})$ is written as in formula* (31)*. For all I we have that $v_{I,3} = 0$.*

Proof. By Proposition 6.15, we have $2a_3(i) + B_3(i) = 0$ and $3a_3(i) + B_3(i) = 0$ for all $i \in \{1,2,3,4\}$. Therefore $a_3(i) = 0$ which immediately implies $v_{L,3} = 0$ for every L such that $|L| < 4$.

Proposition 6.15 also provides $D_0(1) + C_1(1) + B_2(1) + b_3(1) = 0$, $C_1(1) + 2B_2(1) + a_2(1) + 3b_3(1) = 0$ and $2a_2(1) + B_2(1) + 3b_3(1) = 0$. A linear combination of these equations gives us $D_0(1) + a_2(1) + b_3(1) = 0$. Since $D_0(1) = 0$, we have

$$0 = a_2(1) + b_3(1) = -\sum_L (-1)^{1+|L|}(\xi_1 \star \eta_L) \otimes v_{L,2} - \eta_{234} \otimes v_{1234,3} = 0,$$

which implies $v_{1234,3} = 0$. □

Lemma 6.16 implies $a_3 = b_3 = B_3 = C_3 = D_3 = G_3 = 0$ and so all equations in Proposition 6.15 can be significantly simplified. Next result provides a further semplification.

Lemma 6.17. *Let $\vec{m} \in \mathrm{Ind}(F)$ be a singular vector such that $T(\vec{m})$ is written as in formula* (31)*. For all I we have that $v_{I,2} = 0$.*

Proof. By Proposition 6.15 and Lemma 6.16 we know that $D_0(i) + C_1(i) + B_2(i) = 0$, $C_1(i) + 2B_2(i) + a_2(i) = 0$ and $2a_2(i) + B_2(i) = 0$ for all $i \in \{1,2,3,4\}$. Moreover $D_0(i) = 0$ by definition and from these equations we can deduce $a_2(i) = 0$ for all $i \in \{1,2,3,4\}$ and, as in the proof of Lemma 6.16 we can immediately conclude that $v_{L,2} = 0$ for every L such that $|L| < 4$.

We now show that $v_{1234,2} = 0$. By Proposition 6.15 we know that $b_0(123) + G_1(123) = 0$ and since $G_1(123) = 0$ by definition we have

$$0 = b_0(123)$$

$$= \sum_L (-1)^{|L|}\Big(\sum_{i=1}^{4}(\partial_i\xi_{123}\star\partial_i\eta_L)\otimes v_{L,0} + \sum_{r<s}(\partial_{rs}\xi_{123}\star\eta_L)\otimes\xi_{rs}.v_{L,0} + (\xi_4\star\eta_L)\otimes Cv_{L,0}\Big).$$

In this equation the unique term in η_4 is

$$\eta_4 \otimes Cv_{\emptyset,0}$$

and so $Cv_{\emptyset,0}=0$.

By Proposition 6.15 we have that $C_0(1)+B_1(1)+b_2(1)=0$, $B_1(1)+a_1(1)+2b_2(1)=0$ and so $C_0(1)-a_1(1)-b_2(1)=0$. We have:

$$\begin{aligned}
0 &= C_0(1)-a_1(1)-b_2(1)\\
&= \sum_L\sum_{i<j}(-1)^{1+|L|}(\xi_{1ij}\star\eta_L)\otimes\xi_{ij}.v_{L,0} - \sum_L(-1)^{1+|L|}(\xi_{234}\star\eta_L)\otimes Cv_{L,0}\\
&\quad + \sum_L(-1)^{1+|L|}(\xi_1\star\eta_L)\otimes v_{L,1} + \eta_{234}\otimes v_{1234,2}.
\end{aligned}$$

The terms in η_{234} in this expression are

$$\eta_{234}\otimes Cv_{\emptyset,0} + \eta_{234}\otimes v_{1234,2} = 0.$$

Since $Cv_{\emptyset,0}=0$, we conclude that $v_{1234,2}=0$. □

By Lemma 6.17 we can deduce that $a_2=b_2=B_2=C_2=D_2=G_2=0$ for all I.

Lemma 6.18. *Let $\vec{m}\in \mathrm{Ind}(F)$ be a singular vector such that $T(\vec{m})$ is written as in formula* (31)*. For all L such that $|L|\leq 2$, we have that $v_{L,1}=0$.*

Proof. By Proposition 6.15 and Lemmas 6.16 and 6.17, we have $a_1(I)=0$ for all $|I|\geq 3$ which immediately implies $v_{L,1}=0$ for all L such that $|L|\leq 1$.

Let's show the result for $|L|=2$. By Proposition 6.15 and Lemmas 6.16 and 6.17, we know that $B_0(a)+b_1(a)=0$ for all $a\in\{1,2,3,4\}$. Letting $(a)^c=(b,c,d)$ we have

$$\begin{aligned}
0 &= \sum_L(-1)^{1+|L|}\Big((\xi_a\star\eta_L)\otimes t.v_{L,0} + \sum_{i\neq a}\partial_i(\xi_{ai}\star\eta_L)\otimes v_{L,0} - \sum_{j\neq a}(\xi_j\star\eta_L)\otimes\xi_{j,a}.v_{L,0}\Big)\\
&\quad + \sum_{|L|\geq 2}(-1)^{1+|L|}\partial_a\eta_L\otimes v_{L,1}.
\end{aligned}$$

The terms in η_d of $B_0(a)$ are:

$$\eta_d\otimes\xi_{a,d}.v_{\emptyset,0}.$$

We have shown in Lemma 6.17 that $Cv_{\emptyset,0}=0$ and so $v_{\emptyset,0}=0$ if $C\neq 0$. Nevertheless, if $C=0$ the $\lambda-$action in Proposition 5.10 reduces to the $\lambda-$action found in Theorem 4.3 of [1] and one can obtain even in this case $v_{\emptyset,0}=0$ proceeding as in Lemma B.4 of [1].

Therefore, the unique term in η_d in the equation above is

$$\partial_a\eta_{L_0}\otimes v_{L_0,1}=0,$$

where $L_0=ad$ if $a<d$ and $L_0=da$ if $a>d$, and this implies $v_{L_0,1}=0$. □

By Lemmas 6.16, 6.17 and 6.18 and the fact $v_{\emptyset,0}=0$, for a singular vector $\vec{m}$ the expression in (27) can be simplified as

$$T(\vec{m})=\Theta\sum_{|L|\geq 3}\eta_L\otimes v_{L,1}+\sum_{|L|\geq 1}\eta_L\otimes v_{L,0}. \tag{33}$$

Therefore, from (33), we have that there can only be singular vectors of degree 3, 2 and 1. Hence we have showed Theorem 6.9. Following the notation used in [1], we rewrite (33) in the following way: for $|L|=3$, η_L will be written as $\eta_{(i)^c}$, where $(i)^c=L$, $v_{L,1}$ will be renamed as $v_{i,1}$ and $v_{L,0}$ will be renamed as v_i, so that they depend on one index; for $|L|=2$, η_L will be written as $\eta_{(i,j)^c}$, where $(i,j)^c=L$, and $v_{L,0}$ will be renamed as $v_{i,j}$. In particular, by (33), the singular vectors of degree 3, 2 and 1 are such that

degree 3: $T(\vec{m})=\Theta\sum_i\eta_{(i)^c}\otimes v_{i,1}+\sum_i\eta_i\otimes v_{i,0}$,
degree 2: $T(\vec{m})=\Theta\eta_*\otimes v_*+\sum_{i<j}\eta_{(i,j)^c}\otimes v_{i,j}$,
degree 1: $T(\vec{m})=\sum_i\eta_{(i)^c}\otimes v_i$.

By Proposition 6.15 and Lemmas 6.16, 6.17, 6.18 we obtain the following result.

Proposition 6.19. *Let $\vec{m}\in\mathrm{Ind}(F)$ be such that $T(\vec{m})$ is as in formula* (33)*. Using notation* (32)*, we have that:*

(1) *condition **S1** implies that for all $I\in\mathcal{I}_{\neq}$*
$$C_1=D_1=D_0=C_0+B_1=0;$$
(2) *condition **S2** implies that for all $I\in\mathcal{I}_{\neq}$ with $|I|\geq 1$*
$$B_0+b_1=B_1+a_1=0;$$
(3) *condition **S3** implies that for all $I\in\mathcal{I}_{\neq}$ with $|I|\geq 3$*
$$b_0+G_1=a_0+b_1=a_1=0.$$

7. Singular vectors of degree 2

The aim of this section is to classify all singular vectors of degree 2. We have that a singular vector of degree 2 is such that:

$$T(\vec{m})=\Theta\eta_*\otimes v_*+\sum_{i<j}\eta_{(i,j)^c}\otimes v_{i,j}. \tag{34}$$

We will assume for our convenience that $v_{i,i}=0$ and $v_{i,j}=-v_{j,i}$ for all i,j. We write the vector $\vec{m}$ also in the following way:

$$\begin{aligned}\vec{m}=&(\eta_2+i\eta_1)(\eta_4+i\eta_3)\otimes w_1+(\eta_2+i\eta_1)(\eta_4-i\eta_3)\otimes w_2+(\eta_2-i\eta_1)(\eta_4+i\eta_3)\otimes w_3\\&+(\eta_2-i\eta_1)(\eta_4-i\eta_3)\otimes w_4+(\eta_2+i\eta_1)(\eta_2-i\eta_1)\otimes w_5+(\eta_4+i\eta_3)(\eta_4-i\eta_3)\otimes w_6+\Theta\otimes w_7\\=&(-\eta_{13}+i\eta_{14}+i\eta_{23}+\eta_{24})\otimes w_1+(\eta_{13}+i\eta_{14}-i\eta_{23}+\eta_{24})\otimes w_2+(\eta_{13}-i\eta_{14}+i\eta_{23}+\eta_{24})\otimes w_3\\&+(-\eta_{13}-i\eta_{14}-i\eta_{23}+\eta_{24})\otimes w_4+(2\Theta+2i\eta_{12})\otimes w_5+(2\Theta+2i\eta_{34})\otimes w_6+\Theta\otimes w_7.\end{aligned} \tag{35}$$

From these two expressions it follows that

$$\begin{aligned}v_{1,2}&=2iw_5,\\v_{1,3}&=w_1-w_2-w_3+w_4,\\v_{1,4}&=iw_1+iw_2-iw_3-iw_4,\end{aligned} \tag{36}$$

$$v_{2,3} = iw_1 - iw_2 + iw_3 - iw_4,$$
$$v_{2,4} = -w_1 - w_2 - w_3 - w_4,$$
$$v_{3,4} = 2iw_6,$$
$$v_* = 2w_5 + 2w_6 + w_7.$$

Indeed, let us show for example one of the previous equations. In (34), let us consider $\eta_{(1,3)^c} \otimes v_{1,3} = \eta_{24} \otimes v_{1,3}$. We have that $T(\eta_{13}) = -\eta_{24}$. In (35), the terms in η_{13} are:

$$-\eta_{13} \otimes w_1 + \eta_{13} \otimes w_2 + \eta_{13} \otimes w_3 - \eta_{13} \otimes w_4,$$

therefore $v_{1,3} = w_1 - w_2 - w_3 + w_4$.
In the following lemma we write explicitly the relations of Proposition 6.19 for a vector as in formula (34).

Lemma 7.1. *Let $\vec{m} \in \mathrm{Ind}(F)$ be such that $T(\vec{m})$ is as in formula* (34)*. We have that:*
*1) condition **S1** implies (for $I = \emptyset$)*

$$\sum_{i<j} (\xi_{ij} \star \eta_{(i,j)^c}) \otimes \xi_{ij}.v_{i,j} + \eta_* \otimes t.v_* = 0; \tag{37}$$

*2) condition **S2** implies that for all I such that $|I| = 1, 2$*

$$\sum_{i<j} \Big[(\xi_I \star \eta_{(i,j)^c}) \otimes t.v_{i,j} - (-1)^{|I|} \sum_{l=1}^{4} \partial_l (\xi_{Il} \star \eta_{(i,j)^c}) \otimes v_{i,j} + (-1)^{|I|} \sum_{k \neq l} (\partial_k \xi_{Il} \star \eta_{(i,j)^c}) \otimes \xi_{lk}.v_{i,j} \tag{38}$$

$$+ \chi_{|I|=2}\, \varepsilon_I (\xi_{I^c} \star \eta_{(i,j)^c}) \otimes Cv_{i,j} \Big] - (-1)^{|I|} \sum_{i=1}^{4} (\partial_i \xi_I \star \partial_i \eta_*) \otimes v_* + \sum_{r<s} (\partial_{rs} f \star \eta_*) \otimes \xi_{rs}.v_* = 0;$$

*3) condition **S3** implies that for all I such that $|I| \geq 3$*

$$\sum_{i<j} (-1)^{(|I|(|I|+1)/2)} \Big[-(-1)^{|I|} \sum_{l=1}^{4} (\partial_l \xi_I \star \partial_l \eta_{(i,j)^c}) \otimes v_{i,j} + \sum_{r<s} (\partial_{rs} \xi_I \star \eta_{(i,j)^c}) \otimes \xi_{rs}.v_{i,j} \tag{39}$$

$$+ \chi_{|I|=3}\, \varepsilon_I (\xi_{I^c} \star \eta_{(i,j)^c}) \otimes Cv_{i,j} \Big] - \chi_{|I|=4} \varepsilon_I \eta_* \otimes Cv_* = 0.$$

The following result collects the crucial equations that we will use in the classification of singular vectors of degree 2.

Lemma 7.2. *Let $\vec{m} \in \mathrm{Ind}(F)$ be as in* (34)*. Then for any permutation (a, b, c, d) of $\{1, 2, 3, 4\}$ we have*

$$\sum_{j \neq a} (-1)^{a+j} \xi_{ja}.v_{j,a} = v_*; \tag{40}$$

$$t.v_{a,b} - v_{a,b} + \sum_{j \neq a,b} (-1)^{a+j} \xi_{aj}.v_{j,b} = 0; \tag{41}$$

$$\xi_{ab}.v_* + (-1)^{a+b} t.v_{a,b} + \sum_{j \neq a,b} (-1)^{b+j} \xi_{aj}.v_{j,b} - \sum_{j \neq a,b} (-1)^{a+j} \xi_{bj}.v_{j,a} - \varepsilon_{(a,b)} \varepsilon_{(c,d)} C v_{c,d} = 0; \tag{42}$$

$$\sum_{i<j}(-1)^{i+j}\xi_{ij}.v_{i,j}=t.v_*; \tag{43}$$

$$(-1)^{b+c}v_{b,c}-(-1)^{a+c}\xi_{ab}.v_{a,c}+(-1)^{a+b}\xi_{ac}.v_{a,b}+\varepsilon_{(a,b,c)}(-1)^{a+d}Cv_{a,d}=0; \tag{44}$$

$$\sum_{i<j}\xi_{(ij)^c}.v_{i,j}+Cv_*=0. \tag{45}$$

Finally:

$$\begin{aligned}
&e_1.v_*=0 && e_2.v_*=0\\
&e_1.v_{1,2}=-iv_{1,3}+v_{2,3}, && e_2.v_{1,2}=-v_{1,4}-iv_{2,4},\\
&e_1.v_{1,3}=iv_{1,2}, && e_2.v_{1,3}=-iv_{3,4},\\
&e_1.v_{1,4}=-v_{3,4}, && e_2.v_{1,4}=v_{1,2},\\
&e_1.v_{2,3}=-v_{1,2}, && e_2.v_{2,3}=v_{3,4},\\
&e_1.v_{2,4}=-iv_{3,4}, && e_2.v_{2,4}=iv_{1,2},\\
&e_1.v_{3,4}=v_{1,4}+iv_{2,4}, && e_2.v_{3,4}=iv_{1,3}-v_{2,3},
\end{aligned} \tag{46}$$

where e_1 and e_2 are defined by (21) *and* (22).

Proof. We will repeatedly use Lemma 7.1.

- Equation (40). We consider Equation (38) with $I=a$:

$$\sum_{i<j}\Big[(\xi_a\star\eta_{(i,j)^c})\otimes t.v_{i,j}+\sum_{l=1}^{4}\partial_l(\xi_{al}\star\eta_{(i,j)^c})\otimes v_{i,j}+\sum_{l\neq a}(\xi_l\star\eta_{(i,j)^c})\otimes\xi_{al}.v_{i,j}\Big]+\partial_a\eta_*\otimes v_*=0, \tag{47}$$

and, considering the terms in $\eta_{(a)^c}$, we obtain:

$$\begin{aligned}
0&=\sum_{l<a}(\xi_l\star\eta_{(l,a)^c})\otimes\xi_{al}.v_{l,a}+\sum_{l>a}(\xi_l\star\eta_{(a,l)^c})\otimes\xi_{al}.v_{a,l}+\partial_a\eta_*\otimes v_*\\
&=\sum_{l\neq a}(-1)^l\eta_{(a)^c}\otimes\xi_{l,a}.v_{l,a}+(-1)^{a-1}\eta_{(a)^c}\otimes v_*.
\end{aligned}$$

and Equation (40) follows.

- Equation (41). Consider the terms in $\eta_{(b)^c}$ in Equation (47):

$$\begin{aligned}
0&=\eta_a\eta_{(a,b)^c}\otimes t.v_{a,b}-\eta_a\eta_{(a,b)^c}\otimes v_{a,b}-\sum_{a\neq l,\,l<b}(\xi_l\star\eta_{(l,b)^c})\otimes\xi_{la}.v_{l,b}-\sum_{a\neq l,\,l>b}(\xi_l\star\eta_{(b,l)^c})\otimes\xi_{la}.v_{b,l}\\
&=(-1)^{a-1}\eta_{(b)^c}\otimes t.v_{a,b}-(-1)^{a-1}\eta_{(b)^c}\otimes v_{a,b}-\sum_{a\neq l,\,l<b}(-1)^{l-1}\eta_{(b)^c}\otimes\xi_{la}.v_{l,b}\\
&\quad-\sum_{a\neq l,\,l>b}(-1)^{l}\eta_{(b)^c}\otimes\xi_{la}.v_{b,l}\\
&=(-1)^{a-1}\eta_{(b)^c}\otimes t.v_{a,b}-(-1)^{a-1}\eta_{(b)^c}\otimes v_{a,b}+\sum_{l\neq a,b}(-1)^{l}\eta_{(b)^c}\otimes\xi_{la}.v_{l,b},
\end{aligned}$$

and Equation (41) follows.

• Equation (42). We consider Equation (38) with $I = ab$ and we assume $c < d$ with no loss of generality. We have

$$\begin{aligned}0 = &- \eta_* \otimes \xi_{ab}.v_* - (-1)^{a+b}\eta_* \otimes t.v_{a,b} \\ &+ \sum_{i<j}\sum_{l\neq k}(\partial_l \xi_{abk} \star \eta_{(i,j)^c}) \otimes \xi_{kl}.v_{i,j} + \varepsilon_{(a,b)}(\xi_{cd} \star \eta_{(c,d)^c}) \otimes Cv_{c,d}.\end{aligned}$$

The coefficient of $-\eta_*$ in this expression is:

$$0 = \xi_{ab}.v_* + (-1)^{a+b}t.v_{a,b} - \sum_{j\neq a,b}(-1)^{b+j}\xi_{ja}.v_{j,b} + \sum_{j\neq a,b}(-1)^{a+j}\xi_{jb}.v_{j,a} - \varepsilon_{(a,b)}\varepsilon_{(c,d)}Cv_{c,d}.$$

• Equation (43). This follows immediately by Equation (37).
• Equation (44). Equation (39) for $I = abc$ provides

$$0 = \sum_{i<j}\Big(\sum_{l=1}^{4}(\partial_l \xi_{abc} \star \partial_l \eta_{(i,j)^c}) \otimes v_{i,j} + \sum_{r<s}(\partial_{rs}\xi_{abc} \star \eta_{(i,j)^c}) \otimes \xi_{rs}.v_{i,j} + \varepsilon_{(a,b,c)}(\xi_d \star \eta_{(i,j)^c}) \otimes Cv_{i,j}\Big).$$

Considering the coefficients of $(-1)^a\eta_{(a)^c}$ we have:

$$(-1)^{b+c}v_{b,c} - (-1)^{a+c}\xi_{ab}.v_{a,c} + (-1)^{a+b}\xi_{ac}.v_{a,b} + \varepsilon_{(a,b,c)}(-1)^{a+d}Cv_{a,d} = 0.$$

• Equation (45). Equation (39) for $I = 1234$ is:

$$\begin{aligned}0 &= -\sum_{i<j}\sum_{l=1}^{4}(\partial_l \xi_{1234} \star \partial_l \eta_{(i,j)^c}) \otimes v_{i,j} + \sum_{i<j}\sum_{r<s}(\partial_{rs}\xi_{1234} \star \eta_{(i,j)^c}) \otimes \xi_{rs}.v_{i,j} - \eta_* \otimes Cv_* \\ &= -\eta_* \otimes \sum_{i<j}\xi_{(i,j)^c} \otimes v_{i,j} - \eta_* \otimes Cv_*.\end{aligned}$$

• Equations (46). These equations are a consequence of **S0**, i.e. $e_1.\vec{m} = e_2.\vec{m} = 0$. Recall that $T(e_1.\vec{m}) = T((-\xi_{13} + i\xi_{23}).\vec{m}) = -(T(\xi_{13\lambda}\vec{m}))_{|\lambda=0} + i(T(\xi_{23\lambda}\vec{m}))_{|\lambda=0}$ and so it can be easily computed by means of Proposition 5.10. We obtain

$$\begin{aligned}0 &= T(e_1.\vec{m}) \\ &= -\eta_* \otimes e_1.v_* - (\eta_1 - i\eta_2)\eta_4 \otimes v_{1,2} - \eta_3\eta_4 \otimes e_1.v_{1,2} - i\eta_3\eta_4 \otimes v_{1,3} - \eta_2\eta_4 \otimes e_1.v_{1,3} \\ &\quad + \eta_1\eta_2 \otimes v_{1,4} - \eta_2\eta_3 \otimes e_1.v_{1,4} + \eta_3\eta_4 \otimes v_{2,3} - \eta_1\eta_4 \otimes e_1.v_{2,3} \\ &\quad + i\eta_1\eta_2 \otimes v_{2,4} - \eta_1\eta_3 \otimes e_1.v_{2,4} + (-\eta_2\eta_3 - i\eta_1\eta_3) \otimes v_{3,4} - \eta_1\eta_2 \otimes e_1.v_{3,4} \\ &= -\eta_* \otimes e_1.v_* + \eta_1\eta_2 \otimes (v_{1,4} + iv_{2,4} - e_1.v_{3,4}) + \eta_1\eta_3 \otimes (-e_1.v_{2,4} - iv_{3,4}) \\ &\quad + \eta_1\eta_4 \otimes (-v_{1,2} - e_1.v_{2,3}) + \eta_2\eta_3 \otimes (-e_1.v_{1,4} - v_{3,4}) \\ &\quad + \eta_2\eta_4 \otimes (-e_1.v_{1,3} + iv_{1,2}) + \eta_3\eta_4 \otimes (-e_1.v_{1,2} - iv_{1,3} + v_{2,3}).\end{aligned}$$

From the previous equation we obtain relations (46) for e_1.
Equations (46) for e_2 are obtained similarly.

□

Lemma 7.3. *If $\vec{m}$ is a singular vector of degree 2 such that $T(\vec{m})$ is as in (34) then with $v_* = 0$.*

Proof. Let $T(\vec{m}) \in \mathrm{Ind}(F)$ be as in formula (34). We show that relations of Lemma 7.2 lead to $v_* = 0$. Let $a, b \in \{1, 2, 3, 4\}$ with $a < b$ and $(a, b)^c = (c, d)$.
Considering Equation (41) and the same equation for reversed role of a and b we can deduce

$$0 = -(-1)^{a+b}2t.v_{a,b} + 2(-1)^{a+b}v_{a,b} - \sum_{j\neq a,b}(-1)^{b+j}\xi_{aj}.v_{j,b} + \sum_{j\neq a,b}(-1)^{a+j}\xi_{bj}.v_{j,a}.$$

We compare this with Equation (42) and obtain

$$\xi_{ab}.v_* = (-1)^{a+b}t.v_{a,b} + 2(-1)^{a+b+1}v_{a,b} + Cv_{c,d}, \tag{48}$$

since for $a < b$ we have that $\varepsilon_{(a,b)}\varepsilon_{(c,d)} = 1$.
Now consider Equation (42):

$$0 = \xi_{ab}.v_* + (-1)^{a+b}t.v_{a,b} + \sum_{j\neq a,b}[(-1)^{b+j}\xi_{aj}.v_{jb} - (-1)^{a+j}\xi_{bj}.v_{j,a}] - Cv_{c,d}. \tag{49}$$

We also consider (44) with $a = j$, $b = a$, $c = b$ and $d = h$ and we substitute it into (49); we obtain

$$\begin{aligned}\xi_{ab}.v_* =&(-1)^{a+b+1}t.v_{a,b} + 2(-1)^{a+b}v_{a,b} + \sum_{j\neq a,b}\varepsilon_{(j,a,b)}(-1)^{h+j}Cv_{j,h} + Cv_{c,d} \\ =&(-1)^{a+b+1}t.v_{a,b} + 2(-1)^{a+b}v_{a,b} + \sum_{j<a \text{ or } j>b}(-1)^{j}Cv_{j,h} + \sum_{a<j<b}(-1)^{j+1}Cv_{j,h} + Cv_{c,d}.\end{aligned} \tag{50}$$

Combining (50) and (48), we get:

$$(-1)^{a+b}2t.v_{a,b} = 4(-1)^{a+b}v_{a,b} + \sum_{j<a \text{ or } j>b}(-1)^{j}Cv_{j,h} + \sum_{a<j<b}(-1)^{j+1}Cv_{j,h}.$$

Comparing this equation with (50) we obtain

$$2\xi_{ab}.v_* = \sum_{j<a \text{ or } j>b}(-1)^{j}Cv_{j,h} + \sum_{a<j<b}(-1)^{j+1}Cv_{j,h} + 2Cv_{c,d},$$

which simplifies to

$$\xi_{a,b}.v_* = 0 \tag{51}$$

for every $a < b$. This implies that, if $v_* \neq 0$, then $F = \langle v_* \rangle$ has dimension 1 and $\mathfrak{so}(4)$ acts trivially on it. Moreover all the $v_{a,b}$'s are scalar multiple of v_* since $F = \langle v_* \rangle$.
By (40) we also have $v_* = \sum_{j\neq a}(-1)^{a+j}\xi_{ja}.v_{j,a}$ for every $1 \leq a \leq 4$; then, since all the $v_{a,b}$'s are multiple of v_*, we have a contradiction. □

By Lemma 7.3 we know that if $\vec{(m)}$ is a singular vector of degree 2 $T(\vec{m})$ has the following form

$$T(\vec{m}) = \sum_{i<j}\eta_{(i,j)^c} \otimes v_{i,j}. \tag{52}$$

Remark 7.4. Relations (46), by Lemma 7.3 and notation (36) are equivalent to the following:

$$\begin{aligned} e_1.w_1 &= -w_5 - w_6, & e_2.w_1 &= w_5 + w_6, \\ e_1.w_2 &= w_5 - w_6, & e_2.w_2 &= w_5 - w_6, \\ e_1.w_3 &= 0, & e_2.w_3 &= 0, \end{aligned} \tag{53}$$

$$
\begin{aligned}
e_1.w_4 &= 0, & e_2.w_4 &= 0,\\
e_1.w_5 &= w_3 - w_4, & e_2.w_5 &= w_3 + w_4,\\
e_1.w_6 &= -w_3 - w_4, & e_2.w_6 &= -w_3 + w_4.
\end{aligned}
$$

We represent these relations in the following diagrams

$$
\begin{tikzcd}
\langle w_1\rangle \arrow[r, bend left, "e_1"] \arrow[r, bend right, "e_2"'] & \langle w_5+w_6\rangle \arrow[r, bend left, "e_1"] \arrow[r, bend right, "e_2"'] & \langle w_4\rangle,
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle w_2\rangle \arrow[r, bend left, "e_1"] \arrow[r, bend right, "e_2"'] & \langle w_5-w_6\rangle \arrow[r, bend left, "e_1"] \arrow[r, bend right, "e_2"'] & \langle w_3\rangle.
\end{tikzcd}
$$

Proof of Theorem 6.7. Throughout this proof we let $\mu = (m, n, \mu_0, \mu_1)$ where m, n, μ_0, μ_1 denote the highest weights of F with respect to h_x, h_y, t, C respectively. We split the proof in four cases that we number by 1), 2), 3), 4).

1) Let $w_5 = w_6 = 0$.
We immediately have also $w_3 = w_4 = 0$ by (53).

1a) Let $w_1 \neq 0$ and $w_2 = 0$.
By (53), we have that w_1 is a highest weight vector and, by (36),

$$v_{1,2} = v_{3,4} = 0,\ v_{1,3} = w_1,\ v_{1,4} = iw_1,\ v_{2,3} = iw_1,\ v_{2,4} = -w_1.$$

Equation (41) for $a = 1, b = 3$ gives $(t - i\xi_{12} - 1).w_1 = 0$, Equation (44) for $a = 1, b = 2, c = 3$ gives $(C - i\xi_{12} + 1).w_1 = 0$, and Equation (44) for $a = 3, b = 1, c = 4$ gives $(C - i\xi_{34} + 1).w_1 = 0$.
Recalling that $h_x = -i\xi_{12} + i\xi_{34}$ and $h_y = -i\xi_{12} - i\xi_{34}$ we deduce that $\mu = (0, n, 1 - \frac{n}{2}, -1 - \frac{n}{2})$ for some $n \in \mathbb{Z}_{\geq 0}$.
A simple verification shows that these conditions lead to the vector

$$\vec{m}_{2a} = w_{11}w_{21} \otimes y_1^n,$$

in $M(0, n, 1 - \frac{n}{2}, -1 - \frac{n}{2})$ for $n \in \mathbb{Z}_{\geq 0}$ which is indeed a singular vector.

1b) Let $w_1 = 0$ and $w_2 \neq 0$.
By (53) we have that w_2 is a highest weight vector and, by (36),

$$v_{1,2} = v_{3,4} = 0,\ v_{1,3} = -w_2,\ v_{1,4} = iw_2,\ v_{2,3} = -iw_2,\ v_{2,4} = -w_2,\ v_{3,4} = 0.$$

Equation (41) for $a = 1, b = 3$ gives $(t - i\xi_{12} - 1).w_2 = 0$, Equation (44) for $a = 1, b = 2, c = 3$ gives $(C + i\xi_{12} - 1).w_2 = 0$, and Equation (44) for $a = 3, b = 1, c = 4$ gives $(-C + i\xi_{34} + 1).w_2 = 0$.
From these conditions, recalling that $h_x = -i\xi_{12} + i\xi_{34}$ and $h_y = -i\xi_{12} - i\xi_{34}$, we deduce that $\mu = (m, 0, 1 - \frac{m}{2}, 1 + \frac{m}{2})$ with $m \in \mathbb{Z}_{\geq 0}$ and we obtain the singular vector

$$\vec{m}_{2b} = w_{11}w_{12} \otimes x_1^m,$$

in $M(m, 0, 1 - \frac{m}{2}, 1 + \frac{m}{2})$ with $m \in \mathbb{Z}_{\geq 0}$.

1c) Let $w_1 \neq 0$ and $w_2 \neq 0$.
By (53), we have that both w_1 and w_2 are highest weight vectors of F, so that $w_1 = \alpha w_2$ for some $\alpha \neq 0$. By Equation (44) for $a = 3, b = 2, c = 4$ and for

$a = 4, b = 1, c = 3$ we obtain respectively

$$(-\alpha - 1 - i\xi_{34}(-\alpha + 1) + C(-\alpha + 1)).w_2 = 0$$

and

$$(\alpha - 1 - i\xi_{34}(\alpha + 1) + C(\alpha + 1)).w_2 = 0.$$

The sum and the difference of these two equations show an evident contradiction.

2) Let $w_5 \neq 0$ and $w_5 + w_6 = 0$.

Since $e_1.w_5 = w_3 - w_4$ and $e_1.w_6 = -w_3 - w_4$ we deduce that $w_4 = 0$; we also know that $w_2 \neq 0$ since $e_1.w_2 = 2w_5$.

2a) Let $w_1 = 0$ and $w_3 \neq 0$. By Remark 7.4 w_3 is a highest weight vector and Equations (36) provide

$$v_{1,2} = 2iw_5,\ v_{1,3} = -w_2 - w_3, v_{1,4} = iw_2 - iw_3,$$
$$v_{2,3} = -iw_2 + iw_3,\ v_{2,4} = -w_2 - w_3,\ v_{3,4} = -2iw_5.$$

Let us compute the weight of w_2 and w_3.

Equation (41) for $a = 1, b = 3$ gives $t.(-w_2 - w_3) + w_2 + w_3 - \xi_{1,2}.(-iw_2 + iw_3) + \xi_{14}.(-2iw_5) = 0$, and for $a = 2, b = 3$ gives $t.(-iw_2 + iw_3) + iw_2 - iw_3 + \xi_{12}.(-w_2 - w_3) - \xi_{24}.(-2iw_5) = 0$.

Recalling the definition of e_2 in (22), we deduce from these equations that $2it.w_3 - 2iw_3 - 2\xi_{12}.w_3 - 2ie_2.w_5 = 0$ that is equivalent to

$$(t + i\xi_{12} - 2).w_3 = 0.$$

Equation (44) for $a = 1, b = 2, c = 4$ gives $-w_2 - w_3 + \xi_{12}.(iw_2 - iw_3) - \xi_{14}.2iw_5 - C(-w_2 - w_3) = 0$, and for $a = 2, b = 1, c = 4$ gives $-iw_2 + iw_3 + \xi_{12}.(-w_2 - w_3) + \xi_{24}.(2iw_5) - C(-iw_2 + iw_3) = 0$.

By these equations we obtain $-2w_3 - 2i\xi_{12}.w_3 + 2e_2.w_5 + 2Cw_3 = 0$ that is equivalent to

$$(-i\xi_{1,2} + C).w_3 = 0.$$

Equation (44) for $a = 3, b = 1, c = 4$ gives $-i(w_2 - w_3) + 2i\xi_{13}.w_5 + \xi_{34}.(w_2 + w_3) + iC(w_2 - w_3) = 0$, and for $a = 3, b = 2, c = 4$ gives $-w_2 - w_3 - \xi_{2,3}.(-2iw_5) + \xi_{3,4}.(-iw_2 + iw_3) - C(-w_2 - w_3) = 0$.

By these equations we obtain

$$(i\xi_{3,4} + C).w_3 = 0.$$

Hence, we conclude that $\mu = (m, 0, \frac{m}{2} + 2, -\frac{m}{2})$ for some $m \geq 0$ and since $\dim F \geq 3$ (since, e.g., w_2, w_3 and w_5 are linearly independent) we also have $m \geq 2$. All these conditions lead to

$$\vec{m}_{2c} = w_{22}w_{21} \otimes x_1^m + (w_{11}w_{22} + w_{21}w_{12}) \otimes x_1^{m-1}x_2 - w_{11}w_{12} \otimes x_1^{m-2}x_2^2,$$

in $M(m, 0, \frac{m}{2} + 2, -\frac{m}{2})$ with $m \geq 2$, which is indeed a singular vector.

2b) Let $w_1 \neq 0$ or $w_3 = 0$. We show that in this case necessarily $C = 0$ so the $\lambda-$action of Proposition 5.10 reduces to the action found in Theorem 4.3 of [1]; in that case it was shown that there are no singular vectors of degree 2.

Equation (44) for $a = 1, b = 3, c = 4$ gives $2iw_5 + \xi_{13}.(iw_1 + iw_2 - iw_3) + \xi_{14}.(w_1 - w_2 - w_3) - 2iCw_5 = 0$.

Equation (44) for $a = 2, b = 3, c = 4$ gives $2iw_5 + \xi_{23}.(w_1 + w_2 + w_3) - \xi_{24}.(iw_1 -$

$iw_2 + iw_3) - 2iCw_5 = 0$.
We take the sum of these equations and get:

$$\begin{aligned} 0 &= 4iw_5 - ie_1.w_1 - ie_1.w_2 + ie_2.w_1 - ie_2.w_2 - 2if_x.w_3 - 4iCw_5 \\ &= -2if_x.w_3 - 4iCw_5. \end{aligned}$$

If $w_3 = 0$ we can conclude $C = 0$ so we can assume $w_3 \neq 0$ and $w_1 \neq 0$. We observe that w_3 and w_1 are highest weight vectors so that they are scalar multiples of each other. If we take the difference of the two equations above we get $f_x.w_1 = 0$ and so we have $C = 0$ also in this case.

3) Let $w_5 = w_6 \neq 0$.
This condition implies $w_1 \neq 0$ since $e_1.w_1 = -w_5 - w_6$, and $w_3 = 0$ since $e_1(w_5 - w_6) = 2w_3$.

3a) Let $w_2 = 0$ and $w_4 \neq 0$. By Remark 7.4 w_4 is a highest weight vector and Equations (36) reduce to:

$$\begin{aligned} &v_{1,2} = 2iw_5,\ v_{1,3} = w_1 + w_4,\ v_{1,4} = iw_1 - iw_4, \\ &v_{2,3} = iw_1 - iw_4,\ v_{2,4} = -w_1 - w_4,\ v_{3,4} = 2iw_5. \end{aligned}$$

Let us compute the weight of w_4.
Equation (41) for $a = 1$, $b = 3$ gives $t.(w_1 + w_4) - w_1 - w_4 - i\xi_{12}.(w_1 - w_4) + 2i\xi_{14}.w_5 = 0$, and for $a = 2, b = 3$ gives $it.(w_1 - w_4) - i(w_1 - w_4) + \xi_{12}.(w_1 + w_4) - 2i\xi_{24}.w_5 = 0$.
These two equations provide

$$(t + i\xi_{12} - 2).w_4 = 0.$$

Equation(44) for $a = 1, b = 2, c = 4$ gives $-w_1 - w_4 + i\xi_{12}.(w_1 - w_4) - 2i\xi_{14}.w_5 - C(w_1 + w_4) = 0$, and for $a = 2, b = 1, c = 4$ gives $i(-w_1 + w_4) - \xi_{12}.(w_1 + w_4) + 2i\xi_{24}.w_5 - iC(w_1 - w_4) = 0$.
These two equations provide

$$(i\xi_{12} + C).w_4 = 0.$$

Equation (44) for $a = 3, b = 1, c = 4$ gives $-i(w_1 - w_4) - 2i\xi_{13}.w_5 - \xi_{34}.(w_1 + w_4) - iC(w_1 - w_4) = 0$ and for for $a = 3, b = 2, c = 4$ gives $-w_1 - w_4 - 2i\xi_{23}.w_5 + i\xi_{34}.(w_1 - w_4) - C(w_1 + w_4) = 0$.
These two equations provide

$$(i\xi_{34} + C).w_4 = 0.$$

We conclude that $\mu = (0, n, \frac{n}{2} + 2, \frac{n}{2})$ for some $n \geq 0$. Moreover, since w_1, w_5 and w_4 are linearly independent we have $\dim F \geq 3$ and so $n \geq 2$. All this conditions lead to the vector

$$\vec{m}_{2d} = w_{22}w_{12} \otimes y_1^n - (w_{22}w_{11} + w_{21}w_{12}) \otimes y_1^{n-1}y_2 - w_{11}w_{21} \otimes y_1^{n-2}x_2^2,$$

in $M(0, n, \frac{n}{2} + 2, \frac{n}{2})$ with $n \geq 2$ which is indeed a singular vector.

3b) Let $w_2 \neq 0$ or $w_4 = 0$.
We show that in this case necessarily $C = 0$, so the $\lambda-$action of Proposition 5.10 reduces to the action found in Theorem 4.3 of [1] and we already know that there are no singular vectors of degree 2.

Equation (44) for $a=1, b=3, c=4$ gives $-2iw_5+i\xi_{13}.(w_1+w_2-w_4)+\xi_{14}.(w_1-w_2+w_4)-2iCw_5=0$, and for $a=2, b=3, c=4$ gives $-2iw_5+\xi_{23}.(w_1+w_2+w_4)-i\xi_{24}.(w_1-w_2-w_4)-2iCw_5=0$.
Taking the sum of these equations we obtain

$$\begin{aligned}0 &= -4iw_5 - ie_1.w_1 + ie_2.w_1 - ie_1.w_2 \\ &\quad - ie_2.w_2 - i(\xi_{13}-\xi_{24}+i\xi_{14}+i\xi_{23}).w_4 - 4iCw_5 \\ &= -2if_y.w_4 - 4iCw_5.\end{aligned}$$

Therefore if $w_4=0$ we can conclude $C=0$. If $w_4\neq 0$ and so $w_2\neq 0$ and both w_2 and w_4 are highest weight vectors. We take the difference of the previous equations and we obtain:

$$\begin{aligned}0 &= i(\xi_{13}+i\xi_{23}+\xi_{24}-i\xi_{14}).w_1 + i(\xi_{13}+i\xi_{23}-\xi_{24}+i\xi_{14}).w_2 \\ &\quad - i(\xi_{13}+\xi_{24}+i\xi_{14}-i\xi_{2,3}).w_4 \\ &= 2if_x.w_1 + 2if_y.w_2 + i(e_1+e_2).w_4 \\ &= 2if_x.w_1 + 2if_y.w_2.\end{aligned}$$

Since w_2 is a highest weight vector and w_1 is not, these two terms are both 0. In particular, since w_4 is a scalar multiple of w_2 we have that $f_y.w_4=0$ and we conclude $C=0$ by a previous equation.

4) Let $w_5\neq \pm w_6$.
We show that in this case necessarily $C=0$, so the $\lambda-$action of Proposition 5.10 reduces to the action found in Theorem 4.3 of [1] and we already know that there are no singular vectors of degree 2.
Equation (44) for $a=1, b=3, c=4$ gives

$$-2iw_6+i\xi_{13}.(w_1+w_2-w_3-w_4)+\xi_{14}.(w_1-w_2-w_3+w_4)-2iCw_5=0,$$

and for $a=2, b=3, c=4$ gives

$$-2iw_6+\xi_{23}.(w_1+w_2+w_3+w_4)-i\xi_{24}.(w_1-w_2+w_3-w_4)-2iCw_5=0.$$

These equations provide

$$2Cw_5+f_x.w_3+f_y.w_4=0. \tag{54}$$

Equation (44) for $a=4, b=1, c=2$ gives

$$-2iw_5+\xi_{14}.(w_1+w_2+w_3+w_4)-i\xi_{24}.(w_1+w_2-w_3-w_4)-2iCw_6=0,$$

and for $a=3, b=1, c=2$ gives

$$-2iw_5+i\xi_{13}.(w_1-w_2+w_3-w_4)+\xi_{23}.(w_1-w_2-w_3+w_4)-2iCw_6=0.$$

These equations provide

$$2Cw_6+f_x.w_3-f_y.w_4=0 \tag{55}$$

If $w_3=0$ or $w_4=0$ we immediate deduce by (54) and (55) that $C=0$, since $w_5\neq\pm w_6$. If $w_3\neq 0$ and $w_4\neq 0$ they are both highest weight vectors. Applying $e_1=e_x+e_y$ to (54) we obtain

$$2C(w_3-w_4)+h_x.w_3+h_y.w_4=0$$

and applying $e_2 = e_x - e_y$ to (54) we obtain

$$2C(w_3 + w_4) + h_x.w_3 - h_y.w_4 = 0.$$

From these equations we deduce $2C + m = 0$ and so $C \leq 0$, and $-2C + n = 0$ and so $C \geq 0$.

$\square$

8. Singular vectors of degree 3

The aim of this section is to classify all singular vectors of degree 3.
We have that a singular vector $\vec{m}$ of degree 3 is such that:

$$T(\vec{m}) = \Theta \sum_i \eta_{(i)^c} \otimes v_{i,1} + \sum_i \eta_i \otimes v_{i,0}. \tag{56}$$

We write the vector $\vec{m}$ also in the following way

$$\begin{aligned}
\vec{m} =&(\eta_2 + i\eta_1)(\eta_2 - i\eta_1)(\eta_4 + i\eta_3) \otimes w_1 + (\eta_2 + i\eta_1)(\eta_2 - i\eta_1)(\eta_4 - i\eta_3) \otimes w_2 + \\
&(\eta_4 + i\eta_3)(\eta_4 - i\eta_3)(\eta_2 + i\eta_1) \otimes w_3 + (\eta_4 + i\eta_3)(\eta_4 - i\eta_3)(\eta_2 - i\eta_1) \otimes w_4 + \\
&\Theta(\eta_2 + i\eta_1) \otimes w_5 + \Theta(\eta_2 - i\eta_1) \otimes w_6 + \Theta(\eta_4 + i\eta_3) \otimes w_7 + \Theta(\eta_4 - i\eta_3) \otimes w_8 \\
=&(2\Theta i\eta_3 + 2\Theta\eta_4 - 2\eta_1\eta_2\eta_3 + 2i\eta_1\eta_2\eta_4) \otimes w_1 + (-2i\Theta\eta_3 + 2\Theta\eta_4 + 2\eta_1\eta_2\eta_3 + 2i\eta_1\eta_2\eta_4) \otimes w_2 + \\
&(2i\Theta\eta_1 + 2\Theta\eta_2 - 2\eta_1\eta_3\eta_4 + 2i\eta_2\eta_3\eta_4) \otimes w_3 + (-2i\Theta\eta_1 + 2\Theta\eta_2 + 2\eta_1\eta_3\eta_4 + 2i\eta_2\eta_3\eta_4) \otimes w_4 + \\
&\Theta(\eta_2 + i\eta_1) \otimes w_5 + \Theta(\eta_2 - i\eta_1) \otimes w_6 + \Theta(\eta_4 + i\eta_3) \otimes w_7 + \Theta(\eta_4 - i\eta_3) \otimes w_8.
\end{aligned} \tag{57}$$

From these two expressions it follows that

$$\begin{aligned}
v_{1,0} &= 2iw_3 + 2iw_4, \\
v_{2,0} &= 2w_3 - 2w_4, \\
v_{3,0} &= 2iw_1 + 2iw_2, \\
v_{4,0} &= 2w_1 - 2w_2, \\
v_{1,1} &= -2iw_3 + 2iw_4 - iw_5 + iw_6, \\
v_{2,1} &= 2w_3 + 2w_4 + w_5 + w_6, \\
v_{3,1} &= -2iw_1 + 2iw_2 - iw_7 + iw_8, \\
v_{4,1} &= 2w_1 + 2w_2 + w_7 + w_8.
\end{aligned} \tag{58}$$

Indeed, let us show for example one of the previous equations. In (56), let us consider $\eta_2 \otimes v_{2,0}$. We have that η_2 is the Hodge dual of $-\eta_{134}$. In (57), the terms in η_{134} are:

$$-2\eta_{134} \otimes w_3 + 2\eta_{134} \otimes w_4,$$

hence $v_{2,0} = 2w_3 - 2w_4$. Analogously for $v_{1,0}, v_{3,0}$ and $v_{4,0}$. Moreover in (56), let us consider, for example, $\Theta\eta_{(1)^c} \otimes v_{1,1} = \Theta\eta_{234} \otimes v_{1,1}$. We have that $\Theta\eta_{234}$ is the Hodge dual of $-\Theta\eta_1$. In (57), the terms in $\Theta\eta_1$ are:

$$2i\Theta\eta_1 \otimes w_3 - 2i\Theta\eta_1 \otimes w_4 + i\Theta\eta_1 \otimes w_5 - i\Theta\eta_1 \otimes w_6,$$

hence $v_{1,1} = -2iw_3 + 2iw_4 - iw_5 + iw_6$. Analogously for $v_{2,1}, v_{3,1}$ and $v_{4,1}$.
In the following lemma we write explicitly some of the relations of Proposition 6.19 for a vector as in formula (56).

Lemma 8.1. *Let $\vec{m} \in \operatorname{Ind} F$ be a singular vector such that $T(\vec{m})$ is as in formula* (56). *Then*
1) *For all $b \in \{1,2,3,4\}$ we have*

$$0 = \sum_i \Big[\sum_{l<k} (\xi_{blk} \star \eta_i) \otimes \xi_{lk}.v_{i,0} - \varepsilon_b(\xi_{(b)^c} \star \eta_i) \otimes Cv_{i,0} + (\xi_b \star \eta_{(i)^c}) \otimes t.v_{i,1} \tag{59}$$
$$+ \sum_{l \neq k} (\partial_l \xi_{bk} \star \eta_{(i)^c}) \otimes \xi_{lk}.v_{i,1} \Big].$$

2) *For all $s \in \{1,2,3,4\}$ we have*

$$0 = \sum_i \Big[(\xi_s \star \eta_i) \otimes t.v_{i,0} + \sum_{l=1}^{4} \partial_l(\xi_{sl} \star \eta_i) \otimes v_{i,0} + \sum_{l \neq k} (\partial_l \xi_{sk} \star \eta_i) \otimes \xi_{lk}.v_{i,0} \tag{60}$$
$$+ \sum_{l=1}^{4} (\partial_l \xi_s \star \partial_l \eta_{(i)^c}) \otimes v_{i,1} \Big],$$

For all $b \in \{1,2,3,4\}$ we have

$$0 = \sum_i \Big[(\xi_b \star \eta_{(i)^c}) \otimes t.v_{i,1} + \sum_{l \neq k} (\partial_l \xi_{bk} \star \eta_{(i)^c}) \otimes \xi_{lk}.v_{i,1} - (\xi_b \star \eta_{(i)^c}) \otimes v_{i,1} \Big]. \tag{61}$$

3) *For all I such that $|I| = 3$ we have*

$$0 = \sum_i \Big[\partial_i \eta_I \otimes v_{i,0} + \sum_{r<s} (\partial_{rs} \xi_I \star \eta_i) \otimes \xi_{rs}.v_{i,0} + \varepsilon_I (\xi_{I^c} \star \eta_i) \otimes Cv_{i,0} \Big]. \tag{62}$$

For all I such that $|I| = 4$ we have

$$0 = \sum_i \Big[-\partial_i \eta_I \otimes v_{i,0} + \sum_{r<s} (\partial_{rs} f \star \eta_i) \otimes \xi_{rs}.v_{i,0} - \varepsilon_I \eta_{(i)^c} \otimes Cv_{i,1} \Big]. \tag{63}$$

Proof. These are particular cases of Proposition 6.19. In particular we have (59) is $C_0(b) + B_1(b) = 0$, (60) is $B_0(s) + b_1(s) = 0$, (61) is $B_1(b) + a_1(b) = 0$, (62) is $b_0(I) + G_1(I) = 0$ for $|I| = 3$ and (63) is $b_0(I) + G_1(I) = 0$ for $|I| = 4$. □

Lemma 8.2. *Let $\vec{m} \in \operatorname{Ind} F$ be a highest weight singular vector such that $T(\vec{m})$ is as in formula* (56). *Then for every (a,b,c,d) permutation of $\{1,2,3,4\}$ we have*

$$v_{a,1} = (-1)^{a+1} 2Cv_{a,0}. \tag{64}$$
$$t.v_{a,0} - 2v_{a,0} + \xi_{ab}.v_{b,0} = 0 \tag{65}$$
$$v_{c,0} + \xi_{ca}.v_{a,0} + \xi_{cb}.v_{b,0} = 0, \tag{66}$$
$$\xi_{bc}.v_{d,0} + \varepsilon_{(a,b,c)} Cv_{a,0} = 0. \tag{67}$$

Moreover C (resp. t) acts as multiplication by $\pm\frac{1}{2}$ (resp. $\frac{5}{2}$) on F.

Finally:

$$e_1.v_{1,0} = -v_{3,0}, \qquad e_2.v_{1,0} = -iv_{4,0}, \tag{68}$$

$$
\begin{aligned}
&e_1.v_{2,0} = iv_{3,0}, && e_2.v_{2,0} = -v_{4,0},\\
&e_1.v_{3,0} = v_{1,0} - iv_{2,0}, && e_2.v_{3,0} = 0,\\
&e_1.v_{4,0} = 0 && e_2.v_{4,0} = iv_{1,0} + v_{2,0},
\end{aligned}
$$

where e_1 and e_2 are defined by (21) *and* (22).

Proof. • Equation (64). We consider the difference between (59) and (61). We assume $(a,c,d) = (b)^c$. We have that:

$$
-(\xi_b \star \eta_{(b)^c}) \otimes v_{b,1} = \sum_{i=1}^{4} \sum_{l<k} (\xi_{blk} \star \eta_i) \otimes \xi_{lk}.v_{i,0} - \varepsilon_b(\xi_{acd} \star \eta_b) \otimes Cv_{b,0}.
$$

It is equivalent to:

$$
(\xi_b \star \eta_{(b)^c}) \otimes v_{b,1} = -\sum_{l<k} (\xi_{blk} \star \eta_{(b,l,k)^c}) \otimes \xi_{lk}.v_{(b,l,k)^c,0} - \varepsilon_b \eta_{bacd} \otimes Cv_{b,0}. \tag{69}
$$

Let us focus on Equation (60) for $s \neq b$. We have:

$$
0 = \sum_{i=1}^{4} \partial_s \eta_{(i)^c} \otimes v_{i,1} + \sum_{i=1}^{4} (\xi_s \star \eta_i) \otimes t.v_{i,0} + \sum_{i=1}^{4}\sum_{l=1}^{4} \partial_l(\xi_{sl} \star \eta_i) \otimes v_{i,0} + \sum_{i=1}^{4}\sum_{l\neq s} (\xi_l \star \eta_i) \otimes \xi_{sl}.v_{i,0}. \tag{70}
$$

The terms in $\eta_{(s,b)^c}$ of this equation are:

$$
\partial_s \eta_{(b)^c} \otimes v_{b,1} + \sum_{l\neq s,b} (\xi_l \star \eta_{(s,b,l)^c}) \otimes \xi_{sl}.v_{(s,b,l)^c,0} = 0.
$$

We take the sum over $s \neq b$ and, as in [1], using (69) we obtain:

$$
\begin{aligned}
0 =& \sum_{s\neq b} (\xi_s \star \partial_s \eta_{(b)^c}) \otimes v_{b,1} + \sum_{s\neq b}\sum_{l\neq s,b} (\xi_{sl} \star \eta_{(s,b,l)^c}) \otimes \xi_{sl}.v_{(s,b,l)^c,0}\\
=& 3\eta_{(b)^c} \otimes v_{b,1} + 2\sum_{s<l} (\xi_{sl} \star \eta_{(s,b,l)^c}) \otimes \xi_{sl}.v_{(s,b,l)^c,0}\\
=& \eta_{(b)^c} \otimes (v_{b,1} - 2\varepsilon_{(b)} Cv_{b,0}).
\end{aligned}
$$

Equation (64) follows.

• Equation (65). Given $r \neq s \in \{1,2,3,4\}$, the terms in η_{sr} of (70) are:

$$
\eta_{sr} \otimes t.v_{r,0} + \sum_{l\neq s,r} \partial_l(\xi_{sl} \star \eta_r) \otimes v_{r,0} - \eta_{sr} \otimes \xi_{sr}.v_{s,0} = 0.
$$

This condition is equivalent to:

$$
t.v_{r,0} - 2v_{r,0} - \xi_{sr}.v_{s,0} = 0
$$

which is Equation (65).

• Equations (66) and (67).

Let us analyze Equation (62) for $I = abc$. We obtain:

$$
\sum_{i=1}^{4}\sum_{l=1}^{4} (\partial_l \xi_{abc} \star \partial_l \eta_i) \otimes v_{i,0} + \sum_{r<s} (\partial_{rs} \xi_{abc} \star \eta_i) \otimes \xi_{rs}.v_{i,0} + \sum_{i} \varepsilon_{(a,b,c)} (\xi_d \star \eta_i) \otimes Cv_{i,0} = 0.
$$

Looking at the coefficients of η_{ab} and η_{ad} we obtain Equations (66) and (67):

$$v_{c,0} + \xi_{ca}.v_{a,0} + \xi_{cb}.v_{b,0} = 0,$$
$$\xi_{bc}.v_{d,0} + \varepsilon_{(a,b,c)} C v_{a,0} = 0.$$

• $C = \pm 1/2$ and $t = 5/2$.
Using (64), Equation (63) for $I = 1234$ is:

$$\begin{aligned}0 = &-\sum_{i=1}^{4}\sum_{l=1}^{4}(\partial_l \xi_{1234} \star \partial_l \eta_i) \otimes v_{i,0} + \sum_{r<s}\sum_{i=1}^{4}(\partial_{rs}\xi_{1234} \star \eta_i) \otimes \xi_{rs}.v_{i,0} - C\sum_i \eta_{(i)^c} \otimes v_{i,1}\\
=&\eta_{123} \otimes ((1+2C^2)v_{4,0} + \xi_{41}.v_{1,0} + \xi_{42}.v_{2,0} + \xi_{43}.v_{3,0})\\
&- \eta_{124} \otimes ((1+2C^2)v_{3,0} + \xi_{31}.v_{1,0} + \xi_{32}.v_{2,0} + \xi_{34}.v_{4,0})\\
&+ \eta_{134} \otimes ((1+2C^2)v_{2,0} + \xi_{21}.v_{1,0} + \xi_{23}.v_{3,0} + \xi_{24}.v_{4,0})\\
&- \eta_{234} \otimes ((1+2C^2)v_{1,0} + \xi_{12}.v_{2,0} + \xi_{13}.v_{3,0} + \xi_{14}.v_{4,0}).\end{aligned}$$

Therefore for every $a = 1, 2, 3, 4$ we have $(1 + 2C^2)v_{a,0} + \sum_{b \neq a} \xi_{ab}.v_{b,0} = 0$ and by Equation (65) we deduce $(7 + 2C^2 - 3t).v_{a,0} = 0$. This implies that t acts as $\frac{1}{3}(7 + 2C^2)$ on F.

Equation (66), for $a = 2, b = 3, c = 1$, is:

$$v_{1,0} + \xi_{13}.v_{3,0} + \xi_{12}.v_{2,0} = 0.$$

Using (65) and the fact that t acts as $\frac{1}{3}(7 + 2C^2)$, we get

$$\begin{aligned}0 =&v_{1,0} + \xi_{13}.v_{3,0} + \xi_{12}.v_{2,0}\\
=&v_{1,0} - 2\frac{1+2C^2}{3}v_{1,0} = \frac{1-4C^2}{3}v_{1,0}.\end{aligned}$$

From this we deduce that $C = \pm\frac{1}{2}$ and so t acts as $\frac{5}{2}$.

• Equations (68). The fact that $\vec{m}$ is annihilated by $e_1 = -\xi_{13} + i\xi_{23}$ provides:

$$\begin{aligned}0 = &-\sum_i\sum_{l=1}^{4}(\partial_l(-\xi_{13} + i\xi_{23}) \star \partial_l \eta_i) \otimes v_{i,0} + \sum_i\sum_{r<s}(\partial_{rs}(-\xi_{13} + i\xi_{23}) \star \eta_i) \otimes \xi_{rs}.v_{i,0}\\
=&\eta_3 \otimes v_{1,0} - \eta_1 \otimes e_1.v_{1,0} - i\eta_3 \otimes v_{2,0} - \eta_2 \otimes e_1.v_{2,0} + (-\eta_1 + i\eta_2) \otimes v_{3,0} - \eta_3 \otimes e_1.v_{3,0}\\
&- \eta_4 \otimes e_1.v_{4,0}\\
=&-\eta_1 \otimes (e_1.v_{1,0} + v_{3,0}) + \eta_2 \otimes (iv_{3,0} - e_1.v_{2,0}) + \eta_3 \otimes (v_{1,0} - iv_{2,0} - e_1.v_{3,0}) - \eta_4 \otimes e_1.v_{4,0}.\end{aligned}$$

Equations (68) for e_1 follow. Equations for e_2 are obtained similarly. □

Remark 8.3. Let us point out that relations (64) are equivalent to the following, using notation (58):

$$\begin{aligned}-2iw_3 + 2iw_4 - iw_5 + iw_6 &= 2C(2iw_3 + 2iw_4),\\
2w_3 + 2w_4 + w_5 + w_6 &= -2C(2w_3 - 2w_4),\\
-2iw_1 + 2iw_2 - iw_7 + iw_8 &= 2C(2iw_1 + 2iw_2),\\
2w_1 + 2w_2 + w_7 + w_8 &= -2C(2w_1 - 2w_2).\end{aligned}$$

Thus, we obtain:

$$w_5 = -(2 + 4C)w_3, \tag{71}$$

$$
\begin{aligned}
w_6 &= -(2-4C)w_4,\\
w_7 &= -(2+4C)w_1,\\
w_8 &= -(2-4C)w_2.
\end{aligned}
$$

Equations (68) are therefore equivalent to the following, using notation (58):

$$
\begin{aligned}
e_1.w_1 &= w_4, & e_2.w_1 &= -w_4,\\
e_1.w_2 &= w_4, & e_2.w_2 &= w_4,\\
e_1.w_3 &= -w_1 - w_2, & e_2.w_3 &= -w_1 + w_2,\\
e_1.w_4 &= 0, & e_2.w_4 &= 0.
\end{aligned} \tag{72}
$$

We represent these relations in the following diagram

$$
\begin{tikzcd}
 & \langle -w_1 - w_2 \rangle \arrow[dr, "e_1"] & \\
\langle w_3 \rangle \arrow[ur, "e_1"] \arrow[dr, "e_2"'] & & \langle w_4 \rangle. \\
 & \langle w_2 - w_1 \rangle \arrow[ur, "e_2"'] &
\end{tikzcd}
$$

We are now ready to prove the stated classification of singular vectors of degree 3.

Proof of Theorem 6.8. Let $\mu = (m, n, 5/2, C)$, with $C = \pm 1/2$, be the highest weight of F with respect with (h_x, h_y, t, C). We observe that $w_3 \neq 0$ otherwise $\vec{m} = 0$.

1) Let $w_4 = 0$.

1a) Let $w_2 = 0$ and $w_1 \neq 0$. By Equations (72), we have that w_1 is a highest weight vector. By (58) we have:

$$
\begin{aligned}
&v_{1,0} = 2iw_3,\ v_{2,0} = 2w_3,\ v_{3,0} = 2iw_1,\ v_{4,0} = 2w_1,\\
&v_{1,1} = 4iCw_3,\ v_{2,1} = -4Cw_3,\ v_{3,1} = 4iCw_1,\ v_{4,1} = -4Cw_1.
\end{aligned}
$$

Equation (65) for $a = 3, b = 4$ gives $-\xi_{34}.v_{3,0} = (2-t).v_{4,0}$ which is equivalent to

$$(i\xi_{34} - 1/2).w_1 = 0.$$

Equation (67) for $a = 3, b = 1, c = 2$ gives $-\xi_{12}.v_{4,0} - Cv_{3,0} = 0$ which is equivalent to

$$(-i\xi_{12} + C).w_1 = 0.$$

These equations imply $(h_y + C + 1/2).w_1 = 0$ and so $n + C + 1/2 = 0$ which implies $C = -1/2$ (since $n \geq 0$) and $n = 0$. Similarly the same equations imply $m + C - 1/2 = 0$ and so $m = 1$ and hence $\mu = (1, 0, 5/2, -1/2)$.
By Equations (72) we know that $2e_x.w_3 = e_1.w_3 + e_2.w_3 = -2w_1$. Hence $w_3 = -f_x.w_1$. All these conditions lead to the vector

$$\vec{m}_{3a} = w_{11}w_{22}w_{21} \otimes x_1 + w_{21}w_{12}w_{11} \otimes x_2 \in M(1, 0, 5/2, -1/2),$$

which is indeed a singular vector.

1b) Let $w_1 = 0$ and $w_2 \neq 0$. By Equations (72), we have that w_2 is a highest weight vector. From (58) we have:

$$v_{1,0} = 2iw_3,\ v_{2,0} = 2w_3,\ v_{3,0} = 2iw_2,\ v_{4,0} = -2w_2,$$
$$v_{1,1} = 4iCw_3,\ v_{2,1} = -4Cw_3,\ v_{3,1} = 4iCw_2,\ v_{4,1} = 4Cw_2.$$

Equation (65) for $a = 3, b = 4$ gives $-\xi_{34}.v_{3,0} = 2v_{4,0} - t.v_{4,0}$ which is equivalent to

$$(i\xi_{34} + 1/2).w_2 = 0.$$

Equation (67) for $a = 3, b = 1, c = 2$ gives $-\xi_{12}.v_{4,0} - Cv_{3,0} = 0$ which is equivalent to

$$(i\xi_{12} + C).w_2 = 0.$$

These equations imply $(h_x - C + 1/2).w_2 = 0$ and so $(m - C + 1/2) = 0$. Since $m \geq 0$ and $C = \pm 1/2$ we necessarily have $C = 1/2$ and $m = 0$. The same equations also imply $n - C - 1/2 = 0$ and hence we have $n = 1$.
By Equations (72) we know that $2e_y.w_3 = e_1.w_3 - e_2.w_3 = -2w_2$. Hence $w_3 = -f_y.w_2$.
These conditions lead to the vector

$$\vec{m}_{3b} = w_{11}w_{22}w_{12} \otimes y_1 + w_{12}w_{21}w_{11} \otimes y_2 \in M(0, 1, 5/2, 1/2),$$

which is indeed a singular vector.

1c) Let $w_1 \neq 0$, $w_2 \neq 0$. By Equations (72), we know that w_1 and w_2 are highest weight vectors.
Equations (67) for $a = 3, b = 1, c = 2$, using (58) gives

$$-i\xi_{12}.(w_1 - w_2) + C(w_1 + w_2) = 0$$

Equation (67) for $a = 4, b = 1, c = 2$, using (58), gives

$$-i\xi_{12}.(w_1 + w_2) + C(w_1 - w_2) = 0$$

These two equations lead to $C = 0$ which is a contradiction since $C = \pm 1/2$.

1d) We suppose $w_1 = w_2 = 0$. By Equations (72), we know that w_3 is a highest weight vector.
Equation (66) for $a = 2, b = 3, c = 1$, using (58), gives

$$0 = (-2i\xi_{12} + 2).w_3 = (h_x + h_y + 2).w_3$$

which implies $m + n + 2 = 0$, a contradiction.

2) Let $w_4 \neq 0$. By Equations (72), we have that $w_1 \neq 0$, $w_2 \neq 0$, $w_3 \neq 0$ and that w_4 is a highest weight vector.
Equation (66) for $a = 1, b = 3, c = 2$ gives

$$-w_3 + w_4 - i\xi_{23}.(w_1 + w_2) + i\xi_{12}.(w_3 + w_4) = 0,$$

and for $a = 2, b = 3, c = 1$ gives

$$-w_3 - w_4 + i\xi_{12}.(w_3 - w_4) - \xi_{13}.(w_1 + w_2) = 0.$$

These equations imply

$$0 = 2w_4 - e_1.w_1 - e_1.w_2 + 2i\xi_{12}.w_4 = 2i\xi_{12}.w_4.$$

Equation (67) for $a = 1, b = 3, c = 4$ gives

$$-i\xi_{34}.(w_3 - w_4) + C(w_3 + w_4) = 0,$$

and for $a = 2, b = 3, c = 4$ gives

$$-i\xi_{3,4}.(w_3 + w_4) + C(w_3 - w_4) = 0.$$

These equations imply

$$(i\xi_{34} + C).w_4 = 0$$

We deduce that $m + C = 0$ and $-n + C = 0$, hence $C = 0$, a contradiction.

□

9. Singular vectors of degree 1

The aim of this section is to classify singular vectors of degree 1. Let us consider a vector $\vec{m} \in \mathrm{Ind}(F)$ of degree 1 such that $T(\vec{m})$ is of the form:

$$T(\vec{m}) = \sum_i \eta_{(i)^c} \otimes v_i. \tag{73}$$

We write $\vec{m}$ as:

$$\vec{m} = (\eta_2 - i\eta_1) \otimes w_1 + (\eta_4 - i\eta_3) \otimes w_2 + (\eta_2 + i\eta_1) \otimes \widetilde{w}_1 + (\eta_4 + i\eta_3) \otimes \widetilde{w}_2 \tag{74}$$

Hence :

$$v_1 = i(w_1 - \widetilde{w}_1),\ v_2 = w_1 + \widetilde{w}_1,\ v_3 = i(w_2 - \widetilde{w}_2),\ v_4 = w_2 + \widetilde{w}_2. \tag{75}$$

Indeed, let us show one of these relations. In (73), let us consider $\eta_{(1)^c} \otimes v_1$. We have that $\eta_{(1)^c} = \eta_{234}$ is the Hodge dual of $-\eta_1$. In (74), the terms in η_1 are $-i\eta_1 \otimes w_1 + i\eta_1 \otimes \widetilde{w}_1$. Hence $v_1 = i(w_1 - \widetilde{w}_1)$. The other relations in (75) are obtained analogously.
In the following lemma we write explicitly some of the relations of Proposition 6.19 for a vector as in formula (73) that we need.

Lemma 9.1. *Let $\vec{m} \in \mathrm{Ind}(F)$ be a highest weight singular vector such that $T(\vec{m})$ is as in (73). Then for all $a \in \{1, 2, 3, 4\}$ we have*

$$0 = \sum_i \Big[(\xi_a \star \eta_{(i)^c}) \otimes t.v_i + \sum_{l=1}^{4} (\partial_l \xi_{al} \star \eta_{(i)^c}) \otimes v_i + \sum_{l \neq k} (\partial_l \xi_{1k} \star \eta_{(i)^c}) \otimes \xi_{lk}, v_i\Big]. \tag{76}$$

and for every permutation (a, b, c, d) of $\{1, 2, 3, 4\}$ we have

$$0 = \sum_i \Big[\sum_{r<s} (\partial_{rs} \xi_{abc} \star \eta_{(i)^c}) \otimes \xi_{rs}.v_i + \varepsilon_{abc} (\xi_d \star \eta_{(i)^c}) \otimes Cv_i\Big]. \tag{77}$$

Proof. Equation (76) is obtained by $B_0(a) + b_1(a) = 0$ and Equation (77) is obtained by $b_0(abc) + G_1(abc) = 0$ in Proposition 6.19. Note that $b_1(a) = 0$ and $G_1(abc) = 0$ since $\vec{m}$ has degree 1 and so the previous equations reduce to $B_0(a) = 0$ and $b_0(abc) = 0$. □

Lemma 9.2. *Let $\vec{m} \in \mathrm{Ind}(F)$ be a highest weight sungular vector such that $T(\vec{m})$ is as in* (73)*. Then for every permutation (a, b, c, d) of $\{1, 2, 3, 4\}$ we have*

$$(-1)^a t.v_a + \sum_{k \neq a} (-1)^k \xi_{ak}.v_k = 0; \tag{78}$$

$$(-1)^c \xi_{ab}.v_c + (-1)^b \xi_{ca}.v_b + (-1)^a \xi_{bc}.v_a - \varepsilon_{(a,b,c)} (-1)^d C v_d = 0. \tag{79}$$

Moreover

$$\begin{aligned} e_1.v_1 &= -v_3, & e_2.v_1 &= iv_4, \\ e_1.v_2 &= -iv_3, & e_2.v_2 &= -v_4, \\ e_1.v_3 &= v_1 + iv_2, & e_2.v_3 &= 0, \\ e_1.v_4 &= 0, & e_2.v_4 &= -iv_1 + v_2, \end{aligned} \tag{80}$$

where e_1 and e_2 are defined in (21) *and* (22).

Proof. Equation (78) follows by considering the terms in η_{1234} in (76).
For equation (79) we can assume $a < b < c$ with no loss of generality. Equation (76) becomes

$$0 = \eta_c \eta_{(c)^c} \otimes \xi_{ab}.v_c + \eta_b \eta_{(b)^c} \otimes \xi_{ca}.v_b + \eta_a \eta_{(a)^c} \otimes \xi_{bc}.v_a - \varepsilon_{(a,b,c)} \eta_d \eta_{(d)^c} \otimes C v_d,$$

which is equivalent to (79).

The fact that $\vec{m}$ is annihilated by e_1 implies

$$\begin{aligned} 0 = &- \sum_i \sum_{l=1}^{4} (\partial_l(-\xi_{13} + i\xi_{23}) \star \partial_l \eta_{(i)^c}) \otimes v_i + \sum_i \sum_{r<s} (\partial_{rs}(-\xi_{13} + i\xi_{23}) \star \eta_{(i)^c}) \otimes \xi_{rs}.v_i \\ =&\eta_{124} \otimes (v_1 + iv_2 - e_1.v_3) + \eta_{234} \otimes (-v_3 - e_1.v_1) + \eta_{134} \otimes (-iv_3 - e_1.v_2) + \eta_{123} \otimes (-e_1.v_4). \end{aligned}$$

Therefore:

$$\begin{aligned} e_1.v_1 &= -v_3, \\ e_1.v_2 &= -iv_3, \\ e_1.v_3 &= v_1 + iv_2, \\ e_1.v_4 &= 0. \end{aligned}$$

Equations (80) for e_2 follow similarly. □

Remark 9.3. By (75), Equations (80) are equivalent to:

$$\begin{aligned} e_1.w_1 &= 0 & e_2.w_1 &= 0, \\ e_1.w_2 &= w_1 & e_2.w_2 &= w_1 \\ e_1.\widetilde{w}_1 &= w_2 - \widetilde{w}_2 & e_2.\widetilde{w}_1 &= -w_2 - \widetilde{w}_2 \\ e_1.\widetilde{w}_2 &= -w_1 & e_2.\widetilde{w}_2 &= w_1 \end{aligned} \tag{81}$$

We represent these relations in the following diagram

$$\begin{tikzcd} & \langle w_2-\widetilde{w}_2\rangle \arrow[dr, "e_1"] & \\ \langle\widetilde{w}_1\rangle \arrow[ur, "e_1"] \arrow[dr, "e_2"'] & & \langle w_1\rangle. \\ & \langle w_2+\widetilde{w}_2\rangle \arrow[ur, "e_2"'] & \end{tikzcd}$$

By (75), Equations (78) can be rewritten in the following equivalent way

$$\begin{aligned} (t-i\xi_{12}).\widetilde{w}_1 &= -f_x.\widetilde{w}_2+f_y.w_2, && (82)\\ (t-i\xi_{34}).\widetilde{w}_2 &= -f_y.w_1-e_x.\widetilde{w}_1, \\ (t+i\xi_{12}).w_1 &= e_x.w_2-e_y.\widetilde{w}_2, && (83)\\ (t+i\xi_{34}).w_2 &= e_y.\widetilde{w}_1+f_x.w_1. \end{aligned}$$

Proof of Theorem 6.6. As usual we denote by $\mu=(m,n,\mu_0,C)$ the highest weight of the Verma module containing the singular vector $\vec{m}$. Let us first observe that, by Equations (81), we have that if $w_1\neq 0$ then $w_2\neq 0$.

1) Let $w_1=w_2=0$.

By Equations (81), we obtain that if $\widetilde{w}_2\neq 0$, then $\widetilde{w}_1\neq 0$. Hence, there are two subcases.

1a) Let $\widetilde{w}_1\neq 0$ and $\widetilde{w}_2=0$.
By Equations (81) we know that $\widetilde{w}_1$ is a highest weight vector. Let us compute its weight. By (75) we know that $v_1=-i\widetilde{w}_1, v_2=\widetilde{w}_1, v_3=0, v_4=0$.
Equation (79) for $a=1, b=3, c=4$ gives

$$(-i\xi_{3,4}+C).\widetilde{w}_1=0.$$

Equation (82) gives $(t-i\xi_{12}).\widetilde{w}_1=0$. These two conditions imply $\mu=(m,n,-\frac{m+n}{2},\frac{m-n}{2})$ with $m,n\in\mathbb{Z}_{\geq 0}$.
These conditions lead to the vector

$$\vec{m}_{1a}=w_{11}\otimes x_1^m y_1^n\in M(m,n,-\frac{m+n}{2},\frac{m-n}{2}).$$

which is indeed a singular vector.

1b) Let $\widetilde{w}_1\neq 0$ and $\widetilde{w}_2\neq 0$.
Equations (81) imply that $\widetilde{w}_2$ is a highest weight vector, let us compute its weight.
By (75), we know that $v_1=-i\widetilde{w}_1, v_2=\widetilde{w}_1, v_3=-i\widetilde{w}_2, v_4=\widetilde{w}_2$.
Equation (79) for $a=1, b=2, c=3$ gives

$$-i\xi_{12}.\widetilde{w}_2+\xi_{13}.\widetilde{w}_1-i\xi_{23}.\widetilde{w}_1+C\widetilde{w}_2=0,$$

and for $a=1, b=2, c=4$ gives

$$-\xi_{12}.\widetilde{w}_2+\xi_{14}.\widetilde{w}_1-i\xi_{24}.\widetilde{w}_1-iC\widetilde{w}_2=0.$$

These two equations imply

$$0=-2i\xi_{12}.\widetilde{w}_2+2C\widetilde{w}_2-(e_1+e_2).\widetilde{w}_1=2(-i\xi_{12}+C+1).\widetilde{w}_2.$$

that is equivalent to:

$$(-i\xi_{12}+C+1).\widetilde{w}_2=0.$$

Equation (82) provides

$$(t - i\xi_{34}).\widetilde{w}_2 = -e_x.\widetilde{w}_1 = \widetilde{w}_2$$

and so these equations imply $\mu = (m, n, 1 + \frac{m-n}{2}, -\frac{m+n}{2} - 1)$, for $m, n \geq 0$. We point out that $m \in \mathbb{Z}_{>0}$ since $e_x.\widetilde{w}_1 = -\widetilde{w}_2 \neq 0$.

These conditions lead to the vector

$$\vec{m}_{1b} = w_{21} \otimes x_1^m y_1^n - w_{11} \otimes x_1^{m-1} x_2 y_1^n \in M(m, n, -\frac{m+n}{2}, \frac{m-n}{2}),$$

which is indeed a singular vector.

2) Let $w_1 \neq 0$ and $w_2 \neq 0$.
By (81) we have $\widetilde{w}_2 \neq 0$ $\widetilde{w}_1 \neq 0$ and that w_1 is a highest weight vector.
By (75), Equation (79) for $a = 1, b = 3, c = 4$ gives

$$(-\xi_{13} - i\xi_{14}).w_2 + (-\xi_{13} + i\xi_{14}).\widetilde{w}_2 + (i\xi_{34} + C).w_1 + (-i\xi_{34} + C).\widetilde{w}_1 = 0,$$

and for $a = 2, b = 3, c = 4$ gives

$$(-\xi_{23} - i\xi_{24}).w_2 + (-\xi_{23} + i\xi_{24}).\widetilde{w}_2 + (-\xi_{34} + iC).w_1 + (-\xi_{34} - iC).\widetilde{w}_1 = 0.$$

By these equations we deduce

$$(e_1 + e_2).w_2 + (e_1 - e_2).\widetilde{w}_2 + 2(i\xi_{34} + C).w_1 = 2(i\xi_{34} + C).w_1 = 0.$$

Moreover, Equation (83) provides

$$(t + i\xi_{12}).w_1 = 2w_1.$$

These conditions imply that $\mu = (m, n, \frac{m+n}{2} + 2, \frac{n-m}{2})$. Note that $m, n > 0$ since $e_x.w_2 = w_1 \neq 0$ and $e_y.\widetilde{w}_2 = -w_1 \neq 0$.

All these conditions lead to the vector

$$\vec{m}_{1c} = w_{22} \otimes x_1^m y_1^n - w_{12} \otimes x_1^{m-1} x_2 y_1^n - w_{21} \otimes x_1^m y_1^{n-1} y_2 + w_{11} \otimes x_1^{m-1} x_2 y_1^{n-1} y_2$$
$$\in M(m, n, \frac{m+n}{2} + 2, \frac{n-m}{2}),$$

which is indeed a singular vector.

3) Let $w_1 = 0$ and $w_2 \neq 0$. Note that $\widetilde{w}_1 \neq 0$, since $(e_1 - e_2).\widetilde{w}_1 = 2w_2 \neq 0$ by (81).

3a) Let $\widetilde{w}_2 = 0$.
Note that w_2 is a highest weight vector by (81). Let us compute its weight.
Using (75), Equation (79) for $a = 1, b = 2, c = 3$ gives

$$i\xi_{12}.w_2 + \xi_{13}.\widetilde{w}_1 - i\xi_{23}.\widetilde{w}_1 + Cw_2 = 0$$

and Equation (79) for $a = 1, b = 2, c = 4$ gives

$$-\xi_{12}.w_2 + \xi_{14}.\widetilde{w}_1 - i\xi_{24}.\widetilde{w}_1 + iCw_2 = 0.$$

These two equations imply

$$0 = 2i\xi_{12}.w_2 - e_1.\widetilde{w}_1 + e_2.\widetilde{w}_1 + 2Cw_2 = 2(i\xi_{12} - 1 + C).w_2$$

and Equation (83) provides

$$(t + i\xi_{34}).w_2 = w_2.$$

These equations imply $\mu = (m, n, \frac{n-m}{2}+1, \frac{m+n}{2}+1)$, with $m, n \geq 0$. Moreover we have $n > 0$ since, by (81) $e_y.\widetilde{w}_1 = w_2 \neq 0$.

All these conditions lead to the vector

$$\vec{m}_{1d} = w_{12} \otimes x_1^m y_1^n - w_{11} \otimes x_1^m y_1^{n-1} y_2 \in M(m, n, \frac{n-m}{2}+1, \frac{m+n}{2}+1),$$

which is indeed a singular vector.

3b) Let $\widetilde{w}_2 \neq 0$.

By Equations (81), w_2 and $\widetilde{w}_2$ are highest weight vectors. Let us compute their weight. By (75), Equation (79) for $a = 1, b = 2, c = 3$ gives

$$i\xi_{12}.w_2 - i\xi_{12}.\widetilde{w}_2 + \xi_{13}.\widetilde{w}_1 - i\xi_{23}.\widetilde{w}_1 + Cw_2 + C\widetilde{w}_2 = 0,$$

and Equation (79) for $a = 1, b = 2, c = 4$ gives

$$-\xi_{12}.w_2 - \xi_{12}.\widetilde{w}_2 + \xi_{14}.\widetilde{w}_1 - i\xi_{24}.\widetilde{w}_1 + iCw_2 - iC\widetilde{w}_2 = 0$$

These two equations imply

$$(i\xi_{12} - 1 + C).w_2 = 0$$

and

$$(-i\xi_{12} + 1 + C).\widetilde{w}_2 = 0$$

and in particular $C = 0$. But, for $C = 0$, the $\lambda-$action of Proposition 5.10 reduces to the action found in Theorem 4.3 of [1] where the vectors of degree 1 were classified and this case was ruled out.

Acknowledgments. The authors would like to thank Nicoletta Cantarini and Victor Kac for useful comments and suggestions. L.B. was partially supported by the EU Horizon 2020 project GHAIA, MCSA RISE project GA No 777822.

Data availability statement: not applicable.

□

References

[1] Boyallian C., Kac V.G., Liberati, J. *Irreducible modules over finite simple Lie conformal superalgebras of type K*, J. Math. Phys. 51 (2010), 1-37.

[2] Boyallian C., Kac V. G., Liberati J. *Classification of finite irreducible modules over the Lie conformal superalgebra CK_6*, Comm. Math. Phys. 317 (2013), 503-546.

[3] Boyallian C., Kac V. G., Liberati J., Rudakov A. *Representations of simple finite Lie conformal superalgebras of type W and S*, J. Math. Phys. 47 (2006), 1-25.

[4] Cantarini N., Caselli F. *Low Degree Morphisms of E(5, 10)-generalized Verma Modules*, Algebras and Representation Theory 23, (2020) 2131-2165.

[5] Cantarini N., Caselli F., Kac, V.G. *Lie conformal superalgebras and duality of modules over linearly compact Lie superalgebras*, Adv. Math. 378 (2021), 107523.

[6] Cantarini N., Caselli F., Kac, V.G. *Classification of degenerate Verma modules for E(5,10)*, Commun. Math. Phys. (2021). https://doi.org/10.1007/s00220-021-04031-z.

[7] Cheng S., Cantarini N., Kac V. G., *Errata to Structure of Some $\mathbb{Z}$-graded Lie Superalgebras of Vector Fields*, Transf. Groups 9 (2004), 399-400.

[8] Cheng S., Kac, V. G. *Conformal modules*, Asian J. Math. 1, 181 (1997); 2, 153(E) (1998).

[9] Cheng S., Kac V.G. *A new $N = 6$ superconformal algebra*, Comm. Math. Phys. 186 (1997), 219-231.

[10] Cheng S., Kac, V. G. *Structure of some $\mathbb{Z}$-graded Lie superalgebras of vector fields*, Transf. Groups, 4 (1999), 219-272.

[11] Cheng, S., Lam, N. *Finite conformal modules over N=2,3,4 superconformal algebras*, J. Math. Phys. 42 (2001), 906-933.

[12] D'Andrea A, Kac V. G. *Structure theory of finite conformal algebras*, Selecta Math., (N.S.) 4 (1998) 377-418.

[13] Fattori D., Kac V. G. *Classification of finite simple Lie conformal superalgebras*, J. Algebra 258, (2002) 23-59., Special issue in celebration of Claudio Procesi's 60th birthday.

[14] Kac V.G., Van de Leur J.W. *On classification of superconformal algebras*, in: S.J. Gates, et al. (Eds.), Strings 88, World Scientific, Singapore, 1989, 77-106.

[15] Kac V.G. *Vertex algebras for beginners*, Univ. Lecture Ser., Vol. 10, AMS, Providence, RI, 1996, 2nd ed., (1998).

[16] Kac V.G. *Classification of Infinite-Dimensional Simple Linearly Compact Lie Superalgebras*, Advanced in Mathematics 139 (1998), 11-55

[17] Kac V.G., Rudakov A. *Representations of the Exceptional Lie Superalgebra* $E(3,6)$ *II: Four Series of Degenerate Modules*, Comm. Math. Phys. 222, (2001) 611-661 .

[18] Kac V. G., Rudakov A. *Representations of the exceptional Lie superalgebra E(3,6). I. Degeneracy conditions*, Transform. Groups 7, (2002) 67-86.

[19] Kac V. G., Rudakov A. *Complexes of modules over the exceptional Lie superalgebras E(3, 8) and E(5, 10)*, Int. Math. Res. Not. 19 (2002), 1007-1025.

[20] Kac V. G., Rudakov A. *Representations of the exceptional Lie superalgebra E(3, 6). III. Classification of singular vectors*, J. Algebra Appl. 4 (2005), 15-57.

[21] Knapp A. W. *Lie Groups Beyond an Introduction*, Progress in Mathematics Vol. 140 (Birkhäuser, Boston, 1996).

[22] Mac Lane S. *Homology* (fourth printing) Springer-Verlag, (1995).

[23] Martínez C., Zelmanov E., *Irreducible representations of the exceptional Cheng-Kac superalgebra*, Trans. Amer. Math. Soc. 366 (2014), 5853-5876.

[24] Rudakov A. *Morphisms of Verma modules over exceptional Lie superalgebra E(5, 10)*, arXiv 1003.1369v1.

Lucia Bagnoli and Fabrizio Caselli, Dipartimento di matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy

Email address: `lucia.bagnoli4@unibo.it, luciabagnoli93@gmail.com`

Email address: `fabrizio.caselli@unibo.it`

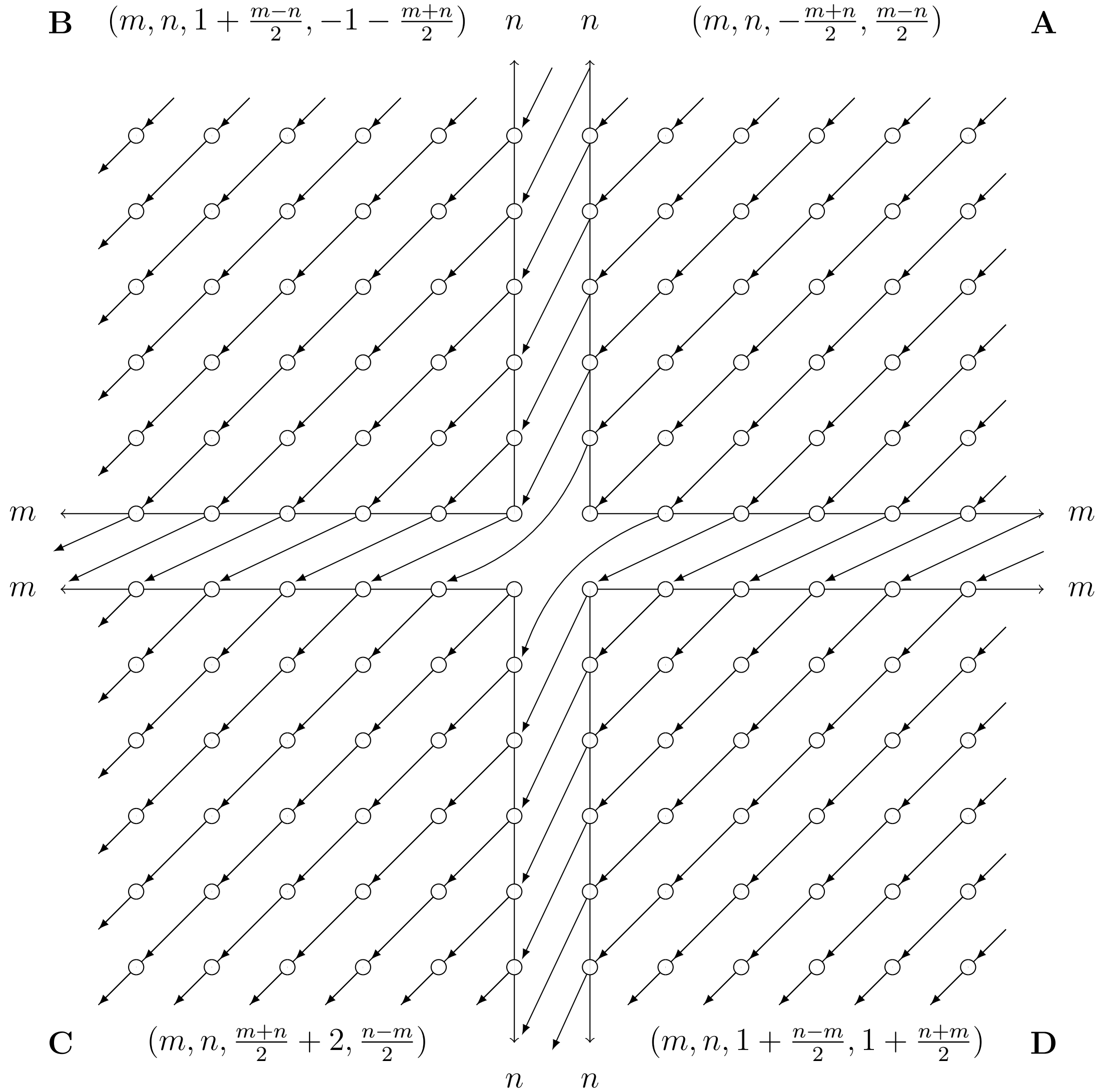
B
$(m, n, 1 + \frac{m-n}{2}, -1 - \frac{m+n}{2})$
n
n
$(m, n, -\frac{m+n}{2}, \frac{m-n}{2})$
A
m
m
m
m
C
$(m, n, \frac{m+n}{2} + 2, \frac{n-m}{2})$
$(m, n, 1 + \frac{n-m}{2}, 1 + \frac{n+m}{2})$
D
n
n

FIGURE 1