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Approximating compact objects in bootstrapped Newtonian gravity: use of the canonical potential

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Abstract We consider compact objects in a classical and non-relativistic generalisation of Newtonian gravity, dubbed bootstrapped Newtonian theory, which includes higher-order derivative interaction terms of the kind generically present in the strong-field regime of gravity. By means of a field redefinition, the original bootstrapped Newtonian action is written in a canonical Newtonian form with non-linear source terms. Exact analytic solutions remain unattainable, but we show that perturbative solutions of the canonical theory can be efficiently used to derive approximate descriptions of compact objects. In particular, using the canonical potential, we can more directly and generally show that the Arnowitt–Deser–Misner mass differs from the (Newtonian) proper mass due to the non-linear couplings in the theory. A few examples of sources with different density profiles are explicitly reanalysed in this framework.

1 Introduction

Despite gravity being the oldest of the forces known to science, many gravitational phenomena and their quantum foundations remain, to a large degree, fully open questions. Roughly speaking, this can be mainly attributed to its weakness (as measured by the gravitational coupling $G_{\rm N} \sim 10^{-11}\,{\rm m}^3\,{\rm kg}^{-1}\,{\rm s}^{-1}$). When Newton's constant is combined with other "small" parameters (like $\hbar$ in a quantum regime), gravitational effects seem to become utterly negligible at laboratory scales. Testing aspects of strong gravity thus requires objects with huge masses (or large compactness) in order to produce sizeable effects. On the other hand, gravity is a non-linear phenomenon at its core and the theoretical study of compact objects is a difficult endeavour due to the lack of general techniques to solve non-linear differential equations.

Most of the known results are obtained perturbatively in the weak-field regime of general relativity, far away from the compact source. Perturbation theory indeed fails in the strong-field regime around compact objects, where an infinite tower of couplings are generated in the non-relativistic approximation given by the expansion of the Einstein-Hilbert action. In such a regime, all terms contribute equally and the infinite series cannot be truncated. Inspired by this result, one could take a bottom-up approach and construct a modified Newtonian theory by including, from the onset, terms of the functional forms which appear at the leading order in the aforementioned expansion. Rather than truncating the series obtained from general relativity, the resulting action is viewed as a new theory, where finitely many terms are treated on the same foot. From this perspective, the model functions as an alternative, rather than an extension, of Newtonian gravity, in very much the same way that Stelle's higher-derivative gravity [1] differs from general relativity. The theory so obtained is called bootstrapped Newtonian gravity [2].

One of the main purposes for devising the bootstrapped Newtonian gravity was to study static (and spherically symmetric) compact sources [2–6]. A major difficulty however remains that the non-linearity of the field equation, and its interplay with the (Newtonian) conservation equation, make it impossible to find analytical solutions. It is therefore hard to derive general results to compare with the predictions of Newtonian physics or general relativity.[1] For this reason, in this work, we extend our previous investigations of compact

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[1] A major difference with respect to general relativity is the absence of a Buchdahl limit for isotropic stars [3].

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sources in bootstrapped Newtonian gravity by applying an idea introduced in Ref. [7], whereby the kinetic term of the bootstrapped Newtonian Lagrangian is put in canonical form by performing a field redefinition. The role of the field redefinition is to replace derivative couplings, which are hard to deal with, by standard (albeit non-linear) couplings. In particular, we shall employ the same Taylor expansion in powers of the radial coordinate r from Refs. [2,3] and show that the error one makes when truncating the transformed solution (to second order in r) is negligible with respect to the truncated solution before the field redefinition. We stress that, because of the non-linear nature of the problem and of the required field redefinition, this is a non-trivial result. It is also practically relevant because working with the redefined canonical field makes it easier to find approximate solutions for describing compact objects in the bootstrapped Newtonian theory.

This paper is organised as follows: in Sect. 2, we review the bootstrapped Newtonian gravity and detail the aforementioned field redefinition that brings the kinetic term to the canonical form; in Sect. 3, we discuss approximate solutions that can be obtained for generic source terms and some particular cases. The main results we will obtain are that the bootstrapped Newtonian solutions are identical (at least) to second order in r and coincide with the Newtonian potential inside a homogeneous source, regardless of the actual density and pressure profiles; terms with odd powers of r must vanish and the difference with respect to the Newtonian potential appears at order r^4 (or higher); the Arnowitt–Deser–Misner (ADM)-like mass [8] M and the (Newtonian) proper mass M_0 are always different. Finally, we will draw some more conclusions in Sect. 4.

2 Bootstrapped Newtonian gravity

We start by recalling that, in its most general form, the Lagrangian for the bootstrapped Newtonian potential $V = V(r)$ for static and spherically symmetric systems is given by [2][2]

$$\begin{aligned} L[V] &= L_{\rm N}[V] - 4\,\pi \int_0^\infty r^2\,{\rm d}r \left[q_V\,\mathcal{J}_V\,V + q_p\,\mathcal{J}_p\,V \right. \\ &\quad \left. + q_\rho\,\mathcal{J}_\rho\,(\rho + q_p\,\mathcal{J}_p)\right] \\ &= -4\,\pi \int_0^\infty r^2\,{\rm d}r \left[\frac{(V')^2}{8\,\pi\,G_{\rm N}}\,(1-4\,q_V\,V)\right. \\ &\quad \left. + (\rho + 3\,q_p\,p)\,V\,(1-2\,q_\rho\,V)\right], \end{aligned} \tag{2.1}$$

where $L_{\rm N}$ is the Lagrangian for the Newtonian potential and $f' \equiv {\rm d}f/{\rm d}r$. The motivation for each additional term was described extensively in previous publications [2–6], so in the next paragraphs we will just briefly recall the meaning and role of each of them.

The standard Newtonian Lagrangian,

$$L_{\rm N}[V] = -4\,\pi \int_0^\infty r^2\,{\rm d}r \left[\frac{(V')^2}{8\,\pi\,G_{\rm N}} + \rho\,V\right], \tag{2.2}$$

yields the Poisson equation

$$r^{-2}\left(r^2\,V'\right)' \equiv \triangle V = 4\,\pi\,G_{\rm N}\,\rho \tag{2.3}$$

for the Newtonian potential $V = V_{\rm N}$ generated by the matter energy density $\rho = \rho(r)$. As it was detailed in Refs. [2,9–11], the gravitational self-coupling contribution is then sourced by the gravitational energy $U_{\rm N}$ per unit volume, to wit

$$\mathcal{J}_V \simeq \frac{{\rm d}U_{\rm N}}{{\rm d}\mathcal{V}} = -\frac{\left[V'(r)\right]^2}{2\,\pi\,G_{\rm N}}, \tag{2.4}$$

which couples to V via the constant q_V in Eq. (2.1). The static pressure $p = p(r)$ becomes very large for compact sources with a compactness [2]

$$X \equiv \frac{G_{\rm N}\,M}{R} \gtrsim 1, \tag{2.5}$$

where M is the ADM-like mass that one would measure when studying orbits [12,13] and R is the radius of the source [6]. For this reason, a corresponding potential energy U_p was added such that

$$\mathcal{J}_p \simeq -\frac{{\rm d}U_p}{{\rm d}\mathcal{V}} = 3\,p, \tag{2.6}$$

which couples to V via the constant q_p in Eq. (2.1). Since the above just adds to ρ, it can be easily included by simply shifting $\rho \to \rho + 3\,q_p\,p$, where q_p is a positive constant which formally allows us to implement the non-relativistic limit as $q_p \to 0$. Upon including these new source terms, and the analogous higher-order term $\mathcal{J}_\rho = -2\,V^2$, which couples with the matter source, we obtain the total Lagrangian (2.1).

The Euler–Lagrange equation for V is then given by

$$\triangle V = 4\,\pi\,G_{\rm N}\,(\rho + 3\,q_p\,p)\,\frac{1-4\,q_\rho\,V}{1-4\,q_V\,V} + \frac{2\,q_V\,(V')^2}{1-4\,q_V\,V}. \tag{2.7}$$

We remark that the (dimensionless) coupling constants q_V, q_p and q_ρ track the effects of each additional contribution and could be related to different specific theories of the interaction between gravity and matter (for similar considerations, see, e.g. Ref. [14]). The Newtonian limit is clearly recovered for $q_V = q_p = q_\rho \to 0$.

[2] We shall use units with $c = 1$.

2.1 Field redefinition

The Lagrangian (2.2) can be generalised to non-static configurations $V = V(x^\mu)$ in flat spacetime, thus yielding the kinetic term [7]

$$K = -\left(1 - 4\,q_V\,V\right)\frac{\partial_\mu V\,\partial^\mu V}{8\,\pi\,G_\mathrm{N}}, \tag{2.8}$$

which is not in canonical form, and neither is V of the canonical dimension for a scalar field. In fact, we can change K into the precise canonical form

$$K = -\frac{1}{2}\,\partial_\mu\psi\,\partial^\mu\psi \tag{2.9}$$

by means of the transformation [7]

$$\psi \equiv \psi(V) = \frac{1}{6\,\alpha}\left[1 - \left(1 - 4\,q_V\,V\right)^{3/2}\right], \tag{2.10}$$

where

$$\alpha = q_V\,\sqrt{G_\mathrm{N}}. \tag{2.11}$$

The inverse relationship is given by

$$V \equiv V(\psi) = \frac{1}{4\,q_V}\left[1 - \left(1 - 6\,\alpha\,\psi\right)^{2/3}\right], \tag{2.12}$$

and, after some algebra, the total Lagrangian (2.2) for static and isotropic configurations $\psi = \psi(r)$ reads

$$L[\psi] = -4\,\pi\int_0^\infty r^2\,\mathrm{d}r\left[\frac{\left(\psi'\right)^2}{8\,\pi} + \left(J_\rho + 3\,q_p\,J_p\right)\xi(\psi)\right], \tag{2.13}$$

in which the matter density was rescaled as

$$J_\rho = \sqrt{G_\mathrm{N}}\,\rho, \tag{2.14}$$

like the pressure contribution

$$J_p = \sqrt{G_\mathrm{N}}\,p. \tag{2.15}$$

The interaction terms, which do not contain any derivatives of the new field ψ, are all included in the non-linear coupling to the sources J_ρ and J_p, that is

$$\xi(\psi) = \frac{1}{4\,\alpha}\left[1 - \left(1 - 6\,\alpha\,\psi\right)^{2/3}\right] \times\left\{1 - \frac{\beta}{2\,\alpha}\left[1 - \left(1 - 6\,\alpha\,\psi\right)^{2/3}\right]\right\}, \tag{2.16}$$

where

$$\beta = q_\rho\,\sqrt{G_\mathrm{N}}. \tag{2.17}$$

The most general form of the Euler–Lagrange equation for the canonically normalised $\psi = \psi(r)$ is finally given by

$$\triangle\psi = 4\,\pi\,J\,\frac{\alpha - \beta\left[1 - \left(1 - 6\,\alpha\,\psi\right)^{2/3}\right]}{\alpha\left(1 - 6\,\alpha\,\psi\right)^{1/3}}, \tag{2.18}$$

where $J \equiv J_\rho + 3\,q_p\,J_p$ is the total effective density. In the following, we shall show that (approximate) solutions of Eq. (2.18) can indeed be efficiently employed in order to determine (approximate) solutions of the original Eq. (2.7).

2.2 Vacuum solution

In the vacuum outside a source of mass M and radius $r = R$, we have $J = 0$ and the above Eq. (2.18) reduces to

$$\triangle\psi = 0. \tag{2.19}$$

Of course, the exact solution for $r > R$ satisfying the proper asymptotic behaviour is the (canonically normalised) Newtonian potential

$$\psi_\mathrm{out} = -\frac{\sqrt{G_\mathrm{N}}\,M}{r}, \tag{2.20}$$

which transforms back to [7]

$$V_\mathrm{out} = V(\psi_\mathrm{out}) = \frac{1}{4\,q_V}\left[1 - \left(1 + 6\,q_V\,\frac{G_\mathrm{N}\,M}{r}\right)^{2/3}\right]. \tag{2.21}$$

This is the exact solution of the bootstrapped Newtonian Eq. (2.7) where $\rho = p = 0$ with the expected asymptotic behaviour. In particular, the Newtonian potential is recovered for $q_V \to 0$ and the first post-Newtonian order of general relativity for $q_V = 1$ [2,12,13].

One important aspect that is not apparent from the above derivation is that the mass M in Eq. (2.21) is not equal to the proper mass M_0 of the source [2,6]. This follows precisely because of the non-linearity of Eq. (2.7) and the equivalent Eq. (2.18). We shall further investigate this aspect for the canonical field ψ and various density profiles in the next sections.

3 Quadratic approximation for the inner canonical potential

To work with simpler equations, in the remainder of the paper we set $q_V = q_\rho$ (equivalent to $\alpha = \beta$), which simplifies the field equation (2.18) to

$$\triangle\psi = 4\,\pi\,J\left(1 - 6\,\alpha\,\psi\right)^{1/3}, \tag{3.1}$$

where the effective density $J = J(r)$ vanishes outside the source of radius $r = R$.

Any solution of Eq. (2.7) for the bootstrapped Newtonian potential $V_\mathrm{in} = V(0 \le r < R)$ needs to match smoothly with the outer vacuum solution V_out in Eq. (2.21) across the boundary $r = R$ of the source. It is very easy to show that identical constraints must then hold for the field $\psi = \psi(V)$ satisfying Eq. (3.1). More precisely

$$V_{\rm in}(R) = V_{\rm out}(R) \equiv V_R \quad \Leftrightarrow \quad \psi_{\rm in}(R) = \psi_{\rm out}(R) \equiv \psi_R, \tag{3.2}$$

and

$$V'_{\rm in}(R) = V'_{\rm out}(R) \equiv V'_R \quad \Leftrightarrow \quad \psi'_{\rm in}(R) = \psi'_{\rm out}(R) \equiv \psi'_R, \tag{3.3}$$

where we defined $\psi_{\rm in} = \psi(0 \leq r \leq R)$ and $\psi_{\rm out} = \psi(R \leq r)$. Furthermore, we are looking for potentials generated by density profiles that are finite in the centre. Therefore, the inner solution also needs to satisfy the regularity condition $V'_{\rm in}(0) = 0$, which in turn means that

$$\psi'_{\rm in}(0) = 0. \tag{3.4}$$

Obtaining exact analytic solutions $V_{\rm in}$ for Eq. (2.7) or $\psi_{\rm in}$ for Eq. (3.1) is not feasible, even for an object with constant density and negligible pressure. Hence, we will here focus on finding a good approximation by Taylor expanding around $r = 0$. Odd powers can be shown to vanish when imposing the constraint in Eq. (3.4) [2], so that the bootstrapped Newtonian potential up to second order reads

$$V_{\rm in} \simeq V_0 + V_2\, r^2. \tag{3.5}$$

This approximation was shown to work well for sources of small and intermediate compactness X by comparing with numerical solutions in Refs. [2,3]. Therefore, we limit this case study to $X \lesssim 1$, which excludes objects hidden behind a horizon.

The same Taylor expansion for the canonical potential reads

$$\psi_{\rm in} \simeq \psi_0 + \psi_2\, r^2, \tag{3.6}$$

and the mapping in Eq. (2.12) will then yield $\tilde{V}_{\rm in} = V(\psi_0 + \psi_2\, r^2)$, which we can compare with $V_{\rm in}$ in Eq. (3.5). Note that the two results will contain different powers of r and their comparison is therefore not straightforward. However, one can estimate quantitatively how close the two approximations are to one another by calculating their relative difference.

3.1 General density and pressure profiles

Let us first consider a generic $J = J(r)$, with the only constraint that $J'(0) = 0$. Upon inserting the Taylor expansion (3.6) in the equation of motion (3.1) and employing the boundary condition (3.3), we find

$$\psi_0 = \frac{1}{3\,\alpha}\left[1 - \frac{27\, G_{\rm N}^{3/2}\, M^3}{R^9\, J_0^3}\right] \tag{3.7}$$

where $J_0 \equiv J(0)$, and

$$\psi_2 = \frac{\sqrt{G_{\rm N}}\, M}{2\, R^3}. \tag{3.8}$$

From the matching condition in Eq. (3.2), one obtains

$$\psi_0 = \frac{3\,\sqrt{G_{\rm N}}\, M}{2\, R}. \tag{3.9}$$

The approximate expression for the inner canonical field thus becomes

$$\psi_{\rm in} \simeq -\frac{\sqrt{G_{\rm N}}\, M}{2\, R}\left(3 - \frac{r^2}{R^2}\right), \tag{3.10}$$

which is in fact the (canonically normalised) Newtonian solution for the inner potential inside a homogeneous source satisfying

$$\triangle\psi = 4\,\pi\, J_0. \tag{3.11}$$

This result shows that, as long as the quadratic approximations (3.5) and (3.6) hold, the canonical $\psi_{\rm in}$ depends only on the mass M and the size R of the source. From the inverse map (2.12), we conclude that the bootstrapped Newtonian potential $V_{\rm in}$ is the same regardless of the density (and pressure) profile, up to terms of order $(r^4)^{2/3}$.

We emphasise that, in the Newtonian case, the ADM mass M equals the (Newtonian) proper mass[3]

$$M_0 = 4\,\pi \int_0^R r^2\, {\rm d}r\, \rho(r). \tag{3.12}$$

In bootstrapped Newtonian gravity, M and M_0 instead differ [2,6], since demanding that $M = M_0$ would over constrain the problem and yield no solution, as will become apparent in the following subsections. Calculating $M = M(M_0)$ is then very instructive, but it can only be done for given $J = J(r)$ and the corresponding expression can be very cumbersome.

3.2 Homogeneous ball with negligible pressure

As a simple application, we directly consider a ball with homogeneous density

$$\rho = \rho_0\, \Theta(R - r) = \frac{3\, M_0}{4\,\pi\, R^3}\, \Theta(R - r), \tag{3.13}$$

where Θ is the Heaviside step function enforcing the density to vanish for $r > R$ and M_0 is the (Newtonian) proper mass defined in Eq. (3.12). We also assume that the pressure be negligible, so that Eq. (3.1) further simplifies to

$$\triangle\psi = 4\,\pi\, J_\rho\, (1 - 6\,\alpha\,\psi)^{1/3} = 4\,\pi\, J_0\, (1 - 6\,\alpha\,\psi)^{1/3}, \tag{3.14}$$

where $J_0 = \sqrt{G_{\rm N}}\, \rho_0$, and we limit the investigation to values of the compactness $X \lesssim 1$, as stated earlier.

[3] We notice that Eq. (3.12) would give the ADM mass of a source in vacuum in general relativity. For this reason, we refer to M_0 as the Newtonian proper mass.

3.2.1 Inner potential via field redefinition

By plugging the Taylor expansion for the canonical $\psi_{\rm in}$ in Eq. (3.6) into the equation of motion (3.14), we find

$$\psi_2 \simeq \frac{\sqrt{G_{\rm N}}\, M_0}{2\, R^3}\, (1 - 6\,\alpha\, \psi_0)^{1/3}\,. \tag{3.15}$$

Eq. (3.3) can next be used to determine

$$\psi_0 \simeq \frac{1}{6\,\alpha}\left(1 - \frac{M^3}{M_0^3}\right). \tag{3.16}$$

The matching condition (3.2) can be used to express the proper mass in terms of M, yielding

$$\tilde{M}_0 = \frac{M}{(1 + 9\, q_V\, G_{\rm N}\, M/R)^{1/3}} = \frac{M}{(1 + 9\, q_V\, X)^{1/3}}. \tag{3.17}$$

Finally, we find the same approximate expression (3.10) corresponding to the Newtonian solution of Eq. (3.11). This shows that, at least in the quadratic approximation, the change in the coupling

$$\xi(V) = 1 \to \xi(\psi) = (1 - 6\,\alpha\, \psi)^{1/3} \tag{3.18}$$

is equivalent to rescaling the Newtonian mass M_0 into the ADM mass M according to Eq. (3.17). One could therefore solve the simpler Newtonian problem (3.11) and just write M instead of M_0 in the solution.

The bootstrapped Newtonian potential is obtained by substituting Eq. (3.10) in Eq. (2.12) and reads

$$\begin{aligned}\tilde{V}_{\rm in} &\simeq \frac{1}{4\, q_V}\left\{1 - \left[1 + 3\, q_V\, X\left(3 - \frac{r^2}{R^2}\right)\right]^{2/3}\right\} \\ &\simeq \frac{1}{4\, q_V}\left\{1 - (1 + 9\, q_V\, X)^{2/3}\left[1 - \frac{2\, q_V\, r^2/R^2}{(1 + 9\, q_V\, X)}\right]\right\}.\end{aligned} \tag{3.19}$$

We can also estimate the relative error for this approximation by replacing the expression (3.10) into Eq. (3.14), from which we obtain

$$\mathcal{E} \equiv \frac{q_V\, X}{1 + 9\, X}\left(\frac{r}{R}\right)^2 \ll 1, \tag{3.20}$$

which is displayed in Fig. 1. Of course, this error vanishes everywhere inside the source in the Newtonian limit $q_V \to 0$ and is proportional to the compactness X otherwise.

3.2.2 Inner bootstrapped Newtonian potential

The approximate solution (3.5) for the homogeneous ball with vanishing pressure was found in Ref. [2] and is given by

$$V_{\rm in} \simeq \frac{1}{4\, q_V}\left[1 - \frac{1 + 2\, q_V\, X\,(4 - r^2/R^2)}{(1 + 6\, q_V\, X)^{1/3}}\right]$$

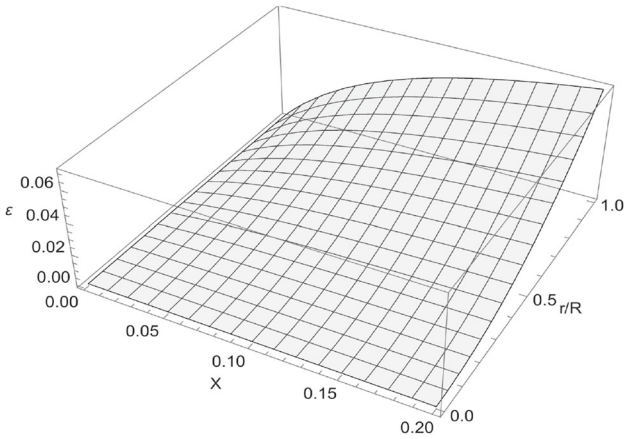

Fig. 1 Relative error from Eq. (3.20) for $q_V = 1$

$$\simeq \frac{1}{4\, q_V}\left\{1 - (1 + 9\, q_V\, X)^{2/3}\left[1 - \frac{2\, q_V\, r^2/R^2}{(1 + 9\, q_V\, X)}\right]\right\}. \tag{3.21}$$

The matching conditions across the surface also yield

$$M_0 = \frac{M}{(1 + 6\, q_V\, X)^{1/3}}, \tag{3.22}$$

and one notices a different numerical factor multiplying q_V in comparison with Eq. (3.17).

3.2.3 Comparing the approximations

The main reason for this analysis is to understand if the field redefinition that brings the kinetic term in a canonical form, thus simplifying the equation of motion, leads to results that are in good agreement with those obtained without this transformation. The relative difference between the approximations in Eqs. (3.21) and (3.19) for small compactness is given by

$$\Delta \equiv \left|\frac{\tilde{V}_{\rm in} - V_{\rm in}}{V_{\rm in}}\right| \simeq \frac{q_V\, X}{6}\left|1 - \frac{5\, r^2}{3\, R^2}\right|. \tag{3.23}$$

which is roughly of the same order as the error (3.20) shown in Fig. 1.

Because of the non-linearity of the field equations, however, the above estimate remains of questionable relevance. In order to asses how reliable the analytical approximations are, we solve Eq. (2.7) numerically with the same boundary conditions (3.2)-(3.4) and denote the numerical solution in the interior as $\hat{V}_{\rm in}$. From the plot in Fig. 2, we see that $\tilde{V}_{\rm in}$, $V_{\rm in}$ and $\hat{V}_{\rm in}$ follow each other very closely, both for small and intermediate values of the compactness. In the lower panels of the same figure one can also see plots of the relative difference Δ for the same values of the compactness. Therefore, even though the approximate analytical expressions obtained

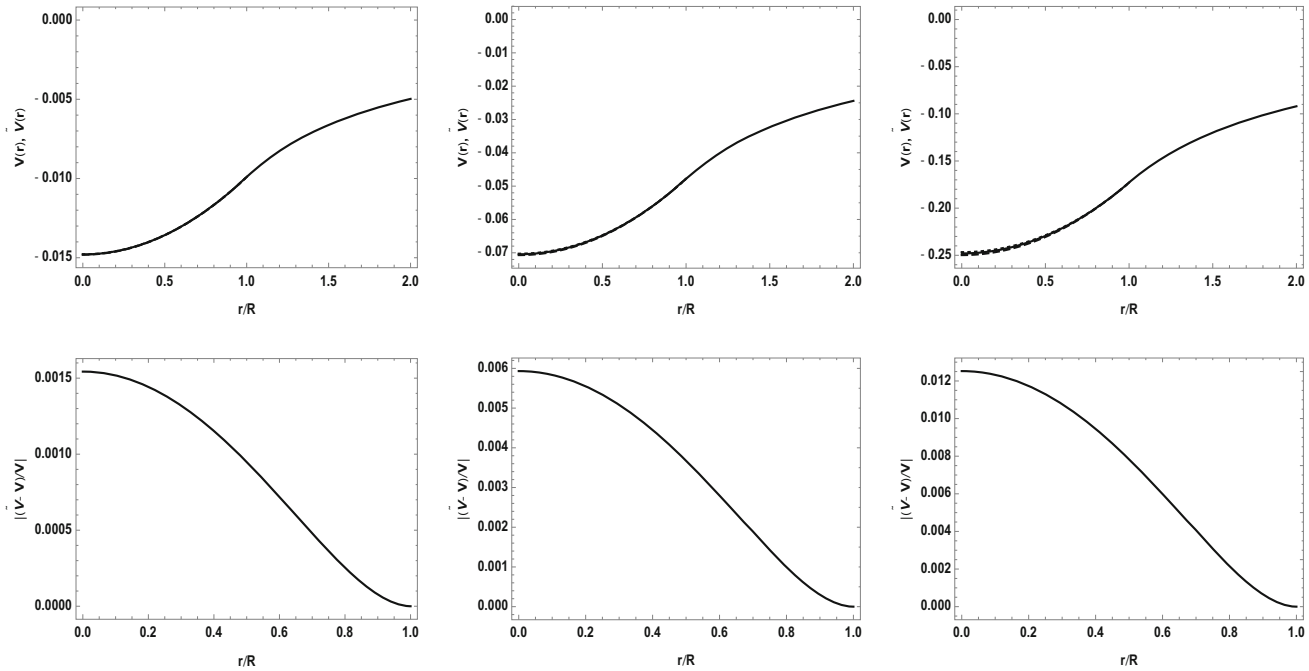

Fig. 2 Upper panels: inner potentials $\tilde{V}_{\rm in}$ (dotted line), $V_{\rm in}$ (dashed line), $\hat{V}_{\rm in}$ (solid line) and outer potential $\tilde{V}_{\rm out}$ (solid line) for $X = 1/100$ (left panel), $X = 1/20$ (center panel) and $X = 1/5$ (right panel). Lower panels: absolute value of the relative difference $\Delta = |(\tilde{V}_{\rm in} - V_{\rm in})/V_{\rm in}|$ for the same compactness as above

for the potential are different, both $\tilde{V}_{\rm in}$ and $V_{\rm in}$ appear to be in very good agreement with the numerical results.

The main difference between the two approximate solutions $\tilde{V}_{\rm in}$ and $V_{\rm in}$ is how $\tilde{M}_0$ and M_0 depend on M. Figure 3 shows the ratios $\tilde{M}_0/M$ and M_0/M as functions of the compactness X for the same values of q_V, respectively as functions of q_V for the same values of X.

The value of the coupling q_V defines different regimes of the theory. In the limit $q_V \to 0$, one recovers Newtonian physics, as it is obvious from Eq. (2.1), while the model is expected to approach General Relativity for $q_V \to 1$. This is why in Fig. 4 we display $|(\tilde{M}_0 - M_0)/M|$ for three different values of q_V. The relative difference increases with the compactness and with q_V. Therefore, it is largest for $q_V = 1$ for the largest compactness considered of $X = 0.2$. One notices that even in this case, the agreement remains fairly good.

3.3 Gaussian polytropic source

As a much less trivial example of relation between the ADM mass M and the (Newtonian) proper mass M_0, we consider a self-gravitating object described by the Gaussian density profile

$$\rho = \rho_0\, e^{-\frac{r^2}{b^2 R^2}}\, \Theta(R - r). \tag{3.24}$$

which was more extensively analysed in Ref. [4], using both numerical techniques and analytical approximations. This source becomes homogeneous for $b \gg 1$, while it is mostly concentrated around the centre for $b \ll 1$. We also assume a polytropic equation of state [4,15]

$$p = \gamma\, \rho(r) \left[\frac{\rho(r)}{\rho_0}\right]^{n-1} \equiv \gamma\, \frac{\rho^n(r)}{\rho_0^{n-1}}, \tag{3.25}$$

where γ and n are the polytropic parameters and the pressure clearly vanishes for $r > R$ due to Eq. (3.24).[4] In this case, the (Newtonian) proper mass (3.12) is given by

$$M_0 = \pi\, b^3\, R^3\, \rho_0 \left[\sqrt{\pi}\, \mathrm{Erf}\left(\frac{1}{b}\right) - \frac{2}{b}\, e^{-1/b^2}\right]. \tag{3.26}$$

In the quadratic approximation of Eq. (3.6), from the canonical field equation (3.1) and the matching condition (3.3), we find the new analytical approximation

$$\psi_{\rm in} \simeq \frac{64\,\pi^3\, R^9\, \rho_0^3\, (1+3\,\gamma)^3 - 27\, M^3}{384\,\pi^3\,\alpha\, R^9\, \rho_0^3\, (1+3\,\gamma)^3} + \frac{\sqrt{G_{\rm N}}\, M}{2\, R^3}\, r^2, \tag{3.27}$$

[4] As was shown in Ref. [4], $p(R) > 0$, but one can choose the polytropic parameters in such a way that the pressure on the surface is negligibly small. Alternatively one could assume that a thin solid crust with a tension that balances the non-vanishing pressure covers the surface of the object.

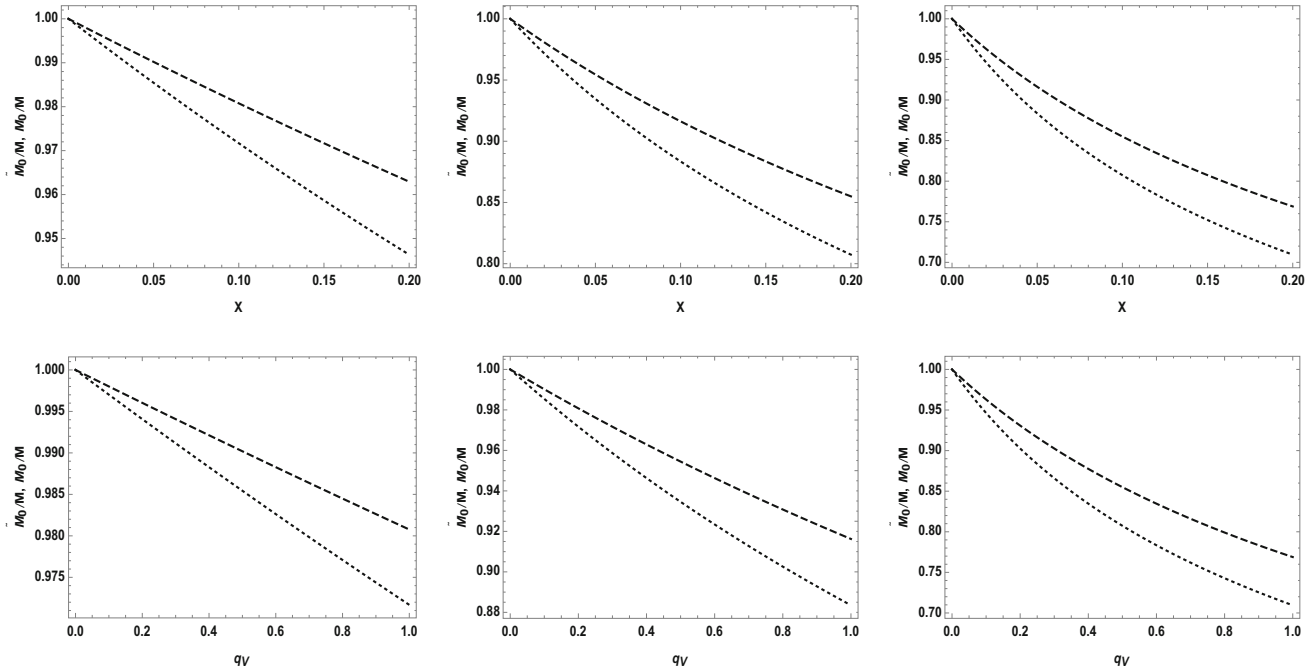

Fig. 3 Upper panels: ratios $\tilde{M}_0/M$ (dotted line) and M_0/M (dashed line) as functions of X for $q_V = 0.1$ (left panel), $q_V = 0.5$ (center panel) and $q_V = 1$ (right panel). Bottom panels: ratios $\tilde{M}_0/M$ (dotted line) and M_0/M (dashed line) as functions of q_V for $X = 1/100$ (left panel), $X = 1/20$ (center panel) and $X = 1/5$ (right panel)

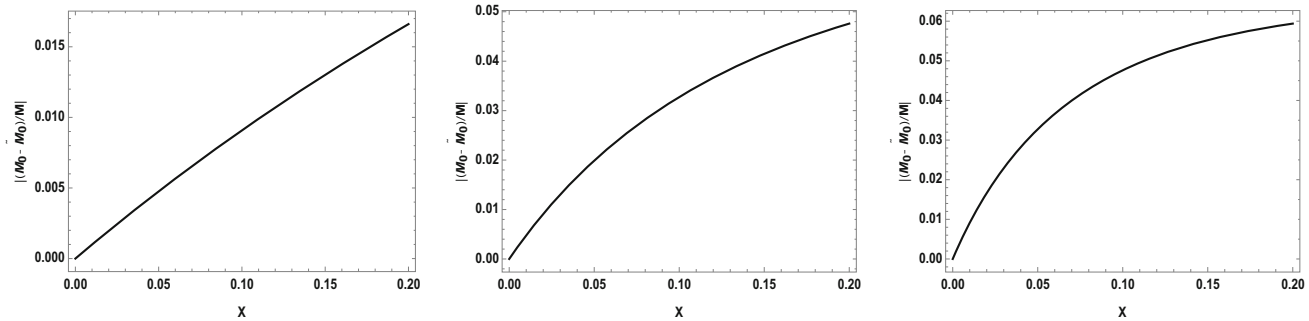

Fig. 4 Relative difference $|(\tilde{M}_0 - M_0)/M|$ as a function of X for $q_V = 0.1$ (left panel), $q_V = 0.5$ (center panel) and $q_V = 1$ (right panel)

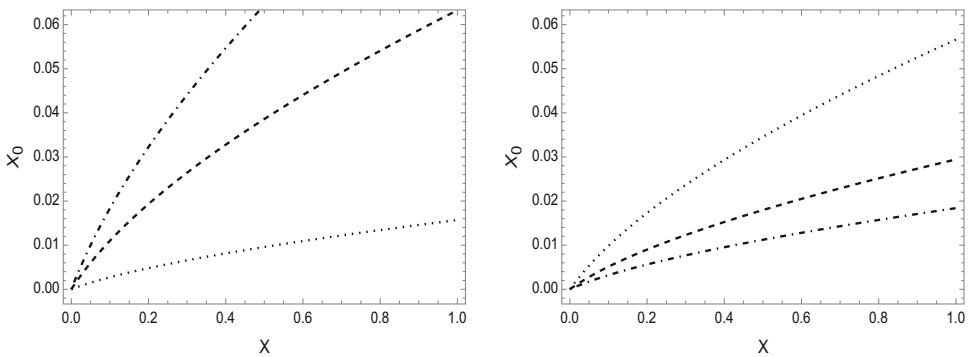

Fig. 5 Plots of X_0 as a function of X for $q_V = 1$. Left panel: $\gamma = 0.5$, respectively $b = 0.4$ (dotted), $b = 0.7$ (dashed) and $b = 1$ (dash-dotted). Right panel: $b = 0.5$, respectively $\gamma = 0.1$ (dotted), $\gamma = 0.5$ (dashed) and $\gamma = 1$ (dash-dotted)

where, from Eq. (3.26), the central density can be written in terms of M_0 as

$$\rho_0 = \frac{M_0}{\pi\, b^2\, R^3 \left[\sqrt{\pi}\, b\, \mathrm{Erf}\,(1/b) - 2\, e^{-1/b^2}\right]}. \tag{3.28}$$

The remaining matching condition (3.2) reads

$$\frac{64\,\pi^3\, R^9\, \rho_0^3\, (1+3\,\gamma)^3 - 27\, M^3}{384\,\pi^3\,\alpha\, R^9\, \rho_0^3\, (1+3\,\gamma)^3} = -\frac{3\,\sqrt{G_\mathrm{N}}\, M}{2\, R^{\frac{3}{2}}}, \tag{3.29}$$

which shows that the solution (3.27) is indeed the same as the one in Eq. (3.10). Moreover, in this approximation, neither ψ_in nor the boundary condition (3.29) depend on the polytropic index n.

The expression (3.29) can be written in terms of the compactness X in Eq. (2.5) and the analogue proper compactness $X_0 = G_\mathrm{N}\, M_0/R$ as

$$\frac{3}{2}\, X = \frac{27\, b^6\, X^3 \left[\sqrt{\pi}\, b\, \mathrm{Erf}\,(1/b) - 2\, e^{-1/b^2}\right]^3 - 64\, X_0^3\, (1+3\,\gamma)^3}{384\, q_V\, X_0^3\, (1+3\,\gamma)^3}, \tag{3.30}$$

which shows the dependence of X on X_0, equivalent to $M = M(M_0)$ for fixed R.[5] The plot of X_0 as a function of X for several particular cases is shown in Fig. 5. It is clear that $X_0 < X$ in all displayed cases, which emphasises once more that M and M_0 are generally different and setting them equal would only leave the trivial solution $M = M_0 = 0$. Figure 5 also shows that the ratio X_0/X increases with the parameter b for constant γ, respectively decreases with γ when b is kept the same.

4 Conclusions

After having explored extensively the bootstrapped Newtonian gravity model in a series of previous papers, we have now tested an alternative approach to finding solutions for various cases within the same model. In its standard form, besides the Laplacian, the Euler–Lagrange equation for the bootstrapped potential contains extra derivatives of the potential. This makes it very cumbersome (when at all possible) to obtain solutions, in addition to hindering the true degrees of freedom of the theory. After performing a field redefinition, one can write the theory in canonical form, which is easier to solve and has a more transparent interpretation.

In order to test the effectiveness of the new formulation in terms of the canonical potential, we solved the canonical equation of motion (2.18) for a general density profile in a quadratic approximation around the centre, and compared with the results obtained by solving the non-canonical equation. We emphasise that the field redefinition (2.12) allowed us to prove some general results: in the approximation (3.5), at least up to terms of order $(r^4)^{2/3}$,[6] the interior bootstrapped Newtonian potential does not depend on the density or pressure profile of the source, but only on its mass and radius. The density profile will, however, determine the relationship between the (Newtonian) proper mass M_0 and the ADM mass M of the source. The other striking and seemingly general property observed in all cases is that these two masses are different, $M_0 \neq M$, regardless of the values for any other parameters. This is a fundamental difference with respect to Newtonian gravity and is indeed expected due to the non-linear nature of the theory. Nonetheless, we remark that the Newtonian M_0 in our approach is different from the proper mass in general relativity, so that the discrepancy we find between M and M_0 could lead to important consequences in cosmology and astrophysics. Some of these phenomenological applications will be considered thoroughly in a separate paper [16].

We should also stress that, in principle, field redefinitions cannot change the physical content of a (classical) theory. Nevertheless, when one considers non-linear redefinitions, such as Eq. (2.12) employed in this paper, approximate solutions of the canonical equations are not in one-to-one correspondence with the approximate solutions of the non-canonical ones. In fact, truncating the solution of either side of the equivalence would require infinitely many terms from the other side. One should thus expect large errors when comparing truncated solutions of each side of the correspondence. We, however, showed that such an error is quite small for the regimes we are interested in, thus opening up the possibility of solving the equations of motion in a much simpler way.

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[5] This relation can be compared with the analogous Eq. (4.10) of Ref. [4], which instead contains the index n.

[6] The power 4 is from the Taylor expansion and the power 2/3 from the field transformation (2.12).

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