# Stabilization of Unstable Distributed Port-Hamiltonian Systems in Scattering Form

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***Abstract*—In this letter, we consider the exponential stabilization of a distributed parameter port-Hamiltonian system interconnected with an unstable finite-dimensional linear system at its free end and control input at the opposite one. The infinite-dimensional system can also have in-domain anti-damping. The control design passes through the definition of a finite-dimensional linear system that "embeds" the response of the distributed parameter model, and that can be stabilized by acting on the available control input. The conditions that link the exponential stability of the latter system with the exponential stability of the original one are obtained thanks to a Lyapunov analysis. Simulations are presented to show the pros and cons of the proposed synthesis methodology.**

***Index Terms*—Distributed-parameter systems, Lyapunov methods, port-Hamiltonian systems.**

## I. Introduction

DISTRIBUTED port-Hamiltonian systems [1], [2] are a framework for modeling, simulation and control design for physical systems described by partial differential equations (PDEs). The most popular control synthesis methodologies deal with the linear case and are based on the so-called energy-shaping *plus* damping injection paradigm, [3]. The control action, usually applied at the boundary of the spatial domain, is designed so that the energy function is modified in particular to shift the equilibrium, and some dissipative effect is added, [4]–[6]. Control design and stability proof rely on the hypothesis that the dynamics is passive or dissipative. This means that the plant has a sort of "stable behavior" since if the control input is set equal to zero, the total energy is not increasing.

In this letter, the stabilization problem of boundary control systems in port-Hamiltonian form that are not dissipative is tackled. The focus is on systems with one-dimensional spatial domain associated to the dynamics of two coupled transport equations, and with an unstable SISO finite-dimensional linear system at the uncontrolled side of the domain. The control input is at the other side. This is a particular case of the framework analyzed in [7] where the regulator has been obtained thanks to a backstepping transformation. The idea is to extend the reduction method illustrated in [8], [9] and applied to finite-dimensional linear systems with input delay to the mixed PDE and ordinary differential equation (ODE) models studied here. The result is a control action in state-feedback form and designed to transform the system into a new one that "embeds" the response of the PDE model and can be made exponentially stable. Such a finite-dimensional system is endowed with the same input of the original model and its dynamics is strongly related to the one of the system interconnected at the free end of the domain. The design procedure is rather simple, and the true question is to determine under which conditions the exponential stability of this latter system implies that the same property holds for the initial one.

The stability analysis is based on Lyapunov techniques for linear hyperbolic systems, see [10] for a complete overview. The Lyapunov functionals proposed in [11] are employed to determine under which conditions the closed-loop system is exponentially stable. Despite its simplicity, the regulator deals with port-Hamiltonian systems for which energy-based techniques fail. For example, passivity or the fact that the system interconnected at the free end has to be exponentially stable are not required. On the other hand, the approach inherits most of the "structural" problems of Lyapunov-based controllers designed for hyperbolic systems. Among them, in-domain cross-coupling between state variables, actuation only at one side of the domain and a finite-dimensional linear system at the other one with a large feedthrough gain limit the applicability of the method. In general, such limitations are not present when the controller is obtained via a backstepping transformation, [7]. However, the price to pay is a noteworthy computational effort to determine the feedback gains.

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## II. Problem Formulation

Let us consider the following infinite-dimensional port-Hamiltonian system, [1], [2]:

$$\frac{\partial}{\partial t}\begin{pmatrix}\xi_1(t,z)\\ \xi_2(t,z)\end{pmatrix} = \begin{pmatrix}-\lambda & 0\\ 0 & \lambda\end{pmatrix}\frac{\partial}{\partial z}\begin{pmatrix}\xi_1(t,z)\\ \xi_2(t,z)\end{pmatrix} + \underbrace{\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}}_{=:M}\begin{pmatrix}\xi_1(t,z)\\ \xi_2(t,z)\end{pmatrix}, \tag{1}$$

where, $z \in [0, \ell]$ is the spatial coordinate, $\xi := (\xi_1, \xi_2) \in L^2(0, \ell; \mathbb{R}^2)$ the state variable, and $\lambda$ a positive real constant. Despite its simplicity, this PDE is quite general and includes models of flexible structures, traveling waves or heat exchangers. In (1), $(\xi_1, \xi_2)$ are the wave or scattering variables [12], with $\xi_1$ associated to the propagation from 0 to $\ell$, and $\xi_2$ to the propagation in the opposite direction. Differently from what is usually done within the port-Hamiltonian framework where the dynamics is formulated in terms of the energy and co-energy variables, this linear hyperbolic form has been preferred because control synthesis and stability proof turn out to be simpler.

The next step deals with the definition of boundary input and output pairs for (1). Let $u_0, u_\ell \in \mathbb{R}$ be the inputs in $z = 0$ and $z = \ell$, respectively, given by

$$\begin{pmatrix}u_0(t)\\ u_\ell(t)\end{pmatrix} := \underbrace{\begin{pmatrix}0 & 0 & w_{01} & w_{02}\\ w_{\ell 1} & w_{\ell 2} & 0 & 0\end{pmatrix}}_{=:W}\begin{pmatrix}\xi_1(t,\ell)\\ \xi_2(t,\ell)\\ \xi_1(t,0)\\ \xi_2(t,0)\end{pmatrix}, \tag{2}$$

where $w_{ij} \in \mathbb{R}$, and so that $w_{01}, w_{\ell 2} \neq 0$. From [2, Th. 4.2], this requirement assures that, when $M + M^{\mathrm{T}} \leq 0$, (1) is a boundary control system in the sense of semi-group theory provided that $u_0(t)$ and $u_\ell(t)$ are of class $C^2$, [13, Definition 3.3.2]. From a physical point of view, the condition on $M$ means that no anti-dissipative effect is present along the spatial domain. The well-posedness result, however, can be extended to the general case, i.e., when in (1) the matrix $M$ is arbitrary, thanks to [13, Th. 3.2.1]. The outputs $y_0, y_\ell \in \mathbb{R}$ in $z = 0$ and $z = \ell$, respectively, are defined in a similar manner:

$$\begin{pmatrix}y_0(t)\\ y_\ell(t)\end{pmatrix} := \underbrace{\begin{pmatrix}0 & 0 & \tilde{w}_{01} & \tilde{w}_{02}\\ \tilde{w}_{\ell 1} & \tilde{w}_{\ell 2} & 0 & 0\end{pmatrix}}_{=:\tilde{W}}\begin{pmatrix}\xi_1(t,\ell)\\ \xi_2(t,\ell)\\ \xi_1(t,0)\\ \xi_2(t,0)\end{pmatrix}. \tag{3}$$

Here, $\tilde{w}_{ij} \in \mathbb{R}$, and so that $\tilde{w}_{02}, \tilde{w}_{\ell 1} \neq 0$. Besides, the matrix $\begin{pmatrix}W\\ \tilde{W}\end{pmatrix}$ has to be invertible.

Let us assume that the finite-dimensional linear system

$$\begin{cases}\dot{\bar{x}}(t) = A\bar{x}(t) + By_\ell(t)\\ u_\ell(t) = C\bar{x}(t) + Dy_\ell(t),\end{cases} \tag{4}$$

with $\bar{x} \in \mathbb{R}^n$ and matrices $A$, $B$, $C$, and $D$ constant and of proper dimensions, is interconnected to (1) in $z = \ell$. In addition, we suppose that such a system is controllable, and this implies that there exists a family of matrices $K \in \mathbb{R}^{1\times n}$ for which $A + BK$ is Hurwitz. The aim of this letter is to determine the control input $u_0(t)$ so that the system resulting from the interconnection of (1) with (4) is exponentially stable. The problem discussed here is a particular case of the one tackled in [7]. However, the main difficulties are still present. Among them, the fact that the PDE (1) could be characterized by the presence of a positive dissipative contribution, that the wave variables $\xi_1$ and $\xi_2$ could be algebraically coupled, i.e., $m_{12}, m_{21} \neq 0$, and that (4) could be unstable or have a positive feedthrough term. The design procedure is illustrated in Section III, while the stability analysis of the corresponding control action is performed in Section IV.

## III. Control Design

The design procedure relies on the definition of a finite-dimensional system whose dynamics is related to (4) and "embeds" the PDE model (1). The main difference with (4) is that the control input is $u_0(t)$, which can be chosen to make such a "new" system asymptotically stable. This methodology is related to the reduction-based approach proposed in [8] to stabilize linear systems with input delay, and extended in [9] to deal with linear, lumped-parameter systems in port-Hamiltonian form. It is easy to check, in fact, that the input delay follows from (1) by imposing $M = 0$, and looking at the dynamics of the $\xi_1$ coordinate only. The stability of the original system, i.e., the one resulting from the coupling of (1) with (4), with the control input $u_0(t)$ computed below is investigated in Section IV.

Let $f_1, f_2 \in L^2(0, \ell; \mathbb{R}^n)$ to be determined later, and let

$$v(t) := \bar{x}(t) + \int_0^\ell \big[f_1(z)\xi_1(t,z) + f_2(z)\xi_2(t,z)\big]\mathrm{d}z. \tag{5}$$

Clearly, $v \in \mathbb{R}^n$. Moreover, we have that

$$\begin{aligned}\dot{v} &= Av + By_\ell - A\int_0^\ell \big[f_1(z)\xi_1(z) + f_2(z)\xi_2(z)\big]\mathrm{d}z\\ &\quad + \int_0^\ell f_1(z)\left[-\lambda\frac{\partial \xi_1}{\partial z}(z) + m_{11}\xi_1(z) + m_{12}\xi_2(z)\right]\mathrm{d}z\\ &\quad + \int_0^\ell f_2(z)\left[\lambda\frac{\partial \xi_2}{\partial z}(z) + m_{21}\xi_1(z) + m_{22}\xi_2(z)\right]\mathrm{d}z,\end{aligned}$$

where the dependence on $t$ has been removed for the sake of clearness. Noticing that

$$\int_0^\ell f_i(z)\frac{\partial \xi_i}{\partial z}(z)\,\mathrm{d}z = -\int_0^\ell \frac{\mathrm{d}f_i}{\mathrm{d}z}(z)\xi_i(z)\,\mathrm{d}z + f_i(\ell)\xi_i(\ell) - f_i(0)\xi_i(0)$$

for $i = 1, 2$, and from (2), (3) and the last relation in (4) we get that

$$w_{\ell 1}\xi_1(\ell) + w_{\ell 2}\xi_2(\ell) = C\bar{x} + D\big[\tilde{w}_{\ell 1}\xi_1(\ell) + \tilde{w}_{\ell 2}\xi_2(\ell)\big],$$

which implies that

$$\xi_2(\ell) = (w_{\ell 2} - D\tilde{w}_{\ell 2})^{-1}\big[C\bar{x} + (D\tilde{w}_{\ell 1} - w_{\ell 1})\xi_1(\ell)\big], \tag{6}$$

and also that

$$y_\ell = \tilde{w}_{\ell 1}\xi_1(\ell) + \tilde{w}_{\ell 2}\xi_2(\ell), \tag{7}$$

where $\xi_2(t,\ell)$ is given by (6). From a physical point of view, $\gamma_2 := w_{\ell 2} - D\tilde{w}_{\ell 2} \neq 0$ in (6) means that the feedthrough term in (4) combined with the input / output pair defined in $z = \ell$ is not akin to a negative damper whose value matches the "impedance" of the infinite-dimensional system (1).

With simple calculations, we obtain that

$$\dot{v} = \bar{A}v + \bar{B}_0\xi_1(0) + \bar{B}_\ell\xi_1(\ell) + \bar{C}_0\xi_2(0) + \mathcal{I}_1 + \mathcal{I}_2, \quad (8)$$

where

$$\begin{aligned}\bar{A} &:= A + \gamma_2^{-1}\big[\tilde{w}_{\ell 2}B + \lambda f_2(\ell)\big]C\\ \bar{B}_0 &:= \lambda f_1(0)\\ \bar{B}_\ell &:= \tilde{w}_{\ell 1}B - \lambda f_1(\ell) + \gamma_1\gamma_2^{-1}\big[\tilde{w}_{\ell 2}B + \lambda f_2(\ell)\big]\\ \bar{C}_0 &:= -\lambda f_2(0)\end{aligned} \quad (9)$$

with $\gamma_1 := D\tilde{w}_{\ell 1} - w_{\ell 1}$, and

$$\begin{aligned}\mathcal{I}_1 &:= \int_0^\ell \xi_1(z)\left[\big(m_{11}I_n - \bar{A}\big)f_1(z) + m_{21}f_2(z) + \lambda\frac{\mathrm{d}f_1}{\mathrm{d}z}(z)\right]\mathrm{d}z\\ \mathcal{I}_2 &:= \int_0^\ell \xi_2(z)\left[\big(m_{22}I_n - \bar{A}\big)f_2(z) + m_{12}f_1(z) - \lambda\frac{\mathrm{d}f_2}{\mathrm{d}z}(z)\right]\mathrm{d}z, \end{aligned} \quad (10)$$

with $I_n$ the $n \times n$ identity matrix. The design procedure is now summarized in the following proposition.

*Proposition 1:* Let us consider (1) equipped with the boundary sensing and actuation signals defined by (2) and (3), and interconnected in $z = \ell$ to (4). As far as the input / output pairs $(u_0, y_0)$ and $(u_\ell, y_\ell)$ are concerned, assume that $w_{01}, \tilde{w}_{02}, \tilde{w}_{\ell 1}, w_{\ell 2} \neq 0$, and that the matrix $\begin{pmatrix} W \\ \tilde{W}\end{pmatrix}$ is invertible. Finally, let

$$\Gamma(\varphi) := \gamma_2^{-1}\big[\tilde{w}_{\ell 2}B + \lambda\varphi\big], \quad \varphi \in \mathbb{R}^n.$$

If there exists $\varphi \in \mathbb{R}^n$ so that $f_1, f_2 \in C^1(0, \ell; \mathbb{R}^n)$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} f_1 \\ f_2\end{pmatrix} = \lambda^{-1}\Phi(\varphi)\begin{pmatrix} f_1 \\ f_2\end{pmatrix} \quad (11)$$

where

$$\Phi(\varphi) := \begin{pmatrix} A + \Gamma(\varphi)C - m_{11}I_n & -m_{21}I_n \\ m_{12}I_n & -A - \Gamma(\varphi)C + m_{22}I_n\end{pmatrix}$$

and with

$$\begin{aligned} f_1(\ell) &= \lambda^{-1}\big[\tilde{w}_{\ell 1}B + \gamma_1\Gamma(\varphi)\big]\\ f_2(\ell) &= \varphi\\ f_2(0) &= 0, \end{aligned} \quad (12)$$

and if there exists $K_\xi \in \mathbb{R}^{1\times n}$ such that

$$\lambda\Big[f_1(0)K_\xi + \gamma_2^{-1}\varphi C\Big] = BK - \gamma_2^{-1}\tilde{w}_{\ell 2}BC \quad (13)$$

for some $K \in \mathbb{R}^{1\times n}$ that makes the matrix $A + BK$ Hurwitz, then (5) goes to 0 exponentially.

*Proof:* This result is a consequence of the previous computations. In fact, if the functions $f_1$ and $f_2$ exist, from (10) and (11) we have that $\mathcal{I}_1 = \mathcal{I}_2 = 0$, while from (9) and (12) that $\bar{B}_\ell = 0$ and $\bar{C}_0 = 0$. Then, for the quantity defined in (5), along the system trajectories we get that (8) holds, which in this case becomes

$$\dot{v}(t) = \Big\{A + \gamma_2^{-1}\big[\tilde{w}_{\ell 2}B + \lambda\varphi\big]C\Big\}v(t) + \lambda f_1(0)\xi_1(t, 0).$$

If (13) holds, by selecting

$$\xi_1(t, 0) = K_\xi v(t), \quad (14)$$

we obtain that $\dot{v} = (A + BK)v$, with $A + BK$ Hurwitz, and so $v(t)$ goes to 0 exponentially. ■

The idea of the previous proposition is to employ a stabilizing law, i.e., a gain $K$, for (4) to obtain the one when the PDE model (1) is present. Even if several choices for $K$ are possible, it is not immediate to find out when (11) is satisfied for a $K_\xi$, in particular when $n > 1$, and because of the generality of the boundary input / output mapping for (1) defined by (2) and (3). However, since the goal is to let $v(t)$ go to zero, we can state the following simpler result.

*Corollary 1:* Under the conditions of Proposition 1 except that the pair $(A, B)$ of (4) is controllable, if the pair $(\bar{A}, \bar{B}_0)$. with $\bar{A}$ and $\bar{B}_0$ defined in (9), is controllable, there exists $K_\xi \in \mathbb{R}^{1\times n}$ such that $\bar{A} + \bar{B}_0K_\xi$ is Hurwitz, and so $v(t)$ goes to 0 exponentially thanks to (14).

The previous results are instrumental for defining a control action in $z = 0$ that makes the overall system exponentially stable. To get the stabilizing $u_0(t)$, from (3) we have $\xi_2(t, 0) = \tilde{w}_{02}^{-1}[y_0(t) - \tilde{w}_{01}\xi_1(t, 0)]$, and then from (2)

$$\begin{aligned} u_0(t) &= w_{01}\xi_1(t, 0) + w_{02}\xi_2(t, 0)\\ &= \Big(w_{01} - w_{02}\tilde{w}_{02}^{-1}\tilde{w}_{01}\Big)\xi_1(t, 0) + w_{02}\tilde{w}_{02}^{-1}y_0(t). \end{aligned}$$

Finally, from (14), the control action turns out to be

$$\begin{aligned} u_0(t) &= K'v(t) + K_y y_0(t)\\ &= K'\bar{x}(t) + K_y y_0(t) + K'\int_0^\ell \big[f_1(z)\xi_1(t, z) + f_2(z)\xi_2(t, z)\big]\mathrm{d}z, \end{aligned} \quad (15)$$

where $K_y := w_{02}\tilde{w}_{02}^{-1}$ and $K' := (w_{01} - \tilde{w}_{01}K_y)K_\xi$. Note that (15) requires the knowledges of the full state of (1), so the development of observers and how to use their estimate in (15) is a fundamental topic to investigate.

*Remark 1:* The main difficulty in the design procedure is to find $\varphi \in \mathbb{R}^n$ so that $f_1$ and $f_2$ exist. This problem cannot be solved explicitly except in some cases (e.g., when $M = 0$ or diagonal). Since the ODE (11) is linear, of order $2n$, and with the conditions in $z = \ell$ for $(f_1, f_2)$ parameterized by $\varphi$, the value of $\varphi$ follows by requiring that $f_2(0) = 0$, which is equivalent to impose $n$ independent constraints. Such a $\varphi$ can be computed numerically with few lines of code.

*Remark 2:* The control law (15) has the same expression of the ones obtained in [5], [6], [14]–[16] thanks to passivity arguments, and in [7] by relying on the backstepping transformation. So, there are no relevant differences as far as its "practical" implementation is concerned. Besides, the gains that appear in (15) are obtained in a simple way, similar as in the design methodologies based on passivity arguments. On the other hand, if compared to approaches that exploit a backstepping transformation, the computations are less demanding, but at the same time, the stability result is not so general, since exponential stability has been proved only for a smaller class of systems.

## IV. Stability Analysis

The aim is to study under which conditions the control law (15) makes the system obtained from the interconnection of (1) with (4) exponentially stable. The analysis is based on Lyapunov arguments, and relies on [11]. Stability is achieved thanks to (15) when two conditions are met. The first one is that in the infinite-dimensional system (1) the $\xi_1$ and $\xi_2$ coordinates are not "strongly coupled" and $\ell$ is not "large". Secondly, when the feedthrough term in (4) combined with the definition of the input and output signals (2) and (3) in $z = \ell$ generates a "sufficiently small" algebraic coupling. More details are given in Remark 3 after the Lyapunov analysis.

Let us consider the Lyapunov function

$$V_1(v(t)) := \frac{1}{2} v^{\mathrm{T}}(t) Q_v v(t), \text{ with } Q_v = Q_v^{\mathrm{T}} > 0. \tag{16}$$

From Proposition 1 and the Lyapunov Theorem for finite-dimensional linear systems, we can select $Q_v$ so that

$$\dot{V}_1(v(t)) \leq -\delta_1 v^{\mathrm{T}}(t) v(t) \tag{17}$$

for some $\delta_1 > 0$. On the other hand, for the PDE model (1), we take the Lyapunov function

$$V_2(\xi_1(t,\cdot), \xi_2(t,\cdot)) = \frac{1}{2}\int_0^\ell \Big[q_1(z)\xi_1^2(t,z) + q_2(z)\xi_2^2(z)\Big]\mathrm{d}z, \tag{18}$$

where $q_1, q_2 \in L^2(0,\ell;\mathbb{R}_{\geq 0})$ are specified later. Here, $\mathbb{R}_{\geq 0}$ denotes the set of positive real numbers. As in [11], we get

$$\begin{aligned}\dot{V}_2 = \lambda \int_0^\ell &\Big[-q_1(z)\xi_1(z)\frac{\partial \xi_1}{\partial z}(z) + q_2(z)\xi_2(z)\frac{\partial \xi_2}{\partial z}(z)\Big]\mathrm{d}z \\ &+ \int_0^\ell \Big[m_{11}q_1(z)\xi_1^2(z) + m_{22}q_2(z)\xi_2^2(z)\Big]\mathrm{d}z \\ &+ \int_0^\ell \big[m_{12}q_1(z) + m_{21}q_2(z)\big]\xi_1(z)\xi_2(z)\,\mathrm{d}z.\end{aligned} \tag{19}$$

By integrating by parts, the first term in (19) becomes

$$\begin{aligned}\frac{\lambda}{2}\int_0^\ell &\Big[\frac{\mathrm{d}q_1}{\mathrm{d}z}(z)\xi_1^2(z) - \frac{\mathrm{d}q_2}{\mathrm{d}z}(z)\xi_2^2(z)\Big]\mathrm{d}z \\ &+ \frac{\lambda}{2}\Big[q_1(0)\xi_1^2(0) - q_1(\ell)\xi_1^2(\ell) - q_2(0)\xi_2^2(0) \\ &\qquad + q_2(\ell)\xi_2^2(\ell)\Big].\end{aligned} \tag{20}$$

Now, from (2), (3), (4) and (5), we get that

$$\begin{aligned}u_\ell &= w_{\ell 1}\xi_1(\ell) + w_{\ell 2}\xi_2(\ell) \\ &= C\Big\{v - \int_0^\ell \big[f_1(z)\xi_1(z) + f_2(z)\xi_2(z)\big]\mathrm{d}z\Big\} + D y_\ell,\end{aligned}$$

which combined with (7) gives

$$\gamma_2\xi_2(\ell) = \gamma_1\xi_1(\ell) + Cv - \int_0^\ell \Big[\tilde{f}_1(z)\xi_1(z) + \tilde{f}_2(z)\xi_2(z)\Big]\mathrm{d}z$$

where $\tilde{f}_i(z) := C f_i(z)$. From the Young's inequality, i.e., $2ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$ for all $a, b, \varepsilon \in \mathbb{R}$ and $\varepsilon > 0$, we obtain

$$\begin{aligned}\gamma_2^2\xi_2^2(\ell) &\leq (1+\varepsilon_1+\varepsilon_2)\gamma_1^2\xi_1^2(\ell) + \Big(1+\varepsilon_1^{-1}+\varepsilon_3^{-1}\Big)(Cv)^2 \\ &+ \Big(1+\varepsilon_2^{-1}+\varepsilon_3\Big) \times \Big\{\int_0^\ell \Big[\tilde{f}_1(z)\xi_1(z) + \tilde{f}_2(z)\xi_2(z)\Big]\mathrm{d}z\Big\}^2,\end{aligned} \tag{21}$$

for any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, where the integral can be bounded by

$$2\int_0^\ell \Big[\tilde{f}_1^2(z)\xi_1^2(z) + \tilde{f}_2^2(z)\xi_2^2(z)\Big]\mathrm{d}z.$$

We are now ready to state the following proposition that provides sufficient conditions for the exponential stability of the closed-loop system.

*Proposition 2:* Given the control law (14)-(15) obtained in Proposition 1, let $q_1, q_2 \in C^1(0,\ell;\mathbb{R}_{\geq 0})$ so that

$$\begin{aligned}&\frac{\mathrm{d}q_1}{\mathrm{d}z}(z) < 0 \quad \frac{\mathrm{d}q_2}{\mathrm{d}z}(z) > 0 \\ &-\lambda^2\frac{\mathrm{d}q_1}{\mathrm{d}z}(z)\frac{\mathrm{d}q_2}{\mathrm{d}z}(z) > \big[m_{12}q_1(z) + m_{21}q_2(z)\big]^2\end{aligned} \tag{22}$$

for all $z \in [0, \ell]$. If there exists $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, such that for the functions $q_1$ and $q_2$ for which (22) holds we have that

$$2\Big(1+\varepsilon_2^{-1}+\varepsilon_3\Big)q_2(\ell)\tilde{f}_i^2(z) < \gamma_2^2(\delta_2 - m_{ii})q_i(z) \tag{23}$$

for $i = 1, 2$ and all $z \in [0, \ell]$, and that

$$\gamma_1^2(1+\varepsilon_1+\varepsilon_2)q_2(\ell) \leq \gamma_2^2 q_1(\ell), \tag{24}$$

then the closed-loop system is exponentially stable.

*Proof:* Based on (16) and (18), we consider the Lyapunov function $V(\bar{x},\xi) := \kappa V_1(v(\bar{x},\xi)) + V_2(\xi)$, where $\kappa \in \mathbb{R}$ is a positive constant. In [11, Sec. 5], the existence of $q_1$ and $q_2$ such that (22) holds for a PDE in the same form as (1) has been proved. So, there exists $\delta_2 > 0$ such that

$$\begin{aligned}&\frac{\lambda}{2}\Big(\frac{\mathrm{d}q_1}{\mathrm{d}z}\xi_1^2 - \frac{\mathrm{d}q_2}{\mathrm{d}z}\xi_2^2\Big) + (m_{12}q_1 + m_{21}q_2)\xi_1\xi_2 \\ &\qquad \leq -\delta_2\Big(q_1\xi_1^2 + q_2\xi^2\Big).\end{aligned} \tag{25}$$

Besides, starting from (19) and (20), we can write that

$$\begin{aligned}\dot{V}_2 \leq \frac{\lambda}{2}q_1(0)\big(K_\xi v\big)^2 &- \int_0^\ell (\delta_2 - m_{11})q_1(z)\xi_1^2(z)\,\mathrm{d}z \\ &- \int_0^\ell (\delta_2 - m_{22})q_2(z)\xi_2^2(z)\,\mathrm{d}z \\ &- \frac{\lambda}{2}\Big[q_1(\ell)\xi_1^2(\ell) - q_2(\ell)\xi_2^2(\ell)\Big],\end{aligned}$$

where (14) has been taken into account. With an eye on (21), we get that

$$\begin{aligned}q_2(\ell)\xi_2^2(\ell) - q_1(\ell)\xi_1^2(\ell) &\leq \big[(1+\varepsilon_1+\varepsilon_2)\gamma_1^2\gamma_2^{-2}q_2(\ell) - q_1(\ell)\big]\xi_1^2(\ell) \\ &+ q_2(\ell)\gamma_2^{-2}\Big(1+\varepsilon_1^{-1}+\varepsilon_3^{-1}\Big)(Cv)^2 + q_2(\ell)\gamma_2^{-2}\Big(1+\varepsilon_2^{-1}+\varepsilon_3\Big) \\ &\times \Big\{\int_0^\ell \Big[\tilde{f}_1(z)\xi_1(z) + \tilde{f}_2(z)\xi_2(z)\Big]\mathrm{d}z\Big\}^2 \\ &\leq q_2(\ell)\gamma_2^{-2}\Big(1+\varepsilon_1^{-1}+\varepsilon_3^{-1}\Big)(Cv)^2 + q_2(\ell)\gamma_2^{-2}\Big(1+\varepsilon_2^{-1}+\varepsilon_3\Big) \\ &\times \Big\{\int_0^\ell \Big[\tilde{f}_1(z)\xi_1(z) + \tilde{f}_2(z)\xi_2(z)\Big]\mathrm{d}z\Big\}^2\end{aligned}$$

because of (24). From (17) and (23), and if $\kappa$ is so that

$$2\frac{\delta_1\kappa}{\lambda} > q_1(0)\left\|K_\xi\right\|^2 + q_2(\ell)\gamma_2^2\left(1+\varepsilon_1^{-1}+\varepsilon_3^{-1}\right)\|C\|^2$$

we get that $\dot{V}(\bar{x}(t), \xi(t)) \leq -\Delta V(\bar{x}(t), \xi(t))$ for some $\Delta > 0$, which finally proves the exponential stability. ■

*Remark 3:* As already pointed out, (22) does not pose particular constraints on the existence of $q_1$ and $q_2$ that appear in the Lyapunov analysis. The challenge is to meet (23) and (24). If $m_{12} = m_{21} = 0$, i.e., the $\xi_1$ and $\xi_2$ coordinates are not algebraically linked, $q_1$ and $q_2$ can be chosen so that (25) is satisfied for an arbitrarily large $\delta_2$. So, (23) holds for all $m_{ii}$, $i = 1, 2$, and this implies that even in-domain instability in the PDE (1) does not cause any particular issue. Besides, as far as relation (24) is concerned, note that (4) can be re-written in terms of the (boundary) input $\xi_1(t, \ell)$ and output $\xi_2(t, \ell)$ as

$$\begin{cases} \dot{x}(t) = A_\xi x(t) + B_\xi \xi_1(t, \ell) \\ \xi_2(t, \ell) = C_\xi x(t) + D_\xi \xi_1(t, \ell), \end{cases} \tag{26}$$

where $A_\xi := A + \tilde{w}_{\ell 2}\gamma_2^{-1}BC$, $B_\xi := (\tilde{w}_{\ell 1} + \tilde{w}_{\ell 2}\gamma_2^{-1}\gamma_1)B$, $C_\xi := \gamma_2^{-1}C$, and $D_\xi := \gamma_2^{-1}\gamma_1$. From (24) and as in [11, Th. 2], there is an upper bound on the absolute value of feedthrough term $D_\xi$ that depends on the functions $q_1$ and $q_2$ evaluated in $z = \ell$, and for which exponential convergence is achieved. More precisely, we get that $D_\xi^2 < q_1(\ell)q_2^{-1}(\ell)$. This fact is intrinsic in all the stability results that rely on quadratic functionals in the form (18). It is important to note that, in general, design methods for linear hyperbolic systems based on the backstepping transformation such as [7] do not suffer from such limitations.

## V. Numerical Examples

In this section, numerical results that illustrate pros and cons of the design methodology discussed in Sections III and IV are presented. Starting from the discussion in Remark 3, the main properties of the control law (15) are better investigated once for the system interconnected in $z = \ell$, the expression (26) is employed. As a matter of fact, the control input takes the simpler expression (14). For simplicity, we assume that $n = 1$, and that the initial condition for the closed-loop system is $(\bar{x}(0), \xi_1(0, z), \xi_2(0, z)) = (\bar{x}_\star, \xi_{1\star}, \xi_{2\star})$, so the state of the PDE is constant along the spatial domain in $t = 0$. Based on the second relation in (26), the condition $\xi_{2\star} = C_\xi \bar{x}_\star + D_\xi \xi_{1\star}$ has to hold.

In the numerical examples presented below, we suppose that $\xi_{1\star} = -\xi_{2\star} = 1$, while for the PDE model (1), we have that $\lambda = 0.33$ and $\ell = 2$. A finite difference scheme has been adopted for its simulation, [17, Ch. 1]. Finally, as far as (26) is concerned, we set $A_\xi = 0.3$, $B_\xi = C_\xi = 0.5$ and $D_\xi = 0.8$. Note that such a system is unstable. In the first simulation, the $\xi_1$ and $\xi_2$ coordinates in (1) are not algebraically linked since $m_{12} = m_{21} = 0$. Besides, we have assumed that $m_{11} = 0.2$ and $m_{22} = -0.1$. As discussed in Corollary 1, since it is obtained that $\bar{A} = 0.30$ and $\bar{B}_0 = 0.2727$, the control gain $K_\xi = -10$ in (14) makes the $v$ dynamics asymptotically stable. The behavior of the closed-loop system is summarized by the graphs in Fig. 1. As pointed out in Remark 3, even if the finite-dimensional linear system is unstable, the PDE model

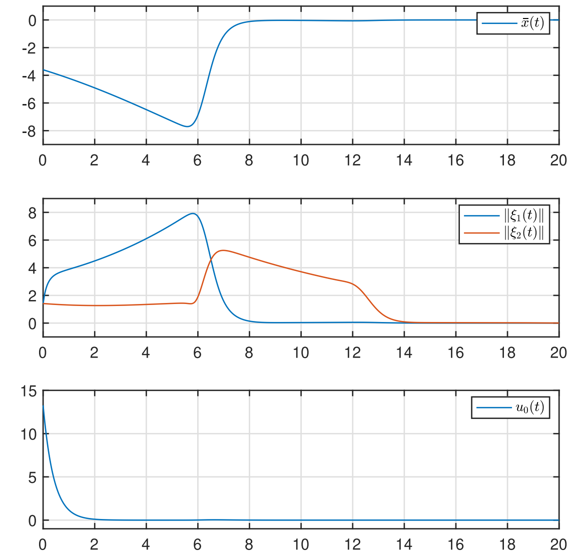


Fig. 1. In (1), we have $m_{12} = m_{21} = 0$, $m_{11} = 0.2$, and $m_{22} = -0.1$, while in (14) we have $K_\xi = -10$.

(1) does not prevent to exponentially stabilize the system due to the lack of cross-coupling between the wave variables. This result is in line with what has been presented, e.g., in [9], where just the pure delay on the input signal has been taken into account.

The case in which the $\xi_1$ and $\xi_2$ coordinates are algebraically linked has been taken into account in a second simulation. All the parameters remain the same as in the previous case, with the exception of $m_{12}$ and $m_{21}$ that have been set equal to 0.35 and $-0.2$, respectively. The response of the closed-loop system with such a new choice in the parameters is reported in Fig. 2. Since $\bar{A} = 0.4467$ and $\bar{B}_0 = 0.1106$, no change in the $K_\xi$ gain is necessary. It is immediate to note how coupling causes a performance degradation: for example, the steady-state is reached after a longer time interval. The anti-stable behavior in the $\xi_1$ coordinate associated to the $m_{11}$ coefficient, prevents to effectively compensate the effect due to the $\xi_2$ dynamics. In fact, if the $m_{12}$ coefficient is increased, e.g., to 0.7, the closed-loop system is unstable, and this fact is in line with the analysis in Remark 3. However, since $\bar{A} = 0.5814$ and $\bar{B}_0 = 0.0505$, asymptotic stability of the $v$ dynamics is guaranteed with $K_\xi = -100$, and then also for the overall system. The results are reported in Fig. 3. The oscillatory behavior, e.g., in the evolution of $\bar{x}$ reveals that the system is close to instability.

## VI. Conclusion and Future Work

This letter contribution is a design procedure of *simple* control actions for one-dimensional, distributed parameter port-Hamiltonian systems associated to the dynamics of coupled transport equations. Such a PDE model is interconnected to an unstable finite-dimensional linear system at its free end, is equipped with a control input at the other side, and can also have in-domain anti-damping. To get the stabilizing law,

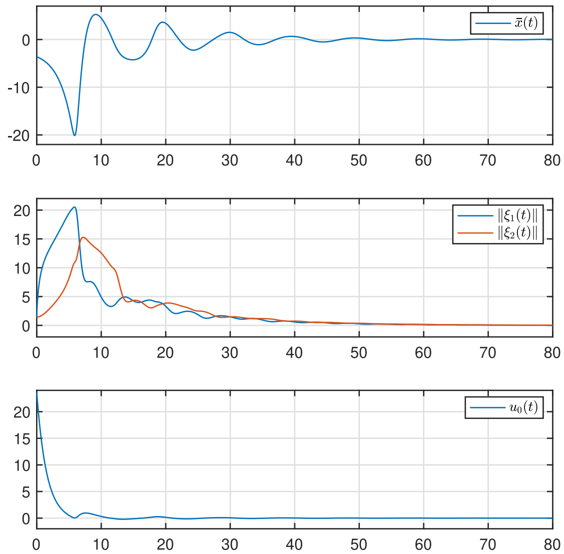


Fig. 2. In (1), we have $m_{12} = 0.35$ and $m_{21} = -0.2$, while $m_{11}$, $m_{22}$ and $K_\xi$ are the same as in Fig. 1.

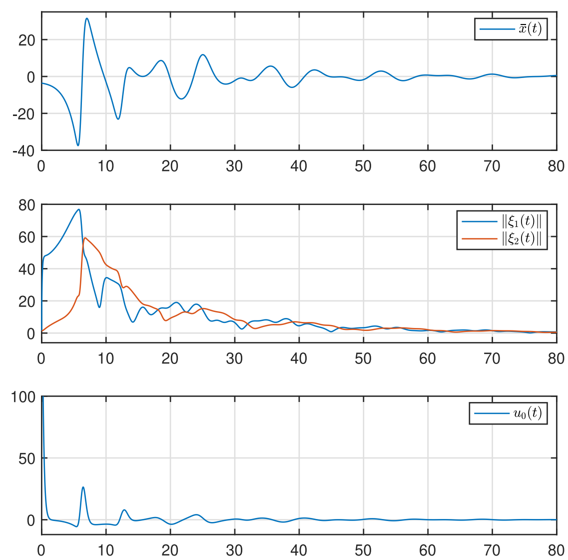


Fig. 3. In (1), we have $m_{12} = 0.7$ and $K_\xi = -100$, while $m_{11}$, $m_{21}$ and $m_{22}$ are the same as in Fig. 2.

a new finite-dimensional system that embeds the response of the infinite-dimensional dynamics, and has the same input of the original one is obtained. The key result has been to show under which conditions the exponential stability of this latter system implies that the same property is valid for the initial one.

Future researches are mainly focused on two topics. Since the feedback law relies on the knowledge of the full state of the PDE (1), the first one deals with observer design whose estimate has to be employed in the control action. The second subject, instead, is about the generalization of the proposed synthesis methodology to a wider class of boundary control systems (possibly nonlinear) in port-Hamiltonian form, for which the instability source is either at the boundary of the spatial domain, and inside the domain itself. Exponential convergence is expected to be achieved under less restrictive conditions than the ones obtained here.

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