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Alessia Cattabriga* and Michele Mulazzani

The complexity of orientable graph manifolds

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Abstract: We give an upper bound for the Matveev complexity of the whole class of closed connected orientable prime graph manifolds; this bound is sharp for all 14502 graph manifolds of the Recogniser catalogue (available at http://matlas.math.csu.ru/?page=search).

Keywords: Graph manifold, complexity.

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1 Introduction

Graph manifolds have been introduced and classified by Waldhausen in [15] and [16]. They are defined as compact 3-manifolds obtained by gluing Seifert fibre spaces along toric boundary components; so they can be described using labelled digraphs, as it will be explained in the next section.

Matveev in [13], see also [11], introduced the notion of complexity for compact 3-dimensional manifolds, as a way to measure how "complicated" a manifold is. Indeed, for closed irreducible and $\mathbb{P}^2$-irreducible manifolds the complexity coincides with the minimum number of tetrahedra needed to construct the manifold, with the only exceptions of S^3, $\mathbb{RP}^3$ and $L(3,1)$, all having complexity zero. Moreover, complexity is additive under connected sums and it is finite-to-one in the closed irreducible case. The last property has been used in order to construct a census of manifolds according to increasing complexity: for the orientable case, up to complexity 12 in the Recogniser catalogue (see http://matlas.math.csu.ru/?page=search), and for the non-orientable case, up to complexity 11 in the Regina catalogue (see https://regina-normal.github.io).

Upper bounds for the complexity of infinite families of 3-manifolds are given in [12] for lens spaces, in [10] for closed orientable Seifert fibre spaces and for orientable torus bundles over the circle, in [5] for orientable Seifert fibre spaces with boundary and in [2] for non-orientable compact Seifert fibre spaces. All the previous upper bounds are sharp for manifolds contained in the above cited catalogues. Furthermore, in [7] and [8] it has been proved that the upper bound given in [12] is sharp for two infinite families of lens spaces. Very little is known for the complexity of graph manifolds: in [6] and [4] upper bounds are given only for the case of graph manifolds obtained by gluing along the boundary two or three Seifert fibre spaces with disk base space and at most two exceptional fibres.

The main goal of this paper is to furnish a potentially sharp upper bound for the complexity of all closed connected orientable prime graph manifolds different from Seifert fibre spaces and orientable torus bundles over the circle. It is worth noting that the upper bounds given in Theorems 1, 2 and 3 are sharp for all 14502 manifolds of this type included in the Recogniser catalogue.

The organisation of the paper is the following. In Section 2 we recall some definitions and results about complexity and skeletons (Subsection 2.1), graph manifolds (Subsection 2.2) and theta graphs (Subsection 2.3). In Section 3 we state the results of the paper and in Section 4 we work out the proofs.

***Corresponding author: Alessia Cattabriga,** Department of Mathematics, University of Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy, email: alessia.cattabriga@unibo.it
Michele Mulazzani, Department of Mathematics and ARCES, University of Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy, email: michele.mulazzani@unibo.it

2 Preliminaries

2.1 Complexity and skeletons. A polyhedron P is said to be *almost simple* if the link of each point $x \in P$ can be embedded into K_4, the complete graph with four vertices. In particular, the polyhedron is called *simple* if the link is homeomorphic to either a circle, or a circle with a diameter, or K_4. A *true vertex* of an (almost) simple polyhedron P is a point $x \in P$ whose link is homeomorphic to K_4. A *spine* of a closed connected 3-manifold is a polyhedron P embedded in M such that $M \setminus P \cong B^3$, where B^3 is an open 3-ball. The *complexity* $c(M)$ of M is the minimum number of true vertices among all almost simple spines of M.

We will construct a spine for a given graph manifold by gluing skeletons of its Seifert pieces. Consider a compact connected 3-manifold M whose boundary either is empty or consists of tori. Following [9] and [10], a *skeleton of* M is a sub-polyhedron P of M such that (i) $P \cup \partial M$ is simple, (ii) $M \setminus (P \cup \partial M) \cong B^3$, (iii) for any component T^2 of ∂M the intersection $T^2 \cap P$ is a non-trivial theta graph[1]. Note that if M is closed then P is a spine of M. Given two manifolds M_1 and M_2 as above with non-empty boundary, let P_i be a skeleton of M_i, for $i = 1, 2$. Take two components $T_1 \subseteq \partial M_1$ and $T_2 \subseteq \partial M_2$ such that $P_i \cap T_i = \theta_i$ and consider a homeomorphism $\varphi : (T_1, \theta_1) \to (T_2, \theta_2)$. Then $P_1 \cup_\varphi P_2$ is a skeleton for $M_1 \cup_\varphi M_2$: we call this operation, as well as the manifold $M_1 \cup_\varphi M_2$, an *assembling* of M_1 and M_2.

2.2 Graph manifolds. We fix some notation for Seifert fibre spaces. We consider only oriented compact connected Seifert fibre spaces with non-empty boundary, described as $S = (g, d, (p_1, q_1), \dots, (p_r, q_r), b)$ where $g \in \mathbb{Z}$ coincides with the genus of the base space if it is orientable and with the opposite if it is non-orientable, $d > 0$ is the number of boundary components of S, (p_j, q_j) are lexicographically ordered pairs of coprime integers such that $0 < q_j < p_j$ for $j = 1, \dots, r$, describing the type of the exceptional fibres of S and $b \in \mathbb{Z}$ can be considered as a (non-exceptional) fibre of type $(1, b)$.

Up to fibre-preserving homeomorphism, we can assume (see [3]) that the Seifert pieces appearing in a graph manifold belong to the set $\mathcal{S}$ of the oriented compact connected Seifert fibre spaces with non-empty boundary that are different from fibred solid tori and from the fibred spaces $S^1 \times S^1 \times I$ and $N \tilde{\times} S^1$ (i.e., the orientable circle bundle over the Moebius strip N, which will be considered with the alternative Seifert fibre structure $(0, 1, (2, 1), (2, 1), b)$).

A Seifert fibre space $S = (g, d, (p_1, q_1), \dots, (p_r, q_r), b) \in \mathcal{S}$, with base space $B = p(S)$, is equipped with coordinate systems on the toric boundary components, as follows (see [11, p. 422]). Let B' be the compact surface obtained from B by removing the interior of $r + 1$ disks and denote with $c_1, \dots, c_{r+1}$ the boundary circles of these disks. Denote with $c_{r+2}, \dots, c_{r+d+1}$ all the remaining circles of $\partial B'$. Consider an orientable S^1-bundle S' over B'. In other words $S' = B' \times S^1$, if B' is orientable and $S' = B' \tilde{\times} S^1$ otherwise. Choose an orientation for S' and a section $s : B' \to S'$ of the projection map $p' : S' \to B'$. On each torus $T_h = p'^{-1}(c_h)$ choose a coordinate system (μ_h, λ_h) taking $s(c_h)$ as μ_h and a fibre $p'^{-1}(\{*\})$ as λ_h, for $h = 1, \dots, r + d + 1$. The orientations of λ_h and μ_h are chosen so that the intersection number of μ_h with λ_h is equal to 1 and the orientation of λ_h is induced by a fixed orientation of S^1 if $S' = B' \times S^1$ and is arbitrarily chosen otherwise. The manifold S is obtained from S' by attaching solid tori $V_h = D^2_h \times S^1$ to S' via homeomorphisms $f_h : \partial V_h \to T_h$, for $1 \le h \le r+1$, so that each f_h takes the meridian $\partial D^2_h \times \{*\}$ of V_h into a curve of type (p_h, q_h) for $1 \le h \le r$ and into the curve of type $(1, b)$ for $h = r+1$. Note that also the remaining boundary tori T_h, with $r+2 \le h \le r+d+1$, of S still possess coordinate systems (μ_h, λ_h).

Consider a finite connected non-trivial digraph $G = (V, E)$, where V is the set of vertices and E is the set of oriented edges of G. Given $e \in E$ denote with v'_e the starting vertex and with v''_e the ending one. Let $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $U = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and associate

- to each vertex $v \in V$ having degree d_v a Seifert fibre space $S_v = (g_v, d_v, (p_1, q_1), \dots, (p_{r_v}, q_{r_v}), b_v) \in \mathcal{S}$ (i.e., the degree of v is equal to the number of components of ∂S_v);

1 A *non-trivial* theta graph θ on a torus T^2 is a subset of T^2 homeomorphic to the theta graph (i.e., the graph with 2 vertices and 3 edges joining them) such that $T^2 \setminus \theta$ is an open disk.

- to each edge $e \in E$ a matrix $A_e = \begin{pmatrix} \alpha_e & \beta_e \\ \gamma_e & \delta_e \end{pmatrix} \in \mathrm{GL}_2^-(\mathbb{Z})$ such that $\beta_e \neq 0$ and $0 \leq \varepsilon_e\alpha_e, \varepsilon_e\delta_e < |\beta_e|$, where $\varepsilon_e = \beta_e/|\beta_e|$. We call a matrix in $\mathrm{GL}_2^-(\mathbb{Z})$ *normalised* if it satisfies these conditions. Moreover,
  (i) $A_e \neq \pm H$ when either $S_{v'_e}$ or $S_{v''_e}$ is the space $(0, 1, (2, 1), (2, 1), -1)$;
  (ii) when $|V| = 2$, $|E| = 1$ and $S_1 = (0, 1, (2, 1), (2, 1), b_1)$, $S_2 = (0, 1, (2, 1), (2, 1), b_2)$,
    (a) if $A = \pm H$ then $(b_1, b_2) \neq (0, 0), (-2, -2)$;
    (b) if $A = \pm\begin{pmatrix} 1 & \beta \\ 1 & \beta-1 \end{pmatrix}$ with $\beta > 1$, then $(b_1, b_2) \neq (-1, -2)$;
    (c) if $A = \pm\begin{pmatrix} \beta-1 & \beta \\ 1 & 1 \end{pmatrix}$ with $\beta > 1$, then $(b_1, b_2) \neq (0, -1)$.

The graph manifold M associated to the above data is obtained by gluing, for each edge $e \in E$ with starting vertex v'_e and ending vertex v''_e, a toric boundary component of $S_{v'_e}$ with one of $S_{v''_e}$, using the homeomorphism represented by A_e with respect to the fixed coordinate systems on the tori.[2] Clearly, M is a closed, orientable and connected graph manifold. On the other hand, each closed connected orientable prime graph manifold different from a Seifert fibre space and an orientable torus bundle over the circle can be obtained in this way; see [3, § 11]. We call G a *decomposition graph* of M.

If $G' = (V, E')$ is a spanning subgraph of a decomposition graph G, we denote by $M_{G'}$ the graph manifold (with boundary if $G' \neq G$) obtained by performing only the attachments corresponding to the elements of E'.

Remark 1. There is no restriction in assuming that all matrices associated to the edges of a decomposition graph are normalised: this is because of the following two operations that do not change the resulting graph manifold (see [3, § 11] and [14]):

1) replacement of the matrix A_e with A_eU^k and of the parameter $b_{v'_e}$ of the Seifert space $S_{v'_e}$ with the parameter $b_{v'_e} + k$;
2) replacement of the matrix A_e with U^kA_e and of the parameter $b_{v''_e}$ of the Seifert space $S_{v''_e}$ with $b_{v''_e} - k$.

Indeed, given a matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2^-(\mathbb{Z})$, let $k = -\lfloor \frac{\alpha}{\beta} \rfloor$ and $h = -\lfloor \frac{\delta}{\beta} \rfloor$ where $\lfloor x \rfloor$ denotes the floor of x. Then the matrix

$$A' = U^hAU^k = \begin{pmatrix} \alpha + k\beta & \beta \\ \gamma + h\alpha + k\delta + kh\beta & \delta + h\beta \end{pmatrix}$$

is normalised. Note that for a normalised matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ the following properties hold:

$$\beta\gamma > 0; \quad \text{if } \beta = \pm 1 \text{ then } A = \beta H; \quad \text{if } A \neq \pm H \text{ then } \beta/\delta > 0.$$

Moreover $A \in \mathrm{GL}_2^-(\mathbb{Z})$ is normalised if and only if $-A$ is normalised.

2.3 Theta graphs and Farey triangulation. Consider the upper half-plane model of the hyperbolic plane $\mathbb{H}^2$ and let $\mathbb{F}$ be the ideal Farey triangulation; see [1]. The vertices of $\mathbb{F}$ coincide with the points of $\mathbb{Q} \cup \{\infty\} \subset \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}^2$; the edges of $\mathbb{F}$ are geodesics in $\mathbb{H}^2$ with endpoints the pairs a/b, c/d such that $ad - bc = \pm 1$, with $\pm 1/0 = \infty$. Let $\Delta_{\frac{a}{b},\frac{c}{d},\frac{e}{f}}$ be the triangle of the Farey triangulation with vertices $a/b, c/d, e/f \in \mathbb{Q} \cup \{\infty\}$ and set $\Delta_+ = \Delta_{\infty,0,1}$, $\Delta_- = \Delta_{\infty,0,-1}$.

Let T^2 be a torus. It is a well-known fact that the vertex set of $\mathbb{F}$ is in bijection with the set of slopes (i.e., isotopy classes of non-contractible simple closed curves) on T^2 via $a/b \leftrightarrow a\mu + b\lambda$, where (μ, λ) is a fixed basis of $H_1(T^2)$. This bijection induces a bijection between the set of triangles of the Farey triangulation and the set $\Theta(T^2)$ of non-trivial theta graphs on T^2, considered up to isotopy. Indeed, given $\theta \in \Theta(T^2)$, consider the three slopes l_1, l_2, l_3 on T^2 formed by the pairs of edges of θ. The triangle associated to θ is Δ_{l_1,l_2,l_3}. Note that this bijection is well defined since the intersection index of l_i and l_j, with $i \neq j$, is always ± 1.

The graph $\mathbb{F}^*$ dual to $\mathbb{F}$ is an infinite tree. Given two triangles Δ and Δ' in $\mathbb{F}$ the distance $d(\Delta, \Delta')$ between them is the number of edges of the unique simple path joining the vertices v_Δ and $v_{\Delta'}$ corresponding to Δ and Δ' in $\mathbb{F}^*$, respectively. Given two theta graphs $\theta, \theta' \in \Theta(T^2)$ it is possible to pass from one to the other by a

2 If $\beta_e = 0$ for some A_e, then the gluing map sends a fibre of $S_{v'_e}$ into a fibre of $S_{v''_e}$. This implies that the decomposition of M in Seifert pieces is not minimal with respect to the number of cutting tori. For the same reason Conditions (i) and (ii) hold (see [3, p. 279]). Observe that we can assume $\beta_e > 0$, for any e belonging to a fixed spanning tree of G.

sequence of flip moves (see Figure 1): the distance on the set of triangles of the Farey triangulation induces a distance on $\Theta(T^2)$ such that $d(\theta, \theta')$ turns out to be the minimal number of flips necessary to pass from θ to θ'; see [10].

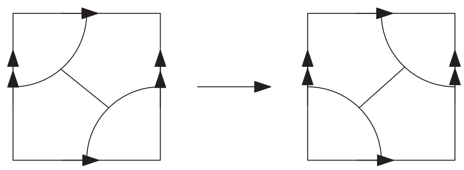

Figure 1: Two theta graphs connected by a flip move.

The group $\mathrm{GL}_2(\mathbb{Z})$ acts on $\mathbb{H}^2$ as isometries and $\mathbb{F}$ is invariant under this action: if we associate to a given triangle $\Delta_{\frac{a}{b},\frac{c}{d},\frac{e}{f}} \in \mathbb{F}$ the matrix $\left(\begin{smallmatrix} a & c & e \\ b & d & f \end{smallmatrix}\right)$, then the group $\mathrm{GL}_2(\mathbb{Z})$ acts on the set of triangles of the Farey triangulation by left multiplication.

The *complexity* c_A of a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ is defined as

$$c_A = \min\{d(A\Delta_-, \Delta_-), d(A\Delta_-, \Delta_+), d(A\Delta_+, \Delta_-), d(A\Delta_+, \Delta_+)\}.$$

Now we state a result about the complexity of normalized matrices. Let $S : \mathbb{Q}^+ \to \mathbb{N}$ be defined by $S(a/b) = a_1 + \cdots + a_k$, where

$$\frac{a}{b} = a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}$$

is the expansion of the positive rational number a/b as a continued fraction, with $a_1, \ldots, a_k > 0$.

Lemma 1. *Let* $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2^-(\mathbb{Z})$ *be a normalised matrix.*

- *If* $A = \pm H$ *then* $c_A = d(A\Delta_-, \Delta_-) = d(A\Delta_+, \Delta_+) = 0$.
- *If* $A \neq \pm H$ *then* $c_A = d(A\Delta_-, \Delta_+) = S(\beta/\delta) - 1$.

Proof. The first statement is straightforward since $\pm H\Delta_\pm = \Delta_\pm$. To prove the second one let $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \neq \pm H$ and so $|\beta| > 1$ (see Remark 1). Let $\mathcal{D}_{\beta/\delta}$ be the set of triangles of the Farey triangulation having a vertex in β/δ. By [9, Lemma 4.3] we have $\min\{d(\Delta, \Delta_+) \mid \Delta \in \mathcal{D}_{\beta/\delta}\} = S(\beta/\delta) - 1$. If A is normalised then

$$0 < \frac{\alpha}{\gamma} < \frac{\beta + \alpha}{\gamma + \delta} < \frac{\beta}{\delta} < \frac{\beta - \alpha}{\delta - \gamma}.$$

Indeed, since $\alpha\delta - \beta\gamma = -1$, we have

$$\frac{\alpha}{\gamma} < \frac{\alpha}{\gamma} + \frac{1}{\delta\gamma} = \frac{\alpha\delta + 1}{\delta\gamma} = \frac{\beta}{\delta}.$$

So,

$$0 < \frac{\alpha}{\gamma} = \frac{\frac{\alpha}{\gamma} + \frac{\alpha}{\delta}}{1 + \frac{\gamma}{\delta}} < \frac{\frac{\beta}{\delta} + \frac{\alpha}{\delta}}{1 + \frac{\gamma}{\delta}} = \frac{\beta + \alpha}{\delta + \gamma} = \frac{\frac{\beta}{\gamma} + \frac{\alpha}{\gamma}}{\frac{\delta}{\gamma} + 1} < \frac{\frac{\beta}{\gamma} + \frac{\beta}{\delta}}{\frac{\delta}{\gamma} + 1} = \frac{\beta}{\delta} = \frac{\frac{\beta}{\gamma} - \frac{\beta}{\delta}}{\frac{\delta}{\gamma} - 1} < \frac{\frac{\beta}{\gamma} - \frac{\alpha}{\gamma}}{\frac{\delta}{\gamma} - 1} = \frac{\beta - \alpha}{\delta - \gamma},$$

where we suppose $\delta - \gamma \neq 0$, otherwise the last inequality is straightforward.

Clearly $A\Delta_- = \Delta_{\frac{\alpha}{\gamma},\frac{\beta}{\delta},\frac{\beta-\alpha}{\gamma-\delta}}$, $A\Delta_+ = \Delta_{\frac{\alpha}{\gamma},\frac{\beta}{\delta},\frac{\beta+\alpha}{\gamma+\delta}}$ and the relative position of the triangles is represented in Figure 2, where for convenience we use the Poincaré disk model of $\mathbb{H}^2$. All triangles of $\mathcal{D}_{\beta/\delta}$ different from $A\Delta_-$ and $A\Delta_+$ are contained in the two hyperbolic half-planes depicted in gray. As a consequence, we have $\min\{d(\Delta, \Delta_+) \mid \Delta \in \mathcal{D}_{\beta/\delta}\} = d(A\Delta_-, \Delta_+)$. Since the path in $\mathbb{F}^*$ going from $v_{A\Delta_+}$ to v_{Δ_-} contains $v_{A\Delta_-}$ and v_{Δ_+}, we have $c_A = d(A\Delta_-, \Delta_+) = S(\beta/\delta) - 1$. □

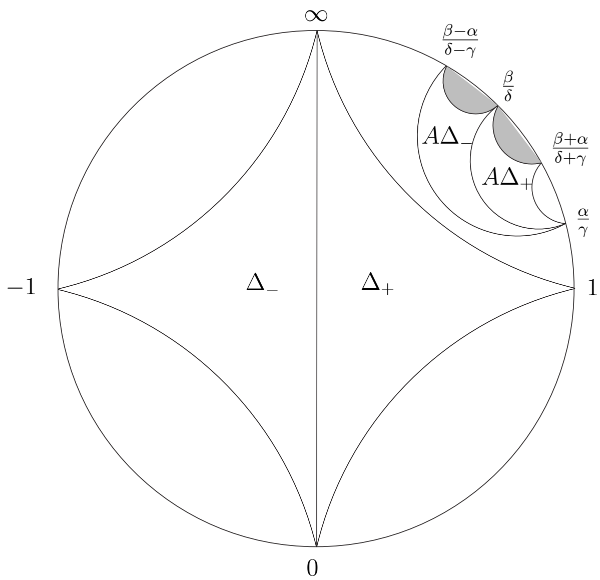

Figure 2: The Farey triangulation in the Poincaré disk model.

3 Complexity upper bounds

In this section we provide an upper bound for the complexity of graph manifolds. The general result is quite technical since it involves two partial colourings of the decomposition graph. So, before stating it, we deal with two special classes of graph manifolds that are interesting by their own. In all cases, the result is achieved by constructing a spine for a graph manifold: the description of the spines, as well as the proofs of the statements, are postponed in the next section.

Denote by E' the subset of E consisting of the edges associated to $\pm H$ (i.e., $A_e = \pm H$) and set $E'' = E \setminus E'$. Given $v \in V$ denote by d_v^+ (respectively d_v^-) the number of edges of E'' stating from (respectively ending in) the vertex v. Moreover, for $v \in V$ set $h_v = 2g_v$ (respectively $-g_v$) if the base space B_v of S_v is orientable (respectively non-orientable) where g_v denotes the genus of B_v.

Finally let $f_{m,M} : \mathbb{Z} \to \mathbb{N}$ defined by

$$f_{m,M}(b) = \begin{cases} m - b & \text{if } b < m \\ 0 & \text{if } m \leq b \leq M \\ b - M & \text{if } b > M \end{cases},$$

for $m, M \in \mathbb{Z}$, $m < M$, $m \leq 1$ and $M \geq -1$ (see the graph in Figure 3).

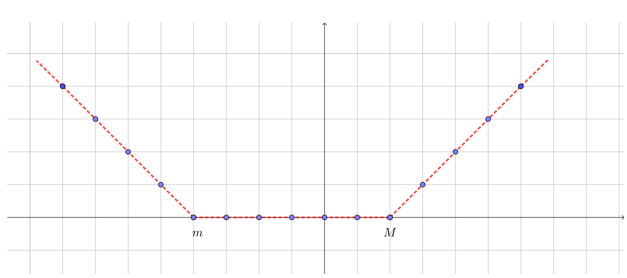

Figure 3: The graph of the function $f_{m,M}$.

The first result deals with the case $E' = \emptyset$, i.e., the one concerning manifolds with decomposition graphs without edges associated to $\pm H$.

Theorem 1. *Let M be a graph manifold associated to a decomposition graph $G = (V, E)$ having no edge associated to the matrices $\pm H$ (i.e., $E' = \emptyset$ and $E'' = E$). Then*

$$c(M) \leq 5(|E| - |V| + 1) + \sum_{e \in E}(S(\beta_e/\delta_e) - 1) + \sum_{v \in V}\Big(3(d_v + r_v + 2h_v - 2) + \sum_{k=1}^{r_v}(S(p_k/q_k) - 2) + f_{m_v,M_v}(b_v)\Big),$$

where $m_v = -r_v - h_v - d_v^- + 1$ and $M_v = h_v + d_v^+ - 1$.

The second case is the one of graph manifolds having a decomposition graph admitting a spanning tree containing all the edges associated to the matrices $\pm H$. This case seems to be rather technical, but it is quite interesting since, up to complexity 12, about 99% of all prime graph manifolds belong to this class (they are exactly 14346 out of 14502). To deal with this case we need to introduce a colouring on the edges of E' (that in this case are all contained in a spanning tree).

Consider $\Psi = \{\psi : E' \to \{+, -\}\}$ and given $\psi \in \Psi$ and $v \in V$ denote by $d^+_{v,\psi}$ (respectively $d^-_{v,\psi}$) the number of edges of E' incident to the vertex v and decorated with $+$ (respectively $-$).

Theorem 2. *Let M be a graph manifold associated to a decomposition graph $G = (V, E)$ such that all the edges associated to the matrices $\pm H$ are contained in a spanning tree of G. Then*

$$c(M) \leq 5(|E| - |V| + 1) + \sum_{e \in E''}(S(\beta_e/\delta_e) - 1) + \sum_{v \in V}\Big(3(d_v + r_v + 2h_v - 2) + \sum_{k=1}^{r_v}(S(p_k/q_k) - 2)\Big) +$$
$$+ \min_{\psi \in \Psi}\Big\{\sum_{v \in V} f_{m_v,M_v}(b_v)\Big\}, \quad \textit{where } m_v = -r_v - h_v - d_v^- - d^-_{v,\psi} + 1 \textit{ and } M_v = h_v + d_v^+ + d^+_{v,\psi} - 1.$$

Denote with $\mathcal{T}_G$ the set of all spanning trees of G and let $\phi : \mathcal{T}_G \to \mathbb{N}$ be the function defined by $\phi(T) = |(E - E_T) \cap E'|$, i.e., ϕ counts the number of edges not belonging to T and associated to the matrices $\pm H$. Let $\Phi(G) = \min\{\phi(T) \mid T \in \mathcal{T}_G\}$. The decomposition graphs of the manifolds involved in the previous result are characterised by the fact that $\Phi(G) = 0$. In the general case, we want to consider the spanning trees that minimise $\Phi(G)$: a spanning tree $T \in \mathcal{T}_G$ is called *optimal* if $\phi(T) \leq \phi(T')$ for any $T' \in \mathcal{T}_G$, that is if it realises the minimum of ϕ. We denote the set of optimal spanning trees of G with $\mathcal{O}_G$ and we decorate the edges associated to the matrices $\pm H$ with two colourings as follows:

$$\Psi_T = \{\psi : E'_T \to \{+, -\}\}, \quad \Psi'_T = \{\psi' : E' \setminus E'_T \to \{++, +, +-, -+, -, --\}\},$$

where E'_T are the edges of E' belonging to T. If $E'_T = E'$ (respectively $E'_T = \emptyset$) we have $\Psi'_T = \emptyset$ (respectively $\Psi_T = \emptyset$). Finally, given $\psi \in \Psi$, $\psi' \in \Psi'$ and $v \in V$ let

- $d^+_{v,\psi,T}$ and $d^-_{i,\psi,T}$ be the numbers of edges in T, incident to v and decorated with $+$ and $-$, respectively;
- $d^+_{v,\psi',T} = 2\left|\{e \in E' \setminus E'_T \mid v'_e = v, \psi'(e) = ++\}\right| + \left|\{e \in E' \setminus E'_T \mid v'_e = v, \psi'(e) = +\}\right| +$ $+ \left|\{e \in E' \setminus E'_T \mid v''_e = v, \psi'(e) = ++\}\right| + 2\left|\{e \in E' \setminus E'_T \mid v''_e = v, \psi'(e) = +\}\right| +$ $+ \left|\{e \in E' \setminus E'_T \mid v'_e = v, \psi'(e) = +-\}\right| + \left|\{e \in E' \setminus E'_T \mid v''_e = v, \psi'(e) = -+\}\right|$;
- $d^-_{v,\psi',T} = 2\left|\{e \in E' \setminus E'_T \mid v'_e = v, \psi'(e) = --\}\right| + \left|\{e \in E' \setminus E'_T \mid v'_e = v, \psi'(e) = -\}\right| +$ $+ \left|\{e \in E' \setminus E'_T \mid v''_e = v, \psi'(e) = --\}\right| + 2\left|\{e \in E' \setminus E'_T \mid v''_e = v, \psi'(e) = -\}\right| +$ $+ \left|\{e \in E' \setminus E'_T \mid v'_e = v, \psi'(e) = -+\}\right| + \left|\{e \in E' \setminus E'_T \mid v''_e = v, \psi'(e) = +-\}\right|$.

We are ready to state the general result.

Theorem 3. *Let M be a graph manifold associated to a decomposition graph $G = (V, E)$. Then*

$$c(M) \leq 5(|E| - |V| + 1) + \Phi(G) + \sum_{e \in E''}(S(\beta_e/\delta_e) - 1) + \sum_{v \in V}\Big(3(d_v + r_v + 2h_v - 2) + \sum_{k=1}^{r_v}(S(p_k/q_k) - 2)\Big) +$$
$$+ \min_{T \in \mathcal{O}_G}\Big\{\min_{\psi \in \Psi_T, \psi' \in \Psi'_T}\{\sum_{v \in V} f_{m_v,M_v}(b_v)\}\Big\},$$

where $m_v = -r_v - h_v - d_v^- - d^-_{v,\psi,T} - d^-_{v,\psi',T} + 1$ and $M_v = h_v + d_v^+ + d^+_{v,\psi,T} + d^+_{v,\psi',T} - 1$.

If, as in case of Theorem 2, there exists a spanning tree containing all the edges associated to $\pm H$, then clearly $\Phi(G) = 0$, $\Psi' = \emptyset$ and $E'_T = E'$, so the formula of the previous theorem reduces to the one of Theorem 2.

The sharpness of the previous upper bound in all known cases justifies the following

Conjecture. The upper bound given in Theorem 3 is sharp for all closed connected orientable prime graph manifolds.

4 Construction of the spines and proofs of the results

The aim of this section is to prove the results stated Section 3. In all cases the result is achieved by constructing a spine for a graph manifold starting from skeletons of its Seifert pieces. The construction of these skeletons is essentially the one described in [2], specialised to our case (i.e., orientable Seifert manifolds) and adapted to take care of the fact that the boundary components of the Seifert pieces will be glued together to obtain a closed graph manifold. Anyway, for the sake of the reader we recall, in the next subsection, how to construct a skeleton for the Seifert pieces. The construction and the number of true vertices of the resulting skeletons depend on some choices: we explain in the proofs of the theorems (see Sections 4.2, 4.3 and 4.4) how to fix them in order to minimise the number of true vertices of the spine.

4.1 Skeletons of Seifert pieces. Consider a Seifert manifold $S = (g, d, (p_1, q_1), \dots, (p_r, q_r), b) \in \mathcal{S}$. Let $S_0 = (g, d, (p_1, q_1), \dots, (p_r, q_r), 0)$ and let $S'_0 = (g, d + r + 1, 0)$ be the space obtained from S_0 by removing $r+1$ open fibred solid tori (with disjoint closures) which are regular neighbourhoods of the exceptional fibres of S_0 and of a regular fibre of type $(1, 0)$ contained in $\mathrm{int}(S_0)$. Then $S_0 \setminus \mathrm{int}(S'_0) = \Phi'_0 \sqcup \Phi_1 \sqcup \dots \sqcup \Phi_r$, where Φ_k (respectively Φ'_0) is a closed solid torus having the k-th exceptional fibre (respectively a regular fibre) as core. Let $p_0 : S_0 \to B_0$ and $p : S \to B$ be the projection maps and set $s = d + r$. Note that if $g \geq 0$ (respectively $g < 0$) then $B'_0 = p_0(S'_0)$ is a disk with $2g + s$ orientable (respectively $-g$ non-orientable and s orientable) handles attached. We recall from Section 3 that $h = 2g$ if $g \geq 0$ and $h = -g$ if $g < 0$.

Let $D = p_0(\Phi'_0)$ and let A_0 be the union of the disjoint arcs properly embedded in B'_0 depicted by thick lines in Figure 4. Then A_0 is non-empty and is composed of h edges with both endpoints in ∂D and s edges with an endpoint in ∂D and the other one in a different component of $\partial B'_0$. By construction, $B'_0 \setminus (A_0 \cup \partial B'_0)$ is homeomorphic to an open disk and the number of points of A_0 belonging to ∂D is at least three, since the conditions on the class $\mathcal{S}$ ensure that $s + 2h > 2$.

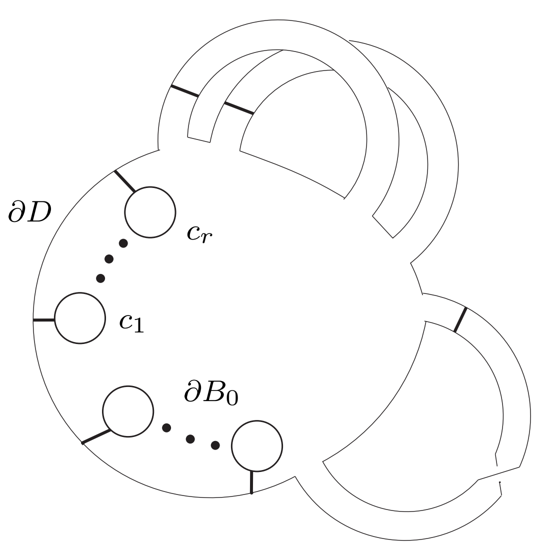

Figure 4: The set $A_0 \subset B'_0 \setminus \mathrm{int}(D)$, with $c_k = p_0(\partial \Phi_k)$.

Let $s'_0 : B'_0 \to S'_0$ be a section of p_0 restricted to S'_0. If $b \neq 0$, it is convenient to replace the fibre of type $(1, b)$ with $|b|$ fibres of type $(1, \mathrm{sign}(b))$. In this way the manifold S is obtained from S_0 by removing $|b|$ open trivially fibred solid tori (with disjoint closures) $\mathrm{int}(\Phi'_1), \ldots, \mathrm{int}(\Phi'_{|b|})$, each being a fibre-neighbourhood of regular fibres $\phi_1, \ldots \phi_{|b|}$ contained in $\mathrm{int}(S_0)$, and by attaching back $|b|$ solid tori $D^2 \times S^1$ via homeomorphisms $\psi_l : \partial(D^2 \times S^1) \to \partial\Phi'_l$ such that $\psi_l(\partial D^2 \times \{*\})$ is a curve of type $(1, \mathrm{sign}(b))$ on $\partial\Phi'_l$, with respect to a positive basis (μ_l, λ_l) of $H_1(\partial\Phi'_l)$, where $\mu_l = s'_0(p_0(\partial\Phi'_l))$ and λ_l is the fibre over a point $*' \in p_0(\partial\Phi'_l)$, for $l = 1, \ldots, |b|$. Referring to Figure 5, it is convenient to take the fibre ϕ_l corresponding to an internal point Q_l of A_0 and to suppose that $p_0(\Phi'_l)$ is a "small" disk intersecting the component δ_l of A_0 containing Q_l in an interval and being disjoint from $\partial B'_0$ and from the other components of A_0. In this way $\delta_l \setminus \mathrm{int}(p_0(\Phi'_l))$ is the disjoint union of two arcs δ'_l and δ''_l. Let $A = A_0 \setminus \cup_{l=1}^{|b|}\mathrm{int}(p_0(\Phi'_l))$ and note that p and p_0 coincide on $S_0 \setminus \cup_{l=1}^{|b|}\mathrm{int}(\Phi'_l)$.

Figure 5: The set $A \subset B'_0$.

Let $\bar{s} : (B'_0 \setminus (\bigcup_{l=1}^{|b|} \mathrm{int}(p_0(\Phi'_l)))) \cup D \to S$ be a section of p restricted to $p^{-1}((B'_0 \setminus (\bigcup_{l=1}^{|b|} \mathrm{int}(p_0(\Phi'_l)))) \cup D)$ and consider the polyhedron $P = \mathrm{Im}(\bar{s}) \cup p^{-1}(A) \cup \partial\Phi'_0 \bigcup_{l=1}^{|b|}(\partial\Phi'_l \cup_{\psi_l} (D^2 \times \{*\})) \subset S$. As represented in the central picture of Figure 6, the set $\mathrm{int}(\bar{s}(A))$ is a collection of quadruple lines in the polyhedron (the link of each point is homeomorphic to a graph with two vertices and four edges connecting them), and a similar phenomenon occurs for $\bar{s}(\partial D \setminus A)$. Therefore we change the polyhedron P performing "small" shifts by moving in parallel the disk $\bar{s}(D)$ along the fibration and the components of $p^{-1}(A)$ as depicted in the left and right pictures of Figure 6. It is convenient to think of the shifts of $p^{-1}(A)$ as performed on the components of A. Moreover, the shifts on δ'_l and δ''_l can be chosen independently.

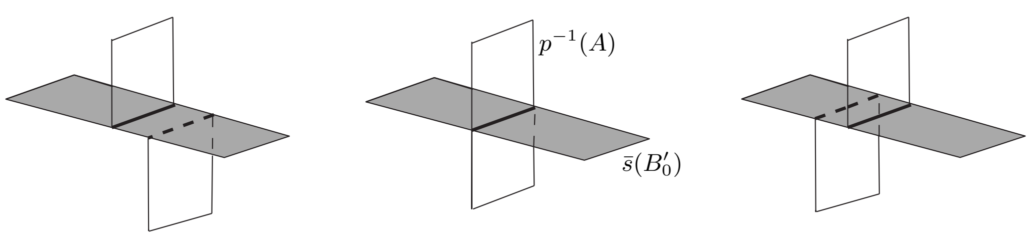

Figure 6: The two possible shifts on a component of $p^{-1}(A)$.

As shown by the pictures, the shift of any component of $p^{-1}(A)$ may be performed in two different ways that are not usually equivalent in terms of complexity of the final spine. On the contrary, the two possible parallel shifts for $\bar{s}(D)$ are equivalent as is evident from Figure 7, which represents the torus $\partial\Phi'_0$. Let

$$P' = \bar{s}\Big(B'_0 \setminus \Big(\bigcup_{l=1}^{|b|}\mathrm{int}(p_0(\Phi'_l))\Big)\Big) \cup D' \cup W' \cup \partial\Phi'_0 \bigcup_{l=1}^{|b|}\Big(\partial\Phi'_l \cup_{\psi_l} (D^2 \times \{*\})\Big)$$

be the polyhedron obtained from P after the shifts, where D' and W' are the results of the shifts of $\bar{s}(D)$ and $p^{-1}(A)$, respectively.

It is easy to see that $P' \cup \partial S \bigcup_{k=1}^{r} \partial\Phi_k$ is simple, P' intersects each component of ∂S and each torus $\partial\Phi_k$ in a non-trivial theta graph and the manifold $S \setminus (P' \cup \partial S \bigcup_{k=1}^{r} \Phi_k)$ is the disjoint union of $|b| + 2$ open balls. So in order to obtain a skeleton P'' for $S \setminus (\bigcup_{k=1}^{r} \mathrm{int}(\Phi_k))$ it is enough to remove a suitable open 2-cell from the torus $T_0 = \partial\Phi'_0$ and one from each torus $T_l = \partial\Phi'_l$, for $l = 1, \ldots, |b|$, connecting in this way the balls.

The graph $\Gamma_l = T_l \cap (\bar{s}(B'_0 \setminus \mathrm{int}(p_0(\Phi'_l))) \cup W' \cup \psi_l(\partial D^2 \times \{*\}))$ (respectively $\Gamma_0 = T_0 \cap (\bar{s}(B'_0) \cup D' \cup W')$) is cellularly embedded in T_l (respectively T_0) and its vertices with degree greater than 2 are true vertices of $P' \cup \partial S \cup_{k=1}^{r} \partial\Phi_k$: we will remove the region R_l (respectively R_0) of $T_l \setminus \Gamma_l$ (respectively $T_0 \setminus \Gamma_0$) having in the boundary the greatest number of vertices of Γ_l, for $l = 1, \dots, |b|$ (respectively Γ_0).

Referring to Figure 7, the graph Γ_0 is composed of two horizontal parallel loops $\xi = \partial(\bar{s}(D))$ and $\xi' = \partial D'$, and an arc with both endpoints on ξ for each boundary point of A belonging to ∂D. Changing the shift of a component of A has the same effect as performing a symmetry along ξ of the correspondent arc(s). A region of $T_0 \setminus \Gamma_0$ has 4 or 6 vertices when the non-horizontal arcs belonging to its boundary are not parallel or 5 vertices otherwise. So, except for the case where all the arcs are parallel there is always a region with 6 vertices.

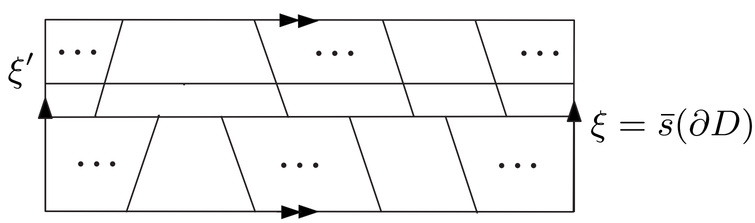

Figure 7: A fragment of the graph Γ_0 embedded in $\partial\Phi'_0$.

When $b \neq 0$, the graph Γ_l, for $l = 1, \dots, |b|$, is depicted in Figure 8 (respectively Figure 9) for a fibre of type $(1, 1)$ (respectively $(1, -1)$), just labelled by + (respectively −) inside the disk. If we take for δ'_l and δ''_l the shifts induced by that of δ_l, then we can choose as region R_l the gray one, containing in its boundary all vertices of Γ_l belonging to $\partial\Phi'_l$ except one (the thick points in the first two pictures). On the contrary, if one of the two shifts is changed as in the third draw of Figures 8 and 9, then R_l can be chosen containing in its boundary all the vertices of Γ_l belonging to $\partial\Phi'_l$.

We remark that changing the shift of a component of A changes the intersection between the corresponding element of W' and ∂S (which is a non-trivial theta graph) by a flip move (see Figure 1). We denote with P'' the skeleton obtained by removing the regions R_0 and R_l from P', for $l = 1, \dots, |b|$.

In order to construct a skeleton for Φ_k for $k = 1, \dots, r$, consider the skeleton P_F depicted in Figure 10: it is a skeleton for $T^2 \times [0, 1]$ with one true vertex and such that $\theta_0 = P_F \cap (T^2 \times \{0\})$ (the graph in the upper face) and $\theta_1 = P_F \cap (T^2 \times \{1\})$ (the graph in the bottom face) are two theta graphs differing for a flip move. Denote with Θ_{p_k/q_k} the subset of $\Theta(T^2)$, consisting of the theta graphs containing the slope corresponding to $p_k/q_k \in \mathbb{Q} \cup \{\infty\}$. Let θ_{p_k/q_k} be the theta graph in Θ_{p_k/q_k} that is closest to θ_+. The skeleton X_k for Φ_k is obtained by assembling several skeletons of type P_F connecting the theta graph $P'' \cap \Phi_k$ to a theta graph which is one step closer to θ_+ than θ_{p_j/q_j}, with respect to the distance on $\Theta(T^2)$; see [5]. The number of the required flips is either $S(p_j, q_j) - 2$ or $S(p_j, q_j) - 1$ depending on the shift chosen for the corresponding component of A used in the construction of the skeleton P''. We call the shift *regular* in the first case and *singular* in the second one (see Figure 11).

The skeleton P_S of S is obtained by assembling P'' with X_k, via the identity, for $k = 1, \dots, r$.

4.2 Proof of Theorem 1. Here we prove our first result. To begin with we need to discuss how to fix the choices in the construction of the skeleton P_S previously described, when the Seifert fibre space $S = (g, d, (p_1, q_1), \dots, (p_r, q_r), b)$ is a piece of a graph manifold having all gluing matrices different from $\pm H$. According to the notation introduced at the beginning of Section 3, we have $d = d^+ + d^-$ since $E' = \emptyset$.

Remark 2. Let θ_+ and θ_- be the theta graphs corresponding, respectively, to Δ_+ and Δ_- in the Farey triangulation. The intersection of each boundary component of S with the skeleton P_S is either θ_+ or θ_-, depending whether the shift of the corresponding component δ of A has been chosen as depicted in the left or right part of Figure 12, respectively.

We always choose these shifts such that exactly d^+ (respectively d^-) components have θ_- (respectively θ_+) as intersection with P_S. Suppose that $m \le b \le M$, where $m = -r - h - d^- + 1$ and $M = h + d^+ - 1$. If $b \le -1$

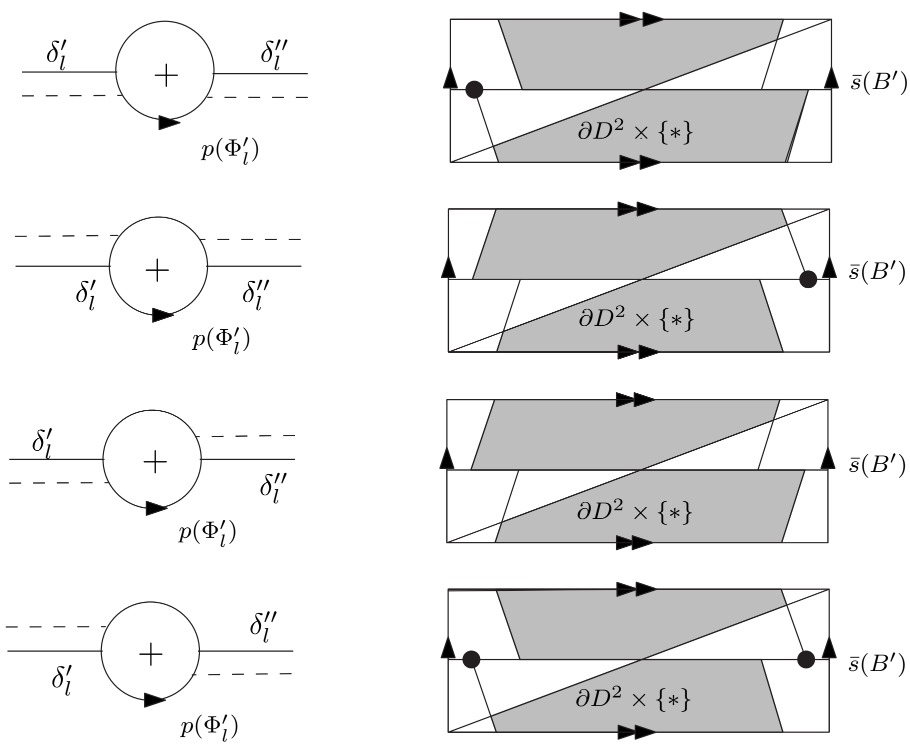

Figure 8: The graph Γ_l, with $b > 0$, embedded in $T_l = \partial\Phi_l'$ with different choices of the shifts for δ_l' and δ_l''.

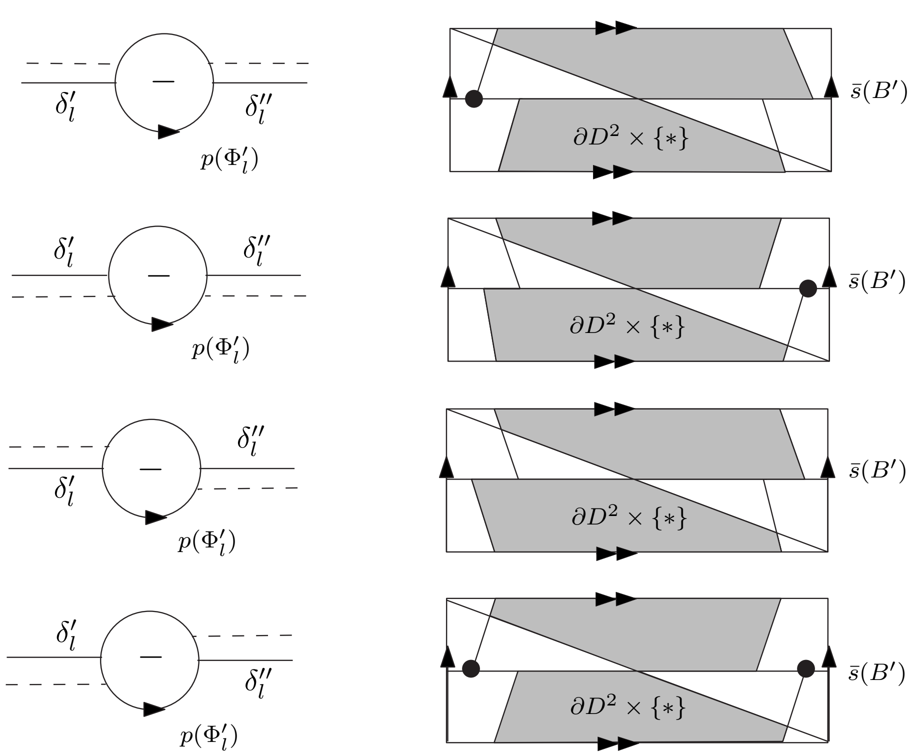

Figure 9: The graph Γ_l, with $b < 0$ embedded in $T_l = \partial\Phi_l'$ with different choices of the shifts for δ_l' and δ_l''.

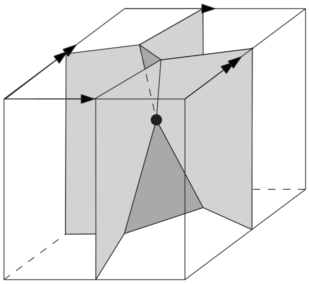

Figure 10: A skeleton for $T^2 \times [0, 1]$ connecting two theta graphs differing by a flip move.

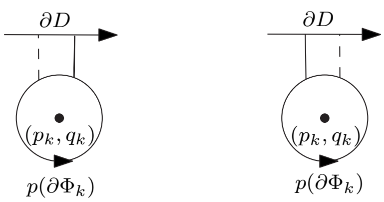

Figure 11: Regular shift (on the left) and singular shift (on the right).

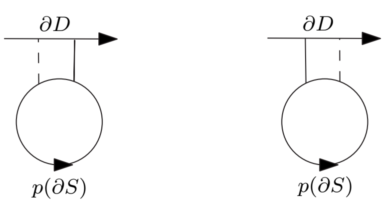

Figure 12: The two possible choices for the shift corresponding to components of ∂S.

we can choose $p_0(\Phi'_1), \dots, p_0(\Phi'_{|b|})$ as $|b|$ disks between those marked with – in Figure 13. In this way: (i) we can remove a region from T_0 containing 6 vertices of Γ_0, (ii) we can remove from T_l a region R_l containing in its boundary all the vertices of the graph Γ_l (as in the third drawing of Figure 9), for each $l = 1, \dots, |b|$, and (iii) we can take all regular shifts in the skeletons X_k corresponding to the exceptional fibres, for $k = 1, \dots, r$. If $b = 0$ we do not have to remove any regular neighbourhood Φ'_l of regular fibres but still (i) and (iii) hold. An analogous situation happens if $b \geq 1$, but in this case in order to satisfy (i), (ii) and (iii) the fibres of type $(1, 1)$ correspond to some of the disks marked with + in Figure 14. As a result, when $m \leq b \leq M$ the polyhedron P_S has $3(d + r + 2h - 2) + \sum_{k=1}^{r}(S(p_k/q_k) - 2)$ true vertices.

If $b < m \leq 0$ (respectively $b > M \geq 0$) then (i) and (iii) hold and there are exactly $m-b$ (respectively $b-M$) tori in which we remove a region R_l containing in its boundary all the vertices of Γ_l except one (see the first two pictures of Figure 8 and 9). Finally, if either $b < m = 1$ or $b > M = -1$ then (i) does not hold so we remove a region from T_0 containing 5 vertices of Γ_0. Moreover, there are exactly $|b|$ tori in which we remove a region R_l containing all the vertices of Γ_l except one and (iii) holds. Summing up, if $b < m$ (respectively $b > M$) then the number of true vertices of P_S increases by $m - b$ (respectively $b - M$) with respect to the case $m \leq b \leq M$.

As a consequence, P_S has $3(d + r + 2h - 2) + \sum_{k=1}^{r}(S(p_k/q_k) - 2) + f_{m,M}(b)$ true vertices.

Now let $T = (V, E_T)$ be a spanning tree of G and consider the graph manifold M_T (with boundary if $T \neq G$). We will construct a skeleton P_{M_T} for M_T by assembling skeletons of its Seifert pieces (constructed as above) with skeletons of thickened tori corresponding to edges of T. More precisely, for each $e \in E_T$ let $T'_e \subset \partial S_{v'_e}$ and $T''_e \subset \partial S_{v''_e}$ be the boundaries attached by A_e. We construct a skeleton P_{A_e} for $T''_e \times I$, and assemble $P_{S_{v'_e}}$ with P_{A_e} using the map $A_e : T'_e \to T''_e = T''_e \times \{0\}$ and P_{A_e} with $P_{S_{v''_e}}$ with the identification $T''_e \times \{1\} = T''_e$.

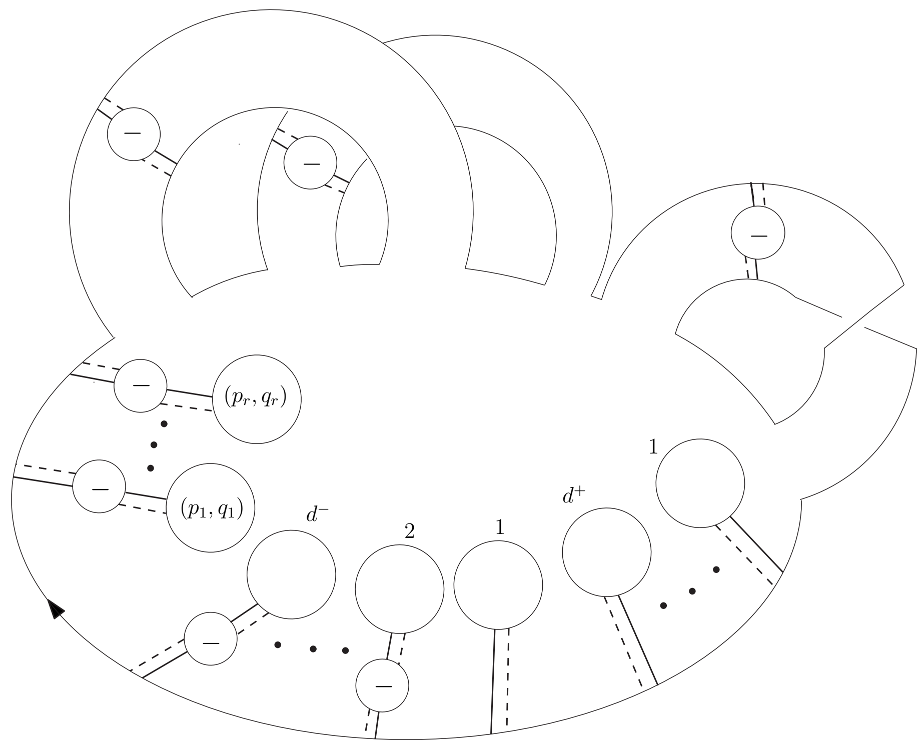

Figure 13: An optimal choice for the shifts corresponding to (1, −1)-fibres, when $b = m \leq 0$.

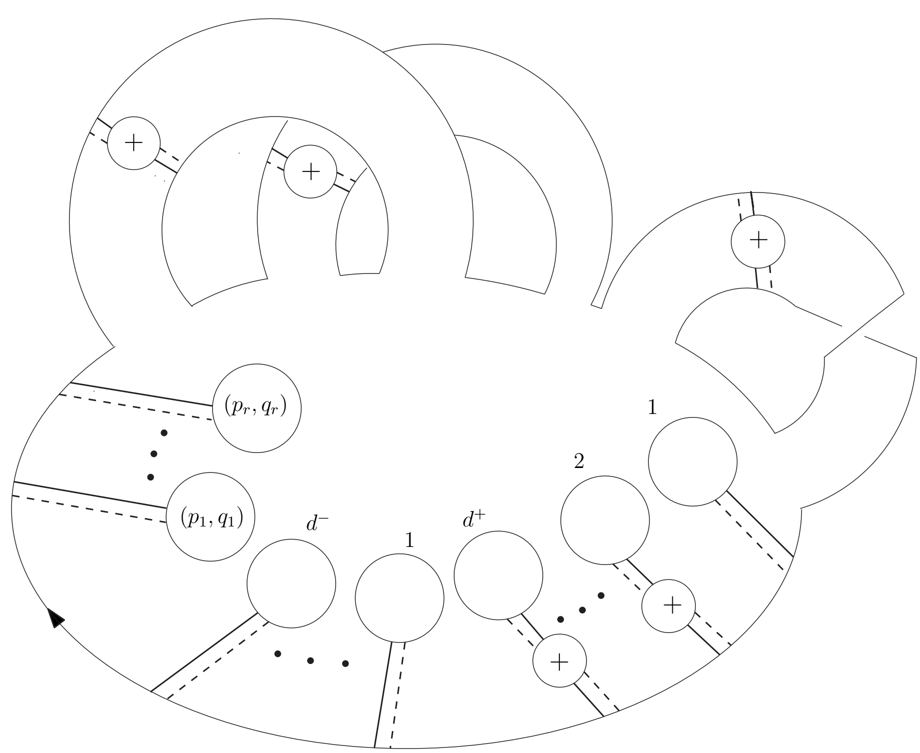

Figure 14: An optimal choice for the shifts corresponding to (1, 1)-fibres, when $b = M \geq 0$.

Given $e \in E_T$, let θ_e be the theta graph corresponding to $A_e\Delta_-$. We construct the skeleton P_{A_e} by assembling flip blocks (see Figure 10) so that (i) $P_{A_e} \cap (T''_e \times \{0\}) = \theta_e$ and (ii) $P_{A_e} \cap (T''_e \times \{1\}) = \theta_+$. By Lemma 1 we have $c_{A_e} = d(A_e\Delta_-, \Delta_+) = S(\beta_e/\delta_e) - 1$, so the number of flip blocks required to construct P_{A_e}, as well as the number of true vertices of P_{A_e}, is $S(\beta_e/\delta_e) - 1$.

By Remark 2, we can construct the skeleton P_{S_v} having $3(d_v + r_v + 2h_v - 2) + \sum_{k=1}^{r_v}(S(p_k/q_k) - 2) + f_{m_v,M_v}(b_v)$ true vertices. So P_{M_T} has

$$\sum_{e\in E_T}(S(\beta_e/\delta_e) - 1) + \sum_{v\in V}\Big(3(d_v + r_v + 2h_v - 2) + \sum_{k=1}^{r_v}(S(p_k/q_k) - 2) + f_{m_v,M_v}(b_v)\Big)$$

true vertices.

For each $e \in E \setminus E_T$, the matrix A_e identifies two boundary components of M_T. Denote with $M_{T\cup e}$ the resulting manifold. Construct P_{A_e} such that (i) $P_{A_e} \cap (T''_e \times \{0\}) = \theta_e$ and (ii) $P_{A_e} \cap (T''_e \times \{1\}) = \theta'$, where θ' is at distance one from θ_+ and is closer to θ_e than θ_+. The graphs θ' and θ_+ differ for a flip, since they correspond to adjacent triangles, and therefore, as shown in Figure 15, we have $i(\theta', \theta_+) = 2$, where $i(\cdot\,,\,\cdot)$ denotes the geometric intersection (i.e., the minimum number of intersection points up to isotopy). Consider the polyhedron $P_{M_T} \cup P_{A_e} \cup (T''_e \times \{1\})$: it is a skeleton for $P_{M_{T\cup e}}$. Since P_{A_e} consists of $S(\beta_e/\delta_e) - 2$ flip blocks and the graph $\theta_+ \cup \theta'$ has 6 vertices of degree greater than 2, the new polyhedron has $5 + S(\beta_e/\delta_e) - 1$ true vertices more than P_{M_T}. By repeating this construction for any $e \in E \setminus E_T$ and observing that $|E \setminus E_T| = |E| - |V| + 1$ we get the statement.

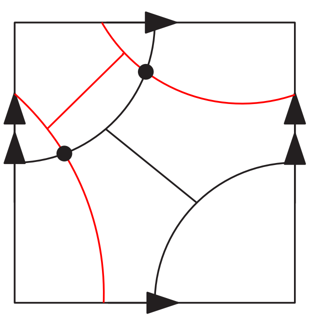

Figure 15: The two intersections of two theta graphs differing by a flip move.

4.3 Proof of Theorem 2. As in the proof of Theorem 1, we start by constructing a skeleton P_{M_T} for the graph manifold M_T (with boundary if $T \neq G$). By Lemma 1 we have $0 = c_{\pm H} = d(\pm H\Delta_-, \Delta_-) = d(\pm H\Delta_+, \Delta_+)$, so whenever $e \in E'_T = E'$, no flip block is required in P_{A_e} and we can assemble directly $P_{S_{v'_e}}$ with $P_{S_{v''_e}}$. So, if T'_e and T''_e denote, respectively, the boundary components of $S_{v'_e}$ and $S_{v''_e}$ glued by A_e, we require that either (i) $P_{S_{v'_e}} \cap T'_e = \theta_+$ and $P_{S_{v''_e}} \cap T''_e = \theta_+$ or (ii) $P_{S_{v'_e}} \cap T'_e = \theta_-$ and $P_{S_{v''_e}} \cap T''_e = \theta_-$.

In order to take care of these two possibilities we use a function $\psi : E' \to \{+, -\}$. If the shifts in the construction of $P_{S_{v'_e}}$ and $P_{S_{v''_e}}$ are chosen so that (i) holds (respectively (ii) holds) set $\psi(e) = -$ (respectively $\psi(e) = +$). Following the construction of Remark 2, with $d^+ = d_v^+ + d_{v,\psi}^+$ and $d^- = d_v^- + d_{v,\psi}^-$, we obtain a skeleton P_{S_v} with $3(d_v + r_v + 2h_v - 2) + \sum_{k=1}^{r_v}(S(p_k/q_k) - 2) + f_{m_v,M_v}(b_v)$ true vertices. Thus the minimum number of true vertices of the skeleton P_{M_T} of M_T is $\sum_{e\in E''_T}(S(\beta_e/\delta_e) - 1) + \sum_{v\in V}(3(d_v + r_v + 2h_v - 2) + \sum_{k=1}^{r_v}(S(p_k/q_k) - 2)) + \min_{\psi\in\Psi}\{\sum_{v\in V} f_{m_v,M_v}(b_v)\}$.

Since all the matrices associated to the edges $e \notin E_T$ are different from $\pm H$, starting from P_{M_T} we can construct a spine for M as described in the proof of Theorem 1. This concludes the proof.

4.4 Proof of Theorem 3. Let $T = (V, E_T) \in \mathcal{O}_G$. Given $\psi \in \Psi_T$, we construct a skeleton P_{M_T} for M_T as described in the proof of Theorem 2. If $e \in E' \setminus E'_T$, then $A_e = \pm H$ and it glues together two toric boundary components $T'_e \subset \partial S_{v'_e}$ and $T''_e \subset \partial S_{v''_e}$ of M_T.

Let $(\theta'_e, \theta''_e) = (P_{S_{v'_e}} \cap T'_e, P_{S_{v''_e}} \cap T''_e)$. Since $A_e\Delta_\pm = \Delta_\pm$ and $i(\theta_+, \theta_+) = i(\theta_-, \theta_+) = i(\theta_-, \theta_-) = 2$, we have to consider all possible cases of $\theta'_e, \theta''_e \in \{\theta_+, \theta_-\}$. If $(\theta'_e, \theta''_e) = (\theta_+, \theta_-)$ (respectively $(\theta'_e, \theta''_e) = (\theta_-, \theta_+)$) then we define $\psi'(e) = -+$ (respectively $\psi'(e) = +-$). If $(\theta'_e, \theta''_e) = (\theta_+, \theta_+)$ or $(\theta'_e, \theta''_e) = (\theta_-, \theta_-)$ we can use Remark 1 in order to obtain a better estimate for $c(M)$. Indeed, if we replace the matrix A_e with $A_eU^{\mp 1}$ (respectively $U^{\pm 1}A_e$), then $b_{v'_e}$ (respectively $b_{v''_e}$) is replaced with $b_{v'_e} \mp 1$ (respectively $b_{v''_e} \mp 1$), but anyway $i(\theta'_-, \theta_-) = i(\theta''_-, \theta_-) = i(\theta'_+, \theta_+) = i(\theta''_+, \theta_+) = 2$, where $\theta'_-, \theta''_-, \theta'_+, \theta''_+$ are the theta graphs associated to the triangles $A_eU^{-1}\Delta_-$, $UA_e\Delta_-$, $A_eU\Delta_+$, $U^{-1}A_e\Delta_+$, respectively. Indeed, $A_eU^{-1}\Delta_-$, $UA_e\Delta_-$ are adjacent to Δ_- and $A_eU\Delta_+$, $U^{-1}A_e\Delta_+$ are adjacent to Δ_+. More precisely, if we take $(\theta'_e, \theta''_e) = (\theta_-, \theta_-)$ and replace A_e with A_eU^{-1} (respectively UA_e) we set $\psi'(e) = ++$ (respectively $\psi'(e) = +$), while if we take $(\theta'_e, \theta''_e) = (\theta_+, \theta_+)$ and replace A_e with A_eU (respectively $U^{-1}A_e$) we set $\psi'(e) = --$ (respectively $\psi'(e) = -$). As a result, the skeleton P_{M_T} determined by $\psi \in \Psi_T$ and $\psi' \in \Psi'_T$ has

$$\sum_{e\in E''_T} (S(\beta_e/\delta_e) - 1) + \sum_{v\in V} \Big(3(d_v + r_v + 2h_v - 2) + \sum_{k=1}^{r_v} (S(p_k/q_k) - 2) \Big) + \sum_{v\in V} f_{m_v, M_v}(b_v)$$

true vertices.

A spine for M is given by the union of P_{M_T} with: (i) the skeleton $P_{A_e} \cup T''_e \times \{1\}$ described in the proof of Theorem 1 and having $5 + (S(\beta_e/\delta_e) - 1)$ true vertices for each $e \in E'' \setminus E''_T$, and (ii) the torus T''_e, containing 6 true vertices for each $e \in E' \setminus E'_T$. Since $|E' \setminus E'_T| + |E'' \setminus E''_T| = |E \setminus E_T| = |E| - |V| + 1$ and $|E' \setminus E'_T| = \Phi(G)$ we get the statement.

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