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AMENABLE CATEGORY AND COMPLEXITY

PIETRO CAPOVILLA, CLARA LÖH, AND MARCO MORASCHINI

Abstract. Amenable category is a variant of the Lusternik-Schnirelman category, based on covers by amenable open subsets. We study the monotonicity problem for degree-one maps and amenable category and the relation between amenable category and topological complexity.

1. Introduction

In applied algebraic topology, various integer-valued invariants associated with covers are considered to model, e.g., sensor coverage problems [63] or motion planning problems [21]. Prototypical invariants of this type are the Lusternik-Schnirelman category $\mathrm{cat}_{\mathrm{LS}}$ and the topological complexity TC, as introduced by Farber [21]: The Lusternik-Schnirelman category $\mathrm{cat}_{\mathrm{LS}}(X)$ of a topological space X is the minimal number of open and in X contractible sets needed to cover X; the topological complexity $\mathrm{TC}(X)$ is the minimal number of open sets that cover $X \times X$ and such that over each member of the cover the path fibration $PX \to X \times X$ (Section 2.2) admits a continuous section.

Relaxing the contractibility condition in the definition of $\mathrm{cat}_{\mathrm{LS}}$ to conditions on the allowed images on the level of π_1 leads to generalised notions of category, e.g., to the amenable category $\mathrm{cat}_{\mathrm{Am}}$ [30] (Section 2.1). The class of amenable groups allows for much bigger flexibility in the covers, such as using "circle-shaped" sets, but is still a class of groups that is often considered as "small" or "negligible" in the context of large-scale topology.

In the present article, we will focus on amenable category and its relation to topological complexity. Our leading questions are:

Question 1 (Question 6.3)**.** *Does the following hold for all oriented closed connected manifolds M and N:*

$$M \geq_1 N \Longrightarrow \mathrm{cat}_{\mathrm{Am}} M \geq \mathrm{cat}_{\mathrm{Am}} N \ ?$$

Here, we write $M \geq_1 N$ if $\dim M = \dim N$ and there exists a continuous map $M \to N$ of degree ± 1.

Question 2 (Question 8.1)**.** *For which topological spaces X do we have*

$$\mathrm{cat}_{\mathrm{Am}}(X \times X) \leq \mathrm{TC}(X) \ ?$$

As main tools we will use bounded cohomology and classifying spaces of families of subgroups.

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1.1. **Small amenable category and the fundamental group.** It is known that closed connected manifolds M with $\operatorname{cat}_{\mathrm{LS}} M = 3$ and $\dim M > 2$ have free fundamental group [17, Theorem 1.1]. In contrast, for each finitely presented group Γ there exists an oriented closed connected 5-manifold M with $\pi_1(M) \cong \Gamma$ and $\operatorname{cat}_{\mathrm{LS}} M = 4$ [17, Theorem 1.3]. For $\operatorname{cat}_{\mathrm{Am}}$ we obtain an analogous picture; as amenable subsets are a richer class of sets than contractible (within the ambient space) sets, a shift by 1 occurs:

Theorem 3 (small values of $\operatorname{cat}_{\mathrm{Am}}$; Proposition 5.1, Corollary 5.4)**.** *Let Γ be a finitely presented group.*

1. *If $n \in \mathbb{N}_{\geq 4}$, then there exists an oriented closed connected n-manifold M with $\pi_1(M) \cong \Gamma$ and $\operatorname{cat}_{\mathrm{Am}} M \leq 3$.*
2. *If Γ is non-amenable, then the following are equivalent:*
   (a) *The group Γ is the fundamental group of a graph of groups whose vertex (and edge) groups are all amenable.*
   (b) *If X is a CW-complex with $\pi_1(X) \cong \Gamma$, then $\operatorname{cat}_{\mathrm{Am}} X = 2$.*
   (c) *There exists an oriented closed connected manifold M that satisfies $\pi_1(M) \cong \Gamma$ and $\operatorname{cat}_{\mathrm{Am}} M = 2$.*

Combining the second part of Theorem 3 with known results on Serre's property FA, bounded cohomology or L^2-Betti numbers leads to many examples of closed manifolds with amenable category bigger than 2 (Corollary 5.5).

1.2. **The monotonicity problem for amenable category.** An interesting open problem about Lusternik-Schnirelmann category is to understand its behaviour under degree-one maps. More precisely, Rudyak asked the following question [58]:

Question 1.1 (monotonicity problem [9, Open Problem 2.48])**.** Does the following hold for all oriented closed connected manifolds M and N:

$$M \geq_1 N \Longrightarrow \operatorname{cat}_{\mathrm{LS}} M \geq \operatorname{cat}_{\mathrm{LS}} N \ ?$$

The main motivation behind the previous question is the fact that in general the domain manifold is "bigger" than the target manifold (Remark 3.11, Proposition 3.10). Rudyak's monotonicity problem for $\operatorname{cat}_{\mathrm{LS}}$ is wide open in full generality. Several partial – positive – results are known [58, 59, 13, 16, 11, 57]. We provide several example classes in which the corresponding question for $\operatorname{cat}_{\mathrm{Am}}$ (Question 1) has a positive answer: It is straightforward to show that oriented closed connected surfaces and target manifolds with positive simplicial volume have this property. Combining previous computations of $\operatorname{cat}_{\mathrm{Am}}$ for 3-manifolds and known descriptions of the relation $\geq_1$ in dimension 3 shows that Question 1 has an affirmative answer in dimension 3 (Theorem 6.6).

Moreover, we explain how bounded cohomology can be used to obtain further positive results (Section 6). An instance of this procedure is:

Theorem 4 (Corollary 6.25)**.** *Let Γ be the fundamental group of an oriented closed connected hyperbolic k-manifold of dimension $k \in \{2, 3\}$. Then, for every $n \geq 2k$ there exists an oriented closed connected n-manifold N with*

$\pi_1(N) \cong \Gamma$ *such that: For all oriented closed connected* n*-manifolds* M *we have*

$$M \geq_1 N \Longrightarrow \operatorname{cat}_{\mathrm{Am}} M \geq \operatorname{cat}_{\mathrm{Am}} N.$$

1.3. **Amenable category and topological complexity.** The topological complexity $\mathrm{TC}(X)$ is related to the category of $X \times X$ with respect to the diagonal family of subgroups of $\pi_1(X \times X)$ [23]. In general, we may not expect a direct connection between this diagonal category and $\operatorname{cat}_{\mathrm{Am}}(X \times X)$. However, it turns out that Question 2 has an affirmative answer in many cases:

Theorem 5 (Theorem 8.5)**.** *The following classes of spaces satisfy the estimate in Question 2:*

1. *Spaces with amenable fundamental group;*
2. *Spaces of type* $B\Gamma$ *where* Γ *is a finitely generated geometrically finite hyperbolic group;*
3. *Spaces of type* $B\Gamma$ *where* $\Gamma = H * H$ *is the free square of a geometrically finite group* H*;*
4. *Manifolds whose fundamental group is the fundamental group of a graph of groups whose vertex (and edge) groups are all amenable.*

Theorem 6 (Corollary 8.8)**.** *Let* Y *be an oriented closed connected* 3*-manifold that is a connected sum of graph manifolds. Then, for every* $n \geq 6$*, there exists an oriented closed connected* n*-manifold* M *with* $\pi_1(M) \cong \pi_1(Y)$ *and such that*

$$\operatorname{cat}_{\mathrm{Am}}(M \times M) \leq \mathrm{TC}(M)\ .$$

1.4. **Generalising the class of subgroups.** In fact, most of our methods and results also apply to classes of groups that are contained in the class of amenable groups and that are closed under isomorphisms, subgroups, and quotients. In the remainder of the article, we will thus usually stick to the more general setup. In particular, in analogy with the corresponding criterion for the category of the trivial group by Eilenberg and Ganea [20], we will show how these generalised category invariants can be expressed in terms of classifying spaces of families of subgroups:

Proposition 7 (Proposition 7.5)**.** *Let* X *be a connected CW-complex and let* $\tilde{X}$ *be its universal covering. Let* Γ *be the fundamental group of* X *and let* F *be a subgroup family of* Γ*. Then,* $\operatorname{cat}_F(X) - 1$ *coincides with the minimal integer* $k \in \mathbb{N}_{\geq 0}$ *such that the classifying map*

$$f_{\tilde{X},\Gamma,F} \colon \tilde{X} \to E_F\Gamma$$

is Γ*-homotopic to a map with values in the* k*-dimensional skeleton* $E_F\Gamma^{(k)}$ *(of any model of* $E_F\Gamma$*).*

This leads to a corresponding lower bound of cat_F in terms of Bredon cohomology (Corollary 7.7). For example, this can be applied in the case of hyperbolic fundamental groups (Example 7.8).

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Organisation of this article. We recall basic notions on category and topological complexity (Section 2) as well as on bounded cohomology (Section 3). In Section 4, we explain how to turn category computations for complexes into manifold examples. Section 5 deals with the influence of the fundamental group on small amenable category and contains a proof of Theorem 3. In Section 6, we discuss an extension of Rudyak's conjecture for degree one maps and LS-category and prove Theorem 4. The description of categorical invariants in terms of classifying spaces is given in Section 7. Finally, Section 8.2 contains the proof of Theorem 5 and of Theorem 6.

2. Category and complexity

We briefly recall the notions of category and topological complexity.

2.1. **Category.**

We will work with the following generalised version of the LS-category [7].

Definition 2.1. Let $\mathcal{A}$ be a class of topological spaces that contains a non-empty space and let X be a topological space. A subset $U \subseteq X$ is called *$\mathcal{A}$-contractible in X* if the inclusion map $i\colon U \to X$ factors homotopically through a space in $\mathcal{A}$, i.e., there exists a space $A \in \mathcal{A}$ and maps $\alpha\colon U \to A$ and $\beta\colon A \to X$ such that $\beta \circ i$ is homotopic to α.

The *$\mathcal{A}$-category* of X, denoted by $\operatorname{cat}_{\mathcal{A}}(X)$, is the minimal $n \in \mathbb{N}_{\geq 1}$ such that there exists an open cover $\{U_1, \dots, U_n\}$ of X by $\mathcal{A}$-contractible sets in X. If such an integer does not exist, we set $\operatorname{cat}_{\mathcal{A}}(X) := \infty$.

Example 2.2. Let $\mathcal{A} = \{*\}$ be the class of spaces containing just the point space. Then $\operatorname{cat}_{\mathcal{A}}(X)$ coincides with the usual notion of LS-category $\operatorname{cat}_{\mathrm{LS}}(X)$ of a topological space X [9]. Notice that we did not normalise the category, whence $\operatorname{cat}_{\mathcal{A}}(X) = 1$ if and only if X is contractible.

Moreover, there is also an algebraic version of category [31]. For the rest of this paper, it will be convenient to stick to the following convention:

Definition 2.3. A class of groups is an *isq-class* if it is non-empty and if it is closed under taking isomorphisms, subgroups, and quotients.

Definition 2.4. Let $\mathcal{G}$ be an isq-class of groups and let X be a topological space. A subset $U \subseteq X$ is called *$\mathcal{G}$-contractible in X* (or, simply, a $\mathcal{G}$-set) if for every $x \in U$ we have

$$\operatorname{im}\big(\pi_1(U \hookrightarrow X, x)\big) \in \mathcal{G}.$$

The *$\mathcal{G}$-category* of X, denoted by $\operatorname{cat}_{\mathcal{G}}(X)$, is then the minimal $n \in \mathbb{N}_{\geq 1}$ such that there exists an open cover $\{U_1, \dots, U_n\}$ of X by $\mathcal{G}$-sets. If such an integer does not exist, we set $\operatorname{cat}_{\mathcal{G}}(X) := \infty$.

By specialising the previous definition to particular isq-classes of groups, we get the definitions of π_1-category [25, 29] and Am-category [30].

Definition 2.5.

- If $\mathcal{G}$ only contains the trivial subgroup $\{e\}$, then $\mathrm{cat}_{\pi_1} \coloneqq \mathrm{cat}_{\{e\}}$ is the *π_1-category*.
- If $\mathcal{G} = \mathrm{Am}$ is the family of amenable groups, then $\mathrm{cat}_{\mathrm{Am}}$ is the *amenable category*.

There is a correspondence between the geometric $\mathcal{A}$-category and the algebraic $\mathcal{G}$-category as follows.

Proposition 2.6 ([31, Proposition 1])**.** *Let $\mathcal{G}$ be an isq-class of groups and let $\mathcal{A}_\mathcal{G}$ denote the class of all topological spaces Y with $\pi_1(Y, y) \in \mathcal{G}$ for every $y \in Y$. Then, we have*

$$\mathrm{cat}_{\mathcal{A}_\mathcal{G}} = \mathrm{cat}_\mathcal{G}\,.$$

The following result generalises a standard estimate for the LS-category.

Proposition 2.7. *Let $\mathcal{G}$ be an isq-class of groups and let X and Y be path-connected topological spaces such that $X \times Y$ is completely normal. Then,*

$$\mathrm{cat}_\mathcal{G}(X \times Y) \leq \mathrm{cat}_\mathcal{G}(X) + \mathrm{cat}_\mathcal{G}(Y) - 1.$$

Proof. One can use the same argument as in the case of LS-category [9, Theorem 1.37] once one notices that disjoint unions of open $\mathcal{G}$-sets are again $\mathcal{G}$-sets. □

Remark 2.8. Let X a topological space and let $\mathcal{F}$ and $\mathcal{G}$ isq-classes of groups with $\mathcal{F} \subset \mathcal{G}$. Then, we have

$$\mathrm{cat}_\mathcal{G}(X) \leq \mathrm{cat}_\mathcal{F}(X) \leq \mathrm{cat}_{\pi_1}(X) \leq \mathrm{cat}_{\mathrm{LS}}(X).$$

If X is a simplicial complex, then $\mathrm{cat}_{\mathrm{LS}}(X) \leq \dim X + 1$ [9, Theorem 1.7].

Remark 2.9. Let $\mathcal{G}$ be an isq-class of groups and let $f\colon X \to Y$ be a continuous map between connected topological spaces whose kernel $\ker \pi_1(f)$ lies in $\mathcal{G}$. Then pulling back open covers of Y shows that

$$\mathrm{cat}_\mathcal{G}\, X \leq \mathrm{cat}_\mathcal{G}\, Y.$$

In particular, this applies if f induces an isomorphism on the level of fundamental groups.

Definition 2.10. We say that path-connected spaces X and Y are *π_1-equivalent*, if there exist continuous maps (called *π_1-equivalences*) $X \to Y$ and $Y \to X$ that induce isomorphisms on the level of fundamental groups.

Lemma 2.11. *If $\mathcal{G}$ is an isq-class of groups, then* $\mathrm{cat}_\mathcal{G}$ *is an invariant of π_1-equivalences.*

Proof. It is sufficient to apply Remark 2.9 twice. □

We conclude this section with a lemma that allows us to make parts of a covering disjoint.

Lemma 2.12 ([9, Lemma A.4])**.** *Let X be a topological space and let $n \in \mathbb{N}$. Let U be an open cover of X of multiplicity $n+1$ with a partition of unity subordinate to the cover. Then there exist and index set B and an open covering*

$$\big\{V_{i\beta} \;\big|\; i \in \{1,\dots,n+1\}, \beta \in B\big\}$$

of X refining U such that $V_{i\beta} \cap V_{i\beta'} = \emptyset$ for all $\beta \neq \beta'$.

2.2. **Topological complexity.** Topological complexity is a categorical invariant introduced by Farber [21] that is motivated by the motion planning problem in robotics. Roughly speaking, it measures the minimal number of continuous rules one needs to move a point in a topological space in an autonomous way. A *motion planning algorithm* is a program that, given a topological space X and a pair of points $(A, B) \in X \times X$, returns a continuous path in X connecting them. In other words, if PX denotes the space of paths $[0,1] \to X$ endowed with the compact-open topology, a motion planning algorithm is a continuous global section of the path fibration

$$\begin{aligned} p\colon PX &\to X \times X \\ \gamma &\mapsto (\gamma(0), \gamma(1)) \ . \end{aligned}$$

Definition 2.13. Let X be a path-connected topological space. The *topological complexity* of X, denoted by $\mathrm{TC}(X)$, is the minimum $n \in \mathbb{N}_{\geq 1}$ such that there exists an open cover $\{U_1,\dots,U_n\}$ of $X \times X$ with the property that for each $i \in \{1,\dots,n\}$, there exists a continuous section $s_i : U_i \to PX$ of the path fibration $p\colon PX \to X \times X$. If such an integer does not exist, we set $\mathrm{TC}(X) := \infty$.

We collect some classical bounds of the topological complexity by means of the LS category [21].

Proposition 2.14. *Let X be a CW-complex and suppose that X is (locally) compact or it has countably many cells. Then the following inequalities hold:*

$$\mathrm{cat}_{\mathrm{LS}}(X) \leq \mathrm{TC}(X) \leq \mathrm{cat}_{\mathrm{LS}}(X \times X) \leq 2 \cdot \mathrm{cat}_{\mathrm{LS}}(X) - 1.$$

In order to compare $\mathrm{TC}(X)$ with the more algebraic $\mathcal{G}$-category, it is convenient to work with families of subgroups.

Definition 2.15. Let Γ be a group. A *subgroup family* of Γ is a non-empty set F of subgroups of Γ that is closed under conjugation and under taking subgroups.

Definition 2.16. Let X be a topological space and let Γ denote its fundamental group. Suppose that F is a subgroup family of Γ. We say that an open set $U \subset X$ is *F-contractible* (or, simply, an F-set) if for every $x \in U$ we have

$$\mathrm{im}\big(\pi_1(U \hookrightarrow X, x)\big) \in F.$$

An open cover $\mathcal{U}$ of X is called an *F-cover* if it is made of F-sets. The *F-category* of X, denoted by $\mathrm{cat}_F(X)$, is the minimal $n \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ such that X admits an open F-cover of cardinality n.

Remark 2.17. If $\mathcal{G}$ is an isq-class of groups and Γ is a group, then we have that

$$\mathcal{G}_\Gamma \coloneqq \{H \leq \Gamma \mid H \in \mathcal{G}\}$$

is a subgroup family of Γ and that all path-connected topological spaces X with fundamental group Γ satisfy

$$\operatorname{cat}_{\mathcal{G}_\Gamma}(X) = \operatorname{cat}_{\mathcal{G}}(X).$$

Moreover, the definition of F-category leads to a variation of the standard topological complexity [23]. This $\mathcal{D}$-topological complexity provides a lower bound for the standard topological complexity; for finite aspherical complexes one even has equality.

Definition 2.18. Let X be a path-connected topological space with fundamental group Γ and let us consider the family $\mathcal{D}$ of subgroups of $\pi_1(X\times X) \cong \Gamma \times \Gamma$ generated by

$$\Delta \coloneqq \{(\gamma,\gamma) \in \Gamma\times\Gamma \mid \gamma \in \Gamma\}.$$

We define the *$\mathcal{D}$-topological complexity* $\mathrm{TC}^{\mathcal{D}}(X)$ as the $\mathcal{D}$-category of $X\times X$.

Lemma 2.19 ([24, Proposition 2.4][23, Lemma 2.3.2])**.** *Let X be a path-connected topological space.*

(1) If X is locally path-connected and semi-locally simply connected, then $\mathrm{TC}^{\mathcal{D}}(X) \leq \mathrm{TC}(X)$.
(2) If X is a finite aspherical CW-complex, then $\mathrm{TC}^{\mathcal{D}}(X) = \mathrm{TC}(X)$.

3. Bounded cohomology

We recall terminology from bounded cohomology, simplicial volume, and their interaction with amenable covers.

3.1. **Bounded cohomology.** Bounded cohomology of groups was first introduced by Johnson [42] and Trauber in the setting of Banach algebras. Later, Gromov [32] extended this notion from groups to topological spaces. Moreover, there is a description in terms of homological algebra available [40]. Given a topological space X, we denote by $(C_\bullet(X),\partial_\bullet)$ and $(C^\bullet(X),\delta^\bullet)$ the *real* singular chain complex and singular cochain complex, respectively. We endow $C^\bullet(X)$ with the *ℓ^∞-norm* as follows: For every $k\in\mathbb{N}$, if $f\in C^k(X)$, we define

$$\|f\|_\infty \coloneqq \sup\big\{|f(\sigma)| \;\big|\; \sigma \text{ is a singular } k\text{-simplex}\big\} \in [0,\infty] \ .$$

A singular cochain $f\in C^\bullet(X)$ is *bounded* if $\|f\|_\infty < +\infty$. Since by linearity the singular coboundary operator $\delta^\bullet$ sends bounded cochains to bounded cochains, we can restrict $\delta^\bullet$ to the subcomplex of *bounded cochains*:

$$C_b^\bullet(X) \coloneqq \{f\in C^\bullet(X) \mid \|f\|_\infty < +\infty\} \ .$$

Definition 3.1. We define the *bounded cohomology* of a topological space X to be the homology of $(C_b^\bullet(X),\delta^\bullet)$ and we denote it by $H_b^\bullet(X)$.

Remark 3.2. Since bounded cohomology is a homotopy invariant, given a discrete group Γ, we can define $H_b^\bullet(\Gamma)$ as the (real) bounded cohomology of any model of $B\Gamma$.

Remark 3.3. Bounded cohomology $H_b^\bullet(X)$ is endowed with a natural seminorm induced by ℓ^∞: For every $\varphi\in H_b^\bullet(X)$, we set

$$\|\varphi\|_\infty \coloneqq \inf\{\|f\|_\infty \mid f\in C_b^\bullet(X) \text{ is a cocycle representing } \varphi\} \ .$$

Notice that the inclusion of the bounded cochain complex $C_b^\bullet(X)$ into $C^\bullet(X)$ induces a map

$$\mathrm{comp}_X^\bullet \colon H_b^\bullet(X) \to H^\bullet(X)$$

in cohomology, the so-called *comparison map*.

We recall some results that we will need in the sequel.

Theorem 3.4 (Gromov's mapping theorem [32][40])**.** *If $f \colon X \to Y$ is a continuous map between path-connected spaces such that $\pi_1(f)$ is an epimorphism with amenable kernel, then*

$$H_b^\bullet(f) \colon H_b^\bullet(Y) \to H_b^\bullet(X)$$

is an isometric isomorphism in all degrees.

Corollary 3.5. *Let $f \colon M \to N$ be a continuous map between oriented closed connected manifolds. Then $H_b^\bullet(f)$ is injective, surjective or an isomorphism, respectively, if and only if $H_b^\bullet(\pi_1(f))$ is injective, surjective or an isomorphism, respectively.*

3.2. **Simplicial volume.** In the case of oriented closed connected manifolds, bounded cohomology can be used to define a numerical invariant called *simplicial volume* [32][27, Section 7.5] (the original definition is formulated differently, but in view of the duality principle these two definitions are equivalent).

Definition 3.6. Let M be an oriented closed connected n-manifold. We define the *simplicial volume* of M as follows:

$$\|M\| := \sup \left\{ \frac{1}{\|\varphi\|_\infty} \,\middle|\, \varphi \in H_b^n(M)\,,\ \langle \mathrm{comp}_M^n(\varphi), [M] \rangle = 1 \right\},$$

where we set $\sup \emptyset := 0$. Here, $[M] \in H_n(M)$ denotes the real fundamental class of M and $\langle \cdot, \cdot \rangle$ the Kronecker product between cohomology and homology.

Remark 3.7. Let M be an oriented closed connected n-manifold. Since $H^n(M)$ is one-dimensional, we have

$$\|M\| > 0 \iff \mathrm{comp}_M^n \text{ is surjective}\,.$$

Example 3.8. Since bounded cohomology of amenable groups vanishes in all positive degree [42, 32, 40], Theorem 3.4 shows that this is also the case for the bounded cohomology of oriented closed connected manifolds with amenable fundamental group. Hence, the previous remark implies that the simplicial volume of all oriented closed connected manifolds with amenable fundamental group (and non-zero dimension) is zero.

Example 3.9. In contrast with the previous example, the following manifolds are known to have positive simplicial volume:

- oriented closed connected hyperbolic manifolds [66, 32]; in particular, surfaces of genus ≥ 2,
- oriented closed connected locally symmetric spaces of non-compact type [6, 45],
- manifolds with sufficiently negative curvature [39, 8].

Moreover, the class of manifolds with positive simplicial volume is closed with respect to connected sums and products.

Proposition 3.10 ([32][27, Section 7.2])**.** *Let M and N be oriented closed connected manifolds with $M \geq_1 N$. Then*

$$\|M\| \geq \|N\| \ .$$

Remark 3.11. Proposition 3.10 is an instance of the philosophy that if $M \geq_1 N$, then M contains more topological information than N. We recall other classical results in this direction [10, Proposition 2.6]: If $f\colon M \to N$ is a continuous map of degree ± 1, then

- $\pi_1(f)\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism,
- $H_*(f;\mathbb{Z})\colon H_*(M;\mathbb{Z}) \to H_*(N;\mathbb{Z})$ is a split epimorphism,
- $H^*(f;\mathbb{Z})\colon H^*(N;\mathbb{Z}) \to H^*(M;\mathbb{Z})$ is a split monomorphism.

3.3. **Amenable covers.** Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset \mathrm{Am}$. Then bounded cohomology and simplicial volume give obstructions to having small $\mathcal{G}$-category.

Theorem 3.12 (Gromov's vanishing theorem [32])**.** *Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset \mathrm{Am}$ and let X be a topological space. Then: If $\mathrm{cat}_{\mathcal{G}}(X) \leq k$, then $\mathrm{comp}_X^s\colon H_b^s(X) \to H^s(X)$ vanishes for all $s \geq k$.*

Remark 3.13. It is worth noticing that usually the previous theorem is stated in a different version, by requiring a control on the multiplicity of the cover instead of its cardinality [32]. More precisely, if X admits an open cover given by amenable sets $\mathcal{U}$ with multiplicity at most k, then $\mathrm{comp}_X^s\colon H_b^s(X) \to H^s(X)$ vanishes for all $s \geq k$. It is immediate to check that the bound on the cardinality of $\mathcal{U}$ in Theorem 3.12 is a priori stronger than the one on the multiplicity of $\mathcal{U}$. On the other hand, by Lemma 2.12, one can check that these two formulations are equivalent if the space X is a CW-complex.

Corollary 3.14. *If M is an oriented closed connected manifold with $\|M\| > 0$, then $\mathrm{cat}_{\mathrm{Am}} M = \dim M + 1$.*

Proof. As mentioned in Remark 3.7, the positivity of the simplicial volume of M is equivalent to the surjectivity of comp_M^n. Hence, Theorem 3.12 implies the claim. □

Moreover, we introduce the family of spaces for which the previous theorem is in fact an *if and only if* statement:

Definition 3.15. Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset \mathrm{Am}$. A topological space X has *efficient $\mathcal{G}$-category* if

$$\mathrm{cat}_{\mathcal{G}}(X) = \max\bigl\{s \in \mathbb{N} \bigm| \mathrm{comp}_X^{s-1} \neq 0\bigr\}.$$

Remark 3.16. In this context it is interesting to consider the following question: Does every oriented closed connected *aspherical* manifold have efficient Am-category?

If this were the case, then the vanishing of simplicial volume of aspherical closed connected manifolds would imply the vanishing of their L^2-Betti numbers [60] and thus of their Euler characteristic. In particular, this would give an affirmative answer to the corresponding question by Gromov [33, p. 232].

We conclude this section by discussing some difficulties in extending classical lower bounds for the LS-category to the amenable setting:

Remark 3.17. A standard lower bound of the LS-category is given by the cup length [9, Proposition 1.5]. A similar lower bound for the amenable category $\mathrm{cat}_{\mathrm{Am}}(X)$ would give new insights into the cup product structure in bounded cohomology. Indeed, given a topological space X, we can define the *bounded cup length* $\mathrm{cup}_b(X)$ of X as the maximal $n \in \mathbb{N}_{\geq 0}$ such that there exist $\alpha_1, \dots, \alpha_n \in H_b^\bullet(X;\mathbb{R})$ of positive degree with $\alpha_1 \smile \cdots \smile \alpha_n \neq 0$. A natural question is now to ask if the bounded cup length still provides a lower bound in the case of Am-category:

$$\mathrm{cup}_b(X) + 1 \leq \mathrm{cat}_{\mathrm{Am}}(X) \ ?$$

A positive answer to this question would have relevant consequences to the computation of higher dimensional bounded cohomology groups.

For instance, if X is a wedge of at least two circles, we have that its *second* and *third* bounded cohomology groups are infinite dimensional vector spaces, since the fundamental group of X is a non-abelian free group [53, 65]. On the other hand, it is an open problem whether the bounded cohomology of X vanishes in higher degrees. A natural approach to investigate the problem is to study the non-vanishing of cup products between cohomology classes in degree 2 or 3. However, recently Heuer [37] and Bucher-Monod [5] proved independently that this approach fails at least for classes induced by Rolli and Brooks quasimorphisms. Hence, a positive answer in this situation to the question above would show that this is in fact the case of *all* cup product in the bounded cohomology of X. Indeed, since $\mathrm{cat}_{\mathrm{Am}}(X) = 2$, we would get $\mathrm{cup}_b(X) \leq 1$, whence the claim.

4. From complexes to manifolds

Using the standard embedding-thickening argument, examples of category computations for finite complexes can be promoted to corresponding examples of closed manifolds.

Lemma 4.1. *Let $\mathcal{G}$ be an isq-class of groups, let $k \in \mathbb{N}_{\geq 2}$, let $n \in \mathbb{N}_{\geq 2k}$, and let X be a connected finite CW-complex of dimension at most k. Then there exists an oriented closed connected n-manifold M with*

$$\mathrm{cat}_{\mathcal{G}} M \leq \mathrm{cat}_{\mathcal{G}} X \qquad \text{and} \qquad \pi_1(M) \cong \pi_1(X).$$

Proof. The CW-complex X is homotopy equivalent to a connected finite simplicial complex Y of dimension at most k [35, Theorem 2C.5]. Then $\pi_1(Y) \cong \pi_1(X)$ and $\mathrm{cat}_{\mathcal{G}}(Y) = \mathrm{cat}_{\mathcal{G}}(X)$ (Proposition 2.11). Therefore, it suffices to prove the claim for Y.

Since Y is a simplicial complex, we can embed it into $\mathbb{R}^n$ for every $n \geq 2\dim(Y) + 1$ [49, Theorem 1.6.1]. Let $W \subset \mathbb{R}^n$ be a regular neighbourhood of Y [56].

Then W is an orientable compact connected n-manifold [56, Proposition 3.10] with non-empty boundary and W deformation retracts onto Y [56, Corollary 3.30]. Let us call this deformation retraction h. In particular, $\pi_1(W) \cong \pi_1(Y)$.

We define

$$M \coloneqq \partial W$$

and we claim that M is our desired oriented closed connected n-manifold. First notice that since $k \geq 2$, the complex Y has codimension at least 3 in W and $\mathbb{R}^n$. This implies that the boundary M is connected and that the boundary inclusion $i_M \colon M \to W$ induces an isomorphism on fundamental groups. Moreover, if we consider the composition

$$h \circ i_M \colon M \to Y$$

we obtain a map from M to Y inducing an isomorphism on the fundamental groups. Hence, we have that $\pi_1(M) \cong \pi_1(Y)$ and $\operatorname{cat}_{\mathcal{G}}(M) \leq \operatorname{cat}_{\mathcal{G}}(Y)$ (Remark 2.9). This finishes the proof. □

Since in most of our applications we will make explicit use of a retraction from M onto Y, we prefer to state the following version in which we construct M as the double of the regular neighbourhood W of Y in $\mathbb{R}^n$. This will produce a shift by 1 on the dimension of the resulting manifold.

Proposition 4.2. *Let $\mathcal{G}$ be an isq-class of groups and let Y be a simplicial complex of finite dimension. Then, for every $n \geq 2\dim(Y) + 1$, there exists an oriented closed connected n-manifold M such that*

(1) $\pi_1(M) \cong \pi_1(Y)$;
(2) There exists a retraction $f \colon M \to Y$, i.e., $f \circ i_Y = \mathrm{id}_Y$, where i_Y is the inclusion of Y into M.

Moreover, if Y is a model for $B\Gamma$ of some finitely presented group Γ or if $\pi_1(Y)$ is Hopfian, we have the following:

(3) $\operatorname{cat}_{\mathcal{G}}(M) \leq \operatorname{cat}_{\mathcal{G}}(Y)$;
(4) If Y has efficient $\mathcal{G}$-category, then M has efficient $\mathcal{G}$-category and $\operatorname{cat}_{\mathcal{G}}(M) = \operatorname{cat}_{\mathcal{G}}(Y)$.

Finally, if Y is an oriented closed connected manifold, then the previous result can be improved by taking $n \geq 2\dim(Y)$.

Proof. Ad 1. We already know that Y can embedded in $\mathbb{R}^n$ for every $n \geq 2\dim(Y) + 1$ [49, Theorem 1.6.1]. Let $W \subset \mathbb{R}^n$ be a regular neighbourhood of Y and let h be the deformation retraction from W to Y.

Since Y has codimension at least 2 in W, we know that the inclusion $i_{\partial W} \colon \partial W \to M$ of the boundary induces an epimorphism on fundamental groups.

We now claim that the double manifold $M \coloneqq D(W)$ is the desired oriented closed connected n-manifold. First, notice that the fundamental group of M is isomorphic to the amalgamated product $\pi_1(W) *_{\pi_1(\partial W)} \pi_1(W)$, where the morphisms $\pi_1(\partial W) \to \pi_1(W)$ are given by $\pi_1(i_{\partial W})$. Since the latter map is an epimorphism, we have that

$$\pi_1(M) \cong \pi_1(W) *_{\pi_1(\partial W)} \pi_1(W) \cong \pi_1(W) \cong \pi_1(Y) \ .$$

Ad 2. By construction, there exists a retraction $r \colon M \to W$, i.e., $r \circ i_W = \mathrm{id}_W$ where i_W denotes the inclusion of W into its double M. Then the composition

$$h \circ r \colon M \to Y \tag{1}$$

is the claimed retraction.

Ad 3. By Remark 2.9, it is sufficient to construct a map $f\colon M \to Y$ that induces an isomorphism on fundamental groups.

Now, we have two cases. Let us suppose that $\pi_1(Y)$ is Hopfian. Then the retraction in Equation (1) induces an epimorphism $\pi_1(M) \cong \pi_1(Y) \to \pi_1(Y)$, whence an isomorphism. This shows that $\mathrm{cat}_{\mathcal{G}}(M) \leq \mathrm{cat}_{\mathcal{G}}(Y)$.

On the other hand, if we assume that Y is a model for $B\Gamma$, there exists a classifying map $f\colon M \to Y$ inducing an isomorphism on fundamental groups. Hence, also in this case we get $\mathrm{cat}_{\mathcal{G}}(M) \leq \mathrm{cat}_{\mathcal{G}}(Y)$. This concludes the proof of the claim.

Ad 4. We already know from the second part that $\mathrm{cat}_{\mathcal{G}}(M) \leq \mathrm{cat}_{\mathcal{G}}(Y)$. Hence, we only have to prove the converse. Since Y has efficient $\mathcal{G}$-category, we can assume that $\mathrm{cat}_{\mathcal{G}}(Y) = k+1$ and comp_Y^k is non-trivial. Let us consider the map $f \coloneqq h \circ r\colon M \to Y$ defined in the proof of the second part. Then we have the following commutative diagram:

$$\begin{array}{ccc} H_b^k(Y) & \xrightarrow{H_b^k(f)} & H_b^k(M) \\ {\scriptstyle \mathrm{comp}_Y^k}\downarrow & & \downarrow{\scriptstyle \mathrm{comp}_M^k} \\ H^k(Y) & \xrightarrow[H^k(f)]{} & H^k(M)\ . \end{array}$$

Since f is a retraction, we have that the two horizontal arrows are monomorphisms. Hence, the non-vanishing of comp_Y^k also implies the non-vanishing of comp_M^k. By applying Theorem 3.12, we get $\mathrm{cat}_{\mathcal{G}}(M) \geq k = \mathrm{cat}_{\mathcal{G}}(Y)$. This finishes the proof of this item.

The last statement comes from the fact that if Y is an orientable closed connected manifold, then the strong Whitney embedding theorem shows that Y can be embedded in $\mathbb{R}^n$ for every $n \geq 2\dim(Y)$. □

Remark 4.3. When using Proposition 4.2, we will always be able to obtain a corresponding version with the improved dimension bound, provided that we use a manifold model of the classifying space as input. Usually, we will not formulate these improved versions explicitly.

Remark 4.4. Notice that by construction the resulting manifold M in Proposition 4.2 is oriented closed connected and *triangulable*.

An oriented closed connected triangulable n-manifold M is *essential* in the sense of Gromov [34, 4.40] if for every map

$$f\colon M \to B\pi_1(M)$$

inducing an isomorphism on fundamental groups, the image $f(M)$ is not contained in the $(n-1)$-skeleton of $B\pi_1(M)$. In particular, this is the case of an oriented closed connected triangulable n-manifold M which admits a classifying map $f\colon M \to B\pi_1(M)$ inducing a non-trivial map

$$H^n(f)\colon H^n(B\pi_1(M)) \to H^n(M)$$

on the *real* n-th cohomology groups.

As introduced by Dranishnikov, Katz, and Rudyak [17, Definition 5.1] one can extend Gromov's definition as follows: Let $k > 1$. Then, an oriented

closed connected triangulable n-manifold M is said to be *strictly k-essential* if there is no map between skeleta

$$f\colon M^{(k)} \to B\pi_1(M)^{(k-1)}$$

inducing an isomorphism on fundamental groups. Clearly, an oriented closed connected triangulable n-manifold is essential if and only if it is strictly n-essential. We then introduce the following definition:

Definition 4.5. Let $k > 1$. An oriented closed connected triangulable n-manifold M is said to be *essential in degree k* if the classifying map $f\colon M \to B\pi_1(M)$ induces a non-trivial map on the real k-th cohomology groups

$$H^k(f)\colon H^k(B\pi_1(M)) \to H^k(M)\ .$$

Remark 4.6. It is immediate to check that if an oriented closed connected triangulable n-manifold is essential in degree $k > 1$, then it is also strictly k-essential. On the other hand, the converse is false.

Corollary 4.7. *Let $\mathcal{G}$ be an isq-class of groups and let Y be an oriented closed connected triangulable n-manifold that is essential in degree $k > 1$. Moreover, suppose that $\pi_1(Y)$ is Hopfian.*

Then, for every $n \geq 2 \cdot \dim(Y)$, there exists an oriented closed connected triangulable n-manifold M essential in degree k such that

$$\operatorname{cat}_{\mathcal{G}}(M) \leq \operatorname{cat}_{\mathcal{G}}(Y) \qquad \textit{and} \qquad \pi_1(M) \cong \pi_1(Y).$$

Proof. Let $g\colon Y \to B\pi_1(Y)$ be the classifying map of Y. Since Y is essential in degree k we have that

$$H^k(g)\colon H^k(B\pi_1(Y)) \to H^k(Y)$$

is non-trivial.

Let us now construct an oriented closed connected triangulable n-manifold M from Y as explained in Proposition 4.2 and Remark 4.4. Using Proposition 4.2.2, we obtain a retraction map $f\colon M \to Y$; in particular, f induces an injection in both bounded and ordinary cohomology. This shows that the composition

$$H^k(g \circ f)\colon H^k(B\pi_1(Y)) \to H^k(M)$$

is non-trivial. Hence, we can conclude that M is essential in degree k if we show that $g \circ f$ is in a fact a classifying map for M. To this end, notice that f is a retraction and $\pi_1(Y)$ is Hopfian, thus f is also a π_1-isomorphism. This shows that the composition $g \circ f$ is a classifying map for M, whence that M is essential in degree k. Then the thesis follows by applying Proposition 4.2.3. □

5. Small amenable category and the fundamental group

We will now prove Theorem 3. We begin with the following statement that corresponds to the first item of Theorem 3.

Proposition 5.1. *Let Γ be a finitely presented group and let $n \in \mathbb{N}_{\geq 4}$. Then there exists an oriented closed connected n-manifold M with $\pi_1(M) \cong \Gamma$ and $\operatorname{cat}_{\mathrm{Am}} M \leq 3$.*

Proof. Let X be the presentation complex of a finite presentation of Γ. Then X is a connected finite CW-complex of dimension 2 with $\pi_1(X) \cong \Gamma$. Applying Lemma 4.1 to X gives a manifold with the claimed properties. □

It is instructive to keep the following example in mind:

Example 5.2. Let $n \in \mathbb{N}_{\geq 3}$ and let T^n denote the n-torus. Then

$$\operatorname{cat}_{\mathrm{Am}}(T^n \# T^n) = 2 \qquad \text{and} \qquad \pi_1(T^n \# T^n) \cong \mathbb{Z}^n * \mathbb{Z}^n.$$

This can be seen as follows: The fundamental group is not amenable; thus, $\operatorname{cat}_{\mathrm{Am}}$ has to be at least 2. Conversely, because $n \geq 3$, the two punctured torus summands give rise to an amenable cover.

Proposition 5.3. *Let X be a connected CW-complex, let F be a subgroup family of $\pi_1(X)$ and let $\operatorname{cat}_F X \leq 2$. Then the fundamental group of X is the fundamental group of a graph of groups whose vertex (and edge) groups are all in F.*

In particular: If $\operatorname{cat}_{\mathrm{Am}} X \leq 2$, then the fundamental group of X is the fundamental group of a graph of groups whose vertex (and edge) groups are all amenable.

Proof. We argue via Bass-Serre theory: Because $\operatorname{cat}_F X \leq 2$, there exists an F-cover U of X by path-connected subsets such that the nerve $N(U)$ is one-dimensional: We decompose an open F-cover of X by two sets into their path-connected components; these sets are open as CW-complexes are locally path-connected.

Let $\tilde{U}$ be the corresponding open cover of the universal covering $\tilde{X}$ of X (where $\pi\colon \tilde{X} \to X$ denotes the universal covering map):

$$\tilde{U} := \big\{ V \subset \tilde{X} \;\big|\; \text{there exists a } W \in U \text{ such that } V \text{ is a path-connected component of } \pi^{-1}(W) \big\}$$

Then the nerve $N(\tilde{U})$ is also one-dimensional. Moreover, $N(\tilde{U})$ is a forest: Because $H_1(\tilde{X}; \mathbb{Z}) \cong 0$, a Leray spectral sequence argument shows that $H_1(N(\tilde{U}); \mathbb{Z}) \cong 0$ [51, Theorem 2.1]; hence, all connected components of $N(\tilde{U})$ are trees.

The fundamental group Γ of X acts simplicially on $N(\tilde{U})$ and the stabiliser groups of the vertices and edges all lie in F [47, Lemma 4.11].

Let T be the barycentric subdivision of a connected component of $N(\tilde{U})$. Then T is a tree that inherits an involution-free simplicial action of Γ whose vertex and edge stabilisers lie in F. Thus, by Bass-Serre theory [61], the group Γ is isomorphic to the fundamental group of the graph of groups on T/Γ, decorated by these stabiliser groups.

Alternatively, one can apply the generalised Seifert and van Kampen theorem for fundamental groupoids [4, 19] and relate the resulting homotopy colimit of groupoids to fundamental groups of graphs of groups via explicit presentations [38]. □

In the manifold case, open amenable covers consisting of exactly two open amenable sets can also be arranged by appropriate closed submanifolds with common boundary [31, Lemma 3].

Corollary 5.4. *Let Γ be a non-amenable group. Then the following are equivalent:*

1. *The group Γ is the fundamental group of a graph of groups whose vertex (and edge) groups are all amenable.*
2. *If X is a model of $B\Gamma$, then $\operatorname{cat}_{\mathrm{Am}} X = 2$.*
3. *If X is a CW-complex with $\pi_1(X) \cong \Gamma$, then $\operatorname{cat}_{\mathrm{Am}} X = 2$.*
4. *There exists a connected CW-complex X that satisfies $\pi_1(X) \cong \Gamma$ and $\operatorname{cat}_{\mathrm{Am}} X = 2$.*

If Γ is finitely presented, then these conditions are also equivalent to:

5. *There exists an oriented closed connected manifold M that satisfies $\pi_1(M) \cong \Gamma$ and $\operatorname{cat}_{\mathrm{Am}} M = 2$.*

Proof. We begin with (1) $\Longrightarrow$ (2). Let Γ be the fundamental group of a graph of groups whose vertex (and edge) groups are all amenable. Then the homotopy colimit X over a corresponding graph of classifying spaces is a model of the classifying space $B\Gamma$ [19, Section 4.1]. Moreover, the underlying graph structure gives an amenable open cover of X consisting of two sets (namely, one associated to the vertices and one associated with the edges). Therefore, $\operatorname{cat}_{\mathrm{Am}} X \leq 2$. Because Γ is non-amenable, we obtain $\operatorname{cat}_{\mathrm{Am}} X = 2$. As $\operatorname{cat}_{\mathrm{Am}}$ is a homotopy invariant (Proposition 2.11), also all other models of $B\Gamma$ have amenable category equal to 2.

The implication (2) $\Longrightarrow$ (3) follows by pulling back amenable open covers of models of $B\Gamma$ along the classifying map (and the fact that Γ is non-amenable).

For (3) $\Longrightarrow$ (4), we only need to notice that for every group Γ there exists a CW-complex whose fundamental group is isomorphic to Γ.

The implication (4) $\Longrightarrow$ (1) is covered by Proposition 5.3.

For the implication (4) $\Longrightarrow$ (5), we apply Lemma 4.1. The implication (5) $\Longrightarrow$ (4) is a consequence of the fact that every compact manifold is homotopy equivalent to a finite CW-complex [44, 62] and that $\operatorname{cat}_{\mathrm{Am}}$ is homotopy invariant (Proposition 2.11). □

This proposition also holds in the case of general isq-classes of groups (instead of Am), under the assumption that Γ is not contained in this class.

Corollary 5.5. *Let M be an oriented closed connected manifold whose fundamental group is in the following list:*

1. *Non-amenable groups with Serre's property FA.*
2. *Fundamental groups of oriented closed connected aspherical manifolds of dimension at least 2 with positive simplicial volume.*
3. *Groups that have a non-zero L^2-Betti number in some degree ≥ 2.*

Then $\operatorname{cat}_{\mathrm{Am}} M \geq 3$.

Proof. Let $\Gamma := \pi_1(M)$ and let X be a model of $B\Gamma$. In view of Corollary 5.4, we only need to show that $\operatorname{cat}_{\mathrm{Am}} X \geq 3$.

(1) This is immediate from the definition of property FA and Bass-Serre theory [61].

(2) In the simplicial volume case, this is a consequence of the vanishing theorem in bounded cohomology (Theorem 3.12).

(3) In the L^2-Betti number case, this is a consequence of Sauer's vanishing theorem [60, Theorem C][34]. □

Corollary 5.5 can, for instance, be applied to infinite groups with property (T) [68], to the examples listed in Example 3.9, and to $F_2 \times F_2$ (which has non-zero second L^2-Betti number).

Using the Berstein class, Dranishnikov and Rudyak showed the following [18, Theorem 5.2]: If $M \geq_1 N$ and $\pi_1(M)$ is free, then also $\pi_1(N)$ is free. In the setting of amenable category, this translates into the following question:

Question 5.6. Let M, N be oriented closed connected manifolds with $M \geq_1 N$ and let $\pi_1(M)$ be the fundamental group of a graph of groups all of whose vertex (and edge) groups are amenable. Is then also $\pi_1(N)$ the fundamental group of a graph of groups all of whose vertex (and edge) groups are amenable?

The strategy of proof of Dranishnikov and Rudyak is based on the characterisation of free groups in terms of cohomological dimension and thus does not seem to have a straightforward counterpart in the case of bounded cohomology and amenable category. However, a positive answer to the generalised monotonicity problem (Question 1) for domain manifolds of amenable category equal to 2 would combine with Corollary 5.4 to give an affirmative answer to Question 5.6.

6. Degree-one maps and categorical invariants

In this section, we focus on Question 1, i.e., on the generalisation of Rudyak's problem (Question 1.1). By now there are several positive results on Rudyak's question [58, 59, 13, 16]. For instance, Rudyak showed that the question has a positive answer for manifolds of dimension at most 4 [59]. Moreover, it is also true if N has LS-category at most 3. Indeed, if $\mathrm{cat}_{\mathrm{LS}}(M) = 2$, then M is a homotopy sphere [41]. This then implies that N is also a homotopy sphere and thus has LS-category 2 [59].

Moreover, a good source of degree-one maps is provided by collapsing maps from the connected sum onto one of its summands. It has been proved by Dranishnikov and Sadykov [13, 15] that Rudyak's question in this situation always has a positive answer. We refer the reader to the literature [59, 16] for examples of higher connected manifolds that satisfy Rudyak's question.

Finally, it is worth mentioning that Rudyak's question is answered affirmatively when the LS-category of N is maximal [58]. This is our source of inspiration for studying the generalised Rudyak's question for classes of groups in the next section. Indeed, by using simplicial volume it is easy to prove that if N has positive simplicial volume, then N satisfies Rudyak's question:

Lemma 6.1. *If M and N are oriented closed connected manifolds with $M \geq_1 N$ and $\|N\| > 0$, then $\mathrm{cat}_{\mathrm{LS}}(M) \geq \mathrm{cat}_{\mathrm{LS}}(N)$.*

Proof. By Corollary 3.14, we have that $\mathrm{cat}_{\mathrm{LS}}(N) = n + 1$. Hence, by using Proposition 3.10 and again Corollary 3.14, it is readily seen that also $\mathrm{cat}_{\mathrm{LS}}(M) = n + 1$, whence the thesis. □

Remark 6.2. Notice that the proof of Lemma 6.1 also applies to the following situation: Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset \mathrm{Am}$. Then, for all oriented closed connected manifolds M and N with $M \geq_1 N$ and $\|N\| > 0$, we have $\mathrm{cat}_{\mathcal{G}}(M) \geq \mathrm{cat}_{\mathcal{G}}(N)$. This observation leads to the study of the generalised monotonicity problem in the following section.

6.1. **The monotonicity problem.** We will now extend Rudyak's question 1.1 to the setting of categories with respect to isq-classes of amenable groups:

Question 6.3 ((generalised) monotonicity problem)**.** Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset \mathrm{Am}$. Does the following hold for all oriented closed connected manifolds M and N:

$$M \geq_1 N \Longrightarrow \mathrm{cat}_{\mathcal{G}} M \geq \mathrm{cat}_{\mathcal{G}} N \ ?$$

Remark 6.4. It is immediate to check that the previous question is always true if $\mathrm{cat}_{\mathcal{G}}(N) \leq 2$. Indeed, if $\mathrm{cat}_{\mathcal{G}}(N) = 1$, there is nothing to prove. On the other hand, if $\mathrm{cat}_{\mathcal{G}}(M) = 1$, then $\pi_1(M) \in \mathcal{G}$. Since $\mathcal{G}$ is closed under quotients, we have that also $\pi_1(N) \in \mathcal{G}$ (Remark 3.11). This shows that $\mathrm{cat}_{\mathcal{G}}(N) = \mathrm{cat}_{\mathcal{G}}(M) = 1$.

6.2. **Low-dimensional manifolds.** We answer the monotonicity problem (Question 1) affirmatively in dimensions 2 and 3.

Proposition 6.5. *Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset \mathrm{Am}$. Then all oriented closed connected surfaces M and N with $M \geq_1 N$ satisfy* $\mathrm{cat}_{\mathcal{G}}(M) \geq \mathrm{cat}_{\mathcal{G}}(N)$.

Proof. We use the classification of surfaces: For $g \in \mathbb{N}$, we write Σ_g for "the" oriented closed connected surface of genus g. Looking at $H_1(\,\cdot\,;\mathbb{Z})$ shows that (Remark 3.11)

$$\forall_{g,h\in\mathbb{N}} \quad \Sigma_g \geq_1 \Sigma_h \Longrightarrow g \geq h.$$

Therefore, the claim is immediate from the following computations:

- $\mathrm{cat}_{\mathcal{G}}(\Sigma_g) = 3$ for all $g \in \mathbb{N}_{\geq 2}$ because $\|\Sigma_g\| > 0$ (Example 3.9, Corollary 3.14) and $\mathcal{G} \subset \mathrm{Am}$.
- $\mathrm{cat}_{\mathcal{G}}(\Sigma_1) \in \{1, 2, 3\}$ by the estimates from Remark 2.8.
- $\mathrm{cat}_{\mathcal{G}}(\Sigma_0) = 1$, because Σ_0 is simply connected and thus a $\mathcal{G}$-set. □

Theorem 6.6. *Let $\mathcal{G}$ is an isq-class of groups with $\mathcal{G} \subset \mathrm{Am}$ that contains the class of solvable groups. Then, for all oriented closed connected 3-manifolds M and N with $M \geq_1 N$, we have* $\mathrm{cat}_{\mathcal{G}}(M) \geq \mathrm{cat}_{\mathcal{G}}(N)$.

Before proving this theorem we need some preparation. First, we mention the behaviour of the $\mathcal{G}$-category under connected sums (see also Corollary 6.9 below).

Proposition 6.7 ([31, Lemma 1])**.** *Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset \mathrm{Am}$ and let $n \geq 3$. Let M_1 and M_2 be oriented closed connected n-manifolds with* $\max\{\mathrm{cat}_{\mathcal{G}}(M_1), \mathrm{cat}_{\mathcal{G}}(M_2)\} \geq 2$. *Then, we have*

$$\mathrm{cat}_{\mathcal{G}}(M_1 \,\#\, M_2) = \max\{\mathrm{cat}_{\mathcal{G}}(M_1), \mathrm{cat}_{\mathcal{G}}(M_2)\} \ .$$

Proof. Notice that the proof [31, Lemma 1] for 3-manifolds such that both $\mathrm{cat}_{\mathcal{G}}(M_1)$ and $\mathrm{cat}_{\mathcal{G}}(M_2) \geq 2$ applies *verbatim* for all manifolds of dimension greater than or equal to 3 such that $\mathrm{cat}_{\mathcal{G}}(M_1), \mathrm{cat}_{\mathcal{G}}(M_2) \geq 2$. Moreover, the very same proof [31, Lemma 1] can be also extended to the case in which either $\mathrm{cat}_{\mathcal{G}}(M_1)$ or $\mathrm{cat}_{\mathcal{G}}(M_2)$ is equal to 1. □

Remark 6.8. If $\max\{\mathrm{cat}_{\mathcal{G}}(M_1), \mathrm{cat}_{\mathcal{G}}(M_2)\} = 1$, then the value of $\mathrm{cat}_{\mathcal{G}}(M_1 \# M_2)$ depends both on $\mathcal{G}$ and the manifolds M_1, M_2. For instance if $\mathcal{G} \subset \mathrm{Am}$ is an isq-class of groups, then $\mathrm{cat}_{\mathcal{G}}(M_1 \# M_2) = 2$ if and only if $\pi_1(M_1) * \pi_1(M_2)$ is a non-trivial free product of amenable groups.

As a corollary, we get some instances for the validity of the monotonicity problem (Question 1), similar to corresponding results for LS-category [13]. Recall that given two oriented closed connected M_1 and M_2 of the same dimension, there always exists a degree-one map $M_1 \# M_2 \to M_i$ for $i \in \{1, 2\}$. Hence, we have the following:

Corollary 6.9. *Let $\mathcal{G}$ be an isq-class of groups and let M_1 and M_2 be oriented closed connected n-manifolds with $n \geq 2$. Then $M_1 \# M_2 \geq_1 M_i$ and $\mathrm{cat}_{\mathcal{G}}(M_1 \# M_2) \geq \mathrm{cat}_{\mathcal{G}}(M_i)$ for both $i \in \{1, 2\}$.*

Proof. The case for $n = 2$ has been already discussed in Proposition 6.5. If $n \geq 3$, by applying Proposition 6.7 we have that $\mathrm{cat}_{\mathcal{G}}(M_1 \# M_2) \geq \mathrm{cat}_{\mathcal{G}}(M_i)$ for both $i \in \{1, 2\}$ if at least one of the manifolds M_1 and M_2 has $\mathcal{G}$-category strictly larger than 1. However, if we have assume that $\mathrm{cat}_{\mathcal{G}}(M_1) = 1 = \mathrm{cat}_{\mathcal{G}}(M_2)$, then the thesis follows trivially. □

As a second ingredient for the proof of Theorem 6.6 we recall the notion of Kodaira dimension of 3-manifolds introduced by Zhang [69]. We follow here the notation by Neofytidis [54], which is slightly different from Zhang's original one. To this end we first divide Thurston's three-dimensional eight geometries into four classes assigning to each of them a value:

- $-\infty$: $S^3, S^2 \times \mathbb{R}$;
- 0: $\mathbb{E}^3, \mathrm{Nil}^3, \mathrm{Sol}^3$;
- 1: $\mathbb{H}^2 \times \mathbb{R}, \widetilde{\mathrm{SL}_2}(\mathbb{R})$;
- $\frac{3}{2}$: $\mathbb{H}^3$.

Given an oriented closed connected 3-manifold M we decompose it via Milnor-Kneser's prime decomposition and then we subdivide each piece in the prime decomposition via a geometric decomposition (i.e., we cut along tori such that each final manifold carries one of the eight geometries and has finite volume). We call the resulting decomposition a *T-decomposition* of M.

Definition 6.10. The *Kodaira dimension* $\kappa^t(M)$ of an oriented closed connected 3-manifold M is defined as follows:

- We set $\kappa^t(M) := -\infty$ if for any T-decomposition of M all its pieces belong to the category $-\infty$ above;
- We set $\kappa^t(M) := 0$ if any T-decomposition of M contains at least one piece lying in category 0, but no pieces in the categories 1 or $\frac{3}{2}$;
- We set $\kappa^t(M) := 1$ if any T-decomposition of M contains at least one piece lying in category 1, but no pieces in the category $\frac{3}{2}$;

- We set $\kappa^t(M) := \frac{3}{2}$ if any T-decomposition of M contains at least one piece lying in category $\frac{3}{2}$.

Remark 6.11. Notice that if a 3-manifold M has $\kappa^t(M) \leq 0$, then all the irreducible pieces in its prime decomposition are closed geometric manifolds [69].

Theorem 6.12 ([54, Theorem 6.1] and [69, Theorem 1.1])**.** *Let M and N be oriented closed connected 3-manifolds with $M \geq_1 N$. Then $\kappa^t(M) \geq \kappa^t(N)$.*

Finally, we will also use previous computations of categorical invariants of 3-manifolds:

Remark 6.13 ([31, Corollary 5])**.** Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset$ Am that contains the class of solvable groups. Gómez-Larrañaga, González-Acuña, and Heil computed the $\mathcal{G}$-category for all prime 3-manifolds. More precisely, if M is prime manifold with $\kappa^t(M) \leq 0$, then the fundamental group is solvable, whence $\mathrm{cat}_{\mathcal{G}}(M) = 1$. If $\kappa^t(M) = 1$, then the fundamental group of M is not solvable and $\mathrm{cat}_{\mathcal{G}}(M) = 3$. Finally, if $\kappa^t(M) = \frac{3}{2}$, we know that that simplicial volume of M is positive [64] and so we have $\mathrm{cat}_{\mathcal{G}}(M) = 4$ (Corollary 3.14).

We are now ready to prove Theorem 6.6:

Proof of Theorem 6.6. Suppose that $M \geq_1 N$. We begin by considering the case in which M is prime. Notice that there are no oriented closed connected prime 3-manifolds with $\mathrm{cat}_{\mathcal{G}}$ equal to 2 [30][31, Corollary 3] (Remark 6.13). So in this situation we have only three cases:

(1) If $\pi_1(M)$ is solvable, then we have that $\mathrm{cat}_{\mathcal{G}}(M) = 1$ because $\mathcal{G}$ contains the class of all solvable groups. This implies that also $\pi_1(N)$ is solvable (Remark 3.11), whence $\mathrm{cat}_{\mathcal{G}}(N) = 1$. In particular, $\mathrm{cat}_{\mathcal{G}}(M) \geq \mathrm{cat}_{\mathcal{G}}(N)$.
(2) If $\mathrm{cat}_{\mathcal{G}}(M) = 3$, then Theorem 3.12 shows that $\|M\| = 0$ and so $\|N\| = 0$ (Proposition 3.10). Hence, we have $\mathrm{cat}_{\mathcal{G}}(N) \leq 3$ [30, Theorem 2][31, Corollary 5] (Remark 6.13). In particular, $\mathrm{cat}_{\mathcal{G}}(M) \geq \mathrm{cat}_{\mathcal{G}}(N)$.
(3) If $\mathrm{cat}_{\mathcal{G}}(M) = 4$, then the desired result $\mathrm{cat}_{\mathcal{G}}(M) \geq \mathrm{cat}_{\mathcal{G}}(N)$ is trivially true by the dimension estimate (Remark 2.8).

We now assume that M is not prime and we argue via Kodaira dimension κ^t. We have the following four cases:

(1) Let $\kappa^t(M) = -\infty$. By Remark 6.11 and Remark 6.13, we know that M is the connected sum of oriented prime geometric manifolds with $\mathrm{cat}_{\mathcal{G}}$ equal to 1. Hence, we have $\mathrm{cat}_{\mathcal{G}}(M) \leq 2$ by Proposition 6.7 and Remark 6.8. If $\mathrm{cat}_{\mathcal{G}}(M) = 1$ we are done as mentioned in the first item above. Let $\mathrm{cat}_{\mathcal{G}}(M) = 2$. By Theorem 6.12, we have $\kappa^t(M) \geq \kappa^t(N)$. This implies that also $\kappa^t(N) = -\infty$. Hence, we have $\mathrm{cat}_{\mathcal{G}}(N) \leq 2 = \mathrm{cat}_{\mathcal{G}}(M)$.
(2) Let $\kappa^t(M) = 0$. By Remark 6.11 and Remark 6.13, we have again that $\mathrm{cat}_{\mathcal{G}}(M) \leq 2$. Since we still have $\kappa^t(M) \geq \kappa^t(N)$, the very same proof as in the previous item applies to this situation.

(3) Let $\kappa^t(M) = 1$. Let $M = M_1 \# \cdots \# M_k$ be a prime decomposition. Up to reordering the pieces, we may assume that M_1 contains a piece belonging to category 1. Then by applying Remark 6.13, we have $\mathrm{cat}_{\mathcal{G}}(M_1) = 3$ [30, Theorem 2]. Hence, Proposition 6.7 shows that $\mathrm{cat}_{\mathcal{G}}(M) = 3$. Then, Theorem 3.12 and Proposition 3.10 imply that $0 = \|M\| \geq \|N\|$. The vanishing of the simplicial volume of N then implies that $\mathrm{cat}_{\mathcal{G}}(N) \leq 3$ (Remark 6.13), whence $\mathrm{cat}_{\mathcal{G}}(M) \geq \mathrm{cat}_{\mathcal{G}}(N)$.

(4) Let $\kappa^t(M) = \frac{3}{2}$. In this situation, we have $\mathrm{cat}_{\mathcal{G}}(M) = 4$ since $\|M\| > 0$ [64]. Hence, the inequality $\mathrm{cat}_{\mathcal{G}}(M) \geq \mathrm{cat}_{\mathcal{G}}(N)$ follows from the dimension estimate (Remark 2.8). □

Remark 6.14. There is also a way to formulate the proof of Theorem 6.6 without using the Kodaira dimension. We are grateful to Dieter Kotschick for sharing such a proof with us. The rough outline is as follows: Using the calculations of Gómez-Larrañaga, González-Acuña, and Heil [30] and the computation of simplicial volume of 3-manifolds, one can derive that we have for all oriented closed connected 3-manifolds M:

- $\mathrm{cat}_{\mathrm{Am}} M = 1$ if and only if $\pi_1(M)$ is amenable.
- If $\mathrm{cat}_{\mathrm{Am}} M = 2$, then M has at least two prime summands and all prime summands of M have amenable fundamental group. The converse also holds except for the pathological case $\mathbb{R}P^3 \# \mathbb{R}P^3$ (which has amenable category 1).
- $\mathrm{cat}_{\mathrm{Am}} M = 4$ if and only if $\|M\| > 0$.
- $\mathrm{cat}_{\mathrm{Am}} M = 3$ if and only if all prime summands of M have vanishing simplicial volume and at least one prime summand has non-amenable fundamental group.

If there exists a map $M \to N$ between oriented closed connected manifolds of non-zero degree, one can then proceed by a case-by-case analysis for the different values of amenable category. The most interesting case is to exclude the option $\mathrm{cat}_{\mathrm{Am}} M = 2$ when $\mathrm{cat}_{\mathrm{Am}} N = 3$. For this case, one uses a result of Wang [67, Lemma 3.4] and basic inheritance properties of amenable groups.

6.3. **Using bounded cohomology.** We will use bounded cohomology and simplicial volume to give sufficient conditions for a positive answer to the monotonicity problem (Question 1). This approach will provide infinite families of target manifolds satisfying the monotonicity problem.

Remark 6.15. It is worth mentioning that working with bounded cohomology and simplicial volume, one can also introduce the notion of ℓ^1-*invisible* manifolds [46]. Recently it has been proved [28, 47, 26] that a sufficient condition for an oriented closed connected n-manifold to be ℓ^1-invisible is having $\mathrm{cat}_{\mathrm{Am}} \leq n$. Hence, an argument similar to the one in Lemma 6.1 (Remark 6.2) shows that manifolds that are not ℓ^1-invisible satisfy the monotonicity problem.

It is known that ℓ^1-invisible manifolds have zero simplicial volume, but the converse implication is still unknown. For manifolds of dimension at most 3, the vanishing of simplicial volume and ℓ^1-invisibility are equivalent;

this can be seen via the computation of amenable category [30] or via classification/geometrisation and the inheritance properties [46, Example 6.7] of ℓ^1-invisibility. In this article, we prefer to stick to the case of simplicial volume because it is of wider interest.

Recall that a degree-one map $f\colon M \to N$ induces a monomorphism in cohomology $H^\bullet(f)\colon H^\bullet(N) \to H^\bullet(M)$ (Remark 3.11). Then, we have the following:

Proposition 6.16. *Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset$ Am and let $f\colon M \to N$ be a degree-one map between oriented closed connected manifolds. Suppose that N has efficient $\mathcal{G}$-category equal to $k+1$. Then, if $H_b^k(f)$ is injective, we have $\mathrm{cat}_{\mathcal{G}}(M) \geq \mathrm{cat}_{\mathcal{G}}(N)$.*

Proof of Proposition 6.16. Let us consider the following commutative diagram:

$$\begin{array}{ccc} H_b^k(N) & \xrightarrow{H_b^k(f)} & H_b^k(M) \\ {\scriptstyle \mathrm{comp}_N^k}\downarrow & & \downarrow{\scriptstyle \mathrm{comp}_M^k} \\ H^k(N) & \xrightarrow[H^k(f)]{} & H^k(M)\ . \end{array}$$

Since f is a degree-one map, $H^\bullet(f)$ is injective (Remark 3.11) and the same holds true for $H_b^k(f)$ by assumption. Thus, the vanishing of comp_M^k implies the vanishing of comp_N^k. However, since by assumption, comp_N^k is non-trivial, we have that comp_M^k is also non-trivial. By Theorem 3.12, this implies that $\mathrm{cat}_{\mathcal{G}}(M) \geq k+1 = \mathrm{cat}_{\mathcal{G}} N$. □

We now describe sufficient conditions such that the map in the previous result $H_b^k(f)$ is injective. First of all we see that this is always the case in degree 2.

Corollary 6.17. *Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset$ Am. If M and N are oriented closed connected manifolds with $M \geq_1 N$ and if N has efficient $\mathcal{G}$-category equal to 3, then $\mathrm{cat}_{\mathcal{G}}(M) \geq \mathrm{cat}_{\mathcal{G}}(N)$.*

Proof. Let $f\colon M \to N$ be a degree-one map. In order to apply Proposition 6.16 we have to show that $H_b^2(f)$ is injective. However, every group epimorphism induces an injective map on bounded cohomology of groups in degree 2 [3] and so this is the case for $H_b^2(\pi_1(f))$. Hence, by applying Corollary 3.5 we have that also $H_b^2(f)$ is injective. □

Remark 6.18. As mentioned in the previous section, Rudyak's conjecture for LS-category is always true if the target manifold has $\mathrm{cat}_{\mathrm{LS}}(N) \leq 3$. This means that the previous result should be interpreted as a (weaker) counterpart of the small-values-case for the generalised monotonicity problem.

Corollary 6.19. *Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset$ Am and let M and N be oriented closed connected manifolds such that there exists a degree-one map $M \to N$ with amenable kernel on the level of π_1. Then, if N has efficient $\mathcal{G}$-category, we have $\mathrm{cat}_{\mathcal{G}}(M) \geq \mathrm{cat}_{\mathcal{G}}(N)$.*

Proof. Let $f\colon M \to N$ be such a map. In order to apply Proposition 6.16, we have to show that $H_b^\bullet(f)$ is injective. However, Theorem 3.4 shows that under the hypothesis on $\ker(f)$ this is always the case. □

Following Bouarich [2], one can study the following family of groups:

Definition 6.20. Let Lex be the family of discrete groups Λ with the following "left exactness" property: For every group Γ and every group epimorphism $f\colon \Gamma \to \Lambda$, the induced map $H_b^\bullet(f)\colon H_b^\bullet(\Lambda) \to H_b^\bullet(\Gamma)$ in bounded cohomology is injective in all degrees.

Example 6.21. The following groups lie in Lex:

(1) *Amenable groups.* Indeed, their bounded cohomology vanishes in all positive degrees [32].
(2) *Free groups.* Given an epimorphism $\varphi\colon \Gamma \to F_n$ onto a free group, there exists a right inverse $\psi\colon F_n \to \Gamma$. Hence, the composition
$$H_b^\bullet(F_n) \xrightarrow{H_b^\bullet(\varphi)} H_b^\bullet(\Gamma) \xrightarrow{H_b^\bullet(\psi)} H_b^\bullet(F_n)$$
induces the identity morphism on bounded cohomology groups in every dimension. This shows that $H_b^\bullet(\varphi)$ is injective as claimed.
(3) The class Lex is stable under the following constructions: quotients by amenable subgroups, extension of an amenable group by an element of Lex and free products of amenable groups [2].
(4) *Fuchsian groups* (e.g., fundamental groups of closed surfaces with negative Euler characteristic) [2, Proposition 3.8 and Corollary 3.9].
(5) *Fundamental groups of a geometric* 3*-manifold* (e.g. Kleinian groups of finite volume). This can be shown by using Bouarich's result [2, Corollary 3.13 and pp. 267] together with Agol's proof of Thurston's Virtual Fibering Conjecture [1].

Corollary 6.22. *Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset$ Am and let M and N be oriented closed connected manifolds with $M \geq_1 N$. If N has efficient $\mathcal{G}$-category and $\pi_1(N) \in$ Lex, then $\operatorname{cat}_{\mathcal{G}}(M) \geq \operatorname{cat}_{\mathcal{G}}(N)$.*

Proof. This follows from the definition of Lex, Corollary 3.5 and Proposition 6.16. □

6.4. **Examples of manifolds with efficient category.** In the previous section we have described some sufficient conditions under which the monotonicity problem has a positive answer. We now investigate how to construct manifolds having efficient $\mathcal{G}$-category, where $\mathcal{G}$ consists of amenable groups. Recall that orientable $\mathrm{PD}^n_{\mathbb{R}}$-groups Γ satisfy $\operatorname{cd}\Gamma = n$ and $H^n(\Gamma) \cong \mathbb{R}$.

Proposition 6.23. *Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset$ Am. Let Γ be a finitely presented group lying in one of the following families:*

(1) hyperbolic orientable $\mathrm{PD}^n_{\mathbb{R}}$-groups for $n \geq 3$;
(2) fundamental groups of aspherical manifolds with positive simplicial volume;
(3) Hopfian fundamental groups of n-manifolds with positive simplicial volume and $\operatorname{gd}\Gamma = n$.

Then, for every $n \geq 2 \cdot \operatorname{gd}\Gamma + 1$, there exists an oriented closed connected n-manifold N with $\pi_1(N) \cong \Gamma$ and efficient $\mathcal{G}$-category.

Proof. By Proposition 4.2.4, in all three cases it suffices to show that $B\Gamma$ has efficient $\mathcal{G}$-category equal to $n+1$. We now consider the three cases:

(1) Let Γ be a hyperbolic orientable $\mathrm{PD}^n_{\mathbb{R}}$-group with $n \geq 3$ and let us consider a model of $B\Gamma$. Then, the comparison map of $B\Gamma$ is surjective in all degrees greater than or equal to 2 [52, Theorem 3]. Hence, since $H^n(B\Gamma) \cong \mathbb{R}$, we have that $\mathrm{comp}^n_{B\Gamma}$ is surjective and non-trivial. Moreover, as $n \geq 3$, we have $\mathrm{gd}\,\Gamma = \mathrm{cd}\,\Gamma = n$. This shows that $B\Gamma$ has efficient $\mathcal{G}$-category equal to $n+1$ (Theorem 3.12 and Remark 2.8).
(2) Let Γ be the fundamental group of an oriented closed connected aspherical manifold with positive simplicial volume. Then, Theorem 3.12 and Corollary 3.14 show that $B\Gamma$ has efficient $\mathcal{G}$-category.
(3) Let Γ be a Hopfian fundamental group of an oriented closed connected n-manifold M with positive simplicial volume. Since $n = \mathrm{gd}\,\Gamma$, there exists a model of $B\Gamma$ with $\mathrm{cat}_{\mathcal{G}}(B\Gamma) \leq n+1$ (Remark 2.8). Moreover, since $\|M\| > 0$, the comparison map $\mathrm{comp}^n_{B\Gamma}$ does not vanish [32, Section 3.1]. Hence, $B\Gamma$ has efficient $\mathcal{G}$-category equal to $n+1$ (Theorem 3.12). □

6.5. **Examples of manifolds that satisfy monotonicity.** We are now ready to produce explicit infinite families of examples of target manifolds for which monotonicity holds. Recall that Lex denotes the family of groups defined in Definition 6.20.

Theorem 6.24. *Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset \mathrm{Am}$. Let $\Gamma \in \mathrm{Lex}$ be a finitely presented group and suppose that Γ lies in one of the families of Proposition 6.23. Then for every $n \geq 2 \cdot \mathrm{gd}\,\Gamma + 1$, there exists an oriented closed connected n-manifold N with $\pi_1(N) \cong \Gamma$ such that for all oriented closed connected n-manifolds M we have*

$$M \geq_1 N \Longrightarrow \mathrm{cat}_{\mathcal{G}} M \geq \mathrm{cat}_{\mathcal{G}} N \ .$$

Proof. Since Γ lies in the families of Proposition 6.23, we have that for every $n \geq 2 \cdot \mathrm{gd}\,\Gamma + 1$, there exists an oriented closed connected n-manifold N that has efficient $\mathcal{G}$-category. Because $\Gamma \in$ Lex we can apply Corollary 6.22. □

As an application of the previous result, we can prove Theorem 4.

Corollary 6.25. *Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset \mathrm{Am}$. Let Γ be the fundamental group of an oriented closed connected hyperbolic k-manifold of dimension $k \in \{2, 3\}$. Then, for every $n \geq 2k$ there exists an oriented closed connected n-manifold N with $\pi_1(N) \cong \Gamma$ such that: For all oriented closed connected n-manifolds M we have*

$$M \geq_1 N \Longrightarrow \mathrm{cat}_{\mathcal{G}} M \geq \mathrm{cat}_{\mathcal{G}} N.$$

Proof. Since Γ is the fundamental group of an oriented closed connected aspherical n-manifold with positive simplicial volume, Γ lies in the classes of Proposition 6.23 and we can use the improved dimension bound (Remark 4.3). Moreover, we have already seen in Example 6.21 that Γ also lies in Lex. Hence, Theorem 6.24 applies. □

7. Bounds via classifying spaces

In this section, we explain how to characterise categorical invariants for classes of groups in terms of classifying spaces of families of subgroups – in analogy with the corresponding statement for cat_1 by Eilenberg and Ganea [20]. In particular, this leads to a corresponding lower bound in terms of Bredon cohomology.

7.1. **Classifying spaces of families.** We recall classifying spaces of families of subgroups [48].

Definition 7.1. Let Γ be a group and let F be a subgroup family of Γ.

- A Γ-CW-complex X has *F-restricted isotropy* if all its isotropy groups lie in F, i.e., for every $x \in X$ one has
$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma \cdot x = x\} \in F.$$
- A *model for the classifying space $E_F\Gamma$ for the family of subgroups F* is a Γ-CW-complex X with F-restricted isotropy with the following universal property: for every Γ-CW-complex Y with F-restricted isotropy there exists a unique (up to Γ-homotopy) Γ-equivariant map $Y \to X$.

We will use the notation $E_F\Gamma$ to denote the choice of a model for the classifying space (it is well defined up to canonical Γ-homotopy equivalence) and we denote by $f_{Y,\Gamma,F} : Y \to E_F\Gamma$ a choice of a map given by the universal property.

When F is the trivial family, then $E_F\Gamma$ is denoted simply by $E\Gamma$ and we recover the usual model of the classifying space of a group Γ.

Remark 7.2. When Γ is a discrete group, a Γ-CW-complex X is simply a CW-complex X endowed with a cellular Γ-action with the following property: For each open cell $e \subset X$ and each $\gamma \in \Gamma$ such that $\gamma \cdot e \cap e \neq \emptyset$, the left multiplication by γ restricts to the identity on e [48, Example 1.5].

This property shows that if a discrete group Γ acts on a simplicial complex K via simplicial automorphisms, it is not true in general that K has the structure of Γ-CW-complex. On the other hand, the induced action by Γ on the first barycentric subdivision K' of K makes K' a Γ-CW-complex.

The remark above suggests the following definition:

Definition 7.3. Let K be a simplicial complex and let Γ be a discrete group acting on K via simplicial automorphisms. We say that K is an *admissible Γ-simplicial complex* if it is a Γ-CW-complex.

Classifying spaces for subgroup families have recently been used to give a new proof of Gromov's Vanishing Theorem (Theorem 3.12) [47]. We will use some results of *loc. cit.* to prove Lemma 7.6.

7.2. **Bounding topological complexity via classifying spaces.** Classifying spaces for subgroup families also play an important role in the computation of the topological complexity of aspherical spaces. The following result shows that topological complexity can be characterised by means of the factorisation of the classifying map $f_{E(\Gamma\times\Gamma),\Gamma\times\Gamma,\mathcal{D}}$, where $\mathcal{D}$ is the subgroup family introduced in Definition 2.18.

Proposition 7.4 ([23, Theorem 3.3])**.** *Let X be a connected finite aspherical CW-complex with fundamental group Γ. Then $\mathrm{TC}(X)-1$ coincides with the minimal integer $k \in \mathbb{N}_{\geq 0}$ such that the classifying map*

$$f_{E(\Gamma\times\Gamma),\Gamma\times\Gamma,\mathcal{D}}\colon E(\Gamma\times\Gamma) \to E_{\mathcal{D}}(\Gamma\times\Gamma)$$

is $(\Gamma\times\Gamma)$-homotopic to a map with values in the k-dimensional skeleton $E_{\mathcal{D}}(\Gamma\times\Gamma)^{(k)}$ (of any model of $E_{\mathcal{D}}(\Gamma\times\Gamma)$).

Using the previous result, Dranishnikov computed the topological complexity of finitely generated geometrically finite hyperbolic groups [12].

7.3. **Bounding category via classifying spaces.** Similarly, we can also characterise F-categories:

Proposition 7.5. *Let X be a connected CW-complex and let $\tilde{X}$ be its universal covering. Let Γ be the fundamental group of X and let F be a subgroup family of Γ. Then, $\mathrm{cat}_F(X)-1$ coincides with the minimal integer $k \in \mathbb{N}_{\geq 0}$ such that the classifying map*

$$f_{\tilde{X},\Gamma,F}\colon \tilde{X} \to E_F\Gamma$$

is Γ-homotopic to a map with values in the k-dimensional skeleton $E_F\Gamma^{(k)}$ (of any model of $E_F\Gamma$).

The proof of Proposition is based on the following lemma.

Lemma 7.6. *Let X be a connected CW-complex and let $n \in \mathbb{N}$. Let F be a subgroup family of the fundamental group Γ of X. Then, $\mathrm{cat}_F(X) \leq n+1$ if and only if there exists a connected Γ-CW-complex L with F-restricted isotropy of dimension at most n and a Γ-map $\tilde{X} \to L$.*

Proof. If we assume that $\mathrm{cat}_F(X) \leq n+1$, then there exists an open F-cover U of X of cardinality at most $n+1$. Let $p\colon \tilde{X} \to X$ denote the universal covering map and let $\tilde{U}$ be the corresponding open cover of $\tilde{X}$ (as in the proof of Proposition 5.3):

$$\tilde{U} := \big\{ V \subset \tilde{X} \;\big|\; \text{there exists a } W \in U \text{ such that } V \text{ is a path-connected component of } p^{-1}(W).\big\}$$

Now we take L to be the nerve of $\tilde{U}$ and $f\colon \tilde{X} \to |L|$ to be a nerve map. Then, $|L|$ is a Γ-CW-complex [47, Lemma 4.5]. Moreover, the nerve map is a Γ-map [47, Lemma 4.8] and clearly we have $\dim(|L|) \leq \dim(N) \leq n$, where N denotes the nerve of U. Finally, $|L|$ has F-restricted isotropy [47, Lemma 4.11].

Conversely, let L be a Γ-CW-complex with F-restricted isotropy of dimension at most n and let $f\colon \tilde{X} \to L$ be a Γ-map. By the equivariant version of simplicial approximation [55, Proposition A.4], we can assume without loss of generality that L is a connected admissible Γ-simplicial complex with F-restricted isotropy and with dimension at most n. Let U be the open cover of L' consisting of the open stars of the barycentric subdivision L' of L, indexed by the dimension of the underlying simplices of L. Grouping the sets of U according to the dimension of the underlying simplices, we can

replace U by an open cover of $n+1$ open sets. Let us pull back U via f and push it down to X via p. We get the following open cover of X:

$$\big\{p(f^{-1}(V)) \;\big|\; V \in U\big\}.$$

By construction, the cardinality of this open cover is again at most $n+1$.

Hence, to conclude it is enough to show that this cover is indeed an F-cover of X. Let $V \in U$. We show now that for every $x \in f^{-1}(V)$ there exists a $y \in L$ such that

$$\mathrm{im}\big(\pi_1(p(f^{-1}(V)) \hookrightarrow X, p(x))\big)$$

is a subgroup (up to conjugation) of the isotropy group Γ_y. The result then easily follows since L has F-restricted isotropy and F is closed under taking subgroups.

So, let $x \in f^{-1}(V)$. Since we need to prove the statement "up to conjugation", we can assume that $\Gamma = \pi_1(X, x)$. Let $\gamma \in \mathrm{im}(\pi_1(p(f^{-1}(V)) \hookrightarrow X, p(x)))$, i.e., $\gamma = [\sigma]$ where $\sigma\colon [0,1] \to p(f^{-1}(V)) \subseteq X$ is a loop based at $p(x)$. By the lifting properties of the covering

$$p_{|f^{-1}(V)} : f^{-1}(V) \to p(f^{-1}(V)),$$

there exists a lift $\tilde{\sigma}\colon [0,1] \to f^{-1}(V) \subseteq \tilde{X}$ such that $p \circ \tilde{\sigma} = \sigma$ and $\tilde{\sigma}(0) = x$. Moreover, by definition of the deck transformation action and by uniqueness of the lift, we have that $\gamma \cdot \tilde{\sigma}(1) = x$. Now, since f is a Γ-map, we also have

$$\gamma \cdot f(\tilde{\sigma}(1)) = f(x).$$

Moreover, both $f(\tilde{\sigma}(1))$ and $f(x)$ are in V and $f \circ \tilde{\sigma}$ is a path connecting them contained in V. Hence they lie in the same path connected component of V and, by construction of the cover U, there exists a vertex v of L' such that $f(\tilde{\sigma}(1)), f(x) \in \mathrm{st}(v)$. Since the action on L' is simplicial, we know that $\gamma \cdot \mathrm{st}(v) = \mathrm{st}(\gamma \cdot v)$. It follows that $\mathrm{st}(v) \cap \mathrm{st}(\gamma \cdot v) \neq \emptyset$. Hence, by definition of open stars, we have that there exists a simplex $\sigma \subset L'$ such that $\gamma \cdot \mathrm{int}(\sigma) \cap \mathrm{int}(\sigma) \neq \emptyset$ and whose set of vertices contains both v and $\gamma \cdot v$. Using the fact that L' is an admissible Γ-simplicial complex, Remark 7.2 readily implies that $\gamma \cdot \sigma = \sigma$. This shows that $\gamma \cdot v = v$, whence $\gamma \in \Gamma_v$, as desired. □

Proof of Proposition 7. The result follows from Lemma 7.6 by using the universal property of classifying spaces for subgroup families and the equivariant cellular approximation theorem [48, Section 1.1]. □

The previous proposition has implications in terms of Bredon cohomology, as shown by Farber, Grant, Lupton, and Oprea [23]. Namely, it provides a lower bound for the topological complexity, via the diagonal category, in terms of a vanishing map in Bredon cohomology [23, Theorem 4.1]. Thanks to Proposition 7.5, this argument also applies to general subgroup families.

Corollary 7.7. *Let X be an aspherical connected CW-complex. Let Γ be the fundamental group of X and let F be a subgroup family of Γ. Suppose that there exists $k \in \mathbb{N}_{\geq 0}$ and a Bredon module M such that the restriction*

$$H^k_F(\Gamma; M) \to H^k(\Gamma; \mathrm{res}^F_{\{1\}} M)$$

is non-zero. Then $\mathrm{cat}_F(X) \geq k+1$.

Proof. Using the factorisation characterisation of cat_F (Proposition 7.5), we can proceed as in the proof for the diagonal family [23, Theorem 4.1]. □

The following application is due to Kevin Li:

Example 7.8. Let M be an oriented closed connected aspherical n-manifold whose fundamental group is hyperbolic and let $n \geq 2$. Then $\mathrm{cat}_{\mathrm{Am}} M = n+1$. Of course, this follows from Mineyev's result on the comparison map in bounded cohomology and Gromov's vanishing theorem (see the proof of Proposition 6.23). Using Corollary 7.7, one can give an alternative argument: Because Γ is hyperbolic (and torsion-free), all amenable subgroups of Γ are virtually cyclic and the construction principle of Juan-Pineda and Leary can be used to obtain a model of $E_{\mathrm{Am}}\Gamma$ from a model of $E\Gamma$ [43, Remark 7]. Therefore, the restriction map

$$H^k_{\mathrm{Am}}(\Gamma;\mathbb{R}) \to H^k(\Gamma;\mathbb{R})$$

is an epimorphism for every $k \in \mathbb{N}_{\geq 2}$ (this is the cohomological analogue of the exact sequence of Juan-Pineda and Leary [43, Proposition 18]). Moreover, M is a model of $B\Gamma$ and so $H^n(\Gamma;\mathbb{R}) \cong H^n(M) \cong \mathbb{R}$. So, Corollary 7.7 shows that $\mathrm{cat}_{\mathrm{Am}} M \geq n+1$.

8. Amenable category vs. topological complexity

We now investigate the following question:

Question 8.1. For which topological spaces X do we have

$$\mathrm{cat}_{\mathrm{Am}}(X \times X) \leq \mathrm{TC}(X) \ ?$$

8.1. **Basic examples.** As warm-up examples behind this question, we consider spaces with amenable fundamental group, wedges of circles, and surfaces.

Example 8.2. As the product of two amenable groups is amenable, we have $\mathrm{cat}_{\mathrm{Am}}(X \times X) = 1$ for all spaces X with amenable fundamental group. Therefore, the property in Question 8.1 holds in this case.

Example 8.3. Let $X = \bigvee_I \mathbb{S}^1$ be a finite wedge of circles. Then we have $\mathrm{TC}(X) \geq 3$ [22, Theorem 7.3]. Hence, by applying the product formula (Proposition 2.7) we get

$$\mathrm{cat}_{\mathrm{Am}}(X \times X) \leq 2 \cdot \mathrm{cat}_{\mathrm{Am}}(X) - 1 \leq 3 \ ,$$

whence the inequality in Question 8.1.

Lemma 8.4. *If S is an oriented connected surface, then we have*

$$\mathrm{cat}_{\mathrm{Am}}(S \times S) \leq \mathrm{TC}(S) \ .$$

Proof. Let us assume first that S is a non-compact surface or a surface with boundary. Then, we know that S has (possibly trivial) free fundamental group and it retracts to a (possibly trivial) wedge of circles. Hence, since both Am-category and TC are homotopy invariants, the desired inequality comes from the computation in Example 8.3.

Let us now assume that S is a closed connected surface. If S has amenable fundamental group, then we are done by Example 8.2.

So we are reduced to consider the case of hyperbolic surfaces Σ_g, with $g \geq 2$. On the one hand, we know that $\mathrm{TC}(\Sigma_g) = 5$ [21, Theorem 9]. On the other hand, by Example 3.9 and Corollary 3.14, we have $\mathrm{cat}_{\mathrm{Am}}(\Sigma_g \times \Sigma_g) = 5$. This shows that $\mathrm{cat}_{\mathrm{Am}}(S \times S) \leq \mathrm{TC}(S)$ is verified also in this case. □

We show in the next section that the results about wedges of circles and surfaces admit natural generalisations.

8.2. **Proof of Theorem 5.** In this section, we will prove the following:

Theorem 8.5. *The following classes of spaces satisfy the estimate in Question 8.1:*

1. *Spaces with amenable fundamental group;*
2. *Spaces of type $B\Gamma$ where Γ is a finitely generated geometrically finite hyperbolic group;*
3. *Spaces of type $B\Gamma$ where $\Gamma = H * H$ is the free square of a geometrically finite group H;*
4. *Manifolds whose fundamental group is the fundamental group of a graph of groups whose vertex (and edge) groups are all amenable.*

Proof. Ad 1. We have already discussed this case in Example 8.2.

Ad 2. Let X be a model of $B\Gamma$, where Γ is a finitely generated geometrically finite hyperbolic group. Then, we have that $2 \cdot \mathrm{gd}\,\Gamma + 1 \leq \mathrm{TC}(X)$ [12, Theorem 3.0.2]. Hence, we get the following chain of inequalities:

$$\begin{aligned} \mathrm{cat}_{\mathrm{Am}}(X \times X) &\leq \mathrm{cat}_{\mathrm{LS}}(X \times X) \\ &\leq 2 \cdot \mathrm{gd}\,\Gamma + 1 \\ &= \mathrm{TC}(X)\ . \end{aligned}$$

Ad 3. Let H be a geometrically finite group and let $\Gamma = H * H$. Assume that X and Y are model of $B\Gamma$ and BH of minimal dimension, respectively. Then, X is homotopy equivalent to $Y \vee Y$. This shows the following

$$\begin{aligned} \mathrm{TC}(X) &= \mathrm{TC}(Y \vee Y) \\ &= \max\{\mathrm{TC}(Y), 2 \cdot \mathrm{gd}\,H + 1\} \qquad \text{by [14, Theorem 2]} \\ &= \max\{\mathrm{TC}(Y), 2 \cdot \mathrm{gd}\,\Gamma + 1\} \\ &\geq 2 \cdot \mathrm{gd}\,\Gamma + 1 \\ &\geq \mathrm{cat}_{\mathrm{LS}}(X \times X) \\ &\geq \mathrm{cat}_{\mathrm{Am}}(X \times X). \end{aligned}$$

Ad 4. Let M be a manifold whose fundamental group Γ is the fundamental group of a graph of groups whose vertex (and edge) groups are all amenable. If Γ is amenable, then we are in the case of the first part.

So, let us assume that Γ is non-amenable. Then, there exists a model X of $B\Gamma$ with $\mathrm{cat}_{\mathrm{Am}}(X) = 2$ (Theorem 3). Hence, given a classifying map $f\colon M \to X$ for M, by Remark 2.9 we get $\mathrm{cat}_{\mathrm{Am}}(M) \leq 2$. This implies that $\mathrm{cat}_{\mathrm{Am}}(X \times X) \leq 3$ by the product formula (Proposition 2.7).

If $\mathrm{cat}_{\mathrm{LS}}(M) \leq 2$, then M is either contractible or a homotopy sphere [41, page 336]. This shows that M has abelian (whence amenable) fundamental

group. As we assumed Γ to be non-amenable, we have $\mathrm{cat}_{\mathrm{LS}}(M) \geq 3$ and thus obtain the following chain of inequalities:

$$\mathrm{cat}_{\mathrm{Am}}(M \times M) \leq 3 \leq \mathrm{cat}_{\mathrm{LS}}(M) \leq \mathrm{TC}(M) \ . \qquad \square$$

Remark 8.6. In the parts (2) and (3) of Theorem 5 we actually showed that $\mathrm{cat}_{\mathrm{LS}}(X \times X) \leq \mathrm{TC}(X)$ holds for the LS-category. However, this does not hold in general: Indeed, if we consider the n-dimensional torus T^n for $n \in \mathbb{N}_{\geq 2}$, we have $\mathrm{cat}_{\mathrm{LS}}(T^n \times T^n) = 2n + 1$, but $\mathrm{TC}(T^n) = n + 1$ [21].

8.3. **Manifolds with 3-manifold fundamental group.** We show that the techniques introduced in Section 4 allow us to construct infinite families of manifolds for whom Question 8.1 is answered positively. In fact, the following results are also true for all isq-classes of groups $\mathcal{G} \subset \mathrm{Am}$.

Theorem 8.7. *Let $\mathcal{G}$ be an isq-class of groups $\mathcal{G} \subset \mathrm{Am}$. Let Y be an oriented closed connected triangulable n-manifold whose fundamental group is Hopfian. Let $\overline{k} > 1$ be the maximal integer such that Y is essential in degree $\overline{k}$. Then, if*

$$\mathrm{cat}_{\mathcal{G}}(Y) \leq \frac{\overline{k}}{2} + \frac{3}{2},$$

there exists for every $n \geq 2\dim(Y)$ an oriented closed connected n-manifold M such that $\pi_1(M) \cong \pi_1(Y)$ and

$$\mathrm{cat}_{\mathcal{G}}(M \times M) \leq \mathrm{TC}(M).$$

Proof. For every $n \geq 2 \cdot \dim(Y)$, by Corollary 4.7, there exists an oriented closed connected triangulable manifold M that is essential in degree $\overline{k}$ with

$$\mathrm{cat}_{\mathcal{G}}(M) \leq \mathrm{cat}_{\mathcal{G}}(Y) \qquad \text{and} \qquad \pi_1(M) \cong \pi_1(Y).$$

Hence, using Proposition 2.7 and the hypothesis on $\mathrm{cat}_{\mathcal{G}}(Y)$, we obtain

$$\mathrm{cat}_{\mathcal{G}}(M \times M) \leq 2 \cdot \mathrm{cat}_{\mathcal{G}}(M) - 1 \leq 2 \cdot \mathrm{cat}_{\mathcal{G}}(Y) - 1 \leq \overline{k} + 2.$$

By Remark 4.6 M is also a strictly $\overline{k}$-essential manifold. Hence, it satisfies $\mathrm{cat}_{\mathrm{LS}}(M) \geq \overline{k}+2$ [17, Theorem 5.2]. Since we already know that $\mathrm{cat}_{\mathrm{LS}}(M) \leq \mathrm{TC}(M)$ by Proposition 2.14, we get

$$\begin{aligned} \mathrm{cat}_{\mathcal{G}}(M \times M) &\leq 2\,\mathrm{cat}_{\mathcal{G}}(M) - 1 \leq 2\,\mathrm{cat}_{\mathcal{G}}(Y) - 1 \leq \overline{k} + 2 \leq \mathrm{cat}_{\mathrm{LS}}(M) \\ &\leq \mathrm{TC}(M). \end{aligned} \qquad \square$$

As an application of the previous result, we obtain Theorem 6.

Corollary 8.8. *Let $\mathcal{G}$ be an isq-class of groups with $\mathcal{G} \subset \mathrm{Am}$. Let Y be an oriented closed connected 3-manifold which is a connected sum of graph manifolds. Then, for every $n \geq 6$, there exists an oriented closed connected n-manifold M with $\pi_1(M) \cong \pi_1(Y)$ and such that*

$$\mathrm{cat}_{\mathcal{G}}(M \times M) \leq \mathrm{TC}(M) \ .$$

Proof. Since Y is an oriented closed connected 3-manifold, its fundamental group is residually finite [36], whence Hopfian. Moreover, Y being a connected sum of aspherical manifolds, is essential in degee 3. Finally,

$\mathrm{cat}_{\mathcal{G}}(Y) \leq 3$ because Y is the connected sum of graph manifolds [30, Theorem 2][31, Corollary 5]. Hence, we have

$$\mathrm{cat}_{\mathcal{G}}(Y) \leq 3 = \frac{3}{2} + \frac{3}{2} = \frac{\dim(Y)}{2} + \frac{3}{2}.$$

This inequality shows that we are in the situation of Theorem 8.7, whence we get the thesis. □

It is a natural question to ask if the previous result can be improved by considering M to be actually a connected sum of graph manifolds. As far as we know, our approach does not lead to this stronger statement, but we refer the reader to recent results by Mescher [50] in this direction.

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