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TOPOLOGICAL VOLUMES OF FIBRATIONS: A NOTE ON OPEN COVERS

CLARA LÖH AND MARCO MORASCHINI

ABSTRACT. We establish a straightforward estimate for the number of open sets with fundamental group constraints needed to cover the total space of fibrations. This leads to vanishing results for simplicial volume and minimal volume entropy, e.g., for certain mapping tori.

1. INTRODUCTION

The Lusternik-Schnirelmann category $\mathrm{cat}_{\mathrm{LS}}(X)$ of a topological space X is the minimal number of open and in X contractible sets necessary to cover X. Despite the first applications of Lusternik-Schnirelmann category were more focused on the study of critical points [36, 14], it is now widely applied also to the study of algorithms' complexity [40, 44] and topological robotics [19].

Relaxing the contractibility condition leads to generalised categorical invariants $\mathrm{cat}_{\mathscr{G}}$ with fundamental group constraints (Section 2): Let $\mathscr{G}$ be a class of groups. A *$\mathscr{G}$-set* in a space X is a subset whose path-connected components all have π_1-image in $\mathscr{G}$. The $\mathscr{G}$-category of X is the minimal number $\mathrm{cat}_{\mathscr{G}}(X)$ of open $\mathscr{G}$-sets needed to cover X. Geometrically relevant classes $\mathscr{G}$ are the class Am of amenable groups or classes of groups with controlled growth (for instance, $\mathrm{Subexp}_{<\delta}$).

The main observation of the present note is the following straightforward adaption of an estimate for the Lusternik-Schnirelmann category [43, 25, 29] to the case of fundamental group constraints:

Theorem 1.1 ($\mathscr{G}$-category of fibrations)**.** *Let $p\colon E \to B$ be a fibration with a path-connected base space. Let $x_0 \in B$ be a non-degenerate basepoint of B and let $F := p^{-1}(x_0)$ denote the fibre. Moreover, let $\mathscr{G}$ be a class of groups that is closed under isomorphisms, subgroups, and quotients. Then:*

$$\mathrm{cat}_{\mathscr{G}}(E) \leq \mathrm{cat}_{\mathscr{G}}(F) \cdot \mathrm{cat}_{\mathrm{LS}}(B)$$

We prove this statement in Theorem 3.1 in terms of categorical invariants of maps. It should be noted that the rough estimate provided by Theorem 1.1, in general, cannot be improved to $\mathrm{cat}_{\mathscr{G}}(E) \leq \mathrm{cat}_{\mathscr{G}}(F) \cdot \mathrm{cat}_{\mathscr{G}}(B)$ (Example 4.15).

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Applications to topological volumes. Simplicial volume and minimal volume entropy are examples of "topological volumes", i.e., of $\mathbb{R}$-valued invariants of manifolds that mitigate between topological properties and Riemannian volume (Section 4.1, Section 6.1). Both simplicial volume and minimal volume entropy admit vanishing theorems in terms of π_1-constrained open covers of small multiplicity. Therefore, Theorem 1.1 gives corresponding vanishing results for fibrations and fibre bundles. For example:

For simplicial volume, we can combine Theorem 1.1 with Gromov's vanishing theorem for bounded cohomology and amenable covers [24] (Theorem 4.6):

Corollary 1.2 (Corollary 4.9)**.** *Let M be an oriented closed connected manifold that is the total space of a fibre bundle $M \to B$ with oriented closed connected fibre N and base B. If*

$$\operatorname{cat}_{\mathrm{Am}}(N) \leq \frac{\dim(M)}{\dim(B)+1},$$

then $\|M\| = 0$.

The fibre collapsing assumption of Babenko and Sabourau can be translated into generalised Lusternik-Schnirelmann category invariants (Section 5). Therefore, combining Theorem 1.1 with the vanishing result of Babenko and Sabourau [4, Theorem 1.3] (Theorem 6.4), we obtain:

Corollary 1.3 (Corollary 6.8)**.** *Let M be an oriented closed connected smooth manifold that is the total space of a fibre bundle $M \to B$ with oriented closed connected smooth fibre N and base B. If*

$$\operatorname{cat}_{\mathrm{Subexp}_{<1/\dim(M)}}(N) \leq \frac{\dim(M)}{\dim(B)+1},$$

then $\operatorname{minent}(M) = 0$.

In particular, Corollary 1.2 and Corollary 1.3 lead to vanishing results for certain mapping tori. In the case of simplicial volume, this complements vanishing results of Bucher and Neofytidis [10] (Remark 4.12).

Moreover, we use generalised Lusternik-Schnirelmann category invariants to generalise a result by Bregman and Clay [8, Proposition 4.1] on the fibre collapsing assumptions and graphs of groups (Remark 5.11) and provide an aspherical version of examples of Babenko and Sabourau [4, Theorem 1.6] of simplicial complexes with large minimal volume entropy and small "simplicial volume" (Proposition 6.11, Corollary 6.12).

Organisation of this article. We recall the generalised Lusternik-Schnirelmann category in Section 2. The proof of Theorem 1.1 is given in Section 3; applications to simplicial volume are contained in Section 4. In Section 5, we recall the fibre (non-)collapsing assumption by Babenko and Sabourau; finally, the applications to minimal volume entropy are located in Section 6.

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2. Generalised LS-category

We recall the definition of the Lusternik-Schnirelmann category (of spaces and maps) and the generalisation to fundamental group constraints [20, 29, 15, 18, 22, 12].

2.1. LS-Category of spaces and maps.

Definition 2.1 (LS-category)**.** Let X be a topological space. The *Lusternik-Schnirelmann category* (or simply LS-*category*) of X, denoted by $\mathrm{cat}_{\mathrm{LS}}(X)$, is the minimal number $n \in \mathbb{N} = \{0, 1, \dots\}$ such that X can be covered with open sets $U_1, \dots, U_n$ that are contractible in X. If such an n does not exist, we set $\mathrm{cat}_{\mathrm{LS}}(X) := +\infty$.

Similarly, we have the definition of LS-category of a continuous map:

Definition 2.2 (LS-category of a map)**.** Let $f\colon X \to Y$ be a continuous map between topological spaces. The LS-*category* of f, denoted by $\mathrm{cat}_{\mathrm{LS}}(f)$, is the minimal number $n \in \mathbb{N}$ such that X can be covered with open sets $U_1, \dots, U_n$ such that the restriction $f|_{U_i}$ is null-homotopic for each $i \in \{1, \dots, n\}$. If such a number n does not exist, we set $\mathrm{cat}_{\mathrm{LS}}(f) := +\infty$.

2.2. $\mathscr{G}$-Category of spaces and maps.

Definition 2.3 ($\mathscr{G}$-sets (for a map))**.** Let $\mathscr{G}$ be a class of groups. Let X be a topological space and let U be a subset of X. We say that U is a $\mathscr{G}$-*set* if for all $x \in U$, we have

$$\mathrm{im}\big(\pi_1(U \hookrightarrow X, x)\big) \in \mathscr{G}.$$

An open cover $\mathcal{U}$ of X is called a $\mathscr{G}$-*cover* if each open subset in $\mathcal{U}$ is a $\mathscr{G}$-set.

Similarly, given a continuous map $f\colon X \to Y$ between topological spaces, we say that an open set $U \subset X$ is a $\mathscr{G}$-*set for* f if for every $x \in U$, we have

$$\pi_1(f)\big(\pi_1(U, x)\big) \in \mathscr{G}.$$

An open cover $\mathcal{U}$ of X is called a $\mathscr{G}$-*cover for* f if each open subset in $\mathcal{U}$ is a $\mathscr{G}$-set for f.

Definition 2.4 ($\mathscr{G}$-category (of a map))**.** Let $\mathscr{G}$ be a class of groups. Let X be a topological space. The $\mathscr{G}$-*category* of X, denoted by $\mathrm{cat}_{\mathscr{G}}(X)$, is the minimal number $n \in \mathbb{N}$ such that X admits an open $\mathscr{G}$-cover of cardinality n. If such an n does not exist, we set $\mathrm{cat}_{\mathscr{G}}(X) := +\infty$.

Similarly, the $\mathscr{G}$-*category of a continuous map* $f\colon X \to Y$ between topological spaces, denoted by $\mathrm{cat}_{\mathscr{G}}(f)$, is the minimal number $n \in \mathbb{N}$ such that X admits an open $\mathscr{G}$-cover for f of cardinality n. If such an n does not exist, we set $\mathrm{cat}_{\mathscr{G}}(f) := +\infty$.

Remark 2.5. Let $\mathscr{G}$ be a class of groups. If $\mathscr{H}$ is a class of groups with $\mathscr{H} \subset \mathscr{G}$, then $\mathrm{cat}_{\mathscr{G}} \leq \mathrm{cat}_{\mathscr{H}}$. In particular: If $\mathscr{G}$ contains all trivial groups and X is a finite-dimensional simplicial complex, then $\mathrm{cat}_{\mathscr{G}}(X) \leq \dim(X) + 1$, as can be seen by the open stars cover of the barycentric subdivision of X (grouped and indexed by the simplices of X).

Remark 2.6. We say that topological spaces X and Y are π_1-*equivalent*, if there exists continuous maps (called π_1-*equivalences*) $X \to Y$ and $Y \to X$

inducing isomorphisms on the level of fundamental groups (these maps are not required to be π_1-inverse to each other).

Similarly, maps $f, g\colon X \to Y$ are said to be π_1-*homotopic* if they induce the same homomorphism on the level of fundamental groups.

Let $\mathscr{G}$ be a class of groups that is closed under isomorphism and let X, Y be π_1-equivalent spaces. Then, $\operatorname{cat}_{\mathscr{G}}(X) = \operatorname{cat}_{\mathscr{G}}(Y)$. Similarly, if $f, g\colon X \to Y$ are π_1-homotopic maps, then $\operatorname{cat}_{\mathscr{G}}(f) = \operatorname{cat}_{\mathscr{G}}(g)$.

We collect some basic properties of the $\mathscr{G}$-category of a map, which are known to hold in the case of LS-category [15, Exercise 1.16].

Lemma 2.7 (properties of $\mathscr{G}$-category)**.** *Let $\mathscr{G}$ be a class of groups that is closed under isomorphisms, subgroups, and quotients. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps between topological spaces. Then, we have the following:*

(1) $\operatorname{cat}_{\mathscr{G}}(f) \leq \min\{\operatorname{cat}_{\mathscr{G}}(X), \operatorname{cat}_{\mathscr{G}}(Y)\}$*;*
(2) $\operatorname{cat}_{\mathscr{G}}(g \circ f) \leq \min\{\operatorname{cat}_{\mathscr{G}}(g), \operatorname{cat}_{\mathscr{G}}(f)\}$*;*
(3) If f is a homotopy equivalence, then $\operatorname{cat}_{\mathscr{G}}(f) = \operatorname{cat}_{\mathscr{G}}(X) = \operatorname{cat}_{\mathscr{G}}(Y)$.

Proof. Ad. 1. Let $\mathcal{U}$ be an open $\mathscr{G}$-cover of X. Then, for each $U \in \mathcal{U}$ and every $x \in U$, we have the following commutative diagram

$$\begin{array}{ccc} \pi_1(U,x) & \xrightarrow{\pi_1(f|_U)} & \pi_1\big(Y, f(x)\big) \\ \downarrow & & \| \\ \pi_1(X,x) & \xrightarrow[\pi_1(f)]{} & \pi_1\big(Y, f(x)\big), \end{array}$$

where the left vertical arrow is induced by the inclusion. Because U is a $\mathscr{G}$-set in X and $\mathscr{G}$ is closed under quotients, the previous diagram shows that U is a $\mathscr{G}$-set for f. Therefore, $\mathcal{U}$ is an open $\mathscr{G}$-cover for f. Taking the infimum over all open $\mathscr{G}$-covers of X shows that $\operatorname{cat}_{\mathscr{G}}(f) \leq \operatorname{cat}_{\mathscr{G}}(X)$.

On the other hand, if $\mathcal{V}$ is an open $\mathscr{G}$-cover of Y, then $(f^{-1}(V))_{V \in \mathcal{V}}$ is an open $\mathscr{G}$-cover for f. Hence, we get $\operatorname{cat}_{\mathscr{G}}(f) \leq \operatorname{cat}_{\mathscr{G}}(Y)$.

Ad. 2. As $\mathscr{G}$ is closed under taking quotients, it is immediate to check that $\operatorname{cat}_{\mathscr{G}}(g \circ f) \leq \operatorname{cat}_{\mathscr{G}}(f)$.

Moreover, if $\mathcal{U}$ is an open $\mathscr{G}$-cover for g of Y, we can consider the pullback $f^{-1}\mathcal{U}$, which is an open $\mathscr{G}$-cover for $g \circ f$. Therefore, $\operatorname{cat}_{\mathscr{G}}(g \circ f) \leq \operatorname{cat}_{\mathscr{G}}(g)$.

Ad. 3. Because X and Y are homotopy equivalent, we have $\operatorname{cat}_{\mathscr{G}}(X) = \operatorname{cat}_{\mathscr{G}}(Y)$ (Remark 2.6). Let $f\colon X \to Y$ be a homotopy equivalence and let $g\colon Y \to X$ be a homotopy inverse. Then, $g \circ f$ is homotopic to id_X. Then Remark 2.6 and the first two parts show that

$$\begin{aligned} \operatorname{cat}_{\mathscr{G}}(X) &= \operatorname{cat}_{\mathscr{G}}(\mathrm{id}_X) = \operatorname{cat}_{\mathscr{G}}(g \circ f) \\ &\leq \min\{\operatorname{cat}_{\mathscr{G}}(f), \operatorname{cat}_{\mathscr{G}}(g)\} \leq \max\{\operatorname{cat}_{\mathscr{G}}(f), \operatorname{cat}_{\mathscr{G}}(g)\} \leq \operatorname{cat}_{\mathscr{G}}(X). \end{aligned}$$

This shows that $\operatorname{cat}_{\mathscr{G}}(Y) = \operatorname{cat}_{\mathscr{G}}(X) = \operatorname{cat}_{\mathscr{G}}(f)$. □

3. Generalised LS-category and fibrations

In this section, we prove Theorem 1.1. Indeed, Lemma 2.7 shows that Theorem 1.1 is a consequence of the following statement:

Theorem 3.1 ($\mathscr{G}$-category of fibrations)**.** *Let $p\colon E \to B$ be a fibration with a path-connected base space. Let $x_0 \in B$ be a non-degenerate basepoint of B, let $F := p^{-1}(x_0)$, and let $\iota\colon F \hookrightarrow E$ denote the inclusion of the fibre. Moreover, let $\mathscr{G}$ be a class of groups that is closed under isomorphisms and subgroups. Then:*

$$\operatorname{cat}_{\mathscr{G}}(E) \leq \operatorname{cat}_{\mathscr{G}}(\iota) \cdot \operatorname{cat}_{\mathrm{LS}}(p)$$

Recall that a basepoint x_0 of a space B is *non-degenerate* if the inclusion $\{x_0\} \to B$ is a cofibration.

Proof. Let $n := \operatorname{cat}_{\mathscr{G}}(\iota)$ and $b := \operatorname{cat}_{\mathrm{LS}}(p)$; without loss of generality, we may assume that they are both finite. Let $(V_i)_{i\in[n]}$ and $(W_j)_{j\in[b]}$ be corresponding open covers of F and E, respectively; here, for $k \in \mathbb{N}$, we abbreviate $[k] := \{1, \dots, k\}$. We construct an open $\mathscr{G}$-cover $(U_{ij})_{i\in[n],j\in[b]}$ of E as follows:

Let $j \in [b]$. As W_j is an LS-set for p (i.e., $p|_{W_j}\colon W_j \to B$ is null-homotopic) and as the basepoint $x_0 \in B$ is non-degenerate, there exists a homotopy $h_j\colon W_j \times [0,1] \to B$ with $h_j(\,\cdot\,,0) = p|_{W_j}$ and $h_j(\,\cdot\,,1) = \mathrm{const}_{x_0}$. By the homotopy lifting property, there exists a homotopy $\tilde{h}_j\colon W_j \times [0,1] \to E$ with $p \circ \tilde{h}_j = h_j$. In particular,

$$\tilde{h}_j(\,\cdot\,,1)(W_j) \subset p^{-1}(x_0) = F.$$

We write $g_j := \tilde{h}_j(\,\cdot\,,1)\colon W_j \to F$.

For all $i \in [n]$ and all $j \in [b]$, we set

$$U_{ij} := g_j^{-1}(V_i) \subset E.$$

By construction, U_{ij} is open in E and $\bigcup_{(i,j)\in[n]\times[b]} U_{ij} = E$.

It remains to show that each U_{ij} is a $\mathscr{G}$-set. Let $U \subset U_{ij}$ be a path-component of U_{ij}, let $i_U\colon U \hookrightarrow E$ be the inclusion and let $y_0 \in U$.

We consider the map

$$k := \tilde{h}_j|_{U\times[0,1]}\colon U \times [0,1] \to E.$$

Then $k(\,\cdot\,,0) = i_U$. Moreover, let $y_1 := k(y_0,1)$ and let $\alpha\colon [0,1] \to E,\ t \mapsto k(y_0,t)$. Then, we obtain the corresponding change of basepoints isomorphism

$$\begin{aligned} \alpha_*\colon \pi_1(E,y_0) &\to \pi_1(E,y_1) \\ [\gamma] &\mapsto [\alpha^{-1} * \gamma * \alpha]. \end{aligned}$$

By construction,

$$\alpha_* \circ \pi_1(i_U) = \pi_1\big(k(\,\cdot\,,1)\big)\colon \pi_1(U,y_0) \to \pi_1(U,y_1).$$

Because α_* is an isomorphism and $\mathscr{G}$ is closed under isomorphisms, it suffices to show that $\Lambda := \pi_1(k(\,\cdot\,,1))(\pi_1(U,y_0))$ lies in $\mathscr{G}$. The commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{k(\,\cdot\,,1)} & E \\ {\scriptstyle\mathrm{incl}}\downarrow & & \uparrow{\scriptstyle\iota} \\ U_{ij} & \xrightarrow[g_j|_{U_{ij}}]{} & F \end{array}$$

shows that Λ is a subgroup of $\pi_1(\iota)(\pi_1(V_i, g_j(y_0))$. The latter group is in $\mathscr{G}$ as V_i is a $\mathscr{G}$-set for ι. Because $\mathscr{G}$ is closed under subgroups, we obtain $\Lambda \in \mathscr{G}$.

Therefore, $(U_{ij})_{i\in[n],j\in[b]}$ is an open $\mathscr{G}$-cover of E and so $\operatorname{cat}_{\mathscr{G}} E \leq n \cdot b$. □

Remark 3.2. Let $p\colon E \to B$ be a fibration with fibre F over the basepoint $x_0 \in B$ and let $i\colon F \hookrightarrow E$ be the inclusion. Let $\mathscr{G}$ be a class of groups that is closed under isomorphisms, subgroups, quotients, and under extensions by Abelian kernels; e.g., $\mathscr{G} = \mathrm{Am}$. Then $\operatorname{cat}_{\mathscr{G}}(i) = \operatorname{cat}_{\mathscr{G}}(F)$: By Lemma 2.7, we already know that $\operatorname{cat}_{\mathscr{G}}(i) \leq \operatorname{cat}_{\mathscr{G}}(F)$. On the other hand, the long exact sequence for p and the closure properties of $\mathscr{G}$ show that $\operatorname{cat}_{\mathscr{G}}(F) \leq \operatorname{cat}_{\mathscr{G}}(i)$.

Corollary 3.3 ($\mathscr{G}$-category of mapping tori)**.** *Let $\mathscr{G}$ be a class of groups that is closed under isomorphisms and subgroups. Let M be a closed connected manifold that admits a fibre bundle $p\colon M \to S^1$ with manifold fibre $i\colon N \hookrightarrow M$. If $2 \cdot \operatorname{cat}_{\mathscr{G}}(i) \leq \dim N + 1$, then*

$$\operatorname{cat}_{\mathscr{G}}(M) \leq \dim M.$$

Proof. Because $\operatorname{cat}_{\mathrm{LS}}(S^1) = 2$, we have $\operatorname{cat}_{\mathrm{LS}}(p) \leq 2$. Applying Theorem 3.1 and the hypothesis on the $\mathscr{G}$-category of i, we obtain

$$\operatorname{cat}_{\mathscr{G}}(M) \leq \operatorname{cat}_{\mathrm{LS}}(p) \cdot \operatorname{cat}_{\mathscr{G}}(i) \leq 2 \cdot \operatorname{cat}_{\mathscr{G}}(i) \leq \dim(N) + 1 = \dim(M). \quad \square$$

4. Applications to simplicial volume

We recall the definition of simplicial volume and bounded cohomology (Section 4.1) and Gromov's vanishing theorem (Section 4.2). In Section 4.3, we derive the vanishing results for simplicial volume and fibrations.

4.1. Bounded cohomology and simplicial volume.

We briefly recall the definition of bounded cohomology for spaces and of simplicial volume. The systematic use of these invariants in geometry was initiated by Gromov [24]. Simplicial volume measures manifolds in terms of singular chains.

Given a topological space X, we denote the real singular chain complex by $(C_\bullet(X;\mathbb{R}), \partial_\bullet)$ and the real singular cochain complex by $(C^\bullet(X;\mathbb{R}), \delta^\bullet)$. These complexes are endowed with norms: For a singular n-chain $c = \sum_{i=1}^k \alpha_i \cdot \sigma_i \in C^n(X;\mathbb{R})$ in reduced form, we define the *ℓ^1-norm* by:

$$|c|_1 := \sum_{i=1}^k |\alpha_i|.$$

Similarly, we endow $C^\bullet(X;\mathbb{R})$ with the *ℓ^∞-norm* given by

$$|\varphi|_\infty := \sup\bigl\{|\varphi(\sigma)| \bigm| \sigma \text{ is a singular } n\text{-simplex in } X\bigr\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

for all $\varphi \in C^n(X;\mathbb{R})$. As the coboundary operator $\delta^\bullet$ maps cochains of finite norm to cochains of finite norm, we obtain the subcomplex $(C_b^\bullet(X;\mathbb{R}), \delta^\bullet)$ of singular cochains φ with $|\varphi|_\infty < \infty$.

Definition 4.1 (simplicial volume)**.** Let M be an oriented closed connected n-dimensional manifold. We define the *simplicial volume* of M to be

$$\|M\| := \inf\bigl\{|c|_1 \bigm| c \in C_n(M;\mathbb{R}) \text{ is a cycle representing } [M]\bigr\},$$

where $[M] \in H_n(M;\mathbb{R})$ denotes the fundamental class of M.

Remark 4.2. More generally, the ℓ^1-norm on the singular chain complex induces a semi-norm $\|\cdot\|_1$ on the whole singular homology with $\mathbb{R}$-coefficients.

A useful tool for detecting positivity/vanishing of simplicial volume is bounded cohomology:

Definition 4.3 (bounded cohomology)**.** Bounded cohomology of spaces is the functor $H_b^\bullet(\,\cdot\,;\mathbb{R}) := H^\bullet(C_b^\bullet(\,\cdot\,;\mathbb{R}))$.

The connection between bounded cohomology and simplicial volume is expressed in terms of the *comparison map* $\mathrm{comp}^\bullet\colon H_b^\bullet(\,\cdot\,;\mathbb{R}) \to H^\bullet(\,\cdot\,;\mathbb{R})$, induced by the inclusion $C_b^\bullet(\,\cdot\,;\mathbb{R}) \hookrightarrow C^\bullet(\,\cdot\,;\mathbb{R})$.

Proposition 4.4 (duality principle)**.** *Let M be an oriented closed connected n-manifold. Then:*

$$\|M\| > 0 \iff \mathrm{comp}^n_M \text{ is surjective}$$

Remark 4.5. By now, many examples of manifolds with non-zero simplicial volume are known. We list some of them:

- oriented closed connected hyperbolic manifolds [42, 24]; in particular, surfaces of genus ≥ 2;
- more generally: oriented closed connected, rationally essential manifolds of dimension ≥ 2 with non-elementary hyperbolic fundamental group [24, mapping theorem] [37];
- oriented closed connected locally symmetric spaces of non-compact type [11, 32];
- manifolds with sufficiently negative curvature [26, 13];
- The class of manifolds with positive simplicial volume is closed with respect to connected sums and products [24].

4.2. **Gromov's vanishing theorem.** A classical application of the duality principle (Proposition 4.4) is to show that the simplicial volume of all oriented closed connected manifolds with amenable fundamental group (and non-zero dimension) is zero. This is a consequence of the vanishing of the bounded cohomology for spaces with amenable fundamental group [24, 28, 21]. More generally, one has [24, 28, 21, 33, 27]:

Theorem 4.6 (Gromov's vanishing theorem)**.** *Let X be a topological space. Then:*

(1) the map $\mathrm{comp}^s_X\colon H_b^s(X) \to H^s(X)$ is zero for all $s \geq \mathrm{cat}_{\mathrm{Am}}(X)$;
(2) we have $\|\alpha\|_1 = 0$ for all $\alpha \in H_s(X;\mathbb{R})$ with $s \geq \mathrm{cat}_{\mathrm{Am}}(X)$.

Remark 4.7. Usually, the vanishing theorem is stated in terms of multiplicity of the cover instead of cardinality. For CW-complexes, the two formulations are indeed equivalent [12, Remark 3.13].

4.3. **Vanishing results for fibrations.** We apply Theorem 1.1 and Theorem 3.2 in the case of the class Am of amenable groups to obtain vanishing results for the comparison map and simplicial volume. The class Am is closed under subgroups, isomorphisms, quotients, and extension by Abelian groups.

Corollary 4.8 (vanishing result for fibrations)**.** *Let $p\colon E \to B$ be a fibration with a path-connected base space. Let $x_0 \in B$ be a non-degenerate basepoint of B and let $F := p^{-1}(x_0)$. Moreover, let $s \geq \mathrm{cat}_{\mathrm{Am}}(F) \cdot \mathrm{cat}_{\mathrm{LS}}(p)$. Then:*

(1) the comparison map $\mathrm{comp}^s_E\colon H^s_b(E) \to H^s(E)$ is zero;
(2) all classes $\alpha \in H_s(E;\mathbb{R})$ have vanishing ℓ^1-seminorm $\|\alpha\|_1 = 0$.

Proof. From Theorem 3.1 and Remark 3.2 we obtain $\mathrm{cat}_{\mathrm{Am}}(E) \leq \mathrm{cat}_{\mathrm{Am}}(F) \cdot \mathrm{cat}_{\mathrm{LS}}(p)$. Thus, the claim follows by applying Gromov's vanishing theorem 4.6. □

Corollary 4.9 (simplicial volume and fibre bundles)**.** *Let M be an oriented closed connected manifold that is the total space of a fibre bundle $p\colon M \to B$ with oriented closed connected fibre N and base B. If*

$$\mathrm{cat}_{\mathrm{Am}}(N) \leq \frac{\dim(M)}{\dim(B)+1},$$

then $\|M\| = 0$.

Proof. In view of Gromov's vanishing theorem (Theorem 4.6), it suffices to show that $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$. Using Theorem 1.1, the fact that $\mathrm{cat}_{\mathrm{LS}}(B) \leq \dim(B)+1$, and the hypothesis on N, we indeed obtain

$$\begin{aligned}\mathrm{cat}_{\mathrm{Am}}(M) &\leq \mathrm{cat}_{\mathrm{Am}}(N) \cdot \big(\dim(B)+1\big)\\ &\leq \dim(M),\end{aligned}$$

as desired. □

Corollary 4.10 (simplicial volume of mapping tori)**.** *Let M be an oriented closed connected manifold that is a mapping torus of a self-homeomorphism of an oriented closed connected manifold N with*

$$2 \cdot \mathrm{cat}_{\mathrm{Am}}(N) \leq \dim(N) + 1\ .$$

Then, we have $\|M\| = 0$.

Proof. This is a special case of Corollary 4.9. □

Example 4.11. A classical question in hyperbolic geometry is to understand when hyperbolic manifolds fiber over the circle [1, 2]. Since the Euler characteristic in even dimension is proportional to the volume of hyperbolic manifolds, it is immediate to see that there are no even dimensional hyperbolic manifolds that fiber over the circle. On the other hand, the question is still open in odd dimension greater than 3.

As hyperbolic manifolds have non-zero simplicial volume (Remark 4.5), Corollary 4.10 shows at least that odd-dimensional hyperbolic manifolds that fiber over the circle cannot have fiber with *small* amenable category.

Remark 4.12. Corollary 4.10 shows that mapping tori over connected sums M of amenable manifolds (of dimension at least 3) have zero simplicial volume (because $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$ [22, Lemma 1][12, Proposition 6.7]). This result may be interpreted as an extension of the classical result about the vanishing of the simplicial volume of manifold fibre bundles with amenable fiber [35, Exercise 14.15].

Bucher and Neofytidis established vanishing results for simplicial volume of certain mapping tori over connected sums M of manifolds with zero simplicial volume [10, Theorem 1.7]. Their approach uses refined information on the structure of the self-glueing homeomorphism $M \to M$. Not all of these vanishing results can be recovered from Corollary 4.10 (which is ignorant of the glueing map) and vice versa.

Remark 4.13. In the situation of mapping tori, the open amenable covers obtained via the proof of Corollary 4.10 coincide with the obvious one obtained by *doubling* an optimal open amenable cover $(U_i)_{i\in I}$ of the fiber: We can "split" the mapping torus of $f\colon N \to N$ into two open overlapping cylinders $N \times J_1$ and $N \times J_2$ that are glued appropriately. Then the mapping torus bundle is trivial over J_1 and J_2 and $(U_i \times J_1)_{i\in I} \cup (U_i \times J_2)_{i\in I}$ gives an amenable open cover of the mapping torus of f consisting of $2 \cdot |I|$ elements.

Remark 4.14. It is tempting to prove Corollary 4.8, Corollary 4.9, and Corollary 4.10 via the Hochschild-Serre spectral sequence for (bounded) cohomology. However, as there is no "five lemma for zero maps", there does not seem to be a direct way to do this.

Example 4.15. The following example shows that in Theorem 1.1, we cannot replace $\mathrm{cat}_{\mathrm{LS}}(B)$ by $\mathrm{cat}_{\mathscr{G}}(B)$: There exist mapping tori M of oriented closed connected hyperbolic surfaces N that are oriented closed connected hyperbolic 3-manifolds. Because $\mathrm{cat}_{\mathrm{Am}}(S^1) = 1$, we then have (Remark 4.5, Theorem 4.6)

- $\mathrm{cat}_{\mathrm{Am}}(N) \cdot \mathrm{cat}_{\mathrm{Am}}(S^1) = 3 \cdot 1$, but
- $\mathrm{cat}_{\mathrm{Am}}(M) = 4$.

5. The fibre (non-)collapsing assumption

In the context of minimal volume entropy, growth conditions on groups naturally occur. We will explain how the fibre collapsing and non-collapsing conditions by Babenko and Sabourau are related to categorical invariants for classes of groups with controlled growth. In Section 6, we will apply our estimates for fibrations to this setting.

5.1. Groups with controlled growth.

To state the fibre (non-)collapsing conditions, we introduce the corresponding classes of groups with controlled growth. Because categorical invariants work better with classes of groups that are closed under subgroups, we consider the following construction:

Remark 5.1. Let $\mathscr{G}$ be a class of groups. We set

$$\overline{\mathscr{G}} := \big\{\Gamma \in \mathrm{Ob}(\mathsf{Group}) \;\big|\; \forall_{\Lambda\leq\Gamma} \quad (\Lambda \text{ finitely generated} \Rightarrow \Lambda \in \mathscr{G})\big\}.$$

Then we have:

(1) The class $\overline{\mathscr{G}}$ is closed under taking subgroups.
(2) If $\mathscr{G}$ is closed under isomorphisms, then $\overline{\mathscr{G}}$ is closed under isomorphisms.
(3) If $\mathscr{G}$ is closed under quotients, then $\overline{\mathscr{G}}$ is closed under quotients.
(4) If $\mathscr{G}$ is closed under taking finitely generated subgroups, then the finitely generated groups in $\overline{\mathscr{G}}$ coincide with the finitely generated groups in $\mathscr{G}$.

Example 5.2 (classes of groups of controlled growth)**.** Let $\delta \in \mathbb{R}_{>0}$. The standard inheritance properties of growth conditions in finitely generated groups show that the following classes of groups are closed with respect to isomorphisms, finitely generated subgroups, and quotients:

- The class $\mathrm{Poly}^{\mathrm{fg}}$ of finitely generated groups of polynomial growth. By the polynomial growth theorem [23], this class coincides with the class of all finitely generated virtually nilpotent groups.
- The class $\mathrm{Subexp}^{\mathrm{fg}}$ of finitely generated groups of subexponential growth.
- The class $\mathrm{Subexp}^{\mathrm{fg}}_{<\delta}$ of finitely generated groups of subexponential growth with subexponential growth rate $< \delta$.

By Remark 5.1, the associated classes

$$\mathrm{Poly} := \overline{\mathrm{Poly}^{\mathrm{fg}}}, \quad \mathrm{Subexp} := \overline{\mathrm{Subexp}^{\mathrm{fg}}}, \quad \mathrm{Subexp}_{<\delta} := \overline{\mathrm{Subexp}^{\mathrm{fg}}_{<\delta}}$$

are closed under isomorphisms, subgroups, and quotients. Moreover, the finitely generated groups in these classes are exactly the groups in $\mathrm{Poly}^{\mathrm{fg}}$, $\mathrm{Subexp}^{\mathrm{fg}}$, $\mathrm{Subexp}^{\mathrm{fg}}_{<\delta}$, respectively.

Let $\mathrm{Exp}^{\mathrm{fg}}_{<\delta}$ be the class of finitely generated groups that admit a finite generating set whose growth rate is at most exponential of exponential growth rate $< \delta$. In other words, a finitely generated group Γ does *not* lie in $\mathrm{Exp}^{\mathrm{fg}}_{<\delta}$ if and only if its uniform exponential growth rate $\mathrm{uexp}(\Gamma)$ is at least δ. It should be noted that $\mathrm{Exp}^{\mathrm{fg}}_{<\delta}$ is *not* closed under taking finitely generated subgroups.

5.2. **The fibre (non-)collapsing assumption.** We recall the fibre (non-) collapsing assumptions by Babenko and Sabourau [4]. For convenience, we formulate the collapsing condition for classes of groups; geometrically relevant choices are the classes Poly, Subexp, and $\mathrm{Subexp}_{<\delta}$.

Definition 5.3 (fibre collapsing assumption; FCA)**.** Let $\mathscr{G}$ be a class of groups.

- Let $k \in \mathbb{N}$. A finite simplicial complex X satisfies the *fibre collapsing assumption with respect to $\mathscr{G}$ in dimension k* if there exists a simplicial map $f\colon X \to P$ to a finite simplicial complex P with $\dim P \leq k$ and such that for all points $p \in P$ (not necessarily vertices), the fibre $f^{-1}(p)$ is a $\mathscr{G}$-subset of X.
- A finite simplicial complex X satisfies the *fibre collapsing assumption with respect to $\mathscr{G}$* if X satisfies the fibre collapsing assumption with respect to $\mathscr{G}$ in dimension $\dim X - 1$.

Remark 5.4 (fundamental groups of fibres are finitely generated)**.** Let X be a finite simplicial complex, let P be a simplicial complex, and let $f\colon X \to P$ be a simplicial map. Then, for every point $p \in P$, the fibre $f^{-1}(p)$ is a finite simplicial complex. In particular, the fundamental groups of all components of $f^{-1}(p)$ are finitely generated.

Remark 5.5. Let $\mathscr{G}$ be a class of groups that is closed under finitely generated subgroups, let X be a finite simplicial complex, and let $k \in \mathbb{N}$. Then X satisfies the fibre collapsing assumption with respect to $\mathscr{G}$ in dimension k if and only if it does so for $\overline{\mathscr{G}}$:

As $\mathscr{G}$ is closed under finitely generated subgroups, we have $\mathscr{G} \subset \overline{\mathscr{G}}$ (Remark 5.1). In particular, the FCA for $\mathscr{G}$ implies the FCA for $\overline{\mathscr{G}}$. For the other implication, we argue as follows: Let P be a simplicial complex and let $f\colon X \to P$ be a simplicial map. Then, for every point $p \in P$, the fundamental groups of all components of $f^{-1}(p)$ are finitely generated (Remark 5.4). As every finitely generated subgroup in $\overline{\mathscr{G}}$ also lies in $\mathscr{G}$, the complex X satisfies the fibre collapsing assumption with respect to $\mathscr{G}$ in dimension k if it does so for $\overline{\mathscr{G}}$.

Definition 5.6 (fibre non-collapsing assumption; FNCA)**.** A finite simplicial complex X satisfies the *fibre non-collapsing assumption* if there exists a $\delta \in \mathbb{R}_{>0}$ with the following property: For each finite simplicial complex P with $\dim P < \dim X$ and for each simplicial map $f\colon X \to P$, there exists a point $p \in P$ and an $x \in f^{-1}(p)$ such that $\pi_1(i)(\pi_1(f^{-1}(p), x))$ has uniformly exponential growth with exponential growth rate at least δ, where $i\colon f^{-1}(p) \hookrightarrow X$ denotes the inclusion map (in other words: $f^{-1}(p)$ is *not* an $\mathrm{Exp}^{\mathrm{fg}}_{<\delta}$-subset of X).

Proposition 5.7. *Let $\pi\colon \overline{X} \to X$ be a simplicial finite-sheeted covering of finite simplicial complexes. If $\overline{X}$ satisfies the fibre non-collapsing assumption, then so does X.*

Proof. Let $\delta \in \mathbb{R}_{>0}$ be such that $\overline{X}$ satisfies the FNCA with uniformly exponential growth rate $\geq \delta$. We write $d \in \mathbb{N}$ for the number of sheets of π and show that X satisfies the FNCA with uniformly exponential growth rate $\geq \sqrt[2d-1]{\delta}$. The basic reason is that uniformly exponential growth is inherited by finite index supergroups.

Let P be a finite simplicial complex with $\dim P < \dim X = \dim \overline{X}$ and let $f\colon X \to P$ be a simplicial map. Then, $\overline{f} := f \circ \pi\colon \overline{X} \to P$ is a simplicial map. As $\overline{X}$ satisfies FNCA, there is a point $p \in P$ and an $\overline{x} \in \overline{f}^{-1}(p) \subset \overline{X}$ such that the image

$$\overline{\Lambda} := \pi_1(\overline{i})\big(\pi_1(\overline{f}^{-1}(p), \overline{x})\big) \subset \pi_1(\overline{X}, \overline{x})$$

has uniform exponential growth rate $\geq \delta$, where $\overline{i}$ denotes the inclusion into $\overline{X}$. As $\pi_1(\pi)$ is injective, the group (where i is the inclusion into X)

$$\Gamma := \pi_1(i)\big(\pi_1(f^{-1}(p), x)\big) \subset \pi_1(X, x)$$

contains a finite index subgroup Λ that is isomorphic to $\overline{\Lambda}$ and has index at most $[\pi_1(\overline{X}, \overline{x}) : \overline{\Lambda}] = d$. Therefore, Γ has uniformly exponential growth with uniformly exponential growth rate at least [39, Proposition 3.3][16, Proposition 2.4]

$$\sqrt[2\cdot[\Gamma:\Lambda]-1]{\mathrm{uexp}(\Lambda)} \geq \sqrt[2\cdot d-1]{\mathrm{uexp}(\overline{\Lambda})} \geq \sqrt[2\cdot d-1]{\delta}. \qquad \square$$

5.3. **F(N)CA via category invariants.** As observed by Babenko and Sabourau [4, Proposition 2.13, Proposition 3.10], the fibre collapsing and non-collapsing assumptions are connected to multiplicity conditions on open covers. We will recast this result in terms of categorical invariants.

Proposition 5.8 (fibre conditions and categorical invariants)**.** *Let $\mathscr{G}$ be a class of groups that is closed under isomorphisms, let X be a finite simplicial complex, and let $k \in \mathbb{N}$. Then:*

(1) If X satisfies the fibre collapsing assumption with respect to $\mathscr{G}$ in dimension k, then

$$\operatorname{cat}_{\mathscr{G}}(X) \leq k+1.$$

(2) If in addition $\mathscr{G}$ is closed under finitely generated subgroups and if $\operatorname{cat}_{\mathscr{G}}(X) \leq k+1$, then there exists an iterated barycentric subdivision of X that satisfies the fibre collapsing assumption with respect to $\mathscr{G}$ in dimension k.

Proof. The proof of Babenko and Sabourau [4, Proposition 2.13] for the connection between the fibre collapsing assumption and multiplicities of open covers with π_1-restrictions also works in the full generality of classes of groups. The condition on the multiplicity of the open covers can be adapted into a condition on $\operatorname{cat}_{\mathscr{G}}$. For the sake of completeness, we recall the arguments:

Ad 1. Let $f\colon X \to P$ be a simplicial map witnessing that X satisfies the fibre collapsing assumption with respect to $\mathscr{G}$ in dimension k. Taking the barycentric subdivision, yields a simplicial map $f'\colon X' \to P'$ between the barycentric subdivisions that witnesses that X' satisfies the fibre collapsing assumption with respect to $\mathscr{G}$ in dimension k (as subsets of the geometric realisations, the fibres of f and f' agree). Because X' is homeomorphic to X, it suffices to show that $\operatorname{cat}_{\mathscr{G}}(X') \leq k+1$.

Let $\mathcal{U} = (U_i)_{i\in I}$ be the open stars cover of P', regrouped and indexed by the dimensions of the underlying simplices in P. Then $|\mathcal{U}| \leq \dim P + 1 = k+1$. We now consider the pull-back cover $\mathcal{V} := (V_i)_{i\in I}$ of X', where $V_i := f'^{-1}(U_i)$ for all $i \in I$. Then $\mathcal{V}$ is an open cover of X' with $|\mathcal{V}| \leq |U| \leq k+1$. It thus suffices to show that each V_i is a $\mathscr{G}$-subset of X'.

Let $i \in I$. Because f' is a simplicial map, there exists a vertex $p_i \in P'$ such that $V_i = f'^{-1}(U_i)$ deformation retracts onto the fibre $f'^{-1}(p_i)$. Let $j_i\colon f'^{-1}(p_i) \hookrightarrow V_i$ and $k_i\colon V_i \hookrightarrow X'$ denote the inclusions. Then j_i is a homotopy equivalence and so

$$\pi_1(k_i)\big(\pi_1(V_i, x)\big) \cong \pi_1(k_i \circ j_i)\big(\pi_1(f'^{-1}(p_i), x)\big)$$

for all $x \in f'^{-1}(p_i)$. Therefore, V_i is a $\mathscr{G}$-subset of X' and we conclude that $\operatorname{cat}_{\mathscr{G}}(X) = \operatorname{cat}_{\mathscr{G}}(X') \leq k+1$.

Ad 2. For the converse implication, we use the nerve construction. Let $\mathcal{U}$ be an open $\mathscr{G}$-cover of X with $|\mathcal{U}| \leq k+1$. Then, the nerve P of $\mathcal{U}$ is a finite simplicial complex with $\dim P = \operatorname{mult}\mathcal{U} - 1 \leq k$. Let Φ be a partition of unity subordinate to $\mathcal{U}$ and let $f\colon X \to P$ be the nerve map associated with Φ. In general, f is not simplicial; this can be handled as follows: By the Lebesgue lemma, there is an iterated barycentric subdivision X' of X such that each simplex of X' is contained in one of the sets in $\mathcal{U}$ and such that f admits a simplicial approximation $f'\colon X' \to P$. If $p \in P$, then $f'^{-1}(p)$ is contained in one of the elements U_i of $\mathcal{U}$. In particular: If $j_i\colon f'^{-1}(p) \hookrightarrow U_i$ and $k_i\colon U_i \to X'$ denote the inclusions, then

$$\pi_1(k_i \circ j_i)\big(\pi_1(f'^{-1}(p), x)\big) \subset \pi_1(k_i)\big(\pi_1(U_i, x)\big)$$

holds for all $x \in f'^{-1}(p)$. Because U_i is a $\mathscr{G}$-subset of X (whence X'), because $\mathscr{G}$ is closed under finitely generated subgroups, and because f' is a simplicial map, also $f'^{-1}(p)$ is a $\mathscr{G}$-subset of X' (Remark 5.4). □

Corollary 5.9 (FCA via cat)**.** *Let X be a finite simplicial complex, let $k \in \mathbb{N}$, and let $\mathscr{G}$ be a class of groups that is closed under isomorphisms and finitely generated subgroups. Then the following are equivalent:*

(1) There exists an iterated barycentric subdivision X' of X that satisfies the fibre collapsing assumption with respect to $\mathscr{G}$ in dimension k.
(2) We have $\operatorname{cat}_{\mathscr{G}}(X) \leq k+1$.
(3) We have $\operatorname{cat}_{\overline{\mathscr{G}}}(X) \leq k+1$.

Proof. Ad 1. $\Longrightarrow$ 2. Let us suppose that there exists an iterated barycentric subdivision X' of X that satisfies the fibre collapsing assumption with respect to $\mathscr{G}$ in dimension k. Because $\mathscr{G}$ is closed under finitely generated subgroups, Proposition 5.8 shows that

$$\operatorname{cat}_{\mathscr{G}}(X') \leq k+1.$$

As X' and X are homeomorphic, we obtain $\operatorname{cat}_{\mathscr{G}}(X) = \operatorname{cat}_{\mathscr{G}}(X') \leq k+1$.

Ad 2. $\Longrightarrow$ 3. This is a direct consequence of the fact that $\mathscr{G} \subset \overline{\mathscr{G}}$.

Ad 3. $\Longrightarrow$ 1. Let $\operatorname{cat}_{\overline{\mathscr{G}}}(X) \leq k+1$. As $\overline{\mathscr{G}}$ is closed under subgroups, by Proposition 5.8, there exists an iterated barycentric subdivision X' of X that satisfies the fibre collapsing assumption with respect to $\overline{\mathscr{G}}$ in dimension k. We can now apply Remark 5.5 to pass to $\mathscr{G}$. □

Example 5.10. Let X be a finite simplicial complex. Then, by Corollary 5.9, $\operatorname{cat}_{\mathrm{Poly}}(X) \leq \dim X$ is equivalent to the existence of an iterated barycentric subdivision of X that satisfies the fibre collapsing condition with polynomial growth.

Remark 5.11 (dimension 2)**.** The following generalisation of a result of Bregman and Clay [8, Proposition 4.1] is an instance of general considerations on categorical invariants: Let $\mathscr{G}$ be a class of groups that is closed under isomorphisms, finitely generated subgroups, and quotients. Let Γ be a group that does *not* lie in $\mathscr{G}$ and let X be a finite simplicial complex with $\pi_1(X) \cong \Gamma$. Then the following are equivalent [12, Corollary 5.4 and the subsequent remark]:

(1) The group Γ is the fundamental group of a graph of groups whose vertex and edge groups all lie in $\overline{\mathscr{G}}$.
(2) We have $\operatorname{cat}_{\overline{\mathscr{G}}}(X) = 2$.

If X is of dimension 2, by Corollary 5.9, these conditions are equivalent to:

(3) There exists an iterated barycentric subdivision of X that satisfies the fibre collapsing assumption with respect to $\mathscr{G}$.
(4) We have $\operatorname{cat}_{\mathscr{G}}(X) = 2$.

For example, this applies to the classes Poly, Subexp, and $\mathrm{Subexp}_{<\delta}$.

Corollary 5.12 (FNCA via cat)**.** *Let X be a finite simplicial complex and let $\delta \in \mathbb{R}_{>0}$. If $\operatorname{cat}_{\mathrm{Exp}^{\mathrm{fg}}_{<\delta}}(X) > \dim X$, then X satisfies the fibre non-collapsing condition (with uniform exponential growth rate δ).*

Proof. This follows from the definition of the fibre non-collapsing condition and the contraposition of the first part of Proposition 5.8. □

It is not clear to us that the converse of Corollary 5.12 also holds (up to subdivision) because $\mathrm{Exp}^{\mathrm{fg}}_{<\delta}$ is not closed under finitely generated subgroups.

For later use, we give an example that slightly generalises an example by Babenko and Sabourau [4, Proposition 3.7]:

Example 5.13. Let $N_1, \dots, N_r$ be oriented closed connected rationally essential smooth manifolds of dimension ≥ 2 with non-elementary hyperbolic fundamental group. Then the product $M := N_1 \times \dots \times N_r$ satisfies the FNCA with respect to every triangulation: We proceed in the following steps:

(0) If Γ is a finitely generated hyperbolic group, then there exists a $\delta_\Gamma \in \mathbb{R}_{>0}$ such that: Every finitely generated subgroup Λ of Γ is virtually cyclic or satisfies $\mathrm{uexp}(\Lambda) \geq \delta_\Gamma$.
(1) There exists a $\delta \in \mathbb{R}_{>0}$ such that: Every finitely generated subgroup Λ of $\pi_1(M)$ is amenable or satisfies $\mathrm{uexp}(\Lambda) \geq \delta$.
(2) $\mathrm{cat}_{\mathrm{Am}}(M) > \dim(M)$.
(3) $\mathrm{cat}_{\mathrm{Exp}_{<\delta}}(M) > \dim(M)$.

The last property implies that M satisfies the FNCA (Corollary 5.12).

Ad 0. This is a result of Delzant an Steenbock [17, Theorem 1.1].

Ad 1. We apply part 0 to the $\pi_1(N_j)$ and set $\delta := \min(\delta_{\pi_1(N_1)}, \dots, \delta_{\pi_1(N_r)})$. Let $\Lambda \subset \pi_1(M)$ be a finitely generated subgroup. We distinguish two cases:

- For all $j \in \{1, \dots, r\}$, the projection $p_j(\Lambda) \subset \pi_1(N_j)$ is virtually cyclic.
- There exists a $j \in \{1, \dots, r\}$ such that $p_j(\Lambda) \subset \pi_1(N_j)$ is not virtually cyclic.

In the first case, Λ is isomorphic to a subgroup of a product of r virtually cyclic groups and thus amenable. In the second case, Λ projects onto a subgroup Λ_j of the non-elementary hyperbolic group $\pi_1(N_j)$ that is not virtually cyclic; thus,

$$\mathrm{uexp}(\Lambda) \geq \mathrm{uexp}(\Lambda_j) \geq \delta_{\pi_1(N_j)} \geq \delta.$$

Ad 2. Let $j \in \{1, \dots, r\}$. Then N_j has non-zero simplicial volume (Remark 4.5). Therefore, also $M = N_1 \times \dots \times N_r$ has non-zero simplicial volume [24]. In particular, $\mathrm{cat}_{\mathrm{Am}}(M) > \dim(M)$ (Theorem 4.6).

Ad 3. By the first part, all finitely generated subgroups of $\pi_1(M)$ that lie in $\mathrm{Exp}_{<\delta}$ are amenable. In combination with the second part, we obtain $\mathrm{cat}_{\mathrm{Exp}_{<\delta}}(M) \geq \mathrm{cat}_{\mathrm{Am}}(M) > \dim(M)$.

6. Applications to minimal volume entropy

We recall the definition of minimal volume entropy (Section 6.1) and the (non-)vanishing results of Babenko and Sabourau (Section 6.2). In Section 6.3, we derive the vanishing results for minimal volume entropy and fibrations. In Section 6.4, we extend the FNCA examples of Babenko and Sabourau.

6.1. **Minimal volume entropy.** The minimal volume entropy measures the minimal possible growth rate of balls.

Definition 6.1 (minimal volume entropy)**.** Let X be a finite connected simplicial complex.

- A *piecewise Riemannian metric* on X is a family of Riemannian metrics on all simplices of X that is compatible along common subsimplices. Let $\mathrm{Riem}(X)$ be the set of all piecewise Riemannian metrics on X.
- Let g be a piecewise Riemannian metric on X. Then the *volume entropy of* (X, g) is defined as
$$\mathrm{ent}(X, g) := \lim_{R\to\infty} \frac{1}{R} \cdot \log \mathrm{vol}\big(B(R, \tilde{x}), \tilde{g}\big),$$
where $\tilde{x}$ is a vertex of the universal covering $\tilde{X}$ of X, where $\tilde{g}$ is the pull-back of g to $\tilde{X}$, and where $B(R, \tilde{x})$ is the ball of radius R around $\tilde{x}$ in $\tilde{X}$. (The choice of $\tilde{x}$ does not matter).
- The *minimal volume entropy of* X is defined by
$$\mathrm{minent}(X) := \inf_{g\in\mathrm{Riem}(X)} \mathrm{ent}(X, g) \cdot \mathrm{vol}(X, g)^{1/\dim(X)}.$$

Remark 6.2 (minimal volume entropy and barycentric subdivisions)**.** Let X be a finite connected simplicial complex.

If X' is the barycentric subdivision of X, then $\mathrm{minent}(X') = \mathrm{minent}(X)$, as can be seen by smooth approximation of piecewise Riemannian metrics on X' by piecewise Riemannian metrics on X.

Inductively, we obtain that if X' is an iterated barycentric subdivision of X, then $\mathrm{minent}(X') = \mathrm{minent}(X)$.

Remark 6.3 (minimal volume entropy of smooth manifolds)**.** Let M be a closed connected smooth manifold. Then the *minimal volume entropy of* M is defined as
$$\mathrm{minent}(M) := \inf_{g\in\mathrm{Riem}(M)} \mathrm{ent}(M, g) \cdot \mathrm{vol}(M, g)^{1/\dim(M)},$$
where $\mathrm{Riem}(M)$ is the set of all actual Riemannian metrics on M. If X is a finite simplicial complex that triangulates M, then $\mathrm{minent}(M) = \mathrm{minent}(X)$ [5, Lemma 2.3]. In fact, minimal volume entropy is a topological invariant of smooth manifolds in a very strong sense [6, 9].

6.2. **The (non-)vanishing theorems by Babenko and Sabourau.**

Theorem 6.4 (FCA and vanishing; [4, Theorem 1.3])**.** *Let X be a finite connected simplicial complex of dimension n. If there is a $k \in \{0, \dots, n-1\}$ such that X satisfies the fibre collapsing assumption for* $\mathrm{Subexp}_{<(n-k)/n}$ *in dimension k, then*
$$\mathrm{minent}(X) = 0.$$

Theorem 6.5 (FNCA and non-vanishing; [4, Theorem 1.5])**.** *Let X be a finite connected simplicial complex that satisfies the fibre non-collapsing assumption. Then*
$$\mathrm{minent}(X) > 0.$$

We can reformulate these (non-)vanishing results in terms of generalised categorical invariants:

Corollary 6.6. *Let X be a finite connected simplicial complex.*

(1) If there is a $k \in \{0, \dots, \dim X - 1\}$ with $\mathrm{cat}_{\mathrm{Subexp}_{<(n-k)/n}}(X) \leq k+1$, then $\mathrm{minent}(X) = 0$.

(2) If there is a $\delta \in \mathbb{R}_{>0}$ with $\mathrm{cat}_{\mathrm{Exp}^{\mathrm{fg}}_{<\delta}}(X) > \dim X$, then $\mathrm{minent}(X) > 0$.

Proof. Ad 1. If $\mathrm{cat}_{\mathrm{Subexp}_{<(n-k)/n}}(X) \leq k+1$, then an interated subdivision X' of X satisfies the fibre collapsing assumption for $\mathrm{Subexp}_{<(n-k)/n}$ in dimension k (Corollary 5.9). Thus, $\mathrm{minent}(X') = 0$, by the vanishing theorem (Theorem 6.4). We then use that $\mathrm{minent}(X) = \mathrm{minent}(X')$ (Remark 6.2).

Ad 2. This is an immediate consequence of Corollary 5.12 and the non-vanishing theorem (Theorem 6.5). □

6.3. **Vanishing results for fibrations.** We now apply Theorem 1.1 in the case of growth classes of groups to obtain vanishing results for minimal volume entropy.

Corollary 6.7 (minimal volume entropy and fibrations)**.** *Let $p\colon E \to B$ be a simplicial fibration of finite connected simplicial complexes. Let $x_0 \in B$ be a vertex and let $F := p^{-1}(x_0)$. If*

$$\mathrm{cat}_{\mathrm{Subexp}_{<1/\dim(X)}}(F) \cdot \mathrm{cat}_{\mathrm{LS}}(B) \leq \dim(X),$$

then $\mathrm{minent}(X) = 0$.

Proof. Under the given hypotheses, Theorem 1.1 shows that

$$\mathrm{cat}_{\mathrm{Subexp}_{<1/\dim(X)}}(X) \leq \mathrm{cat}_{\mathrm{Subexp}_{<1/\dim(X)}}(F) \cdot \mathrm{cat}_{\mathrm{LS}}(B) \leq \dim(X).$$

Therefore, Corollary 6.6 implies $\mathrm{minent}(X) = 0$. □

Corollary 6.8 (minimal volume entropy and fibre bundles)**.** *Let M be an oriented closed connected smooth manifold that is the total space of a fibre bundle $M \to B$ with oriented closed connected smooth fibre N and base B. If*

$$\mathrm{cat}_{\mathrm{Subexp}_{<1/\dim(M)}}(N) \leq \frac{\dim(M)}{\dim(B)+1},$$

then $\mathrm{minent}(M) = 0$.

Proof. Triangulating M, B, and N and subdividing often enough, we may assume that we have a simplicial fibration between finite simplicial complexes of the corresponding dimensions. Moreover, the notions of minimal volume entropy for smooth manifolds and their triangulations coincide (Remark 6.3). Using the estimate $\mathrm{cat}_{\mathrm{LS}}(B) \leq \dim(B) + 1$, the result follows from Corollary 6.7. □

Corollary 6.9 (minimal volume entropy of mapping tori)**.** *Let X be a finite connected simplicial complex of dimension n that fibres as a fibre bundle over the circle, with (simplicial) fibre F. If*

$$2 \cdot \mathrm{cat}_{\mathrm{Subexp}_{<1/n}}(F) \leq \dim F + 1,$$

then $\mathrm{minent}\, X = 0$.

Proof. This is a special case of Corollary 6.7. □

6.4. **FNCA, minimal volume entropy, and simplicial volume.** For manifolds, minimal volume entropy is an upper bound for simplicial volume (up to a dimension constant) [24] [7, Théorème D]; in particular, Remark 4.5 thus leads to examples of positive minimal volume entropy. Conversely, the following is an open problem [4]:

Question 6.10. Let M be an oriented closed connected smooth manifold with $\|M\| = 0$. Does this imply that $\mathrm{minent}(M) = 0$?

By now, we know that the previous question admits a positive answer in dimension 2 [30] and for all oriented closed connected geometric manifolds in dimension 3 [38] (whence all by Perelemann's proof of Thurston's geometrisation conjecture) and 4 [41]. Even though we do not expect this question to have a positive answer in full generality; a particularly interesting special case to study would be aspherical oriented closed connected manifolds.

As shown by Babenko and Sabourau, for finite simplicial complexes, FNCA (and thus positive minimal volume entropy) does not necessarily imply the non-vanishing of "simplicial volume" (interpreted appropriately) [4, Theorem 1.6]. Using a variation of their construction, we obtain aspherical examples of this type:

Proposition 6.11. *Let $n \in \mathbb{N}_{\geq 2}$. Then, there exists a finite connected simplicial complex X with the following properties:*

(1) The space X is aspherical.
(2) The complex X satisfies the fibre non-collapsing assumption.
(3) We have $\dim X = n$ and $H_n(X;\mathbb{Z}) \cong 0$.

Proof. As $n \geq 2$, there exists a $k \in \mathbb{N}$ with $n = 2 \cdot (k+1)$ or $n = 3 + 2 \cdot k$. Let N be the product of k oriented closed connected hyperbolic surfaces. In the first case, let Σ be a non-orientable closed connected hyperbolic surface; in the second case, we take a non-orientable closed connected hyperbolic 3-manifold. We then set

$$M := \Sigma \times N$$

and consider the orientation double covering $p\colon \overline{M} \to M$ of M. Moreover, we triangulate $\Sigma \times N$ and take the induced triangulation of $\overline{M}$.

Then $\overline{M}$ satisfies FNCA (Example 5.13). Therefore also M satisfies FNCA (Proposition 5.7). By construction, M is aspherical, n-dimensional, and $H_n(M;\mathbb{Z}) \cong 0$ (as M is non-orientable).

Alternatively, one can also carry out the same argument when M is a non-orientable closed connected hyperbolic n-manifold; such manifolds indeed exist [34, Theorem 1.2] [31, Section 4.2]. The argument above has the advantage that it does not need existence theorems of such manifolds in higher dimensions. □

Corollary 6.12. *Let $n \in \mathbb{N}_{\geq 2}$ and let $c \in \mathbb{R}_{>0}$. Then, there exists a finite connected simplicial complex X with the following properties:*

(1) The space X is aspherical.
(2) We have $\mathrm{minent}(X) > c$.

(3) *We have* $\dim X = n$ *and* $H_n(X;\mathbb{Z}) \cong \mathbb{Z}$ *as well as*

$$\forall_{\alpha \in H_n(X;\mathbb{R})} \quad \|\alpha\|_1 = 0.$$

Proof. Let Y be an aspherical finite simplicial complex of dimension n as provided by Proposition 6.11. In particular, $\operatorname{minent}(Y) > 0$ because of FNCA (Theorem 6.5). Taking the wedge of a large enough number m of copies of Y results in a finite aspherical simplicial complex $Z := \bigvee_m Y$ of dimension n with [3, Theorem 2.6]

$$\operatorname{minent}(Z) \geq m \cdot \operatorname{minent}(Y) > c \qquad \text{and} \qquad H_n(Z;\mathbb{Z}) \cong 0.$$

Then $X := (S^1)^{\times n} \vee Z$ is a finite aspherical simplicial complex of dimension n with

$$\operatorname{minent}(X) \geq \operatorname{minent}\bigl((S^1)^{\times n}\bigr) + \operatorname{minent}(Z) > c \qquad \text{and} \qquad H_n(X;\mathbb{Z}) \cong \mathbb{Z}.$$

The fundamental class of $(S^1)^{\times n}$ pushes forward to a generator α of $H_n(X;\mathbb{R})$. As the n-torus has simplicial volume 0, it follows that $\|\alpha\|_1 = 0$. □

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