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A Marshall-Olkin type multivariate model with underlying dependent shocks

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Abstract

In this paper we study the distributional properties of a vector of lifetimes modeled as the first arrival time between an idiosyncratic shock and a common systemic shock. Despite unlike the classical multidimensional Marshall-Olkin model here only a unique common shock affecting all the lifetimes is assumed, some dependence is allowed between each idiosyncratic shock arrival time and the systemic one. The dependence structure of the resulting distribution is studied through the analysis of its singularity, its associated survival copula function and conditional hazard rates. Finally, some possible applications to actuarial and credit risk financial products are proposed.

Keywords: Marshall-Olkin distribution; Kendall's distribution function; Kendall's tau; systemic risk; conditional hazard rates; copula

Mathematics Subject Classification (2020): 62H10; 62N05; 91B05; 91G45

1 Introduction

In this paper we consider a particular generalization of the multidimensional Marshall-Olkin distribution (Marshall and Olkin, 1967) in the specific case in which, in addition to the idiosyncratic ones, only one common shock is considered whose occurrence causes the simultaneous end of all lifetimes. More specifically, if $(X_0, X_1, \ldots, X_d)$ are positive random variables that represent the arrival times of some shocks, then we consider, as resulting lifetimes, the random variables $T_1, \ldots, T_d$ defined as $T_j = \min(X_0, X_j)$, $j = 1, \ldots, d$.

In the Marshall-Olkin model the underlying shocks arrival times are assumed to be independent and exponentially distributed. Many extensions exist in the literature in order to consider marginal distributions different from the exponential one and to include some dependence among the underlying shocks arrival times, even in the more general case where additional systemic shocks involving subsets of the lifetimes $T_1, \ldots, T_d$ are assumed. The approach of considering general marginal distributions in place of the exponential one, even preserving the independence, is studied in Li and Pellerey (2011) in the bivariate case and

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extended to the multidimensional case in Lin and Li (2014). As for models allowing for some dependence among the underlying shocks, the scale-mixture of the Marshall-Olkin distribution, introduced in Li (2009), is obtained by scaling, through a positive random variable, a random vector distributed according to the Marshall-Olkin distribution: this is equivalent to assume that the underlying shocks arrival times have a dependence structure given by an Archimedean copula with a generator that is the Laplace transform of the mixing variable. Scale-mixtures of the Marshall-Olkin distributions have also been considered in Mai et al. (2013) where, in the exchangeable case, a different construction, involving Lévy subordinators, is presented, while the case of Archimedean dependence among the underlying shocks with a fully general generator is analyzed in Mulinacci (2018). The combination of Marshall-Olkin and Archimedean dependence structures is also studied in Charpentier et al (2014). The main weakness of these extensions is that they assume an underlying exchangeable dependence. Aiming at considering an asymmetric underlying dependence, in Pinto and Kolev (2015), when $d = 2$, the case in which X_1 and X_2 are dependent, while the external shock X_0 is independent of (X_1, X_2) is studied: this model has been further investigated and applied to life-insurance pricing in Gobbi et al. (2019).

The specific generalization of the Marshall-Olkin distribution presented in this paper is characterized by an asymmetric dependence in the vector $(X_0, \ldots, X_d)$ that goes in the opposite direction with respect to the one considered in Pinto and Kolev (2015): $X_1, \ldots, X_d$ are assumed to be independent while a particular pairwise dependence is assumed between each X_j, $j = 1, \ldots, d$ and X_0. The pairwise dependence results from the assumption that $X_0 = \min_{j=0,1,\ldots,d} Y_j$ where $Y_0, \ldots, Y_d$ are mutually independent while each Y_j is correlated with the idiosyncratic shock arrival time X_j, $j = 1, \ldots, d$.

A possible branch of application of this model is in the reliability modeling of mechanical or electronic systems, and consequently, in the modeling of the resulting operational and actuarial risk. Consider for example d working machines (or electronic components) M_j, $j = 1, \ldots, d$, all separately connected with a same machine M_0 so that if M_0 stops to working, immediately the same occurs for all the other machines. Assuming the classical Marshall-Olkin model, the failure of a single machine M_j, $j = 1, \ldots, d$ does not influence the failure of the machine M_0 or of the remaining M_i, $i = 1, \ldots, d, i \neq j$. Conversely, in our model, the failure of one of the M_j, $j = 1, \ldots, d$ can influence the probability of failure of M_0, and, consequently, of the collapse of the whole system. Consider, for example, the case in which a component M_j fails to works correctly: being connected to M_0, this fact could worsen or even interrupt the functioning of M_0.

Another branch of application of this model is credit risk. There is a wide literature on applications of the Marshall-Olkin model and its generalizations to credit risk (see, among the others, Giesecke, 2003, Lindskog and McNeil, 2003, Elouerkhaoui, 2007, Mai and Scherer, 2009, Baglioni and Cherubini, 2013 and Bernhart et al., 2013). Given the specific type of assumed dependence, the probabilistic model analyzed in this paper looks particularly suitable for the analysis of the joint lifetimes of the so called Systemically Important Financial

Institutions (SIFI) for which the default (or the proximity to it) of one of them, is directly correlated with the collapse of the whole system.

In this paper, we first discuss the survival distribution of the underlying vector of shock arrival times $(X_0, \ldots, X_d)$: we study the associated survival copula function and recover expressions for the Kendall's distribution function and Kendall's tau of the pairs (X_0, X_j), $j = 1, \ldots, d$. Then we focus on the resulting joint survival distribution of the lifetimes $(T_1, \ldots, T_d)$: we analyze the probability of simultaneous end of all lifetimes (that is the singularity of the distribution) and the dependence properties through the analysis of the pairwise Kendall's distribution function and Kendall's tau and the expressions of the conditional hazard rates. We do not make, in principle, any assumption on the marginal distributions of the underlying shocks arrival times and on the underlying dependence structure: however, in order to obtain closed formulas, we restrict the analysis to particular classes of marginal distributions (that include the exponential one as a particular case) and to Archimedean bivariate copulas. Finally, some specific case is numerically analyzed in order to study the impact of the parameters governing the model on the resulting dependence structure and practical credit and actuarial applications are considered.

The paper is organized as follows. In section 2 we present and analyze the underlying shocks arrival times model. In section 3 we derive the distribution of the resulting, subject to shocks, lifetimes which is, by construction, singular: we compute the probability of the singularity and we analyze the dependence structure through the identification of the pairwise Kendall's distribution function and Kendall's tau formulas and the conditional hazard rates. Section 4 is focused on a particular specification of the general model previously presented, for which it is possible to obtain specific and meaningful formulas. In section 5 we perform a sensitivity analysis in the two-dimensional case and we briefly present possible applications to actuarial and credit risk products while section 6 concludes.

2 The shocks arrival times model

In this section we introduce and study the probabilistic model of the underlying shocks arrival times that define the lifetimes, $T_1, \ldots, T_d$, of the d components of a system.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathbf{Y}, \mathbf{X}) = (Y_0, Y_1, \ldots, Y_d, X_1, \ldots, X_d)$, be a $2d+1$-dimensional random vector with strictly positive elements. In the sequel, we will interpret each random variable X_j $(j = 1, \ldots, d)$ in $\mathbf{X}$ as the arrival time of a shock causing the default of only the j-th element in the system, while all random variables Y_j $(j = 0, 1, \ldots, d)$ in $\mathbf{Y}$ represent the arrival times of shocks causing the default of the whole system.

We assume that each X_j in the subvector $\mathbf{X}$ has a survival distribution function, denoted by $\bar{F}_{X_j}$, for $j = 1, \ldots, d$, strictly decreasing on $(0, +\infty)$. The survival distribution functions of the random variables in the subvector $\mathbf{Y}$, denoted by $\bar{F}_{Y_j}$ for $j = 0, \ldots, d)$, can be either strictly decreasing or identically equal to 1 on $(0,+\infty)$: nevertheless, we assume that there exists at leat one $j \in \{0,\dots,d\}$ such that $\bar{F}_{Y_j}$ is not identically equal to 1 on $(0,+\infty)$.

As for the dependence structure, the random variables in the subvector $\mathbf{X}$ are assumed to be mutually independent, as well as those in the subvector $\mathbf{Y}$, while each pair (Y_j, X_j) for $j=1,\dots,d$ has a survival dependence structure given by a bivariate survival copula $\hat{C}_j$. More precisely, the copula associated to the survival distribution of the vector $(\mathbf{Y},\mathbf{X})$ is

$$\hat{C}_{(\mathbf{Y},\mathbf{X})}(u_0,u_1,\dots,u_d,v_1,\dots,v_d) = u_0 \prod_{j=1}^{d} \hat{C}_j\left(u_j,v_j\right).$$

It follows that the joint survival distribution function of $(\mathbf{Y},\mathbf{X})$ is

$$\bar{F}_{(\mathbf{Y},\mathbf{X})}(y_0,y_1,\dots,y_d,x_1,\dots,x_d) = \bar{F}_{Y_0}(y_0) \prod_{j=1}^{d} \hat{C}_j\left(\bar{F}_{Y_j}(y_j), \bar{F}_{X_j}(x_j)\right).$$

Starting from the above setup, we define the arrival time of the systemic shock (that is the shock causing the collapse of the whole system) as

$$X_0 = \min_{j=0,1,\dots,d} Y_j.$$

Clearly, its survival distribution function is

$$\bar{F}_{X_0}(x) = \prod_{j=0}^{d} \bar{F}_{Y_j}(x)$$

and, by the assumptions of the model, it is strictly decreasing on $(0,+\infty)$.

Let us now consider the $d+1$-dimensional random vector $\mathbf{S} = (X_0, X_1, \dots, X_d)$ whose joint survival distribution function is, for $(x_1,\dots,x_d) \in (0,+\infty)^d$,

$$\begin{aligned}\bar{F}_{\mathbf{S}}(x_0,x_1,\dots,x_d) &= \mathbb{P}\left(Y_0 > x_0, Y_1 > x_0, \dots, Y_d > x_0, X_1 > x_1, \dots, X_d > x_d\right) = \\ &= \bar{F}_{Y_0}(x_0) \prod_{j=1}^{d} \hat{C}_j\left(\bar{F}_{Y_j}(x_0), \bar{F}_{X_j}(x_j)\right).\end{aligned}$$

Thanks to Sklar's theorem, the induced survival dependence structure is given by the survival copula

$$\hat{C}_{\mathbf{S}}(u_0,u_1,\dots,u_d) = \bar{F}_{Y_0} \circ \bar{F}_{X_0}^{-1}(u_0) \prod_{j=1}^{d} \hat{C}_j\left(\bar{F}_{Y_j} \circ \bar{F}_{X_0}^{-1}(u_0), u_j\right) \tag{1}$$

where here and in the sequel the symbol "$\circ$" denotes functions composition.

Notice that the vector $\mathbf{S}$ is obtained through a particular implementation of the construction proposed by Liebscher (2008) in his Lemma 2.1, where this technique is applied in order to construct asymmetric copulas. It follows that (1) is a particular specification of the copula family defined in Theorem 2.1 in Liebscher (2008). In fact, let, for $j = 0, \ldots, d$, $g_j = \bar{F}_{Y_j} \circ \bar{F}_{X_0}^{-1}$ and consider $d+1$-dimensional copulas C_j for $j = 0, \ldots, d$, so that, for $j = 1, \ldots, d$ if $\mathbf{u} = (u_0, u_1, \ldots, u_d)$ with $u_i = 1$ for $i \neq 0, j$, then $C_j(\mathbf{u}) = \hat{C}_j(u_0, u_j)$. Clearly,

$$\hat{C}_{\mathbf{S}}(u_0, u_1, \ldots, u_d) = C_0\left(g_0(u_0), 1, \ldots, 1\right) \cdot C_1\left(g_1(u_0), u_1, 1, \ldots, 1\right) \cdots C_d\left(g_d(u_0), 1, \ldots, 1, u_d\right).$$

Since, for $j = 0, \ldots, d$, $g_j = \bar{F}_{Y_j} \circ \bar{F}_{X_0}^{-1} : [0,1] \to [0,1]$ is strictly increasing or identically equal to 1 and $\prod_{j=0}^{d} g_j(v) = v$, then (1) represents a particular specification of the family of copulas obtained according to the methodology proposed in Theorem 2.1 in Liebscher (2008). As a consequence, the results about the type of induced dependence structure studied in that paper apply. Since ours is a very particular specification of Liebscher (2008) copula family, for copulas given in (1) we are able to find explicit formulas for the Kendall's distribution function and the Kendall's tau of the pairs of type (X_0, X_i), as we are going to show in next subsection.

2.1 The dependence structure between the systemic shock arrival time and each idiosyncratic one

The joint survival distribution of the arrival time of each idiosyncratic shock X_j and that of the systemic one X_0 is

$$\bar{F}_{(X_0,X_i)}(x_0, x_i) = \hat{C}_i\left(\bar{F}_{Y_i}(x_0), \bar{F}_{X_i}(x_i)\right) \frac{\bar{F}_{X_0}(x_0)}{\bar{F}_{Y_i}(x_0)}$$

and the corresponding bivariate survival copulas are

$$\hat{C}_{0,i}(u_0, u_i) = \hat{C}_i\left(\bar{F}_{Y_i} \circ \bar{F}_{X_0}^{-1}(u_0), u_i\right) \frac{u_0}{\bar{F}_{Y_i} \circ \bar{F}_{X_0}^{-1}(u_0)}. \quad (2)$$

In order to analyze the dependence structure induced by (2) we compute the Kendall's distribution function of a copula $\tilde{C}$ of type

$$\tilde{C}(u, v) = C\left(g(u), v\right) \frac{u}{g(u)} \quad (3)$$

where $g : [0,1] \to [0,1]$ is strictly increasing.

We remind that the Kendall's distribution function of a bivariate copula $C(u, v)$ is defined as the cumulative distribution function of the random variable $C(U, V)$ where the random variables U and V are uniformly distributed on the interval $[0,1]$ and their joint distribution function is given by the considered copula $C(u, v)$. More precisely the Kendall's distribution function of a bivariate copula C is a function $K_C : [0,1] \to [0,1]$ defined as

$$K_C(t) = \mathbb{P}\left(C(U, V) \leq t\right), \quad \text{for } t \in [0,1]$$

(see Nelsen, 2003), where $\mathbb{P}$ is the probability induced by C. The relevance of this notion relies on the fact that it induces, through the corresponding one-dimensional stochastic ordering, a partial ordering in the set of bivariate copulas: notice in particular that if $C_1(u,v) \leq C_2(u,v)$ for all $(u,v) \in [0,1]^2$, then $K_{C_1}(t) \geq K_{C_2}(t)$ for all $t \in [0,1]$ (see Nelsen, 2003, for more details).
In order to simplify the notation, we set $\partial_1 C(u,v)) = \frac{\partial}{\partial u} C(u,v)$ for any copula $C(u,v)$.

Proposition 2.1. *Let $g : [0,1] \to [0,1]$ be strictly increasing and differentiable and the copula $C(u,v)$ be strictly increasing with respect to v for any u. Then the Kendall's distribution function of a copula $\tilde{C}$ of type (3) is*

$$K(t) = K_0(t) + t \ln(g(t)) + \int_t^1 \partial_1 C(g(u), l_t(u)) \frac{g'(u)}{g(u)} u \, du$$

where $l_t(u)$ solves $\tilde{C}(u, l_t(u)) = t$ and $K_0(t) = t - t \ln t$ is the Kendall's distribution function of the product copula.

Proof. Since, for a given u, $\tilde{C}(u,v)$ is strictly increasing with respect to v, the inverse function $l_t(u)$ with respect to v is well defined for all $t \in (0,u]$ and satisfies $C(g(u), l_t(u)) = \frac{g(u)}{u} t$. Since

$$K(t) = t + \int_t^1 \partial_1 \tilde{C}(u, l_t(u)) du$$

(this is (6) in Genest and Rivest (2001)), after straightforward computations we have that

$$K(t) = t - t \ln\left(\frac{t}{g(t)}\right) + \int_t^1 \partial_1 C(g(u), l_t(u)) \frac{g'(u)}{g(u)} u du.$$

□

The Kendall's distribution function is strictly related to the widely used concordance measure known as Kendall's tau (see Nelsen, 2003). In fact, the Kendall's tau τ can be obtained from the Kendall's distribution function through

$$\tau = 3 - 4 \int_0^1 K(t) dt.$$

Hence, under the assumptions of Proposition 2.1, the Kendall's tau of a copula of type (3) is

$$\tau = -4 \int_0^1 t \ln(g(t)) \, dt - 4 \int_0^1 \int_t^1 \partial_1 C(g(u), l_t(u)) \frac{g'(u)}{g(u)} u \, du dt.$$

Example 2.1. Archimedean copulas case
Let us now assume that the bivariate survival copula functions $\hat{C}_j$, for $j = 1, \ldots, d$ are of Archimedean type, with strict generator ϕ_j, that is $\phi_j : [0, +\infty) \to (0,1]$ satisfies $\phi_j(0) =$

1, $\lim_{x\to+\infty}\phi_j(x) = 0$ *and it is strictly decreasing and convex on* $[0,+\infty)$ *(see McNeal and Nešlehová, 2009). Hence, from (3), we get*

$$\tilde{C}(u,v) = \phi\left(\phi^{-1}\left(g(u)\right) + \phi^{-1}(v)\right)\frac{u}{g(u)}. \tag{4}$$

The expression of the Kendall's distribution function of a copula of this type can be immediately recovered from Proposition 2.1, taking into account that, now, $l_t(u) = \phi\left(\phi^{-1}\left(\frac{g(u)}{u}t\right) - \phi^{-1}(g(u))\right)$. *In fact it is a straightforward computation to verify that*

$$K(t) = t - t\ln t + t\ln\left(g(t)\right) + \int_t^1 \frac{h\left(\frac{g(u)}{u}t\right)}{h\left(g(u)\right)}\frac{g'(u)}{g(u)}udu$$

with $h(x) = \phi' \circ \phi^{-1}(x)$.

If we further assume the function g *in (4) be of type* $g(v) = v^\theta$, *with* $\theta \in (0,1]$ *(this specific case was firstly introduced in Khoudraji, 1995), we get*

$$K(t) = t - t\ln t + \theta t\ln t + \theta\int_t^1 \frac{h\left(u^{\theta-1}t\right)}{h\left(u^\theta\right)}du$$

and

$$\tau = \theta - 4\theta\int_0^1\int_t^1 \frac{h\left(u^{\theta-1}t\right)}{h\left(u^\theta\right)}dudt.$$

In particular,

- *Clayton case, that is* $\phi(x) = (1+x)^{-\frac{1}{\beta}}$, *with* $\beta \geq 0$*: since* $h(y) = -\frac{1}{\beta}y^{1+\beta}$, *we have*

$$K(t) = t\left(1+\frac{\theta}{\beta}\right) - (1-\theta)t\ln t - \frac{\theta}{\beta}t^{1+\beta} \tag{5}$$

  and

$$\tau = \frac{\beta}{\beta+2}\theta = \tau_\beta^C\,\theta \tag{6}$$

  where τ_β^C *is the Kendall's tau of the Clayton copula with parameter* β*;*

- *Gumbel case, that is* $\phi(x) = e^{-x^{\frac{1}{\beta}}}$, *with* $\beta \geq 1$*: since* $h(y) = -\frac{1}{\beta}y(-\ln y)^{1-\beta}$, *we have*

$$K(t) = t - t\ln t\left[1 - (\beta-1)\left(\frac{\theta}{1-\theta}\right)^\beta\int_{\frac{\theta}{1-\theta}}^{+\infty}\frac{1}{z^\beta(z+1)}dz\right]$$

  and

$$\tau = \left(1-\frac{1}{\beta}\right)\left[\beta\left(\frac{\theta}{1-\theta}\right)^\beta\int_{\frac{\theta}{1-\theta}}^{+\infty}\frac{1}{z^\beta(z+1)}dz\right] = \tau_\beta^G\left[\beta\left(\frac{\theta}{1-\theta}\right)^\beta\int_{\frac{\theta}{1-\theta}}^{+\infty}\frac{1}{z^\beta(z+1)}dz\right]$$

  where τ_β^G *is the Kendall's tau of the Gumbel copula with parameter* β*.*

3 The lifetimes model

In this section we study the joint distribution of the observed lifetimes $(T_1, T_2, \ldots, T_d)$, each defined as the first arrival time between the corresponding idiosyncratic shock and the systemic one. More precisely, for $j = 1, \ldots, d$, let

$$T_j = \min(X_j, X_0)$$

be the lifetime of the j-th element in the system. If $Z_j = \min(Y_j, X_j)$, then

$$T_j = \min\left(\min_{\substack{i=0,\ldots,d \\ i\neq j}} Y_i, Z_j\right) \tag{7}$$

and we have that each T_j can also be modeled as the first arrival time among $d+1$ independent shocks arrival times.
The survival distribution of each T_j is, for $x \geq 0$,

$$\bar{F}_{T_j}(x) = \hat{C}_j\left(\bar{F}_{Y_j}(x), \bar{F}_{X_j}(x)\right)\frac{\bar{F}_{X_0}(x)}{\bar{F}_{Y_j}(x)},$$

while the joint survival distribution function of $\mathbf{T} = (T_1, \ldots, T_d)$ is

$$\bar{F}_{\mathbf{T}}(t_1, \ldots, t_d) = \bar{F}_{Y_0}\left(\max_{i=1,\ldots,d} t_i\right)\prod_{j=1}^{d}\hat{C}_j\left(\bar{F}_{Y_j}\left(\max_{i=1,\ldots,d} t_i\right), \bar{F}_{X_j}(t_j)\right) \tag{8}$$

for $(t_1, \ldots, t_d) \in (0, +\infty)^d$.

Relation (7) implies that the random vector $(T_1, \ldots, T_d)$ is built again following the same construction introduced in Lemma 2.1 in Liebscher (2008) and the associated survival copula function

$$\hat{C}_{\mathbf{T}}(u_1, \ldots, u_d) = \min_{i=1,\ldots,d}\bar{F}_{Y_0}\circ\bar{F}_{T_i}^{-1}(u_i)\prod_{j=1}^{d}\hat{C}_j\left(\min_{i=1,\ldots,d}\bar{F}_{Y_j}\circ\bar{F}_{T_i}^{-1}(u_i), \bar{F}_{X_j}\circ\bar{F}_{T_j}^{-1}(u_j)\right)$$

is of the same type as the copula family defined in his Theorem 2.1. In fact, considering the d-dimensional copulas

$$K_1(u_1, \ldots, u_d) = \hat{C}_1\left(\min\left\{\bar{F}_{Y_1}\circ\bar{F}_{Z_1}^{-1}(u_1), u_2, \ldots, u_d\right\}, \bar{F}_{X_1}\circ\bar{F}_{Z_1}^{-1}(u_1)\right)$$

$$\vdots$$

$$K_d(u_1, \ldots, u_d) = \hat{C}_d\left(\min\left\{u_1, u_2, \ldots, \bar{F}_{Y_d}\circ\bar{F}_{Z_d}^{-1}(u_d)\right\}, \bar{F}_{X_d}\circ\bar{F}_{Z_d}^{-1}(u_d)\right)$$

associated to the d-dimensional vectors $(Z_1, Y_1, \ldots, Y_1), \ldots, (Y_d, \ldots, Y_d, Z_d)$, respectively, and the upper Fréchet bound copula $K_0(u_1, \ldots, u_d) = \min_{i=1,\ldots,d} u_i$, then

$$\hat{C}_{\mathbf{T}}(u_1, \ldots, u_d) = \prod_{j=0}^{d} K_j \left(g_{j1}(u_1), \ldots, g_{jd}(u_d)\right)$$

where $g_{jj}(u_j) = \bar{F}_{Z_j} \circ \bar{F}_{T_j}^{-1}(u_j)$ and $g_{ji}(u_i) = \bar{F}_{Y_j} \circ \bar{F}_{T_i}^{-1}(u_i)$ for $i \neq j$.

3.1 The probability of simultaneous default

By construction, the distribution of $\mathbf{T}$ has a singularity generated by the occurrence of the simultaneous default of more than one element in the system. In this subsection we will analyze the distribution of the time of the simultaneous end of all the considered lifetimes. Let

$$S(\omega) = \begin{cases} T_i(\omega), & \text{if } T_1(\omega) = \cdots = T_d(\omega) \\ -\infty, & \text{otherwise} \end{cases} \tag{9}$$

be the time of occurrence of the simultaneous default of all the elements in the system.

Proposition 3.1. *If the random vector $\mathbf{T}$ has a survival distribution of type (8), then the survival distribution function of S, for $t \geq 0$ is given by*

$$\bar{F}_S(t) = \sum_{j=0}^{d} H_j(t) \tag{10}$$

where

$$H_0(t) = -\int_t^{+\infty} \prod_{j=1}^{d} \bar{F}_{Z_j}(x) d\bar{F}_{Y_0}(x) \tag{11}$$

and, for $j = 1, \ldots, d$,

$$H_j(t) = -\int_t^{+\infty} \bar{F}_{Y_0}(x) \left(\prod_{i \neq j} \bar{F}_{Z_i}(x)\right) \partial_1 \hat{C}_j \left(\bar{F}_{Y_j}(x), \bar{F}_{X_j}(x)\right) d\bar{F}_{Y_j}(x). \tag{12}$$

Proof. The conclusion immediately follows taking into account that, after some algebra, it can be shown that

$$P\left(S > t, S = Y_j\right) = \mathbb{E}\left[\mathbb{P}\left(X_k > Y_k, \forall k = 1, \ldots, d, Y_i > Y_j > t, \forall i = 0, \ldots, d, i \neq j | Y_j\right)\right]$$

coincides with $H_j(t)$ given by (11), if $j = 0$, and (12), otherwise. □

It follows that $\mathbb{P}\left(T_1 = T_2 = \cdots = T_d\right) = \sum_{j=0}^{d} H_j(0)$.

3.2 The pairwise Kendall's distribution function

In this section we recover the formula of the pairwise Kendall's distribution function of the bivariate subvectors extracted from $\mathbf{T}$.

In order to simplify the notation we set, for $i, k = 1, \ldots, d$, $i \neq k$,

$$P_{i,k}(x) = \frac{\bar{F}_{X_0}(x)}{\bar{F}_{Y_i}(x)\bar{F}_{Y_k}(x)}.$$

It can be easily checked that the survival distributions of the pairs (T_i, T_k) are

$$\bar{F}_{i,k}(t_i, t_k) = P_{i,k}\left(\max(t_i, t_k)\right) \prod_{j=i,k} \hat{C}_j\left(\bar{F}_{Y_j}(\max(t_i, t_k)), \bar{F}_{X_j}(t_j)\right).$$

Proposition 3.2. *Let us assume that $\hat{C}_i$ and $\hat{C}_k$ are strictly increasing with respect to each argument. If $K_{i,k}$ is the Kendall's distribution function associated to the pair (T_i, T_k), then, for $t \in [0, 1]$,*

$$\begin{aligned} K_{i,k}(t) = t - t&\left(\ln\left(\frac{(\bar{F}_{Z_i} \cdot P_{i,k}) \circ \bar{F}^{-1}_{T_i}(t)}{(\bar{F}_{Z_i} \cdot P_{i,k})(z_t)}\right) + \ln\left(\frac{(\bar{F}_{Z_k} \cdot P_{i,k}) \circ \bar{F}^{-1}_{T_k}(t)}{(\bar{F}_{Z_k} \cdot P_{i,k})(z_t)}\right)\right) + \\ &- \int_{z_t}^{\bar{F}^{-1}_{T_i}(t)} \bar{F}_{Z_i} P_{ik}(x)\, \partial_1 \hat{C}_k\left(\bar{F}_{Y_k}(x), \bar{F}_{X_k}(h_t(x))\right) d\bar{F}_{Y_k}(x) + \\ &- \int_{z_t}^{\bar{F}^{-1}_{T_k}(t)} \bar{F}_{Z_k} P_{ik}(x)\, \partial_1 \hat{C}_i\left(\bar{F}_{Y_i}(x), \bar{F}_{X_i}(g_t(x))\right) d\bar{F}_{Y_i}(x) \end{aligned}$$

where z_t is the solution of $\bar{F}_{i,k}(z_t, z_t) = t$, $h_t(\cdot)$ solves $\bar{F}_{i,k}(x, h_t(x)) = t$ for $z_t < x \leq \bar{F}^{-1}_{T_i}(t)$ and $g_t(\cdot)$ solves $\bar{F}_{i,k}(g_t(y), y) = t$ for $z_t < y \leq \bar{F}^{-1}_{T_k}(t)$.

Proof. Since $\bar{F}_{i,k}(x, x) = \bar{F}_{Z_i}(x)\bar{F}_{Z_k}(x)P_{i,k}(x)$ is strictly decreasing, given any $t \in [0, 1]$, the solution of $\bar{F}_{i,k}(x, x) = t$, denoted by z_t, is well defined.
If we restrict to $t_i > t_k$, then

$$\bar{F}_{i,k}(t_i, t_k) = \bar{F}_{Z_i}(t_i)\hat{C}_k\left(\bar{F}_{Y_k}(t_i), \bar{F}_{X_k}(t_k)\right) P_{i,k}(t_i) \tag{13}$$

which is strictly decreasing with respect to $t_k \in [0, t_i)$ for any given t_i. Hence, for $x \in (z_t, \bar{F}^{-1}_{T_i}(t)]$ and for any $t \in [0, 1]$, the function h_t satisfying $\bar{F}_{i,k}(x, h_t(x)) = t$ is well defined. By similar arguments, the function g_t of the statement is also well defined.

If we rewrite the Kendall's distribution function $K_{i,k}$ in terms of the joint survival distribution function, we get

$$\begin{aligned} K_{i,k}(t) &= \mathbb{P}(\bar{F}_{i,k}(T_i, T_k) \leq t) = \\ &= \bar{F}_{T_i}(z_t) - \mathbb{P}((T_i, T_k) \in \mathcal{D}_1) + \bar{F}_{T_k}(z_t) - \mathbb{P}((T_i, T_k) \in \mathcal{D}_2) - t \end{aligned}$$

where

$$\mathcal{D}_1 = \{(t_i, t_k) : z_t < t_i \leq \bar{F}_{T_i}^{-1}(t), 0 \leq t_k \leq h_t(t_i)\}$$

and

$$\mathcal{D}_2 = \{(t_i, t_k) : z_t < t_k \leq \bar{F}_{T_k}^{-1}(t), 0 \leq t_i \leq g_t(t_k)\}.$$

Let us start computing $\mathbb{P}((T_i, T_k) \in \mathcal{D}_1)$. Since here (13) holds true, thanks to the definitions of z_t and h_t, we have

$$\begin{aligned}
\mathbb{P}\left((T_i, T_k) \in \mathcal{D}_1\right) &= \int_{z_t}^{\bar{F}_{T_i}^{-1}(t)} \left(\mathbb{P}(T_k > h_t(x)|T_i = x) - 1\right) d\bar{F}_{T_i}(x) = \\
&= \int_{z_t}^{\bar{F}_{T_i}^{-1}(t)} \mathbb{P}(T_k > h_t(x)|T_i = x) d\bar{F}_{T_i}(x) - t + \bar{F}_{T_i}(z_t) = \\
&= \int_{z_t}^{\bar{F}_{T_i}^{-1}(t)} t \cdot \frac{d(\bar{F}_{Z_i} \cdot P_{ik})(x)}{\bar{F}_{Z_i} \cdot P_{ik}(x)} + \\
&+ \int_{z_t}^{\bar{F}_{T_i}^{-1}(t)} \bar{F}_{Z_i} \cdot P_{ik}(x) \cdot \partial_1 \hat{C}_k \left(\bar{F}_{Y_k}(x), \bar{F}_{X_k}(h_t(x))\right) d\bar{F}_{Y_k}(x) - t + \bar{F}_{T_i}(z_t) = \\
&= t \ln \left(\frac{(\bar{F}_{Z_i} \cdot P_{i,k}) \circ \bar{F}_{T_i}^{-1}(t)}{(\bar{F}_{Z_i} \cdot P_{i,k})(z_t)} \right) + \\
&+ \int_{z_t}^{\bar{F}_{T_i}^{-1}(t)} \bar{F}_{Z_i} P_{ik}(x) \cdot \partial_1 \hat{C}_k \left(\bar{F}_{Y_k}(x), \bar{F}_{X_k}(h_t(x))\right) d\bar{F}_{Y_k}(x) - t + \bar{F}_{T_i}(z_t).
\end{aligned}$$

Since $\mathbb{P}((T_i, T_k) \in \mathcal{D}_2)$ can be computed similarly, we get the conclusion. □

3.3 The lack of memory property

Since the lack of memory property characterizes the Marshall-Olkin distribution and its marginals, we investigate if and when this property is satisfied by the distribution of $(T_1, \ldots, T_d)$.

Given a random vector $\mathbf{Z}$ of dimension d we define the residual lifetimes vector at time $t \geq 0$ as the vector

$$\mathbf{Z}_t = \left[Z_1 - t, \ldots, Z_d - t | Z_1 > t, \ldots, Z_d > t\right].$$

We have that $\mathbf{Z}$ satisfies the weak lack of memory property (WLMP) if, for all $t \geq 0$,

$$\bar{F}_{\mathbf{Z}_t}(\mathbf{z}) = \bar{F}_{\mathbf{Z}}(\mathbf{z}), \qquad \forall \mathbf{z} \in \mathbb{R}^d.$$

Proposition 3.3. *If $(\mathbf{Y}, \mathbf{X})$ satisfies the WLMP then also $\mathbf{T}$ does.*

Proof. It is an immediate consequence of the fact that for all $t \geq 0$,

$$\bar{F}_{\mathbf{T}_t}(x_1, \ldots, x_d) = \bar{F}_{(\mathbf{Y},\mathbf{X})_t}(\hat{x}, \ldots, \hat{x}, x_1, \ldots, x_d)$$

where $\hat{x} = \max_{i \in \{1, \ldots d\}} x_i$. $\square$

Proposition 3.4. $(\mathbf{Y}, \mathbf{X})$ *satisfies the WLMP if and only if each pair* (Y_j, X_j)*, for* $j = 1, \ldots, d$ *does and* Y_0 *is exponentially distributed.*

Proof. The result immediately follows from the fact that the WLMP for the vector $(\mathbf{Y}, \mathbf{X})$ can be written as

$$\frac{\bar{F}_{Y_0}(y_0 + t)}{\bar{F}_{Y_0}(t)} \prod_{j=1}^{d} \frac{\bar{F}_{Y_j, X_j}(y_j + t, x_j + t)}{\bar{F}_{Y_j, X_j}(t, t)} = \bar{F}_{Y_0}(y_0) \prod_{j=1}^{d} \bar{F}_{Y_j, X_j}(y_j, x_j).$$

$\square$

Thanks to Theorem 3.2 in Mulero and Pellerey (2010), (Y_j, X_j) satisfies the weak lack of memory property when the associated survival copula is Clayton and the marginal survival distributions are of type $\bar{F}_{Y_j}(x) = \left(1 + \frac{e^{cx}-1}{a_j}\right)^{-\frac{1}{\beta_j}}$ and $\bar{F}_{X_j}(x) = \left(1 + \frac{e^{cx}-1}{b_j}\right)^{-\frac{1}{\beta_j}}$ with $c, a_j, b_j > 0$, $\frac{1}{a_j} + \frac{1}{b_j} = 1$ and β_j is the parameter of the associated bivariate Clayton survival copula $\hat{C}_j$. In this case, if Y_0 is exponentially distributed with parameter γ_0, we get the class of joint survival distribution functions

$$\bar{F}_{\mathbf{T}}(t_1, \ldots, t_d) = e^{-\gamma_0 \hat{t}} \prod_{j=1}^{d} \left(\frac{e^{c\hat{t}}}{a_j} + \frac{e^{ct_j}}{b_j} \right)^{-\frac{1}{\beta_j}}$$

where $\hat{t} = \max_{i=1,\ldots,d} t_i$.

3.4 Conditional hazard rates

In order to point out short or long-term dependence of one lifetime with respect to the end of another one, we analyze the conditioned hazard rates associated to (8)
In this subsection we assume that all bivariate survival copulas $\hat{C}_1, \ldots, \hat{C}_d$ are absolutely continuous, with densities $\hat{c}_1, \ldots, \hat{c}_d$, respectively.

Given a positive random variable W, and an event A, we define the conditional hazard rate given A

$$\lambda_W(t|A) = \lim_{h \to 0^+} \frac{\mathbb{P}(t < W \leq t + h | A)}{h}.$$

In order to simplify the notation, we set, for $i = 1, \ldots, d$,

$$\lambda_i(t) = \lambda_{T_i}(t | T_j > t, \forall j = 1, \ldots, d)$$

and, for $i, k = 1, \ldots, d$, $i \neq k$ and $t \geq t_k$,

$$\lambda_{i|k}(t|t_k) = \lambda_{T_i}(t | T_k = t_k, T_j > t, \forall j = 1, \ldots, d, \ j \neq k).$$

Clearly,

$$\lambda_i(t) = -\frac{\partial_i^+ \bar{F}_{\mathbf{T}}(t,t,\ldots,t)}{\bar{F}_{\mathbf{T}}(t,t,\ldots,t)} \quad \text{and} \quad \lambda_{i|k}(t|t_k) = -\frac{\partial_{i,k}^2 \bar{F}_{\mathbf{T}}(x_k(t,t_k))}{\partial_k \bar{F}_{\mathbf{T}}(x_k(t,t_k))}.$$

where ∂_i^+ denotes the right partial derivative with respect to the i-th argument, ∂_k the partial derivative with respect to the k-th variable, $\partial_{i,k}^2$ the mixed second partial derivative with respect to i-th and k-th argument and $x_k(t,z)$ the d-dimensional vector with the k-th entry equal to z and all remaining entries equal to t.
After some algebra, we get

$$\begin{aligned}\lambda_i(t) &= \frac{f_{Y_0}(t)}{\bar{F}_{Y_0}(t)} + \frac{f_{Z_i}(t)}{\bar{F}_{Z_i}(t)} + \sum_{\substack{j=1\\ j\neq i}}^{d} \frac{f_{Y_j}(t)}{\bar{F}_{Z_j}(t)} \cdot \partial_1 \hat{C}_j\left(\bar{F}_{Y_j}(t), \bar{F}_{X_j}(t)\right) = \\ &= \lambda_{Y_0}(t|Y_0>t) + \lambda_{Z_i}(t|Z_i>t) + \sum_{\substack{j=1\\ j\neq i}}^{d} \lambda_{Y_j}(t|Y_j>t, X_j>t)\end{aligned}$$

and

$$\begin{aligned}\lambda_{i|k}(t|t_k) &= \frac{f_{Y_0}(t)}{\bar{F}_{Y_0}(t)} + \frac{f_{Z_i}(t)}{\bar{F}_{Z_i}(t)} + \frac{\hat{c}_k\left(\bar{F}_{Y_k}(t), \bar{F}_{X_k}(t_k)\right)}{\partial_2 \hat{C}_k\left(\bar{F}_{Y_k}(t), \bar{F}_{X_k}(t_k)\right)} f_{Y_k}(t) + \sum_{\substack{j=1\\ j\neq i,k}}^{d} \frac{\partial_1 \hat{C}_j\left(\bar{F}_{Y_j}(t), \bar{F}_{X_j}(t)\right)}{\hat{C}_j\left(\bar{F}_{Y_j}(t), \bar{F}_{X_j}(t)\right)} f_{Y_j}(t) = \\ &= \lambda_{Y_0}(t|Y_0>t) + \lambda_{Z_i}(t|Z_i>t) + \lambda_{Y_k}(t|Y_k>t, X_k=t_k) + \sum_{\substack{j=1\\ j\neq i,k}}^{d} \lambda_{Y_j}(t|Y_j>t, X_j>t).\end{aligned}$$

As expected, the conditional hazard rates are the sum of the conditional hazard rates of the independent random variables that determine T_i (see 7). It follows that

$$\begin{aligned}\lambda_{i|k}(t|t_k) &= \lambda_i(t) + \left(\frac{\hat{c}_k\left(\bar{F}_{Y_k}(t), \bar{F}_{X_k}(t_k)\right)}{\partial_2 \hat{C}_k\left(\bar{F}_{Y_k}(t), \bar{F}_{X_k}(t_k)\right)} - \frac{\partial_1 \hat{C}_k\left(\bar{F}_{Y_k}(t), \bar{F}_{X_k}(t)\right)}{\bar{F}_{Z_k}(t)}\right) f_{Y_k}(t) = \\ &= \lambda_i(t) + \lambda_{Y_k}(t|Y_k>t, X_k=t_k) - \lambda_{Y_k}(t|Y_k>t, X_k>t).\end{aligned} \tag{14}$$

From (14) we have that $\lambda_{i|k}(t|t_k) \geq \lambda_i(t)$ if and only if $\frac{\hat{c}_k\left(\bar{F}_{Y_k}(t), \bar{F}_{X_k}(t_k)\right)}{\partial_2 \hat{C}_k\left(\bar{F}_{Y_k}(t), \bar{F}_{X_k}(t_k)\right)} \geq \frac{\partial_1 \hat{C}_k\left(\bar{F}_{Y_k}(t), \bar{F}_{X_k}(t)\right)}{\bar{F}_{Z_k}(t)}$: notice that, if $\frac{\hat{c}_k(u,v)}{\partial_2 \hat{C}_k(u,v)} \geq \frac{\partial_1 \hat{C}_k(u,v)}{\hat{C}_k(u,v)}$ for $(u,v) \in [0,1]^2$, by integrating both sides, we get $\hat{C}_k(u,v) \geq uv$.

Moreover, again from (14), $\lambda_{i|k}(t|t_k)$ is an increasing (decreasing) function of $t_k \in [0,t]$ if and only if $\frac{\hat{c}_k(u,v)}{\partial_2 \hat{C}_k(u,v)}$ is a decreasing (increasing) function of v: this is in line with the results presented in Spreeuw (2006) where conditional hazard rates in copula-based models for joint residual lifetimes distributions are analized. As a consequence, the performance of short and long-term dependence is induced by the survival copula function associated to the pair (Y_k, X_k): this was expected, since the lifetime T_k influences the other lifetimes through Y_k.

4 Archimedean dependence and proportional cumulative hazard rates of shocks arrival times

In this section we will focus on a particular specification of the model presented and studied in previous sections and we will provide the corresponding expressions for the dependence quantities there analyzed.

In line with the classical Marshall-Olkin model, we assume that all the independent random variables involved in defining the observed lifetimes belong to the same parametric family of distributions: we suppose that their cumulative hazard rates differ for a multiplicative parameter. Moreover, we restrict the analysis to the case in which the bivariate survival copulas $\hat{C}_j$ are of Archimedean type. More precisely, we assume that

- $\bar{F}_{Y_j}(x) = G^{\gamma_j}(x)$, $j = 0, \ldots, d$ and $\bar{F}_{Z_j}(x) = G^{\eta_j}(x)$,$j = 1, \ldots, d$, where G is the survival distribution function of a strictly positive continuous random variable with support $(0, +\infty)$, $\gamma_j \geq 0$ (with at least one j for which $\gamma_j > 0$) and $\eta_j > 0$;
- $\hat{C}_j$ is Archimedean with strict generator ϕ_j.

Since $\bar{F}_{Z_j}(x) \leq \bar{F}_{Y_j}(x)$ we have that $\lambda_j = \eta_j - \gamma_j \geq 0$, for $j = 1, \ldots, d$. If $\lambda_0 = \sum_{j=0}^{d} \gamma_j$ we have that

$$\bar{F}_{X_0}(x) = G^{\lambda_0}(x) \text{ and } \bar{F}_{T_j}(x) = G^{\lambda_0+\lambda_j}(x), x \geq 0. \tag{15}$$

Moreover, $\bar{F}_{X_j}(x) = \phi_j\left(\phi_j^{-1}\left(G^{\eta_j}(x)\right) - \phi_j^{-1}\left(G^{\gamma_j}(x)\right)\right)$, from which (see (8))

$$\bar{F}_{\mathbf{T}}(t_1, \ldots, t_d) = G^{\gamma_0}\left(\max_{i=1,\ldots,d} t_i\right) \prod_{j=1}^{d} \phi_j\left(\phi_j^{-1}\left(G^{\gamma_j}\left(\max_{i=1,\ldots,d} t_i\right)\right) + \phi_j^{-1}\left(G^{\eta_j}(t_j)\right) - \phi_j^{-1}\left(G^{\gamma_j}(t_j)\right)\right). \tag{16}$$

Applying Sklar's theorem, we obtain the associated survival copula function

$$\hat{C}_{\mathbf{T}}(u_1, \ldots, u_d) = \min_{i=1,\ldots d} u_i^{\frac{\gamma_0}{\lambda_0+\lambda_i}} \prod_{j=1}^{d} \phi_j\left[\phi_j^{-1}\left(\min_{i=1,\ldots d} u_i^{\frac{\gamma_j}{\lambda_0+\lambda_i}}\right) + \phi_j^{-1}\left(u_j^{\frac{\eta_j}{\lambda_0+\lambda_j}}\right) - \phi_j^{-1}\left(u_j^{\frac{\gamma_j}{\lambda_0+\lambda_j}}\right)\right].$$

Let us set, for $j = 1, \ldots, d$,

$$\alpha_j = \frac{\lambda_0}{\lambda_0 + \lambda_j},$$

which represents the ratio between the systemic shock intensity and the marginal one, and, for $j = 0, 1, \ldots, d$,

$$\theta_j = \frac{\gamma_j}{\lambda_0},$$

which represents, for $j = 1, \ldots, d$, the contribution percentage of each bank to the systemic shock intensity, while for $j = 0$ is the percentage of contribution of some completely independent exogenous shock.

Then, we can rewrite the survival copula as

$$\hat{C}_{\mathbf{T}}(u_1, \ldots, u_d) = \min_{i=1,\ldots d} u_i^{\alpha_i \theta_0} \prod_{j=1}^{d} \phi_j \left[\phi_j^{-1} \left(\min_{i=1,\ldots d} u_i^{\alpha_i \theta_j} \right) + \phi_j^{-1} \left(u_j^{1-\alpha_j(1-\theta_j)} \right) - \phi_j^{-1} \left(u_j^{\alpha_j \theta_j} \right) \right]. \tag{17}$$

In particular, setting $H(t) = \ln(G(t))$,

- if ϕ_j is for all $j = 1, \ldots, d$ the Gumbel generator with parameter $\beta_j \geq 1$, then $\bar{F}_{X_j}(x) = \exp\left\{ \left(\eta_j^{\beta_j} - \gamma_j^{\beta_j} \right)^{1/\beta_j} H(x) \right\}$, from which we get

$$\bar{F}_{\mathbf{T}}(t_1, \ldots, t_d) = \exp\left\{ -\gamma_0 \left(\max_{i=1,\ldots,d} t_i \right) - \sum_{j=1}^{d} \left[a_j H^{\beta_j} \left(\max_{i=1,\ldots,d} t_i \right) + b_j H^{\beta_j}(t_j) \right]^{1/\beta_j} \right\}, \tag{18}$$

  where $a_j = \gamma_j^{\beta_j}$ and $b_j = \eta_j^{\beta_j} - \gamma_j^{\beta_j} \geq 0$, and

$$\hat{C}_{\mathbf{T}}(u_1, \ldots, u_d) = \min_{i=1,\ldots d} u_i^{\alpha_i \theta_0} \exp\left\{ - \sum_{j=1}^{d} \left[\theta_j^{\beta_j} \max_{i=1,\ldots,d} \{-\alpha_i \ln u_i\}^{\beta_j} + \sigma_j (-\ln u_j)^{\beta_j} \right]^{\frac{1}{\beta_j}} \right\} \tag{19}$$

  where $\sigma_j = (1 - \alpha_j(1-\theta_j))^{\beta_j} - \alpha_j^{\beta_j} \theta_j^{\beta_j}$;

- if ϕ_j is for all $j = 1, \ldots, d$ the Clayton generator with parameter $\beta_j > 0$, then $\bar{F}_{X_j}(x) = \left(1 + e^{\eta_j \beta_j H(x)} - e^{\gamma_j \beta_j H(x)}\right)^{-\frac{1}{\beta_j}}$ from which

$$\bar{F}_{\mathbf{T}}(t_1, \ldots, t_d) = e^{-\gamma_0 H\left(\max_{i=1,\ldots,d}(t_i)\right)} \prod_{j=1}^{d} H \left(e^{a_j \left(\max_{i=1,\ldots,d}(t_i)\right)} + e^{a_j H(t_j)} \left(e^{b_j H(t_j)} - 1 \right) \right)^{-1/\beta_j}, \tag{20}$$

  where $a_j = \gamma_j \beta_j$ and $b_j = \lambda_j \beta_j$, and

$$\hat{C}_{\mathbf{T}}(u_1, \ldots, u_d) = \min_{i=1,\ldots d} u_i^{\alpha_i \theta_0} \prod_{j=1}^{d} \left[\left(\max_{i=1,\ldots,d} u_i^{-\alpha_i} \right)^{\theta_j \beta_j} + u_j^{-(1-\alpha_j(1-\theta_j))\beta_j} - u_j^{-\beta_j \alpha_j \theta_j} \right]^{-\frac{1}{\beta_j}}. \tag{21}$$

From (17) the survival copula associated to (T_i, T_k) is

$$\hat{C}_{T_i,T_k}(u_i, u_k) =$$
$$= (\min(u_i^{\alpha_i}, u_k^{\alpha_k}))^{1-\theta_i-\theta_k} \prod_{j=i,k} \phi_j \left(\phi_j^{-1} \left((\min(u_i^{\alpha_i}, u_k^{\alpha_k}))^{\theta_j} \right) + \phi_j^{-1} \left(u_j^{1-\alpha_j(1-\theta_j)} \right) - \phi_j^{-1} \left(u_j^{\alpha_j \theta_j} \right) \right)$$

from which, setting $\alpha_i = 1$, we recover the survival copula associated to (X_0, T_k)

$$\hat{C}_{X_0,T_k}(u_i, u_k) = \frac{\min(u_i, u_k^{\alpha_k})}{(\min(u_i, u_k^{\alpha_k}))^{\theta_k}} \phi_k \left(\phi_k^{-1} \left((\min(u_i, u_k^{\alpha_k}))^{\theta_k} \right) + \phi_k^{-1} \left(u_k^{1-\alpha_k(1-\theta_k)} \right) - \phi_k^{-1} \left(u_k^{\alpha_k \theta_k} \right) \right).$$

Notice that, from (1), we get

$$\hat{C}_{\mathbf{S}}(u_0, u_1, \ldots, u_d) = u_0^{\theta_0} \prod_{j=1}^{d} \phi_j \left(\phi_j^{-1} \left(u_0^{\theta_j} \right) + \phi_j^{-1} (u_j) \right)$$

and, in particular,

$$\hat{C}_{X_0,X_j}(u_0, u_j) = u_0^{1-\theta_j} \phi_j \left(\phi_j^{-1} \left(u_0^{\theta_j} \right) + \phi_j^{-1} (u_j) \right).$$

The expressions of the corresponding Kendall's distribution function and Kendall's tau are provided in Example 2.1.

Remark 4.1. *If $\hat{C}_j(u, v) = uv$, for all $j = 1, \ldots, d$, we get*

$$\bar{F}_{\mathbf{T}}(t_1, \ldots, t_d) = G^{\lambda_0} \left(\max_{i=1,\ldots d} t_i \right) \prod_{j=1}^{d} G^{\lambda_j}(t_j),$$

which is a particular specification of the generalized Marshall-Olkin distribution (see Li and Pellerey, 2011 and Lin and Li, 2014) with only one independent shock arrival time X_0 and the associated survival copula is of Marshall-Olkin type.

4.1 The probability of simultaneous default

Under the considered specific model assumptions, (11) and (12) are given by

$$H_0(t) = \frac{\gamma_0}{\hat{\lambda}} G^{\hat{\lambda}}(t)$$

and

$$H_j(t) = \gamma_j \int_0^{G(t)} y^{\hat{\lambda}-\lambda_j-1} \frac{h_j(y^{\eta_j})}{h_j(y^{\gamma_j})} dy,$$

respectively, where $\hat{\lambda} = \sum_{i=0}^{d} \lambda_j$ and $h_j = \phi_j' \circ \phi_j^{-1}$. It follows that

$$\mathbb{P}\left(T_1 = T_2 = \cdots = T_d\right) = \frac{\gamma_0}{\hat{\lambda}} + \sum_{j=1}^{d} \gamma_j \int_0^1 y^{\hat{\lambda}-\lambda_j-1} \frac{h_j\left(y^{\eta_j}\right)}{h_j\left(y^{\gamma_j}\right)} dy. \tag{22}$$

Example 4.1. Clayton and Gumbel cases.

- **In** *Clayton case (that is* $\phi_j(x) = (1+x)^{-\frac{1}{\beta_j}}$ *with* $\beta_j > 0$ *and* $h_j(x) = -\frac{1}{\beta_j} x^{1+\beta_j}$*, for* $j = 1, \ldots, d$*), we have that the survival distribution of* S *given in (9) is*

$$\bar{F}_S(t) = \frac{\gamma_0}{\hat{\lambda}} G^{\hat{\lambda}}(t) + \sum_{j=1}^{d} \frac{\gamma_j}{\hat{\lambda} + \lambda_j \beta_j} G^{\hat{\lambda}+\lambda_j\beta_j}(t) \tag{23}$$

and

$$\begin{aligned}\mathbb{P}\left(T_1 = T_2 = \cdots = T_d\right) &= \frac{\gamma_0}{\hat{\lambda}} + \sum_{j=1}^{d} \frac{\gamma_j}{\hat{\lambda} + \lambda_j \beta_j} = \\ &= \frac{\theta_0}{\sum_{i=1}^{d} \alpha_i^{-1} - (d-1)} + \sum_{j=1}^{d} \frac{\theta_j}{\sum_{i=1}^{d} \alpha_i^{-1} - d + \beta_j(\alpha_j^{-1} - 1)}.\end{aligned}$$

- **In** *Gumbel case (that is* $\phi_j(x) = e^{-x^{\frac{1}{\beta_j}}}$ *with* $\beta_j \geq 1$ *and* $h_j(x) = -\frac{1}{\beta_j} x(-\ln x)^{1-\beta_j}$*, for* $j = 1, \ldots, d$*), we have*

$$\bar{F}_S(t) = \left(\frac{\gamma_0}{\hat{\lambda}} + \frac{1}{\hat{\lambda}} \sum_{j=1}^{d} \gamma_j \left(1 + \frac{\lambda_j}{\gamma_j}\right)^{1-\beta_j}\right) G^{\hat{\lambda}}(t) \tag{24}$$

and

$$\begin{aligned}\mathbb{P}\left(T_1 = T_2 = \cdots = T_d\right) &= \frac{\gamma_0}{\hat{\lambda}} + \frac{1}{\hat{\lambda}} \sum_{j=1}^{d} \gamma_j \left(1 + \frac{\lambda_j}{\gamma_j}\right)^{1-\beta_j} = \\ &= \frac{\theta_0}{\sum_{i=1}^{d} \alpha_i^{-1} - (d-1)} + \sum_{j=1}^{d} \frac{\theta_j}{\sum_{i=1}^{d} \alpha_i^{-1} - (d-1)} \left(1 + \frac{1-\alpha_j}{\alpha_j \theta_j}\right)^{1-\beta_j}.\end{aligned}$$

4.2 Kendall's distribution function and Kendall's tau

By Proposition 3.2, since $z_t = G^{-1}\left(t^{1/(\lambda_0+\lambda_i+\lambda_k)}\right)$, after some algebra we recover

$$K_{i,k}(t) = t - t\ln t\left(\frac{\alpha_i(1-\alpha_k)(1-\alpha_i\theta_k)}{\alpha_i+\alpha_k-\alpha_i\alpha_k} + \frac{\alpha_k(1-\alpha_i)(1-\alpha_k\theta_i)}{\alpha_i+\alpha_k-\alpha_i\alpha_k}\right) -$$

$$-\theta_k\int_{t^{\frac{\alpha_i\alpha_k}{\alpha_i+\alpha_k-\alpha_i\alpha_k}}}^{t^{\alpha_i}} y^{\frac{1-\alpha_i}{\alpha_i}}\frac{h_k\left(ty^{-\frac{1-\theta_k\alpha_i}{\alpha_i}}\right)}{h_k(y^{\theta_k})}dy - \theta_i\int_{t^{\frac{\alpha_i\alpha_k}{\alpha_i+\alpha_k-\alpha_i\alpha_k}}}^{t^{\alpha_k}} y^{\frac{1-\alpha_k}{\alpha_k}}\frac{h_i\left(ty^{-\frac{1-\theta_i\alpha_k}{\alpha_k}}\right)}{h_k(y^{\theta_i})}dy,$$

where $h_j = \phi_j' \circ \phi_j^{-1}$, for $j = i, k$.

Example 4.2. *Let us consider Clayton and Gumbel survival copulas specific cases.*

1. *Clayton case ($\phi_j(x) = (1+x)^{\frac{1}{\beta_j}}$, $\beta_j > 0$, $j = i, k$).*
   If we set
   $$\tau_{ik}^{MO} = \frac{\alpha_k\alpha_i}{\alpha_k+\alpha_i-\alpha_k\alpha_i}$$
   which is the Kendall's tau of the Marshall-Olkin bivariate copula with parameters α_i and α_k and
   $$\rho_{rs} = \frac{1-\alpha_s}{\alpha_s}\tau_{rs}^{MO}, r, s = i, j$$
   we get
   $$K_{i,k}(t) = t\left(1+\frac{\theta_k}{\beta_k}\alpha_i + \frac{\theta_i}{\beta_i}\alpha_k\right) - t\ln t\left((1-\theta_k\alpha_i)\rho_{ik} + (1-\theta_i\alpha_k)\rho_{ki}\right) +$$
   $$-\frac{\theta_k}{\beta_k}\alpha_i t^{\rho_{ik}\beta_k+1} - \frac{\theta_i}{\beta_i}\alpha_k t^{\rho_{ki}\beta_i+1}.$$
   The above Kendall's distribution function can be decomposed in
   $$K_{i,k}(t) = K_{ik}^0(t) + K_{0,k}^{(i)} + K_{0,i}^{(k)} - 2K^I(t),$$
   where
   $$K_{ik}^0(t) = t - \left(1-\tau_{ik}^{MO}\right)t\ln t, \tag{25}$$
   $$K_{0,k}^{(i)} = t\left(1+\frac{\theta_k\alpha_i}{\beta_k}\right) - (1-\theta_k\alpha_i\rho_{ik})t\ln t - \frac{\theta_k\alpha_i}{\beta_k}t^{1+\beta_k\rho_{ik}}, \tag{26}$$
   $$K_{0,i}^{(k)} = t\left(1+\frac{\theta_i\alpha_k}{\beta_i}\right) - (1-\theta_i\alpha_k\rho_{ki})t\ln t - \frac{\theta_i\alpha_k}{\beta_i}t^{1+\beta_i\rho_{ki}} \tag{27}$$
   and
   $$K^I(t) = t - t\ln t. \tag{28}$$
   Notice that: (25) is the Kendall's distribution function of the Marshall-Olkin copula with parameters α_i and α_k; (26) is a Kendall's distribution function of type (5) with

parameters $\theta = \theta_k \alpha_i \rho_{ik}$ *and* $\beta = \beta_k \rho_{ik}$ *(that represents the effect of the dependence between* Y_k *and* X_k *on the resulting dependence structure of* (T_i, T_k)*); simmetrically, (27) is a Kendall's distribution function of type (5) with parameters* $\theta = \theta_i \alpha_k \rho_{ki}$ *and* $\beta = \beta_i \rho_{ki}$*; (28) is the Kendall's distribution function of the independence copula. As a consequence, we get a very meaningful decomposition of the Kendall's tau:*

$$\tau_{ik} = \tau_{ik}^{MO} + \bar{\tau}_{0,k}^{(i)} + \bar{\tau}_{0,i}^{(k)},$$

where

$$\bar{\tau}_{0,k}^{(i)} = \alpha_i \rho_{ik} \theta_k \frac{\rho_{i,k}\beta_k}{\rho_{i,k}\beta_k + 2} \text{ and } \bar{\tau}_{0,i}^{(k)} = \alpha_k \rho_{ki} \theta_i \frac{\rho_{k,i}\beta_i}{\rho_{k,i}\beta_i + 2}$$

are Kendall's tau of type (6) with suitably modified parameters.

It follows that

$$\begin{aligned} \tau_{T_k,X_0} &= \alpha_k + (1-\alpha_k)\theta_k \frac{(1-\alpha_k)\beta_k}{(1-\alpha_k)\beta_k + 2} = \\ &= \tau_{T_k,X_0}^{MO} + \tau_{0,k}^{*} \end{aligned} \tag{29}$$

where τ_{T_k,X_0}^{MO} *is the Kendall's tau between the observed lifetime and the systemic shock arrival time in the Marshall-Olkin model and* $\tau_{0,k}^{*}$ *is a Kendall's tau of type (6) with parameters rescaled by the coefficient* $1-\alpha_k$*.*

2. *Gumbel case* ($\phi_j(x) = e^{-x^{\frac{1}{\beta_j}}}$, $\beta_j \geq 1$, $j = i,k$*).*

If

$$A_{ik} = \tau_{ik}^{MO} \left(\frac{1-\alpha_k}{\alpha_k}(1-\alpha_i\theta_k) + \frac{1-\alpha_i}{\alpha_i}(1-\alpha_k\theta_i) \right) +$$
$$+ (\theta_k\alpha_i)^{\beta_k - 1} \int_{\theta_k \tau_{ik}^{MO}}^{\alpha_i\theta_k} \left(\frac{z}{\theta_k\alpha_i - z(1-\theta_k\alpha_i)} \right)^{\beta_k - 1} dz + (\theta_i\alpha_k)^{\beta_i - 1} \int_{\theta_i \tau_{ik}^{MO}}^{\alpha_k\theta_i} \left(\frac{z}{\theta_i\alpha_k - z(1-\theta_i\alpha_k)} \right)^{\beta_i - 1} dz,$$

then

$$K_{i,k}(t) = t - A_{ik} t \ln t$$

and

$$\tau_{i,k} = 1 - A_{ik}.$$

Even if, as in the Clayton case, we can recognize that the resulting dependence is the sum of the Marshall-Olkin one and two different contributions arising from the assumed dependence between Y_i *and* X_i *and between* Y_k *and* X_k*, unlike that case, the latter ones cannot be written in a closed form* **in terms of** *of the corresponding Kendall's tau given in Example 2.1.*

Moreover, we have

$$\tau_{T_k,X_0} = 1 - (1-\alpha_k)(1-\theta_k) - \theta_k^{\beta_k - 1} \int_{\theta_k\alpha_k}^{\theta_k} \left(\frac{z}{\theta_k - z(1-\theta_k)} \right)^{\beta_k - 1} dz.$$

4.3 Hazard rates

Let $M(t)$ and $\mu(t)$ be the cumulative hazard function and the hazard rate of $G(t)$. Formulas for conditional hazard rates are now given by

$$\lambda_i(t) = \mu(t) \left\{ \gamma_0 + \eta_i + \sum_{\substack{j=1 \\ j \neq i}}^{d} \frac{\gamma_j}{G^{\lambda_j}(t)} \cdot \frac{h_j\left(G^{\eta_j}(t)\right)}{h_j\left(G^{\gamma_j}(t)\right)} \right\}$$

and

$$\lambda_{i|k}\left(t|t_k\right) = \mu(t) \left\{ \gamma_0 + \eta_i + \gamma_k \frac{\phi_k''\left(D_k(t,t_k)\right)}{\phi_k'\left(D_k(t,t_k)\right)} \cdot \frac{G^{\gamma_k}(t)}{h_k\left(G^{\gamma_k}(t)\right)} + \sum_{\substack{j=1 \\ j \neq k,i}} \frac{\gamma_j}{G^{\lambda_j}(t)} \cdot \frac{h_j\left(G^{\eta_j}(t)\right)}{h_j\left(G^{\gamma_j}(t)\right)} \right\},$$

where, for $h = 1, \ldots, d$,

$$D_h(x,y) = \phi_h^{-1}\left(G^{\gamma_h}(x)\right) + \phi_h^{-1}\left(G^{\eta_h}(y)\right) - \phi_h^{-1}\left(G^{\gamma_h}(y)\right).$$

Moreover,

$$\lambda_{i|k}\left(t|t_k\right) = \lambda_i(t) + \gamma_k \mu(t) \left\{ \frac{\phi_k''\left(D_k(t,t_k)\right)}{\phi_k'\left(D_k(t,t_k)\right)} \cdot \frac{G^{\gamma_k}(t)}{h_k\left(G^{\gamma_k}(t)\right)} - \frac{1}{G^{\lambda_k}(t)} \cdot \frac{h_k\left(G^{\eta_k}(t)\right)}{h_k\left(G^{\gamma_k}(t)\right)} \right\}.$$

Example 4.3. • *Clayton case:*

$$\lambda_i(t) = \mu(t) \left\{ \gamma_0 + \eta_i + \sum_{\substack{j=1 \\ j \neq i}}^{d} \gamma_j G^{\beta_j \lambda_j}(t) \right\}$$

and

$$\lambda_{i|k}\left(t|t_k\right) = \lambda_i(t) + \gamma_k \mu(t) \left[\left(1+\beta_k\right) G^{-\gamma_k\beta_k}(t) \left(G^{-\gamma_k\beta_k}(t) + G^{-\beta_k\eta_k}(t_k) - G^{-\gamma_k\beta_k}(t_k) \right)^{-1} - G^{\lambda_k\beta_k}(t) \right].$$

• *Gumbel case:*

$$\lambda_i(t) = \mu(t) \left\{ \gamma_0 + \eta_i + \sum_{\substack{j=1 \\ j \neq i}}^{d} \gamma_j^{\beta_j} \eta_j^{1-\beta_j} \right\}$$

and

$$\lambda_{i|k}\left(t|t_k\right) = \lambda_i(t) + \gamma_k^{\beta_k} \mu(t) \left[\frac{\left(\gamma_k^{\beta_k} M^{\beta_k}(t) + \left(\eta_k^{\beta_k} - \gamma_k^{\beta_k} \right) M^{\beta_k}(t_k) \right)^{\frac{1}{\beta_k}} + \beta_k - 1}{\gamma_k^{\beta_k} M^{\beta_k}(t) + \left(\eta_k^{\beta_k} - \gamma_k^{\beta_k} \right) M^{\beta_k}(t_k)} M^{\beta_k - 1}(t) - \eta_k^{1-\beta_k} \right].$$

4.4 Probability density function

The fact that the probability of simultaneous default is positive, implies that the distributions considered in this paper are not absolutely continuous with respect to the Lebesgue measure. Nevertheless, they admit a density with respect to a measure dominating the Lebesgue one: this approach is formally described in Asimit et al. (2016) for a family of non absolutely continuous multivariate Pareto distributions. We will illustrate it in the simplified case $d = 2$, where the reference dominating measure on $\mathbb{R}^2$ is given by the sum of the Lebesgue measure on $\mathbb{R}^2$ and that of the Lebesgue measure on $\mathbb{R}$ concentrated on the straight line $\{(t_1, t_2) \in \mathbb{R}^2 : t_1 = t_2\}$. In particular, we will compute the densities for the survival distributions given in (18) and (20), that, for $d = 2$, can be, respectively, rewritten as

$$\begin{aligned}
&\bar{F}_{\mathbf{T}}(t_1, t_2) = \\
&= \exp\left\{-\gamma_0\left(\max\left(t_1, t_2\right)\right) - \sum_{j=1}^{2}\left[a_j H^{\beta_j}\left(\max\left(t_1, t_2\right)\right) + b_j H^{\beta_j}\left(t_j\right)\right]^{1/\beta_j}\right\} = \\
&= \begin{cases} \exp\left\{-(\gamma_0 + (a_1 + b_1)^{1/\beta_1})H(t_1) - \left[a_2 H^{\beta_2}\left(t_1\right) + b_2 H^{\beta_2}\left(t_2\right)\right]^{1/\beta_2}\right\}, & \text{if } t_1 \geq t_2 \\ \exp\left\{-(\gamma_0 + (a_2 + b_2)^{1/\beta_2})H(t_2) - \left[a_1 H^{\beta_1}\left(t_2\right) + b_1 H^{\beta_1}\left(t_1\right)\right]^{1/\beta_1}\right\}, & \text{if } t_1 < t_2 \end{cases},
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
&\bar{F}_{\mathbf{T}}(t_1, t_2) = \\
&= e^{-\gamma_0 H(\max(t_1, t_2))} \prod_{j=1}^{2}\left(e^{a_j H(\max(t_1, t_2))} + e^{a_j H(t_j)}\left(e^{b_j H(t_j)} - 1\right)\right)^{-1/\beta_j} = \\
&= \begin{cases} e^{-\left(\gamma_0 + \frac{a_1 + b_1}{\beta_1}\right)H(t_1)}\left(e^{a_2 H(t_1)} + e^{a_2 H(t_2)}\left(e^{b_2 H(t_2)} - 1\right)\right)^{-1/\beta_2}, & \text{if } t_1 \geq t_2 \\ e^{-\left(\gamma_0 + \frac{a_2 + b_2}{\beta_2}\right)H(t_2)}\left(e^{a_1 H(t_2)} + e^{a_1 H(t_1)}\left(e^{b_1 H(t_1)} - 1\right)\right)^{-1/\beta_1}, & \text{if } t_1 < t_2 \end{cases}.
\end{aligned} \tag{31}$$

Using (24), and after some straightforward algebra, it can be proved that the density associated to (30) is

$$f(t_1, t_2) = \begin{cases} \bar{F}(t_1, t_2)H'(t_1)H'(t_2)g(t_1, t_2), & t_1 \neq t_2 \\ \left(\gamma_0 + \sum_{j=1}^{2} a_j\left(a_j + b_j\right)^{\frac{1}{\beta_j} - 1}\right) e^{-\left(\gamma_0 + \sum_{j=1}^{2}(a_j + b_j)^{\frac{1}{\beta_j}}\right)H(t)} H'(t), & t = t_1 = t_2 \end{cases}$$

where

$$\begin{aligned}
&g\left(t_1, t_2\right) = \\
&= \begin{cases} b_2 A_2^{\frac{1}{\beta_2} - 2}(t_1, t_2)H^{\beta_2 - 1}(t_2)\left[c_1 A_2(t_1, t_2) + a_2\left(A_2^{\frac{1}{\beta_2}}(t_1, t_2) + \beta_2 - 1\right)H^{\beta_2 - 1}(t_1)\right], & t_1 > t_2 \\ b_1 A_1^{\frac{1}{\beta_1} - 2}(t_1, t_2)H^{\beta_1 - 1}(t_1)\left[c_2 A_1(t_1, t_2) + a_1\left(A_1^{\frac{1}{\beta_1}}(t_1, t_2) + \beta_1 - 1\right)H^{\beta_1 - 1}(t_2)\right], & t_1 < t_2 \end{cases}
\end{aligned}$$

with, for $i = 1, 2$, $c_i = \gamma_0 + (a_i + b_i)^{\frac{1}{\beta_i}}$ and $A_i(t_1, t_2) = a_i H^{\beta_i}(t_{3-i}) + b_i H^{\beta_i}(t_i)$.

Similarly, using (23), we can calculate the density associated to (31), which is

$$f(t_1, t_2) =$$

$$= \begin{cases} e^{-c_1 H(t_1)} H'(t_1) C_2(t_2) B_2(t_1, t_2) \left[\left(c_1 + a_2 \frac{1+\beta_2}{\beta_2} \right) e^{a_2 H(t_1)} + c_1 e^{a_2 H(t_2)} \left(e^{b_2 H(t_2)} - 1 \right) \right], & t_1 > t_2 \\ e^{-c_2 H(t_2)} H'(t_2) C_1(t_1) B_2(t_1, t_2) \left[\left(c_2 + a_1 \frac{1+\beta_1}{\beta_1} \right) e^{a_1 H(t_2)} + c_1 e^{a_1 H(t_1)} \left(e^{b_1 H(t_1)} - 1 \right) \right], & t_1 < t_2 \\ H'(t) \left[\gamma_0 e^{-\hat{\lambda} H(t)} + \sum_{j=1}^{2} \frac{a_j}{\beta_j} e^{-(\hat{\lambda} + b_j) H(t)} \right], & t = t_1 = t_2 \end{cases}$$

with $\hat{\lambda} = \gamma_0 + \sum_{j=1}^{2} \frac{a_j + b_j}{\beta_j}$ and $c_i = \gamma_0 + \frac{a_i + b_i}{\beta_i}$, $C_i(t_i) = \frac{1}{\beta_i} e^{a_i H(t_i)} H'(t_i) \left[(a_i + b_i) e^{b_i H(t_i)} - a_i \right]$, $B_i(t_1, t_2) = \left(e^{a_i H(t_{3-i})} + e^{a_i H(t_i)} \left[e^{b_i H(t_i)} - 1 \right] \right)^{-\frac{\beta_i + 2}{\beta_i}}$, for $i = 1, 2$.

5 Sensitivity analysis and applications

In this section we will analyze the impact of the parameters on marginal distributions and on the dependence structure considering specific examples in the framework of Section 4. Next, we will discuss how to use the model in practical applications in reliability, credit and insurance.

5.1 Sensitivity analysis

5.1.1 Fixed marginal intensities

Let us consider the exchangeable bivariate case with $\gamma_1 = \gamma_2 = 0.1$, $\gamma_0 = 0.01$ and $\lambda_1 = \lambda_2 = 0.1$. Under these assumptions, the marginal survival distributions (see (15)) are fixed to be $\bar{F}_{T_j}(x) = G^{0.31}(x)$ for $j = 1, 2$. Moreover we assume that the survival copulas $\hat{C}_j$ associated to the survival distribution of (Y_j, X_j) are the same for $j = 1, 2$. As a consequence, the vector (T_1, T_2) is exchangeable. In next Tables 1 and 2, we compare the effect of Gumbel and Clayton survival copulas on the dependence quantities introduced in previous sections varying the Kendall's tau value τ of the pairs (Y_j, X_j), $j = 1, 2$. We remark that the case $\tau = 0$ corresponds to the classical Marshall-Olkin model with only one systemic shock and that the considered quantities do not depend on the choice of the survival distribution function G.

In both cases, the singularity mass, $\mathbb{P}(T_1 = T_2)$, as well as the single contribution to it due to the occurrence of each Y_i, with $i = 1, 2$ (that is $H_i(0)$ given by (12)) is decreasing with respect to the dependence induced by the bivariate survival copulas. This was expected since, as the dependence between Y_j and X_j increases, being the distribution of $Z_j = \min(Y_j, X_j)$ and Y_j fixed (depending on $\eta_j = \lambda_j + \gamma_j$ and γ_j, respectively), the survival distribution of X_j decreases, implying an increase in the probability of the event $\{X_j < Y_j\}$. We notice that that this effect is much stronger for the Gumbel survival copula with respect to the Clayton

	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.99$
$\mathbb{P}(T_1 = T_2)$	0.5121951	0.4115612	0.2682927	0.08536585	0.02439024
$H_0(0)$	0.02439024	0.02439024	0.02439024	0.02439024	0.02439024
$H_i(0)$	0.2439024	0.1935855	0.1219512	0.0304878	0
τ_{T_1,T_2}	0.5121951	0.5294137	0.5569266	0.6048447	0.6674419
τ_{T_i,X_0}	0.6774194	0.6936507	0.7196973	0.7655784	0.8287377

Table 1: Gumbel case with Kendall's tau τ. From top to bottom: Singularity mass, contribution to the singularity mass due to the occurrence of Y_0, contribution to the singularity mass due to the occurrence of Y_i, $i = 1, 2$, Kendall's tau between T_1 and T_2, Kendall's tau between each T_i and X_0.

	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.99$
$\mathbb{P}(T_1 = T_2)$	0.5121951	0.4439707	0.3522591	0.22241	0.03428633
$H_0(0)$	0.02439024	0.02439024	0.02439024	0.02439024	0.02439024
$H_i(0)$	0.2439024	0.2097902	0.1639344	0.0990099	0.004948046
τ_{T_1,T_2}	0.5121951	0.5240264	0.5430493	0.5786837	0.6632939
τ_{T_i,X_0}	0.6774194	0.6923329	0.7148852	0.7529652	0.8263652

Table 2: Clayton case with Kendall's tau τ. From top to bottom: Singularity mass, contribution to the singularity mass due to the occurrence of Y_0, contribution to the singularity mass due to the occurrence of Y_i, $i = 1, 2$, Kendall's tau between T_1 and T_2, Kendall's tau between each T_i and X_0.

one. On the other hand, $H_0(0)$, given in (11) is constant since it doesn't depend on the survival copulas $\hat{C}_j$. As for the Kendall's tau parameters associated to (T_1, T_2) and (T_i, X_0), instead, as expected, they do increase with the dependence modeled by $\hat{C}_i$.

The different impact of the choice of the Gumbel and Clayton survival copulas is shown in Figure 1 where scatter plots corresponding to Kendall's tau 0.5 and 0.75 are provided.

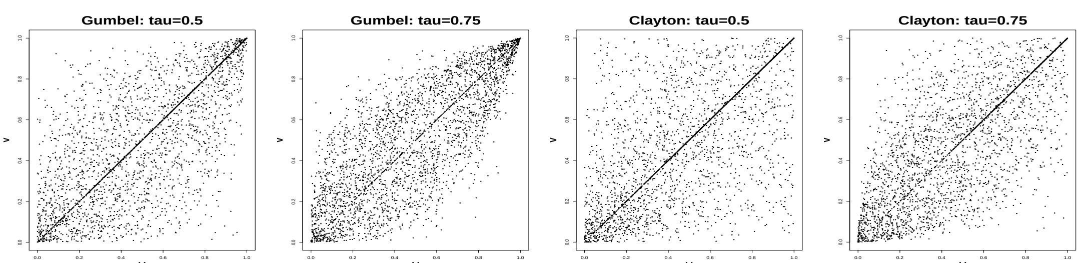

Figure 1: Scatter plots: from left to right, Gumbel and Clayton cases with parameters considered in subsection 5.1.1

5.1.2 Fixed intensities of the hidded shocks and Gumbel survival copulas

Let us now consider again the bivariate exchangeable case but focusing on $\hat{C}_j$ of Gumbel type (with $\beta_1 = \beta_2$). In this case the survival distribution of X_j is given by $\bar{F}_{X_j}(x) = G^{\mu_j}(x)$

with $\mu_j = \left(\eta_j^{\beta_j} - \gamma_j^{\beta_j}\right)^{1/\beta_j}$. We will perform the same analysis done above in the following cases:

1. $\gamma_0 = \gamma_1 = \gamma_2 = 0.01$, and $\mu_1 = \mu_2 = 0.1$: results are shown in Table 3;
2. $\gamma_0 = 0.01$ and $\gamma_1 = \gamma_2 = 0.1$, and $\mu_1 = \mu_2 = 0.1$: results are shown in Table 4;
3. $\gamma_0 = 0.01$ and $\gamma_1 = \gamma_2 = 0.2$, and $\mu_1 = \mu_2 = 0.1$: results are shown in Table 5.

	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.99$
ω_i	0.13	0.1234614	0.1204988	0.1200025	0.12
$\mathbb{P}(T_1 = T_2)$	0.1304348	0.08841155	0.05682567	0.04771314	0.04761905
$H_0(0)$	0.04347826	0.04609936	0.04739392	0.04761791	0.04761905
$H_i(0)$	0.04347826	0.02115609	0.004715872	4.761434e-05	0
τ_{T_1,T_2}	0.1304348	0.1631895	0.1911915	0.2081436	0.2141442
τ_{T_i,X_0}	0.2307692	0.31505	0.3952031	0.460692	0.4988739

Table 3: Gumbel case with Kendall's tau τ and with $\gamma_i = 0.01$, $i = 0,1,2$ and $\mu_i = 0.1$, $i = 1,2$. From top to bottom: Marginal intensity, singularity mass, contribution to the singularity mass due to the occurrence of Y_0, contribution to the singularity mass due to the occurrence of Y_i, $i = 1,2$, Kendall's tau between T_1 and T_2, Kendall's tau between each T_i and X_0.

	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.99$
ω_i	0.31	0.2781793	0.2514214	0.2289207	0.2106956
$\mathbb{P}(T_1 = T_2)$	0.5121951	0.5144359	0.517074	0.5201742	0.5236528
$H_0(0)$	0.02439024	0.02887181	0.03414802	0.04034838	0.04730568
$H_i(0)$	0.2439024	0.242782	0.241463	0.2399129	0.2381736
τ_{T_1,T_2}	0.5121951	0.6180447	0.7346508	0.8620343	0.9942827
τ_{T_i,X_0}	0.6774194	0.7643964	0.8473775	0.9260153	0.9971333

Table 4: Gumbel case with Kendall's tau τ and with $\gamma_0 = 0.01$, and $\gamma_i = \mu_i = 0.1$, $i = 1,2$. From top to bottom: Marginal intensity, singularity mass, contribution to the singularity mass due to the occurrence of Y_0, contribution to the singularity mass due to the occurrence of Y_i, $i = 1,2$, Kendall's tau between T_1 and T_2, Kendall's tau between each T_i and X_0.

In all cases, since μ_j, $j = 1,2$ and γ_j, $j = 0,1,2$ are given, the marginal distributions of T_1 and T_2 are not fixed, but they vary with the parameter of the underlying Gumbel survival copulas: in fact they are given by $\bar{F}_{T_j}(x) = G^{\omega_j}(t)$ with

$$\omega_j = \sum_{i=0}^{2} \gamma_j + \left(\mu_j^{\beta_j} + \gamma_j^{\beta_j}\right)^{\frac{1}{\beta_j}} - \gamma_j.$$

In particular, being $\bar{F}_{X_j}$ and $\bar{F}_{Y_j}$ fixed, since survival copulas $\hat{C}_j$ increase with β_j, we have that

	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.99$
ω_i	0.51	0.4669759	0.4336068	0.4130543	0.41
$\mathbb{P}(T_1 = T_2)$	0.6721311	0.7213187	0.8043743	0.9425901	1
$H_0(0)$	0.01639344	0.01908573	0.02187162	0.02403219	0.02439024
$H_i(0)$	0.3278689	0.3511165	0.3912514	0.459279	0.4878049
τ_{T_1,T_2}	0.6721311	0.7863127	0.8992995	0.9854788	1
τ_{T_i,X_0}	0.8039216	0.8804045	0.946985	0.9926863	1

Table 5: Gumbel case with Kendall's tau τ and with $\gamma_0 = 0.01$ and $\gamma_i = 0.2$ and $\mu_i = 0.1$, $i = 1, 2$. From top to bottom: Marginal intensity, singularity mass, contribution to the singularity mass due to the occurrence of Y_0, contribution to the singularity mass due to the occurrence of Y_i, $i = 1, 2$, Kendall's tau between T_1 and T_2, Kendall's tau between each T_i and X_0.

$\bar{F}_{Z_j}(x)$ increases with β_j and, consequently, $\eta_j = \left(\mu_j^{\beta_j} + \gamma_j^{\beta_j}\right)^{\frac{1}{\beta_j}}$ decreases. As a consequence ω_j decreases while $H_0(0)$ increases with β_j.

From Tables 3, 4 and 5, we notice that when the distributions of the random variables X_j and Y_j, $j = 1, 2$, are fixed, the effect of an increase in the dependence in the vectors (Y_j, X_j) results always in an increment of the overall dependence between T_1 and T_2 and between each T_j, $j = 1, 2$ and X_0, while the effect on the singularity is different depending on the relative magnitude of the intensities of X_j and Y_j: this facts are also shown in Figure 2 where scatter plots from the survival copulas in the three cases are shown in the specific case $\tau = 0.5$.

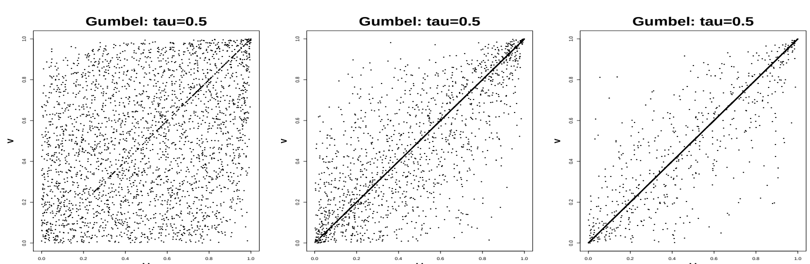

Figure 2: Scatter plots: the three cases considered in subsection 5.1.2 with underlying Gumbel survival copulas with Kendall's tau equal to 0.5. From left to right: $\gamma_1 = \gamma_2 = 0.01$, $\gamma_1 = \gamma_2 = 0.1$, $\gamma_1 = \gamma_2 = 0.2$

5.1.3 Non-exchangeable case

So far, we have considered the exchangeable case, but the observed effects on the singularity can combine in different ways in the non-exchangeable case. In Table 6 we consider again the Gumbel case with $\beta_1 = \beta_2$ but with $\gamma_0 = 0.01$, $\gamma_1 = 0.01$, $\gamma_2 = 0.2$, and $\mu_1 = \mu_2 = 0.1$. While, as expected, all Kendall's taus increase with the dependence induced by the underlying Gumbel survival copulas, single contributions to the singularity behave in opposite way with the increase of the underlying Gumbel dependence due to the different magnitudes of the single common shock intensities (γ_j) with respect to the idiosyncratic ones (μ_j). In Figure 3 scatter plots of the resulting survival copulas corresponding to different Kendall's tau are shown.

	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 0.99$
ω_1	0.32	0.3134614	0.3104988	0.3100025	0.31
ω_2	0.32	0.2769759	0.2436068	0.2230543	0.22
$\mathbb{P}(T_1 = T_2)$	0.5238095	0.536008	0.5683248	0.6424391	0.6774194
$H_0(0)$	0.02380952	0.02699512	0.02993066	0.03194308	0.03225806
$H_1(0)$	0.02380952	0.01238871	0.002978212	3.194069e-05	0
$H_2(0)$	0.47619048	0.49662418	0.535415937	0.6104641	0.6451613
τ_{T_1,T_2}	0.5238095	0.6012654	0.6688531	0.7148461	0.7228528
τ_{T_1,X_0}	0.6875	0.7068935	0.7179654	0.7219115	0.7228528
τ_{T_2,X_0}	0.6875	0.8016304	0.9077599	0.986585	1

Table 6: Gumbel casewith Kendall's tau τ and with $\gamma_0 = \gamma_1 = 0.01$, $\gamma_2 = 0.2$ and $\mu_i = 0.1$, $i = 1, 2$.. From top to bottom: Marginal intensities, singularity mass, contribution to the singularity mass due to the occurrence of Y_0, contribution to the singularity mass due to the occurrence of Y_1 and Y_2, Kendall's tau between T_1 and T_2, Kendall's tau between T_1 and X_0 and between T_2 and X_0.

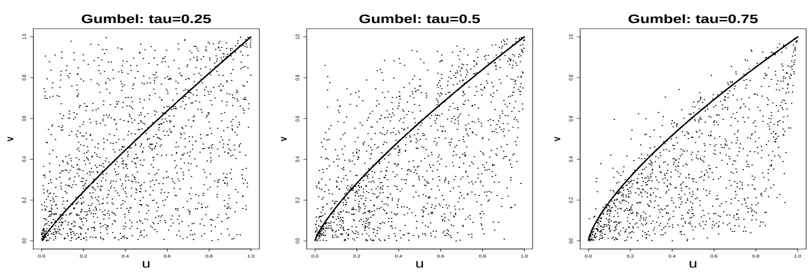

Figure 3: Scatter plots: the Gumbel asymmetric case of subsection 5.1.3

5.2 Reliability, Credit and Actuarial applications

The model introduced and studied in this paper can be used to model joint residual lifetimes in different frameworks.

In reliability theory, one can consider a system with d electronic or mechanical elements whose residual lifetimes are modeled by the random variables $T_1, T_2, \ldots, T_d$, that is assuming that the residual lifetime of each element i in the system is governed by an idiosyncratic component X_i and by some systematic component modeled by X_0 which is dependent on X_i. If the elements are serially connected, then the failure time of the whole system is given by $M_d = \min(T_1, \ldots, T_d)$. The model is also suitable to describe the situation in which a component c_0, whose residual lifetime is denoted by X_0, is serially connected with a system of d devices with residual lifetimes $X_1, \dots, X_d$ connected in a parallel way: the failure of the component c_0 will cause the failure of the whole system.

As for credit applications, consider the case of a group of d firms or banks sharing a same systemic risk they are correlated to. We assume that the time to default of each name is given by an idiosyncratic component, represented by X_i and a systemic one, represented by X_0. Given the features of the model studied in this paper, it results to be particularly suited to the situation in which contagion among the names takes place only through a systemic shock causing the collapse of the whole system, which can be represented by the case of Too-Big-To-Fail institutions.

Insurance products aiming at protecting against the cost due to the failure of some device in a system and credit products against the loss due to the default of an institution, are largely available in the market. Just to show how the model presented in this paper can be applied for such risk protection purposes, we will analyze two examples: the First-to-default and the simultaneous system collapse cases.

5.3 First-to-default (failure) product

Let us consider the case of an insurance product paying a lump sum C at the time of occurrence, by a term T, of the first default (or failure), that is at time $M_d = \min(T_1, \dots, T_d)$. In case of a system composed by d devices serially connected, the risk event covered by the insurance product is represented by the loss due to the failure of the system. In credit risk theory, this is the so called first-to-default swap, that is commonly used in the market practice and that provides a conventional payment for the loss given to the occurrence of the first credit event among the firms or financial institutions in a basket.

Assuming the model studied in this paper, the first failure can be that of only one element in the system or that of the whole system. Notice that $M_d = \min(Y_0, Z_1, \dots, Z_d)$ with $Z_j = \min(Y_j, X_j)$ and $Y_0, Z_1, \dots, Z_d$ independent. Assuming a constant interest rate $r > 0$ and a continuously compounding discounting rule, the present value at time 0 of the amount C paid at time M_d is $C \cdot e^{-rM_d}$. The corresponding expected present value is

$$\begin{aligned}\mathbb{E}\left[C\, e^{-rM_d}\mathbf{1}_{\{M_d \le T\}}\right] &= C\left(1 - e^{-rT}\bar{F}_{M_d}(T) - r\int_0^T e^{-rx}\bar{F}_{M_d}(x)dx\right) = \\ &= C\left(1 - e^{-rT}\bar{F}_{Y_0}(T)\prod_{j=1}^d \bar{F}_{Z_j}(T) - r\int_0^T e^{-rx}\bar{F}_{Y_0}(x)\prod_{j=1}^d \bar{F}_{Z_j}(x)dx\right).\end{aligned}$$

In common practice, such contracts provide for a unique premium P paid at the contract issue time or for a stream of regular premium payments p along the life of the contract conditional on the non occurrence of the insured event. In case of a unique premium, then $P = C\mathbb{E}\left[e^{-rM_d}\mathbf{1}_{\{M_d \le T\}}\right]$. In case of a regular stream of constant premiums p paid at times

$0 = t_0 < t_1 < \ldots < t_n < T$, we have that, according to the equivalence principle,

$$\mathbb{E}\left[\sum_{i=0}^{n} p e^{-rt_i} \mathbf{1}_{\{M_d > t_i\}}\right] = C\mathbb{E}\left[e^{-rM_d}\mathbf{1}_{\{M_d \le T\}}\right],$$

that is

$$p = \frac{C\mathbb{E}\left[e^{-rM_d}\mathbf{1}_{\{M_d \le T\}}\right]}{\sum_{i=0}^{n} e^{-rt_i}\bar{F}_{M_d}(t_i)} = \frac{C\left(1 - e^{-rT}\bar{F}_{Y_0}(T)\prod_{j=1}^{d}\bar{F}_{Z_j}(T) - r\int_0^T e^{-rx}\bar{F}_{Y_0}(x)\prod_{j=1}^{d}\bar{F}_{Z_j}(x)dx\right)}{\sum_{i=0}^{n} e^{-rt_i}\bar{F}_{Y_0}(t_i)\prod_{j=1}^{d}\bar{F}_{Z_j}(t_i)}.$$

Remark 5.1. *In the model specification of section 4 one gets, in the case of a unique premium,*

$$P = C\mathbb{E}\left[e^{-rM_d}\mathbf{1}_{\{M_d \le T\}}\right] = C\left(1 - e^{-rT}G^{\hat{\lambda}}(T) - r\int_0^T e^{-rx}G^{\hat{\lambda}}(x)dx\right),$$

where $\hat{\lambda} = \sum_{j=0}^{d}\lambda_j = \gamma_0 + \sum_{j=0}^{d}\eta_j$ *and, in the case of a periodical premium payment system,*

$$p = C\frac{1 - e^{-rT}G^{\hat{\lambda}}(T) - r\int_0^T e^{-rx}G^{\hat{\lambda}}(x)dx}{\sum_{i=0}^{n} e^{-rt_i}G^{\hat{\lambda}}(t_i)}.$$

In the exponential case, that is $G(x) = e^{-x}$*, and* T *integer with* $t_i = i$*,* $i = 0, \ldots, T-1$*, we recover*

$$P = C\frac{\hat{\lambda}}{r+\hat{\lambda}}\left(1 - e^{-(r+\hat{\lambda})T}\right)$$

and

$$p = C\frac{\hat{\lambda}}{r+\hat{\lambda}}\left(1 - e^{-(r+\hat{\lambda})}\right),$$

and in the whole life case, that is $T = +\infty$*,*

$$P = C\frac{\hat{\lambda}}{r+\hat{\lambda}} \quad \textit{and} \quad p = C\frac{\hat{\lambda}}{r+\hat{\lambda}}\left(1 - e^{-(r+\hat{\lambda})}\right).$$

Notice that P *is an increasing function of the maturity* T *while* p *doesn't depend on the maturity. Moreover,* P *and* p *are obviously always increasing functions of the parameter* $\hat{\lambda}$*.*

5.4 Simultaneous collapse of the whole system insurance product

Let us consider an insurance type contract against the systemic risk of the simultaneous collapse of the considered system paying the lump sum C at the time of occurrence, by a term T, of the simultaneous default (or failure) of all the d components of the system, that it at the time S given by (9). Assuming a constant interest rate $r > 0$ and following the same reasoning of the first-to-default example, the correspondig expected present value is

$$\mathbb{E}\left[C\, e^{-rS}\mathbf{1}_{\{S\le T\}}\right] = -C\int_0^T e^{-rx}d\bar{F}_S(x) = -C\sum_{j=0}^{d}\int_0^T e^{-rx}dH_j(x),$$

where H_j are given by (11), when $j=0$, and by (12) when $j=1,\dots,d$. Hence, the unique premium P paid at the issue time can be written as $P=\sum_{j=0}^d P_j$, where

$$P_j = \mathbb{E}\left[C\, e^{-rS}\mathbf{1}_{\{S=Y_j\le T\}}\right] = -C\int_0^T e^{-rx}dH_j(x)$$

represents the amount of the premium due to the occurrence of j-th systemic shock. In case of a stream of periodical premiums, following the same computations and notation as in the First-to-default case, we get

$$p = \frac{-C\sum_{j=0}^d\int_0^T e^{-rx}dH_j(x)}{\sum_{i=0}^n e^{-rt_i}\bar{F}_S(t_i)}$$

and again this periodical premium can be split in the contributions of each Y_j, $j=0,\dots,d$, that is $p=\sum_{j=0}^d p_j$ where

$$p_j = \frac{P_j}{\sum_{i=0}^n e^{-rt_i}\bar{F}_S(t_i)}.$$

From a credit risk perspective, this can be seen as a contract paying the lump sum C in case of collapse of the whole considered financial system due to the occurrence of a systemic event causing the simultaneous default of all the names. Above formulas allow to split the premium in sub-premia: those corresponding to $j=1,\dots,d$ can be charged to the corresponding name, while that corresponding to $j=0$, being completely exogenous to the system, can be equally shared by the whole system.

Example 5.1. *In the model specification of section 4, in the particular case in which $G(t)=e^{-t}$, one gets*

$$P_0 = \gamma_0\frac{1-\exp(-(r+\hat\lambda)T)}{r+\hat\lambda}$$

and, for $j=1,\dots,d$,

- *Clayton case with parameters β_j, $j=1,\dots,d$: $p_0=\frac{P_0}{\sum_{j=0}^d\frac{\gamma_j}{\hat\lambda+\lambda_j\beta_j}\frac{1-\exp(-(r+\hat\lambda+\lambda_j\beta_j)T)}{1-\exp(-(r+\hat\lambda+\lambda_j\beta_j))}}$, $P_j=\gamma_j\frac{1-\exp(-(r+\hat\lambda+\lambda_j\beta_j)T)}{r+\hat\lambda+\lambda_j\beta_j}$, $j=1,\dots,d$ and $p_j=\frac{P_j}{\sum_{j=0}^d\frac{\gamma_j}{\hat\lambda+\lambda_j\beta_j}\frac{1-\exp(-(r+\hat\lambda+\lambda_j\beta_j)T)}{1-\exp(-(r+\hat\lambda+\lambda_j\beta_j))}}$ where we have assumed $\beta_0=0$;*

- *Gumbel case with parameter* β_j, $j = 1, \ldots, d$*:* $p_0 = \frac{\gamma_0 \hat{\lambda}\left(1-\exp(-(r+\hat{\lambda}))\right)}{(r+\hat{\lambda})\left(\gamma_0+\sum_{j=1}^{d}\gamma_j\left(1+\frac{\lambda_j}{\gamma_j}\right)^{1-\beta_j}\right)}$, $P_j = \gamma_j\left(1+\frac{\lambda_j}{\gamma_j}\right)^{1-\beta_j}\frac{1-\exp(-(r+\hat{\lambda})T)}{r+\hat{\lambda}}$ *and* $p_j = \frac{\hat{\lambda}\gamma_j\left(1+\frac{\lambda_j}{\gamma_j}\right)^{1-\beta_j}\left(1-\exp(-(r+\hat{\lambda}))\right)}{(r+\hat{\lambda})\left(\gamma_0+\sum_{j=1}^{d}\gamma_j\left(1+\frac{\lambda_j}{\gamma_j}\right)^{1-\beta_j}\right)}$.

In Figure 4 we show the plots of the constant and periodical premiums with respect to different values of the Kendall's tau associated to the underlying Gumbel survival copulas in the three cases considered in section 5.1.2: being the resulting model exchangeable, we have $P_1 = P_2$ *and* $p_1 = p_2$ *and so only* P_1 *and* p_1 *are plotted.*

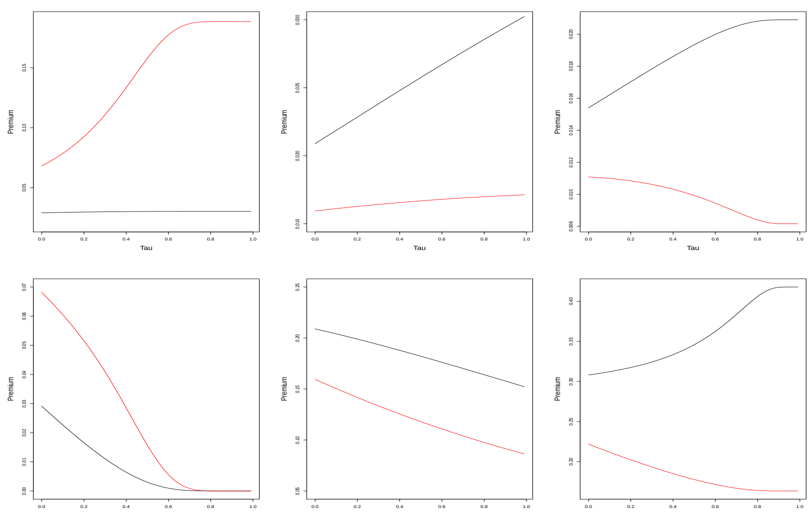

Figure 4: Plots of the premiums dynamics against the underlying Gumbel dependence in the cases of section 5.1.2. First column: $\gamma_1 = \gamma_2 = 0.01$; top: P_0 (black) and p_0 (red); bottom: P_1 (black) and p_1 (red). Second column: $\gamma_1 = \gamma_2 = 0.1$; top: P_0 (black) and p_0 (red); bottom: P_1 (black) and p_1 (red). Third column: $\gamma_1 = \gamma_2 = 0.2$; top: P_0 (black) and p_0 (red); bottom: P_1 (black) and p_1 (red).

5.5 First-to-default-Simutaneous default insurance product

As noticed in section 5.3, the first default can be that of a single element or coincide with that of the whole system: clearly, the loss due to the two cases is different. Hence, we can consider the case of a product paying a different lump sum depending on which of the two events will occur as first. Let C_0 be the lamp sum in the case of the collapse of the whole system, that is the case $M_d = S$, and C that in the case in which only one element in the system fails (or defaults). As a consequence, the expected present value is

$$C_0\mathbb{E}\left[e^{-rS}\mathbf{1}_{\{S\le T\}}\right]+C\mathbb{E}\left[e^{-rM_d}\mathbf{1}_{\{M_d\neq S, M_d\le T\}}\right] = C\mathbb{E}\left[e^{-rM_d}\mathbf{1}_{\{M_d\le T\}}\right]+(C_0-C)\,\mathbb{E}\left[e^{-rS}\mathbf{1}_{\{S\le T\}}\right].$$

Using the formulas obtained in sections 5.3 and 5.4, the unique and the periodical premiums can be calculated following the same approach.

6 Conclusions

In this paper we have introduced a generalization of the Marshall-Olkin distribution in which a specific non-exchangeable dependence among the underlying shocks arrival times is assumed. More specifically, we have supposed that each lifetime is the first arrival time between an idiosyncratic and a systemic shock and that, unlike the standard Marshall-Olkin model, lifetimes influence each other only through the systemic shock arrival being their idiosyncratic component dependent on it. The resulting joint distribution is investigated: its singularity analyzed and its dependence properties studied through the induced survival copula function, the associated pairwise Kendall's distribution function and Kendall's tau and the conditional hazard rates. Results show that the dependence structure is the composition of a Marshall-Olkin type dependence and the assumed dependence of each idiosyncratic component with the systemic shock arrival time. The model is considered for insurance pricing purposes with a particular focus an a product whose aim is to cover losses due to the occurrence of the systemic shock: in this case a very useful premium formula is provided that allows to decompose the premium according to the dependence of each element in the system with the systemic event.

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