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SYMMETRIZATION OF A CAUCHY-LIKE KERNEL ON CURVES

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Abstract. Given a curve $\Gamma \subset \mathbb{C}$ with specified regularity, we investigate boundedness and positivity for a certain three-point symmetrization of a Cauchy-like kernel K_Γ whose definition is dictated by the geometry and complex function theory of the domains bounded by Γ. Our results show that $\mathtt{S}[\mathrm{Re}K_\Gamma]$ and $\mathtt{S}[\mathrm{Im}K_\Gamma]$ (namely, the symmetrizations of the real and imaginary parts of K_Γ) behave very differently from their counterparts for the Cauchy kernel previously studied in the literature. For instance, the quantities $\mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z})$ and $\mathtt{S}[\mathrm{Im}K_\Gamma](\mathbf{z})$ can behave like $\frac{3}{2}c^2(\mathbf{z})$ and $-\frac{1}{2}c^2(\mathbf{z})$, where $\mathbf{z}$ is any three-tuple of points in Γ and $c(\mathbf{z})$ is the Menger curvature of $\mathbf{z}$. For the original Cauchy kernel, an iconic result of M. Melnikov gives that the symmetrized forms of the real and imaginary parts are each equal to $\frac{1}{2}c^2(\mathbf{z})$ for all three-tuples in $\mathbb{C}$.

1. Introduction

Given a complex-valued function $K(w,z)$ defined on a subset of $\mathbb{C}^2$ except possibly the diagonal $\{(z,z) : z \in \mathbb{C}\}$, we consider the following symmetric form associated with K:

$$\mathtt{S}\,[K](\mathbf{z}) := \sum_{\sigma \in S_3} K(z_{\sigma(1)},\, z_{\sigma(2)})\,\overline{K(z_{\sigma(1)},\, z_{\sigma(3)})} \tag{1.1}$$

where S_3 is the group of permutations over three elements, and $\mathbf{z} = \{z_1, z_2, z_3\}$ is any three-tuple of points in $\mathbb{C}$ for which the above expression is meaningful. The above definition of course also makes sense for real-valued $K(w,z)$.

A primary reference is the symmetric form of the kernel

$$K_0(w,z) \;:=\; \frac{1}{w-z}, \qquad z, w \in \mathbb{C},\ z \neq w \tag{1.2}$$

along with three seminal identities discovered by M. Melnikov [19]; these are

$$\mathtt{S}\,[K_0](\mathbf{z}) = c^2(\mathbf{z}), \tag{1.3}$$

$$\mathtt{S}\,[\mathrm{Re}K_0](\mathbf{z}) = \frac{1}{2}\,c^2(\mathbf{z}), \tag{1.4}$$

$$\mathtt{S}\,[\mathrm{Im}K_0](\mathbf{z}) = \frac{1}{2}\,c^2(\mathbf{z})\,. \tag{1.5}$$

Here $\mathbf{z} = \{z_1, z_2, z_3\}$ is any three-tuple of distinct points in $\mathbb{C}$; $\mathrm{Re}K_0$ (resp. $\mathrm{Im}K_0$) is the real (resp. imaginary) part of (1.2), and $c(\mathbf{z})$ denotes the **Menger curvature** associated with $\mathbf{z}$; we recall that $c(\mathbf{z})$ is defined to be either zero or the reciprocal of the radius of the unique circle passing through z_1, z_2 and z_3 according to whether the three points are, or are not,

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collinear. Identities (1.3)-(1.5) are often referred to as "Melnikov's miracle" in recognition of the profound influence they have had on the theory of singular integral operators [25, p. 2].

Because the kernel K_0 in (1.2) is defined in the maximal setting of $(z, w) \in \mathbb{C} \times \mathbb{C} \setminus \{z = w\}$ we will henceforth refer to it as the **universal Cauchy kernel** or the **original Cauchy kernel**. In this paper we study the effects of the symmetrization (1.1) on the **Γ-restricted Cauchy kernel**

$$K_\Gamma(w, z) := \frac{1}{2\pi i}\frac{\mathbf{t}_\Gamma(w)}{w - z}, \qquad z, w \in \Gamma,\ z \neq w \tag{1.6}$$

where $\Gamma \subset \mathbb{C}$ is a given rectifiable curve, and $\mathbf{t}_\Gamma(w)$ is the unit tangent vector to Γ at w in the counterclockwise direction.

At first glance K_Γ would appear to be only a minor variant of K_0 because $\mathtt{S}\,[K_\Gamma](\mathbf{z}) = \mathtt{S}\,[K_0](\mathbf{z})$ for *any* Γ and any three-tuple of distinct points. But the presence of $\mathbf{t}_\Gamma(w)$ renders the function $K_\Gamma(w, z)$ non-homogeneous (unless Γ is a line): in this respect the Γ-restricted Cauchy kernel (1.6) is substantially different in nature from the restriction to Γ of the universal Cauchy kernel (1.2). This distinction is especially relevant in complex analysis, where the numerator $\mathbf{t}_\Gamma(w)$ in (1.6) is indispensable already for characterizing the holomorphic Hardy spaces $H^p(\Gamma, \sigma)$. Here and throughout σ will denote the arc-length measure for Γ.

Our long-term goal is to employ kernel-symmetrization techniques to study the complex function theory of the domains bounded by Γ. As a first step in this direction, here we establish base-line results for $\mathtt{S}[K](\mathbf{z})$ where K is any of $\mathrm{Re}K_\Gamma$ and $\mathrm{Im}K_\Gamma$ (the real and imaginary parts of K_Γ). To be precise, we consider the following two basic features of K_0

- `(i)` boundedness relative to $c^2(\mathbf{z})$ of each of $\mathtt{S}\,[K_0](\mathbf{z})$, $\mathtt{S}\,[\mathrm{Re}\,K_0](\mathbf{z})$ and $\mathtt{S}\,[\mathrm{Im}K_0](\mathbf{z})$;
- `(ii)` positivity of each of $\mathtt{S}\,[K_0](\mathbf{z})$, $\mathtt{S}\,[\mathrm{Re}\,K_0](\mathbf{z})$ and $\mathtt{S}\,[\mathrm{Im}K_0](\mathbf{z})$

which are automatically granted by (1.3)-(1.5), and it is these features that we explore here for $\mathtt{S}\,[\mathrm{Re}K_\Gamma](\mathbf{z})$ and $\mathtt{S}\,[\mathrm{Im}K_\Gamma](\mathbf{z})$.

1.1. **The historical context: a brief review.** In 1995 Melnikov and Verdera [20] discovered that (1.3) leads to a new proof of the celebrated $L^2(\Gamma, \sigma)$-regularity of the **Cauchy transform** for a planar Lipschitz curve [2, 3]:

$$f \mapsto \text{p.v.} \int_\Gamma f(w) K_0(w, z)\, d\sigma(w)\,. \tag{1.7}$$

As noted in [25, Section 3.7.4], this new method of proof combined with (1.4) also shows that L^2-regularity of the Cauchy transform is, in effect, *equivalent* to the regularity of its real part alone that is, of the operator obtained by replacing $K_0(w, z)$ in (1.7) with $\mathrm{Re}K_0(w, z)$; of course (1.5) gives an analogous statement for $\mathrm{Im}K_0$.

This new approach was amenable to a large class of measures μ and ultimately led to groundbreaking progress towards the resolution of long-standing open problems in geometric measure theory known as the Vitushkin's conjecture and the Painlevè problem [18, 26, 27]. At the core of this work is the discovery that Melnikov's miracle (1.3)-(1.5) brings to the fore deep connections between the $L^2(\mu)$-boundedness of the Cauchy transform; the notion of curvature of the reference measure μ, [19]; and the rectifiability of the support of μ. This circle of ideas is often referred to as "the curvature method for K_0"; we defer to the excellent surveys [21] and [25] for detailed reviews of the extensive literature.

The interplay of analysis and geometry as manifested in (1.3)-(1.5), in fact already at the level of the basic features `(i)` and `(ii)`, has inspired the energetic pursuit of analogous connections for other Calderón-Zygmund kernels K and other notions of curvature, with diverse objectives and mixed success. It has been observed (see for instance [25, Section 3.7.4]) that condition `(ii)` fails for most kernels K, thereby ruling out the possibility of a non-negative curvature method valid in a broad context. For instance, Farag [12] shows that there is no higher-dimensional analogue of Menger-like curvatures stemming from Riesz transforms with integer exponents. Prat [22] on the other hand shows that $\mathtt{S}[K]$ *is* non-negative for fractional signed Riesz kernels K with homogeneity $-\alpha$, $0 < \alpha < 1$, using it to prove unboundedness of associated Riesz transforms on certain measure spaces. Lerman and Whitehouse [16, 17] introduce discrete and continuous variants of Menger-type curvatures in a real separable Hilbert space. Their definition of curvature uses general simplices instead of three-tuples of points; curvatures such as these have applications to problems in multiscale geometry. A number of recent articles, notably Chousionis-Mateu-Prat-Tolsa [4, 5]; Chousionis-Prat [6]; Chunaev [7], and Chunaev-Mateu-Tolsa [8, 9], explore curvature methods for various kinds of singular integral operators; certain aspects of the techniques developed in those papers are relevant to the present work and are discussed in section 5.

1.2. **Our context.** While Menger curvature has been employed primarily to study the $L^2(\mu)$-regularity of Calderòn-Zygmund operators and their implications to geometric measure theory, our long term goal is to investigate the connection between Menger curvature and the **Cauchy-Szegő projection** $\mathcal{S}_\Gamma$, which is the unique, orthogonal projection of $L^2(\Gamma, \sigma)$ onto $H^2(\Gamma, \sigma)$, for a given rectifiable curve Γ. The Cauchy-Szegő projection is a singular integral operator whose integration kernel is almost never known in explicit form and in general it is not Calderòn-Zygmund; what's more, $\mathcal{S}_\Gamma$ is trivially bounded on $L^2(\Gamma, \sigma)$ whereas proving boundedness in L^p for $p \neq 2$ is a difficult problem known as "the L^p-regularity problem for $\mathcal{S}_\Gamma$". On the other hand, the Cauchy-Szegő projection bears an intimate connection with the **Kerzman-Stein operator** $\mathcal{A}_\Gamma$ whose integration kernel is

$$A_\Gamma(w, z) := K_\Gamma(w, z) - \overline{K_\Gamma(z, w)}, \qquad z, w \in \Gamma,\ z \neq w. \tag{1.8}$$

The connection with the Cauchy-Szegő projection transpires whenever the Kerzman-Stein operator satisfies finer properties than L^p-regularity: for instance, compactness in $L^p(\Gamma, \sigma)$. Such connection is one reason for our interest in K_Γ and its real and imaginary parts. In fact K_Γ is relevant to the analysis of various reproducing kernel Hilbert spaces (the holomorphic Hardy space $H^2(\Gamma, \sigma)$ being one such instance see e.g., [12, p. 376]), whereas $\mathrm{Re}K_\Gamma$ is of particular interest in potential theory since it is the integration kernel of the **double layer potential operator** [14, (1.14)]. We defer to [1, 11, 13, 15] for the precise definitions and the statements of the main results on these topics, and for references to the extensive literature.

We focus on the restricted setting of a curve parametrized as a graph

$$\Gamma = \{z = x + iA(x),\ x \in J = (a, b) \subseteq \mathbb{R}\} \tag{1.9}$$

where the function $A(x)$ is of class C^1 or better (as specified in the statement of each result below) and we represent the righthand side of (1.6) in parametric form, giving

$$K_\Gamma(w,z) = \frac{1}{2\pi}\frac{A'(x)-i}{(1+(A'(x))^2)^{1/2}\,[x-y+i(A(x)-A(y))]} \tag{1.10}$$

where $w = x+iA(x),\ z = y+iA(y)$, and $x, y \in J$ with $x \neq y$.

The case when J is the entire real line has special relevance in complex analysis because in this case K_Γ agrees with the integration kernel of the **Cauchy integral operator** associated with the domain

$$\Omega := \{y < A(x),\ x \in J\}, \tag{1.11}$$

namely the operator

$$f \mapsto \frac{1}{2\pi i}\int_{b\Omega} f(w)\,\frac{dw}{w-z}, \quad z \notin \mathrm{Supp}(f). \tag{1.12}$$

As is well known [11, 13, 15], the Cauchy integral operator produces and reproduces[1] functions in the holomorphic Hardy space $H^2(b\Omega,\sigma)$ (more generally, functions in $H^p(b\Omega,\sigma), 1 \le p \le \infty$) and it plays a distinguished role in the analysis of the Cauchy-Szegő projection and the Kerzman-Stein operator (1.8).

To see the connection between (1.12) and the kernel (1.10), we first recall that dw in (1.12) is shorthand for the pull-back j^*dw where $j : \Gamma \hookrightarrow \mathbb{C}$ is the inclusion map. With this in place, and again writing $x + iA(x)$ for $w \in \Gamma$, we have

$$\begin{aligned}\frac{1}{2\pi i}\, j^*dw &= \frac{1}{2\pi i}\, d(x+iA(x))\\ &= \frac{1}{2\pi i}\,(1+iA'(x))\,dx\\ &= \frac{1}{2\pi}\,\frac{A'(x)-i}{\sqrt{1+(A'(x))^2}}\,d\sigma(w)\,,\end{aligned}$$

where σ is the arc-length measure for $b\Omega \equiv \Gamma$ whose density is $d\sigma(w) = s(x)dx$, with $s(x) = \sqrt{1+(A'(x))^2}$. It is now clear that the kernel in (1.12) interpreted as an integral with respect to the arc-length measure for Γ, agrees with K_Γ and we will henceforth ignore the constant factor $1/2\pi$.

Note that the restriction to Γ of the universal Cauchy kernel, namely the function j^*K_0, corresponds to the case when A is constant, that is the situation when Γ is a horizontal line. On the other hand, for general Γ we have $j^*K_0 \neq K_\Gamma$. The distinction between these two kernels is especially significant in the context of holomorphic Hardy space theory; for instance, the analysis of the Cauchy-Szegő projection performed in e.g., [13] and [15, Theorem 2.1] relies upon a cancellation of singularities that is enjoyed by the Kerzman-Stein kernel (1.8) but is *not* enjoyed by $j^*K_0(w,z) - \overline{j^*K_0(z,w)}$ unless Γ is a horizontal line. Already in the example of the parabola $\Gamma := \{x + ix^2\,, x \in \mathbb{R}\}$, it is easy to see that $j^*K_0(w,z) - \overline{j^*K_0(z,w)}$ has same principal singularity as $j^*K_0(w,z)$, whereas the Kerzman-Stein kernel (1.8) is in fact a *smooth* function of $(w,z) \in \Gamma\times\Gamma$, even along the diagonal $\{w = z\}$.

[1] that is, it is a *projection*: $L^p(b\Omega,\sigma) \to H^p(b\Omega,\sigma)$.

1.3. **Main results.** We establish results of two kinds: local on Γ, valid for three-tuples of distinct points on Γ that are near a point $z_0 \in \Gamma$ of non-vanishing curvature; and global on Γ (for any three-tuple of distinct points in Γ).

1.3.1. *Sharp local estimates on* Γ. Here we require Γ to be of class C^3; we show that each of $\mathtt{S}\,[\mathrm{Re}K_\Gamma](\mathbf{z})$ and $\mathtt{S}\,[\mathrm{Im}K_\Gamma](\mathbf{z})$ is locally relatively bounded near any point in Γ with non-zero signed curvature, but only the former will be non-negative, in fact strictly positive and forcing the latter to be strictly negative.

Theorem 1.1. *Suppose that A is of class C^3 (i.e., A is thrice continuously differentiable), and that $x_0 \in J$ is such that $A''(x_0) \neq 0$. Then for any $\epsilon > 0$, there exists $\delta = \delta(x_0, A, \epsilon) > 0$ such that for*

$$I = (x_0 - \delta, x_0 + \delta) \subset J \qquad \textit{and} \qquad \Gamma(I) = \{z = x + iA(x) : x \in I\}$$

the following statements hold for any three-tuple $\mathbf{z}$ of distinct points on

$$\Gamma(I)^3 = \Gamma(I) \times \Gamma(I) \times \Gamma(I).$$

(a) If $\kappa_0 = A''(x_0)/s(x_0)^3$ denotes the curvature of Γ at $z_0 = x_0 + iA(x_0)$, then

$$c^2(\mathbf{z}) = \kappa_0^2 + r(\mathbf{z}), \quad \textit{with } |\, r(\mathbf{z})\,| < \epsilon\,. \tag{1.13}$$

(b) We have

$$\left| \mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z}) - \frac{3}{2}c^2(\mathbf{z}) \right| < \epsilon, \quad \left| \mathtt{S}[\mathrm{Im}K_\Gamma](\mathbf{z}) + \frac{1}{2}c^2(\mathbf{z}) \right| < \epsilon. \tag{1.14}$$

Combining the two conclusions of Theorem 1.1, we arrive at the following corollary.

Corollary 1.2. *With same notations and hypotheses as in Theorem 1.1 if, furthermore, $\tilde{\epsilon} > 0$ is sufficiently small then*

$$\left| \frac{\mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z})}{c^2(\mathbf{z})} - \frac{3}{2} \right| < \frac{\tilde{\epsilon}}{\kappa_0^2 - \tilde{\epsilon}}\,; \quad \left| \frac{\mathtt{S}[\mathrm{Im}K_\Gamma](\mathbf{z})}{c^2(\mathbf{z})} + \frac{1}{2} \right| < \frac{\tilde{\epsilon}}{\kappa_0^2 - \tilde{\epsilon}} \tag{1.15}$$

for any three-tuple $\mathbf{z}$ of non-collinear points in $\Gamma(\tilde{I})^3$ where $\tilde{I} = (x_0 - \tilde{\delta}, x_0 + \tilde{\delta})$ is obtained by applying Theorem 1.1 to $\tilde{\epsilon}$.

Remarks:

(i) Theorem 1.1 gives that $\mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z})$ satisfies the positivity condition `(ii)` when $\mathbf{z}$ is taken in $\Gamma(I)^3$, but more is true: the proof will show that $\mathtt{S}[\mathrm{Re}K_\Gamma]$ manifests a phenomenon of "local superpositivity" in the sense that for any $\mathbf{z} \in \Gamma(I)^3$, $\mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z})$ is given by the sum of three positive terms, each comparable to $\frac{1}{2}\kappa_0^2$. On the other hand $\mathtt{S}[\mathrm{Im}K_\Gamma]$ is strictly negative on $\Gamma(I)^3$, in stark contrast with the situation for $\mathtt{S}[\mathrm{Im}(K_0)]$ (that is, when Γ is a horizontal line).

(ii) This leads to the following remarkable fact. Recall that for *any* kernel function $K(z, w)$ and for *any* three-tuple of distinct points $\mathbf{z}$, the quantity $\mathtt{S}[K](\mathbf{z})$ admits the basic split

$$\mathtt{S}[K](\mathbf{z}) \;=\; \mathtt{S}[\mathrm{Re}K](\mathbf{z}) + \mathtt{S}[\mathrm{Im}K](\mathbf{z}), \tag{1.16}$$

see e.g., [14, (2.5)].

However, whereas the split of $\mathtt{S}[K_0](\mathbf{z})$ is perfectly balanced between the real and imaginary parts of K_0 i.e.,

$$\mathtt{S}[\mathrm{Re}K_0](\mathbf{z}) \;=\; \frac{1}{2}\mathtt{S}[K_0](\mathbf{z}) \;=\; \mathtt{S}[\mathrm{Im}K_0](\mathbf{z})$$

see (1.4) – (1.5), the split for $\mathtt{S}[\,K_\Gamma](\mathbf{z})$ with $\mathbf{z}\in\Gamma(I)^3$ is roughly speaking $3/2$ and $-1/2$, respectively, i.e.

$$\mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z}) \approx \frac{3}{2}\mathtt{S}[\,K_\Gamma](\mathbf{z}) \quad \text{and} \quad \mathtt{S}[\mathrm{Im}K_\Gamma](\mathbf{z}) \approx -\frac{1}{2}\mathtt{S}[\,K_\Gamma](\mathbf{z})\,.$$

(iii) Results analogous to Theorem 1.1 continue to hold if $A''(x_0)=0$ but some higher order derivative of A is non-vanishing at x_0. The proof modifies with very little changes and we have chosen to omit it here.

(iv) Corollary 1.2 says that both $\mathtt{S}[\mathrm{Re}K_\Gamma]$ and $\mathtt{S}[\mathrm{Im}K_\Gamma]$ satisfy the relative boundedness condition `(i)` in $\Gamma(\tilde{I})^3$ (and in fact are themselves locally bounded, but see Theorem 1.3 below for a stronger statement).

(v) In general, the inclusion $\tilde{I}\subset J$ in Corollary 1.2 is strict, and in the absence of the localization: $\mathbf{z}\in\Gamma(\tilde{I})^3$ there are no definitive results pertaining to condition (1.15). In Section 4.1 we give an example of a relatively compact, smooth curve Γ; a point $z_0\in\Gamma$ with non-zero signed curvature, and three-tuples $\{\mathbf{z}_\lambda=(z^1_\lambda;z^2_\lambda;z^3_\lambda)\}_\lambda$ of distinct points on Γ such that $z^2_\lambda\to z_0$ but $\mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z}_\lambda)/c^2(\mathbf{z}_\lambda)$ and $\mathtt{S}[\mathrm{Im}K_\Gamma](\mathbf{z}_\lambda)/c^2(\mathbf{z}_\lambda)$ are unbounded.

1.3.2. *Qualitative global estimates on* Γ. We further consider two settings:

• *Curves of class* $C^{1,1}$. This means that A is once differentiable with Lipschitz continuous derivative. We show that each of $c^2(\mathbf{z})$; $\Big|\mathtt{S}\,[\mathrm{Re}K_\Gamma](\mathbf{z})\Big|$ and $\Big|\mathtt{S}\,[\mathrm{Im}K_\Gamma](\mathbf{z})\Big|$ admits a global upper bound valid for any three-tuple $\mathbf{z}$ of distinct points on Γ. Since $c(\mathbf{z})$ can vanish, this result does not imply relative boundedness. See Theorem 1.3 and Corollary 1.6 below.

• *Curves of class* C^2 *with fixed concavity.* We prove global non-negativity of $\mathtt{S}\,[\mathrm{Re}K_\Gamma](\mathbf{z})$ and provide examples to show that there are no definitive results pertaining to the global signature of $\mathtt{S}\,[\mathrm{Im}K_\Gamma](\mathbf{z})$. See Theorem 1.7 and the examples in section 4.2 below.

Theorem 1.3. *Suppose that A is of class $C^{1,1}$ i.e., there exists a constant $M>0$ such that*

$$|A'(x)-A'(y)|\le M|x-y| \quad \text{all } x,y\in J. \tag{1.17}$$

Then we have that

$$\Big|\mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z})\Big| \le \frac{3}{2}M^2$$

for any three-tuple $\mathbf{z}$ of distinct points on Γ.

The order of magnitude of the Lipschitz constant for A' (that is, the quantity M in (1.17)) is optimal, as indicated by the following

Lemma 1.4. *Let $A(x)=x^3$. Then there are $0<\delta_0=\delta_0(A)$ and $0<c_0<1$, $c_0=c_0(A)$ such that*

$$\mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z})\ge c_0\,M_\epsilon^2$$

for $\mathbf{z}=(-\epsilon\alpha-i\epsilon^3\alpha^3,\,0,\,\epsilon\beta+i\epsilon^3\beta^3)$, for any $\alpha,\beta\in[1/2,1]$ and any $0<\epsilon<\min\{1,\delta_0\}$. Here M_ϵ is the Lipschitz constant for the restriction of $A'(x)$ to the interval $(-\epsilon,\epsilon)$.

Note that Lemma 1.4 also shows that Theorem 1.1 gives a sufficient, but not necessary condition for the local positivity of $\mathtt{S}[\mathrm{Re}K_\Gamma]$.

Lemma 1.5. *With same hypotheses as Theorem 1.3, we have that*

$$c^2(\mathbf{z}) \leq 8M^2$$

for any three-tuple $\mathbf{z}$ *of distinct points on* Γ*.*

Corollary 1.6. *With same hypotheses as Theorem 1.3, we have that*

$$\left|\mathtt{S}[\mathrm{Im}K_\Gamma](\mathbf{z})\right| \leq \left(8+\frac{3}{2}\right)M^2$$

for any three-tuple $\mathbf{z}$ *of distinct points on* Γ*.*

Theorem 1.7. *Suppose that* A *is of class* C^2*, and that* A'' *does not change sign on* J *e.g.,* $A''(x) \geq 0$ *for all* $x \in J$ *(alt.* $A''(x) \leq 0$ *for all* $x \in J$*). Then*

$$\mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z}) \geq 0 \tag{1.18}$$

for any three-tuple $\mathbf{z}$ *of distinct points on* Γ*.*

Remarks:

(i) In view of conclusion (1.14) in Theorem 1.1, it makes sense to ask whether the inequality: $\mathtt{S}[\mathrm{Im}K_\Gamma](\mathbf{z}) \leq 0$ can hold for any three-tuple $\mathbf{z}$ of distinct points on a curve satisfying the hypotheses of Theorem 1.7: in section 4.2 below we answer this question in the negative by showing that the parabola $\Gamma = \{x + i\,x^2/2, x \in \mathbb{R}\}$ admits three-tuples $\mathbf{z}$ of distinct points such that $\mathtt{S}[\mathrm{Im}K_\Gamma](\mathbf{z}) > 0$.

(ii) The assumption of fixed concavity in Theorem 1.7 is necessary: in section 4.3 we show that the cubic $\Gamma = \{x + ix^3,\ x \in \mathbb{R}\}$ admits three-tuples $\mathbf{z}$ of distinct points for which $\mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z}) < 0$.

(iii) While our global results hold for three-tuples that lie on the curve Γ, the original Melnikov miracles (1.4) and (1.5) are in fact universally global in that they are valid for three-tuples that may lie *anywhere* in $\mathbb{C}$. It is thus meaningful to ask whether analogues of Theorem 1.3, Corollary 1.6, or Theorem 1.7 can be stated that hold for arbitrary three-tuples in $\mathbb{C}$. To this end we consider the following family of kernels $\{K_h\}_h$

$$K_h(w,z) := \frac{e^{ih(w)}}{w-z} \tag{1.19}$$

parametrized by globally defined, continuous functions $h : \mathbb{C} \to \mathbb{R}$. The family $\{K_h\}_h$ is especially interesting to us because for any C^1-smooth curve Γ the kernel K_Γ is realized as j^*K_h for at least one such h. To see this, note that the domain Ω determined by Γ as in (1.9) and (1.11), is contained in the complement of a ray, thereby granting the existence of a continuous branch of the logarithm of the function

$$x \mapsto \frac{A'(x) - i}{(1 + (A'(x))^2)^{1/2}}, \quad x \in J.$$

Applying such logarithm gives a continuous function: $\Gamma \to \mathbb{R}$ which we call φ. Any extension of φ to a continuous[2] $h : \mathbb{C} \to \mathbb{R}$ produces a kernel K_h in the family (1.19) whose restriction to Γ agrees with K_Γ.

It turns out that $\mathtt{S}[\mathrm{Re}K_h]$ and $\mathtt{S}[\mathrm{Im}K_h]$ satisfy no universally global phenomena for any continuous $h : \mathbb{C} \to \mathbb{R}$, not even for h chosen so that $j^*K_h = K_\Gamma$ where Γ satisfies the stronger hypotheses of class $C^{1,1}$ (Theorem 1.3) or class C^2 (Theorem 1.7): this is proved with techniques that are similar in spirit to recent work of Chousionis-Pratt [6] and Chunaev [7]. The precise statements are given in section 5, see Theorem 5.3 - Theorem 5.8.

An open problem. *Does the stronger assumption:*

$$A''(x) \geq c > 0 \quad \textit{for all} \;\; x \in J \quad (\textit{alt. } A''(x) \leq c < 0 \textit{ for all } x \in J)$$

give that

$$\mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z}) \geq \alpha\, c^2(\mathbf{z}) \tag{1.20}$$

for some $\alpha = \alpha(\Gamma) > 0$ and for all three-tuples $\mathbf{z}$ of distinct points on Γ?

This statement seems much harder to prove than Theorem 1.7 (whose proof is remarkably simple). Note that an answer in the positive would shed some light on the signature of $\mathtt{S}[\mathrm{Im}K_\Gamma](\mathbf{z})$ because it would imply that

$$\mathtt{S}[\mathrm{Im}K_\Gamma](\mathbf{z}) \leq (1-\alpha)c^2(\mathbf{z})$$

for all three-tuples $\mathbf{z}$ of distinct points on Γ. In the example of the parabola: $\Gamma = \{x + ix^2/2, x \in \mathbb{R}\}$, an elementary but non-trivial calculation[3] gives that (1.20) is true with $\alpha = 1/2$; the general case remains unanswered.

1.4. **Conclusion.** The extensive literature on this subject indicates that most kernels do not satisfy basic estimates such as `(i)` and `(ii)` globally in $\mathbb{C}$: the families $\{\mathrm{Re}K_h\}_h$ and $\{\mathrm{Im}K_h\}_h$ are no exception. On the other hand, given any rectifiable curve Γ with regularity prescribed as in our main results, it turns out that certain members in $\{\mathrm{Re}K_h\}_h$ (those for which $j^*K_h = K_\Gamma$) do satisfy Γ-restricted versions of `(i)` and `(ii)`, whereas their counterparts in $\{\mathrm{Im}K_h\}_h$ will satisfy Γ-restricted versions of `(i)` though not necessarily of `(ii)`.

As mentioned earlier, symmetrization techniques have so far been used primarily to study L^2-regularity of Calderòn-Zygmund operators. One would like to know whether effective curvature methods can be developed to prove certain finer properties of the Kerzman-Stein operator (such as compactness in $L^p(\Gamma, \sigma)$ [23]) and of various Kerzman-Stein-like operators that are known to bear upon the L^p-regularity problem for the Cauchy-Szegő projection, the Bergman projection and other holomorphic singular integral operators: the Γ-restricted estimates obtained here are a first step in this direction; we plan to pursue the subsequent steps elsewhere.

One last remark. *What if in place of the symmetrized form* (1.1) *one had considered the following variant:*

$$\tilde{\mathtt{S}}\,[K](\mathbf{z}) := \sum_{\sigma \in S_3} K(z_{\sigma(2)},\, z_{\sigma(1)})\, \overline{K(z_{\sigma(3)},\, z_{\sigma(1)})}, \tag{1.21}$$

[2]for instance, extending A to a C^1-function: $\tilde{A} : \mathbb{R} \to \mathbb{R}$ and then letting $h(x+iy) := \varphi(x + i\tilde{A}(x))$ gives a continuous function $h : \mathbb{C} \to \mathbb{R}$ that is constant along each vertical line.

[3]we are grateful to M. Putinar, E. Wegert and A. Weideman for assisting with these computations.

whose choice is also legitimate because $\tilde{\mathtt{S}}\,[K_0](\mathbf{z}) = \mathtt{S}\,[K_0](\mathbf{z})$*?*

Setting $K^*(w,z) := \overline{K(z,w)}$ it is easy to see that $\tilde{\mathtt{S}}\,[K](\mathbf{z}) = \mathtt{S}\,[K^*](\mathbf{z})$. In section 5.3 we prove failure of the basic estimate `(i)` for the family $\{K^*_h\}_h$ globally in $\mathbb{C}$, see Proposition 5.7 and Theorem 5.8. The analysis in the Γ-restricted setting will be the object of forthcoming work.

1.5. **Organization of this paper.** In section 2 we collect a few auxiliary facts needed to prove the main results, whose proofs are given in section 3. All examples pertaining to the sharpness of the main results are detailed in section 4. Finally, section 5 is an appendix where we collect all the relevant statements for the family of kernels (1.19).

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2. Preliminaries

We begin by recording representations for $\mathtt{S}\,[\mathrm{Re}K_\Gamma](\mathbf{z})$, $\mathtt{S}[\,\mathrm{Im}K_\Gamma](\mathbf{z})$ and $c(\mathbf{z})$ that hold when $\mathbf{z}$ is a three-tuple of distinct points on Γ, that is for $\mathbf{z}=(z_1,z_2,z_3)$ with $z_j = x_j + iA(x_j)\in\Gamma$, $j=1,2,3$ and distinct x_1,x_2,x_3.

Lemma 2.1. *A be of class C^1. Then the symmetrized forms of* $\mathrm{Re}K_\Gamma(\mathbf{z})$ *and* $\mathrm{Im}K_\Gamma(\mathbf{z})$ *admit the following representations at any three-tuple* $\mathbf{z}$ *of distinct points on* Γ*:*

$$\begin{aligned}\mathtt{S}[\,\mathrm{Re}K_\Gamma\,](\mathbf{z}) = 2\sum_{\substack{j=1\\k<l}}^{3} \frac{1}{s^2(x_j)\ell_k^2\,\ell_l^2}&\Big[A'(x_j)(x_j-x_k) - \Big(A(x_j)-A(x_k)\Big)\Big]\times\\ &\Big[A'(x_j)(x_j-x_l) - \Big(A(x_j)-A(x_l)\Big)\Big]\,;\end{aligned}\tag{2.1}$$

$$\begin{aligned}\mathtt{S}[\,\mathrm{Im}K_\Gamma\,](\mathbf{z}) = 2\sum_{\substack{j=1\\k<l}}^{3} \frac{1}{s^2(x_j)\ell_k^2\,\ell_l^2}&\Big[(x_k-x_j) + A'(x_j)\Big(A(x_k)-A(x_j)\Big)\Big]\times\\ &\Big[(x_l-x_j) + A'(x_j)\Big(A(x_l)-A(x_j)\Big)\Big].\end{aligned}\tag{2.2}$$

As in [14], here we have set $\{j,l,k\}=\{1,2,3\}$ and $l,k\in\{1,2,3\}\setminus\{j\}$, and we have adopted the shorthand

$$s^2(x_j) = 1 + (A'(x_j))^2\,;\quad \ell_j^2 = (x_l-x_k)^2 + (A(x_l)-A(x_k))^2\,.$$

Proof. First we recall that if $H(w, z)$ is *real-valued*, then

$$\mathtt{S}\,[H](\mathbf{z}) = 2\sum_{\substack{j=1\\k<l}}^{3} H(z_j,\, z_k)\, H(z_j,\, z_\ell)\,, \tag{2.3}$$

see e.g., [14]. Next we note that (1.10) gives

$$\begin{aligned}\mathrm{Re}K_\Gamma(w, z) \;&=\; \frac{A'(x)(x-y) - \Big(A(x) - A(y)\Big)}{s(x)|w-z|^2} \quad\text{and}\\ \mathrm{Im}K_\Gamma(w, z) \;&=\; \frac{y - x + A'(x)\Big(A(y) - A(x)\Big)}{s(x)|w-z|^2}\end{aligned}$$

for distinct points $w = x + iA(x), z = y + iA(y)$ in Γ. The conclusion now follows by plugging these expressions in (2.3). $\square$

Lemma 2.2. *Let A be continuous, and let $\mathbf{z} = \Big(u + iA(u); x + iA(x); v + iA(v)\Big)$ be any three-tuple of distinct points on Γ. Then the Menger curvature of $\mathbf{z}$ admits the following representation:*

$$c^2(\mathbf{z}) = \frac{4\,\left[A(u)(x-v) + A(x)(v-u) + A(v)(u-x)\right]^2}{\ell_u^2\,\ell_x^2\,\ell_v^2}. \tag{2.4}$$

As before, here we have adopted the shorthand: $\ell_u^2 = (v-x)^2 + (A(v) - A(x))^2$, etc.

Proof. If the three distinct points are collinear then the conclusion is immediate because each side of (2.4) is easily seen to be equal to zero. Suppose next that the three points are not collinear: by the invariance of the numerator of (2.4) under the permutations of $\{u, x, v\}$ we may assume without loss of generality that $u < x < v$. Then there are two cases to consider, depending on whether the point $(x, A(x))$ lies below or above the line segment joining $(u, A(u))$ and $(v, A(v))$. In either case we may assume without loss of generality, that

$$A(u) > 0;\;\; A(x) > 0;\;\; A(v) > 0.$$

(This is because Menger curvature is invariant under translations, and the above condition is achieved by a translation along the vertical axis.) The desired conclusion then follows by employing the well-known formula [25, (3.1)]

$$c(\mathbf{z}) \;=\; \frac{4\,\mathrm{Area}(\Delta(\mathbf{z}))}{\ell_a\ell_b\ell_c}, \tag{2.5}$$

and by expressing the area of the triangle $\Delta(\mathbf{z})$ as an appropriate linear combination of areas of parallelograms whose vertices belong to the set

$$\{(u, 0); (x, 0); (v, 0); (u, A(u)); (x, A(x)); (v, A(v))\}.$$

$\square$

Next we provide an elementary lemma that rules out the possibility of collinearity for three-tuples $\mathbf{z}$ of distinct points on Γ in the vicinity of points with non-zero signed curvature.

Lemma 2.3. *Let A be of class C^2. Then any three-tuple $\mathbf{z}$ of distinct points on Γ that are in the vicinity of a point $z_0 \in \Gamma$ whose curvature κ_0 is non-zero, are non-collinear.*

Proof. We need to show that for any $x_0 \in J$ such that $A''(x_0) \neq 0$ there is $\delta > 0$ with the property that for any $u, x, v \in I := (x_0 - \delta, x_0 + \delta)$ the points $u + iA(u); x + iA(x); v + iA(v)$ are not collinear. Suppose, by contradiction, that there are $x_0 \in J$ and $u_n < v_n < w_n \to x_0$ such that the points $P_n := (u_n, A(u_n)); Q_n := (v_n, A(v_n))$ and $R_n := (w_n, A(w_n))$ are collinear. Then the slopes of the line segments joining any two such points must be equal, giving us

$$\frac{A(v_n) - A(u_n)}{v_n - u_n} = \frac{A(w_n) - A(v_n)}{w_n - v_n} \quad \text{for all } n.$$

By the mean value theorem it follows that there are x_n and y_n with $u_n < x_n < v_n < y_n < w_n$ and such that

$$A'(x_n) = A'(y_n) \quad \text{for all } n.$$

Applying Rolle's theorem to $f(x) := A'(x)$ we conclude that for each n there is z_n with $x_n < z_n < y_n$ and such that

$$A''(z_n) = 0 \quad \text{for all } n,$$

leading us to a contradiction since $A''(z_n) \to A''(x_0) \neq 0$. □

Remark. The same strategy of proof also gives the following global version of Lemma 2.3: *If A: is of class C^2 and $A''(x) \neq 0$ for all $x \in J$, then any three-tuple of distinct points on Γ are non-collinear.*

In closing this section we detail a few lemmas that help to keep track of the effect of the assumed regularity of $A(x)$ in the proofs of our main results.

Lemma 2.4. *Let A be of class C^2. Then for any $x_0 \in J$ and $\epsilon > 0$ there is $\delta > 0$ such that*

$$|A(v) - A(u) - A'(x_0)(v - u)| \leq \epsilon |v - u| \tag{2.6}$$

whenever $u, v \in I = (x_0 - \delta, x_0 + \delta)$.

Proof. Taylor's theorem grants the existence of $\delta_1 > 0$ (which we may take to be finite) such that

$$A(v) - A(u) - A'(x_0)(v - u) = R_1(v) - R_1(u) \tag{2.7}$$

for any $u, v \in I_{\delta_1}(x_0)$, where

$$R_1(y) = \int_{x_0}^{y} (y - t) A''(t)\, dt. \tag{2.8}$$

We claim that there is $0 < \delta \leq \delta_1$ such that

$$|R_1(v) - R_1(u)| \leq \epsilon |v - u| \quad \text{for any} \quad u, v \in I_\delta(x_0). \tag{2.9}$$

The claim is trivial when $u = v$ and we henceforth assume that $u < v$. It follows from (2.8) that

$$R_1(v) - R_1(u) = (v - u) \int_{x_0}^{u} A''(t) dt + \int_{u}^{v} (v - t) A''(t) dt.$$

Integrating the second integral by parts, and then applying the mean-value theorem give that

$$R_1(v) - R_1(u) = (v - u) \int_{x_0}^{u} A''(t) dt \; - A'(u)(v - u) + A'(\xi)(v - u)$$

for some ξ with $u < \xi < v$ and for any $u, v \in I_{\delta_1}(x_0)$. Now for

$$\|A''\|_\infty := \|A''\|_{L^\infty(I_{\delta_1}(x_0))}$$

we see that the above gives

$$|R_1(v) - R_1(u)| \leq |v-u| \|A''\|_\infty 2\delta \leq \epsilon |v-u|$$

as soon as we choose $0 < \delta \leq \min\{\delta_1, \epsilon/(2\|A''\|_\infty)\}$. □

Corollary 2.5. *Let A be of class C^3 and suppose that $A''(x_0) \neq 0$. Then for any $\epsilon > 0$ there is $\delta > 0$ such that*

$$A'(u) - A'(v) = A''(x_0)(u-v)\,(1 + \mu(u,v)) \quad \text{with} \quad |\,\mu(u,v)| < \epsilon\,, \tag{2.10}$$

whenever $u, v \in I = (x_0 - \delta, x_0 + \delta)$.

Proof. Applying Lemma 2.4 to the function $A'(x)$ (which is of class C^2) we obtain $\delta > 0$ such that

$$A'(u) - A'(v) = A''(x_0)(u-v) + \tilde{R}_1(v) - \tilde{R}_1(u)$$

where

$$|\tilde{R}_1(v) - \tilde{R}_1(u)| \leq \epsilon |v-u|$$

for any $u, v \in I_\delta(x_0)$; see (2.7) and (2.9). Thus the conclusion holds with

$$\mu(u,v) = \frac{\tilde{R}_1(v) - \tilde{R}_1(u)}{A''(x_0)(v-u)}.$$

□

Lemma 2.6. *Let A be of class C^3 and suppose that $A''(x_0) \neq 0$. Then for any $\epsilon > 0$ there is $\delta > 0$ such that*

$$\int_u^v \Big(A'(t) - A'(u)\Big) dt = A''(x_0)\frac{(u-v)^2}{2}\Big(1 + R(u,v)\Big) \text{ with } |R(u,v)| < \epsilon \tag{2.11}$$

whenever $u, v \in I = I_\delta(x_0) := (x_0 - \delta, x_0 + \delta)$.

Proof. We may assume without loss of generality that $u \neq v$ and further, that $u < v$ Applying Lemma 2.4 to the function $A'(x)$ (which is of class C^2) we obtain $\delta > 0$ such that for any $t, u \in I_\delta(x_0)$ we have

$$A'(t) - A'(u) = A''(x_0)(t-u) + \tilde{R}_1(t) - \tilde{R}_1(u) \tag{2.12}$$

and

$$|\tilde{R}_1(t) - \tilde{R}_1(u)| \leq \epsilon |t-u|\,; \tag{2.13}$$

see (2.7) and (2.9). Next we take $v \in I_\delta(x_0)$, and integrate both sides of (2.12) over the sub-interval $(u, v) \subset I_\delta(x_0)$:

$$\int_u^v (A'(t) - A'(u))dt = \frac{A''(x_0)}{2}(v-u)^2 + \int_u^v \Big(\tilde{R}_1(t) - \tilde{R}_1(u)\Big)dt\,.$$

It follows from (2.13) that

$$\left|\int_u^v \Big(\tilde{R}_1(t) - \tilde{R}_1(u)\Big)dt\right| \leq \epsilon \int_u^v |t-u|\,dt = \frac{\epsilon}{2}(v-u)^2$$

for any $u, v \in I_\delta(x_0)$. Thus the conclusion holds with

$$R(u,v) = \frac{\int_u^v \left(\tilde{R}_1(t) - \tilde{R}_1(u)\right) dt}{A''(x_0)(v-u)^2}.$$

□

Lemma 2.7. *Let A be of class C^2. Then for any $x_0 \in J$ and any $\epsilon > 0$ there is $\delta > 0$ such that for any $u, v \in I = (x_0 - \delta, x_0 + \delta)$ we have*

$$\frac{(v-u)^2}{[(v-u)^2 + (A(v) - A(u))^2]} = \frac{1}{s^2(x_0)} (1 + \rho(u,v)) \tag{2.14}$$

where

$$|\rho(u,v)| < \epsilon(1 + 2|A'(x_0)|).$$

Proof. A straightforward application of Lemma 2.4 gives $\delta > 0$ such that

$$\frac{(v-u)^2 + (A(v) - A(u))^2}{(v-u)^2} = s^2(x_0)\left(1 + \frac{S(u,v)}{(v-u)^2})\right)$$

for any $u, v \in I_\delta(x_0)$, where (with the same notations as the proof of Lemma 2.4)

$$S(u,v) := (R_1(v) - R_1(u))^2 + 2A'(x_0)(v-u)(R_1(v) - R_1(u))$$

has $|S(u,v)| \leq \epsilon(1 + 2|A'(x_0)|)(v-u)^2$. Taking reciprocals, we obtain the desired conclusion by chosing

$$\rho(u,v) := -\frac{S(u,v)}{1 + \frac{S(u,v)}{(v-u)^2}}.$$

□

Lemma 2.8. *Let A be of class C^2. Then for any $x_0 \in J$ and any $\epsilon > 0$ there is $\delta > 0$ such that*

$$\frac{1}{s^2(x)} = \frac{1}{s^2(x_0)} (1 + \beta(x)) \text{ with } |\beta(x)| < \epsilon, \tag{2.15}$$

whenever $x \in I = I_\delta(x_0) := (x_0 - \delta, x_0 + \delta)$.

Proof. Taylor's theorem gives $\delta_1 > 0$ (which we may choose to be finite) such that for any $x \in I_{\delta_1}(x_0)$ we have

$$f(x) = f(x_0)\left(1 + \frac{R_0(x)}{f(x_0)}\right)$$

with $|R_0(x)| \leq \|f'\|_{L^\infty(I)}|x - x_0|$, where $f(x) = 1/s^2(x)$. Since

$$\|f'\|_{L^\infty(I)} \leq 2\|A'\|_{L^\infty(I)}\|A''\|_{L^\infty(I)},$$

the conclusion holds if we choose $\beta(x) = R_0(x)s^2(x_0)$ and

$$0 < \delta \leq \min\left\{\delta_1,\ \frac{1}{2}\frac{s^2(x_0)}{\|A'\|_{L^\infty(I)}\|A''\|_{L^\infty(I)}}\right\}.$$

□

3. Proofs of the main results

3.1. **Proof of Theorem 1.1.** The proof of conclusion (a) will follow from finitely many applications of Lemmas 2.3 through 2.8. We henceforth set $\delta > 0$ to be the minimum among the positive numbers δ obtained in those lemmas. Let $I = (x_0 - \delta, x_0 + \delta)$ and let $\mathbf{z} := (u + iA(u), x + iA(x), v + iA(v))$ be any three-tuple of distinct points on $\Gamma(I)^3$. By Lemma 2.3 such points are non-collinear, thus $c^2(\mathbf{z})$ is strictly positive and it admits the representation (2.4). We may assume without loss of generality that $u < x < v$. We write $v - u = (v - x) + (x - u)$ and obtain that the numerator in the righthand side of (2.4) equals 4 times

$$\left[(v-x)\int_u^x A'(t)dt \;-\; (x-u)\int_x^v (A'(t))dt\right]^2 .$$

Adding and subtracting the quantity $A'(u)$ from the first integral, and the quantity $A'(x)$ from the second integral, leads us to the following expression for the numerator in the righthand side of (2.4):

$$\left[(v-x)\int_u^x (A'(t) - A'(u))dt \;-(v-x)(x-u)\Big(A'(x) - A'(u)\Big) \;-(x-u)\int_x^v (A'(t) - A'(x))dt\right]^2 .$$

Applying Lemma 2.6 to the each of the two integral terms, and Corollary 2.5 to the remaining term, we see that the above quantity equals

$$\begin{aligned}
&\Big[(v-x)(x-u)A''(x_0)\Big]^2 \times \\
&\qquad\qquad \Big[(x-u)\Big(1 + R(u,x)\Big) - 2(x-u)\Big(1 + \mu(u,x)\Big) - (v-x)\Big(1 + R(x,v)\Big)\Big]^2 \\
=&\Big[(v-x)(x-u)(v-u)A''(x_0)\Big]^2 \\
&\qquad\qquad + \Big[C^2(u,x,v) - 2(v-u)\,C(u,x,v)\Big][(v-x)(x-u)A''(x_0)]^2
\end{aligned}$$

where

$$C(u,x,v) \;:=\; (x-u)\Big[R(u,x) - 2\mu(u,x)\Big] - (v-x)R(x,v)\,, \tag{3.1}$$

and $|\mu(u,x)| < \epsilon$ and $|R(u,x)| < \epsilon$. Furthermore, each of $(x-u)$ and $(v-x)$ is less than $(v-u)$, thus

$$|C^2(u,x,v) - 2(v-u)\,C(u,x,v)| \leq 8\epsilon(1+2\epsilon)(v-u)^2. \tag{3.2}$$

Plugging the above in (2.4) we obtain

$$c^2(\mathbf{z}) = E_1(u,x,v) \;+\; (A''(x_0))^2 \times$$

$$\times \left[\frac{(v-x)^2}{(v-x)^2 + (A(v)-A(x))^2}\right]\left[\frac{(x-u)^2}{(x-u)^2 + (A(x)-A(u))^2}\right]\left[\frac{(v-u)^2}{(v-u)^2 + (A(v)-A(u))^2}\right]$$

with

$$E_1(u,x,v) := (A''(x_0))^2 \times$$

$$\times \left[\frac{(v-x)^2}{(v-x)^2 + (A(v)-A(x))^2}\right]\left[\frac{(x-u)^2}{(x-u)^2 + (A(x)-A(u))^2}\right]\left[\frac{C^2(u,x,v) - 2(v-u)\,C(u,x,v)}{(v-u)^2 + (A(v)-A(u))^2}\right].$$

We now apply Lemma 2.7 to each of the fractional factors in $E_1(u,x,v)$ (use (3.2) to deal with the third factor) and obtain that

$$|E_1(u,x,v)| \leq \left(\frac{A''(x_0)}{s^3(x_0)}\right)^2 \Big|(1+\rho(x,v))(1+\rho(u,x))(1+\rho(u,v))\Big| 8\epsilon(1+2\epsilon) \leq \epsilon'.$$

One more application of Lemma 2.7 gives

$$\begin{aligned} c^2(\mathbf{z}) &= \left(\frac{A''(x_0)}{s^3(x_0)}\right)^2 \Big(1+\rho(x,v)\Big)\Big(1+\rho(u,x)\Big)\Big(1+\rho(u,v)\Big) \;+\; E_1(u,x,v) \\ &= \kappa_0^2 + E_2(u,x,v)\,, \quad \text{with } |E_2(u,x,v)| < \epsilon\,. \end{aligned}$$

The proof of part (a) in Theorem 1.1 is concluded.

To prove conclusion (b), we begin by making the following claim:

$$\mathtt{S}[\,\mathrm{Re}K_\Gamma\,](\mathbf{z}) = \frac{3}{2}\kappa_0^2 \;+\; \lambda(\mathbf{z}) \quad \text{with } |\lambda(\mathbf{z})| < \epsilon \tag{3.3}$$

whenever $\mathbf{z} \in \Gamma(I)^3$. To see this, recall that Lemma 2.1 gives

$$\begin{aligned} \mathtt{S}[\,\mathrm{Re}K_\Gamma\,](\mathbf{z}) &= 2\sum_{\substack{j\\ k<l}} \frac{1}{s^2(x_j)\ell_k^2\,\ell_l^2} \Big[A'(x_j)(x_j-x_k) - \Big(A(x_j)-A(x_k)\Big)\Big] \\ &\qquad\qquad \times \Big[A'(x_j)(x_j-x_l) - \Big(A(x_j)-A(x_l)\Big)\Big] \\ &= 2\sum_{\substack{j\\ k<l}} \frac{1}{s^2(x_j)\ell_k^2\,\ell_l^2} \left[\int_{x_k}^{x_j} (A'(t)-A'(x_j))dt\right] \times \left[\int_{x_\ell}^{x_j} (A'(t)-A'(x_j))dt\right]. \end{aligned}$$

By Lemmas 2.6, 2.7 and 2.8 the latter equals

$$\begin{aligned} &\frac{1}{2}(A''(x_0))^2 \sum_{\substack{j\\ k<l}} \frac{1}{s^2(x_j)} \frac{(x_j-x_k)^2}{\ell_\ell^2} \frac{(x_j-x_\ell)^2}{\ell_k^2} (1+R_{jk})(1+R_{j\ell}) \\ &\qquad = \frac{1}{2}\left(\frac{A''(x_0)}{s^2(x_0)}\right)^2 \sum_{\substack{j\\ k<l}} \frac{1}{s^2(x_j)} (1+\rho_{jk})(1+\rho_{j\ell})(1+R_{jk})(1+R_{j\ell}) \\ &\qquad = \frac{1}{2}\left(\frac{A''(x_0)}{s^3(x_0)}\right)^2 \sum_{\substack{j\\ k<l}} (1+\beta_j)(1+\rho_{jk})(1+\rho_{j\ell})(1+R_{jk})(1+R_{j\ell}) \\ &\qquad = \frac{1}{2}\kappa_0^2\,[3+\eta(\mathbf{z})] \quad \text{with } |\eta(\mathbf{z})| < \epsilon \end{aligned}$$

whenever $\mathbf{z} \in \Gamma(I)^3$. This ends the proof of (3.3). The conclusion of the proof of part (b) is now an immediate consequence of (3.3) along with part (a) and the familiar split (1.16).

3.2. **Proof of Theorem 1.3.** For notational simplicity, we write $z_j \in \Gamma^3$ as $z_j = x_j + iA(x_j) = x_j + iA_j$ and similarly set $A'_j = A'(x_j),\ s_j = s(x_j)$. Recall from Lemma 2.1 that

$$\begin{aligned}
\frac{1}{2}\mathtt{S}[\mathrm{Re}(K_\Gamma)](\mathbf{z}) &= \sum_{\substack{j=1\\k<l}}^{3} \frac{1}{\ell_k^2\ell^2 s_j^2}\Big[A'_j(x_j - x_k) - (A_j - A_k)\Big] \times \Big[A'_j(x_j - x_l) - (A_j - A_\ell)\Big]\\
&= \frac{1}{\ell_1^2\ell_2^2\ell_3^2}\sum_{\substack{j=1\\k<l}}^{3} \frac{\ell_j^2}{s_j^2}\Big[A'_j(x_j - x_k) - (A_j - A_k)\Big] \times \Big[A'_j(x_j - x_l) - (A_j - A_l)\Big]\\
&= \frac{1}{\ell_1^2\ell_2^2\ell_3^2}\sum_{\substack{j=1\\k<l}}^{3} \frac{\ell_j^2}{s_j^2}\Big[\int_{x_j}^{x_k}(A'(x) - A'(x_j))\,dx\Big] \times \Big[\int_{x_j}^{x_l}(A'(x) - A'(x_j))\,dx\Big].
\end{aligned}$$

Now the hypothesis (1.17) along with the fact that $(x_k - x_j)^2 \le (x_k - x_j)^2 + (A_k - A_j)^2 = \ell_l^2$ (and similarly, $(x_l - x_j)^2 \le \ell_k^2$) lead us to

$$\begin{aligned}
\frac{1}{2}\Big|\mathtt{S}[\mathrm{Re}(K_\Gamma)](\mathbf{z})\Big| &\le \frac{M^2}{\ell_1^2\ell_2^2\ell_3^2}\sum_{j=1}^{3}\frac{\ell_j^2}{s_j^2}\frac{(x_k - x_j)^2}{2} \times \frac{(x_l - x_j)^2}{2}\\
&\le \frac{M^2}{\ell_1^2\ell_2^2\ell_3^2}\sum_{j=1}^{3}\frac{\ell_j^2\ell_k^2\ell_l^2}{4\,s_j^2}\\
&= \frac{M^2}{4}\sum_{j=1}^{3}\frac{1}{s_j^2}\\
&\le \frac{3}{4}M^2,
\end{aligned}$$

where the last inequality follows from the trivial bound $s_j^2 = 1 + (A'_j)^2 \ge 1$ for all j. The proof is concluded.

3.3. **Proof of Lemma 1.4.** Applying Lemma 2.1 to $A(x) = x^3$ and $x_1 = -\epsilon\alpha$; $x_2 = 0$; $x_3 = \epsilon\beta$ with $\alpha, \beta \in [0,1]$ and $\epsilon \in (0,1)$ we find that

$$\begin{aligned}
(3.4)\quad &\frac{1}{2}\,\mathtt{S}[\mathrm{Re}(K_h)](\mathbf{z})\\
&= \frac{\epsilon^2\Big(4(\alpha-\beta)^2 + 3\alpha\beta\Big) + \epsilon^4 X(\alpha,\beta) + \epsilon^6 Y(\alpha,\beta) + \epsilon^8 W(\alpha,\beta) + \epsilon^{10} Z(\alpha,\beta)}{(1+9\epsilon^2\alpha^2)(1+9\epsilon^2\beta^2)(1+\epsilon^4\alpha^4)(1+\epsilon^4\beta^4)[1+\epsilon^4(\beta^2-\alpha\beta+\alpha^2)^2]},
\end{aligned}$$

where each of X, Y, W and Z is a real-valued polynomials in $\mathbb{R}^2$ and thus achieves a minimum for $(\alpha,\beta) \in [1/2,1]^2$, which we call X_0, Y_0, W_0 and Z_0, respectively. Since the Lipschitz constant of $A'(x) = 3x^2$ in the interval $|x| < \epsilon$ is $M_\epsilon = 6\epsilon$, it follows that the numerator in the expression above is bounded below by the quantity

$$\frac{M_\epsilon^2}{24} + 2\epsilon^4 X_0 + 2\epsilon^6 Y_0 + 2\epsilon^8 W_0 + 2\epsilon^{10} Z_0$$

for all $(\alpha,\beta) \in [1/2,1]^2$ and for each $\epsilon > 0$.

On the other hand, the denominator in the righthand side of (3.4) is easily seen to be bounded above by the quantity

$$10^2 \cdot 2^2 \cdot \left(1 + \frac{25}{16}\right) = 5^2 \cdot 41$$

for all $(\alpha, \beta) \in [1/2, 1]^2$ and for each $0 < \epsilon < 1$.

One now considers various cases, depending on the sign of each of X_0, Y_0, W_0 and Z_0: if these are all non-negative, then it is clear from the above that

$$\mathtt{S}[\mathrm{Re}(K_h)](\mathbf{z}) \geq c_0 M_\epsilon^2 \quad \text{with} \quad c_0 := \frac{1}{24 \cdot 5^2 \cdot 41} \quad \text{and for all} \quad 0 < \epsilon < \delta_0 := 1.$$

Suppose next that, say, $X_0 < 0$ whereas Y_0, W_0 and Z_0 are all non-negative: in this case it follows that

$$\mathtt{S}[\mathrm{Re}(K_h)](\mathbf{z}) \geq \frac{M_\epsilon^2}{24 \cdot 5^2 \cdot 41} - \epsilon^2 \frac{2|X_0|}{24 \cdot 5^2 \cdot 41},$$

which gives that

$$\mathtt{S}[\mathrm{Re}(K_h)](\mathbf{z}) \geq c_0 M_\epsilon^2 \quad \text{with} \quad c_0 := \frac{1}{48 \cdot 5^2 \cdot 41} \quad \text{and for all } 0 < \epsilon^2 < \min\{1, \delta_0^2\},\ \delta_0^2 := \frac{3}{8|X_0|}.$$

(Here we have used the fact that $M_\epsilon = 6\epsilon$.)

Similarly, the case: $X_0 < 0, Y_0 < 0$ and $W_0 \geq 0, Z_0 \geq 0$ leads to

$$c_0 := \frac{1}{48 \cdot 5^2 \cdot 41} \quad \text{and} \quad \delta_0^4 := \frac{3}{8(|X_0| + |Y_0|)};$$

etc. The proof of the Lemma is concluded.

Proof of Lemma 1.5. Let $\mathbf{z} = (u + iA(u); x + iA(x); v + iA(v))$ be a three-tuple of distinct points in Γ. Without loss of generality we may assume that

$$u < x < v\,.$$

Lemma 2.2 gives

$$c^2(\mathbf{z}) = \frac{4\ [A(x)(v-u) - A(u)(v-x) - A(v)(x-u)]^2}{\ell_u^2\, \ell_x^2\, \ell_v^2}.$$

Since $\ell_u^2 = (v-x)^2 + (A(v) - A(x))^2$, etc., we have that the denominator in the representation formula for $c^2(\mathbf{z})$ is bounded below by the quantity

$$\ell_u^2\, \ell_x^2\, \ell_v^2 \geq (v-x)^2 (x-u)^2 (x-u)^2\,. \tag{3.5}$$

On the other hand, the numerator in the formula for $c^2(\mathbf{z})$ equals

$$4[(A(x) - A(u))(v-x) - (A(v) - A(x))(x-u)]^2 =$$

$$4\left[\int_u^x (A'(t) - A'(u))dt\,(v-x) - (A'(v) - A'(u))(x-u)(v-x) - \int_x^v (A'(t) - A'(v))dt\,(x-u)\right]^2.$$

But the latter is bounded above by the quantity

$$4M^2\left[\frac{(x-u)^2}{2}(v-x) + (v-u)(x-u)(v-x) + \frac{(v-x)^2}{2}(x-u)\right]^2$$

and since each of $(x-u)$ and $(v-x)$ is less than $(v-u)$, the quantity above is further bounded by

$$8M^2(v-x)^2(x-u)^2(x-u)^2.$$

Combining the latter with (3.5) we obtain the desired conclusion.

3.4. **Proof of Theorem 1.7.** Let us recall from Lemma 2.1 that

$$\mathtt{S}[\,\mathrm{Re}\,K_h\,](\mathbf{z}) = \sum_{\substack{j\\ k<l}} \frac{1}{s^2(x_j)\ell_k^2\,\ell_l^2}\Big\{A'(x_j)(x_j-x_k) - \Big(A(x_j)-A(x_k)\Big)\Big\} \times \Big\{A'(x_j)(x_j-x_l) - \Big(A(x_j)-A(x_l)\Big)\Big\}$$

for any three-tuple $\mathbf{z} = \{z_1, z_2, z_3\}$ of distinct points on Γ. We claim that each of the three terms in the above summation is non-negative by the assumed fixed concavity of Γ, that is by the hypothesis that

$$A''(x) \geq 0 \;\text{ for every } x \in J\,, \quad (\text{alt. } A''(x) \leq 0 \text{ for every } x \in J). \tag{3.6}$$

To see this, we assume (without loss of generality) that the three distinct points $\{z_1, z_2, z_3\}$ have been labeled so that

$$x_1 < x_2 < x_3\,, \quad \text{where } z_j = x_j + iA(x_j). \tag{3.7}$$

Now examining for instance the term corresponding to $j = 2$ we find that

$$\Big\{A'(x_2)(x_2-x_1) - \Big(A(x_2)-A(x_1)\Big)\Big\}\Big\{A'(x_2)(x_2-x_3) - \Big(A(x_2)-A(x_3)\Big)\Big\} = \tag{3.8}$$

$$= \left(\int_{x_1}^{x_2}\Big(A'(x_2)-A'(x)\Big)dx\right)\left(\int_{x_3}^{x_2}\Big(A'(x_2)-A'(x)\Big)dx\right) =$$

$$= \left(\int_{x_1}^{x_2}\Big(A'(x_2)-A'(x)\Big)dx\right)\left(\int_{x_2}^{x_3}\Big(A'(x_2+x_3-t)-A'(x_2)\Big)dt\right)$$

and it is immediate to see that the latter is non-negative because of (3.6) and (3.7). The remaining two terms are dealt with in a similar fashion. The proof is concluded.

4. Examples

4.1. **Failure of global relative boundedness for $\mathtt{S}[\mathrm{Re}K_\Gamma]$ and $\mathtt{S}[\mathrm{Im}K_\Gamma]$).** Let $A : (-1, 2) \to \mathbb{R}$ be a smooth function obeying the following constraints:

$$A(0) = 0; \quad A\left(\frac{1}{2}\right) = 0; \quad A(1) = 0;$$

$$A'(0) = 0; \quad A'\left(\frac{1}{2}\right) = -1; \quad A'(1) = 0\,.$$

For instance, the function

$$A(x) = \chi(x)\,\sin 2\pi x \tag{4.1}$$

where χ is in $C_0^\infty\Big((0,1)\Big)$ and has $\chi(1/2) = 1/(2\pi)$, satisfies all of the conditions above. If we further require that $\chi'(1/2) \neq 0$ then $A''(1/2) \neq 0$ and therefore the curve $\Gamma = \{x + iA(x),\ x \in J = (-1, 2)\}$ has nonzero curvature κ_0 at the the point

$$z_0 := \frac{1}{2} \in \Gamma\,.$$

We claim that for such Γ and for $\tilde\epsilon > 0$ as in Corollary 1.2, the interval

$$\tilde I = \left(\frac{1}{2} - \tilde\delta, \frac{1}{2} + \tilde\delta\right)$$

that was obtained there is strictly contained in $J = (-1, 2)$. To see this, we argue by contradiction and suppose that (1.15) were to hold for any $\mathbf{z} \in \Gamma^3$. Invoking the conclusions and notation of [14, Proposition 2.2], it is easy to see that the presumed validity of any of the two inequalities displayed in (1.15) is equivalent to the requirement that

$$\Big|1 - \mathcal{R}_h(\mathbf{z})\Big| \le \frac{\tilde\epsilon}{\kappa_0^2 - \tilde\epsilon} \quad \text{for all non-collinear three-tuples } \; \mathbf{z} \in \Gamma^3.$$

But the latter would obviously imply that

$$|\mathcal{R}_h(\mathbf{z})| \le C \quad \text{for any non-collinear three-tuple } \mathbf{z} \in \Gamma^3 \tag{4.2}$$

which is, in fact, not possible. To see this, consider ordered three-tuples of of the form

$$\mathbf{z}_\lambda = \Big(0;\; \lambda + iA(\lambda);\; 1\Big) \in \Gamma^3, \quad 0 < \lambda \le \frac{1}{2}\,.$$

Such three-tuples are *admissible* in the sense of [14, Definition 2.1], and the triangles with vertices at $\mathbf{z}_\lambda$ have the following properties as $\lambda \to 1/2$:

$$\theta_{j,\lambda} \to 0,\; j = 1, 2;\;\; \theta_{3,\lambda} \to \pi;\;\; \ell_{j,\lambda} \to \frac{1}{2},\; j = 1, 2;\; \ell_{3,\lambda} = 1.$$

Furthermore, with the notations of [14, (2.6)] for each such triangle we have $\alpha_{21} = 0$. Thus [14, Proposition 2.2] gives that

$$\mathcal{R}_h(\mathbf{z}_\lambda) \;=$$

$$= \frac{\ell_{1,\lambda}}{4\ell_{2,\lambda}\sin^2\theta_{1,\lambda}}\Big(\ell_{1,\lambda}\cos(2h(0)-\theta_{1,\lambda})+\ell_{2,\lambda}\cos(2h(1)-\theta_{2,\lambda})-\ell_{3,\lambda}\cos(2h(\lambda+iA(\lambda))+\theta_{2,\lambda}-\theta_{1,\lambda})\Big).$$

Since $A'(0) = A'(1) = 0$,

$$e^{ih(0)} = e^{ih(1)} = -i = e^{i\pi}\,, \quad \text{thus } \; h(0) = h(1) = \pi\,.$$

Also

$$e^{ih(\frac{1}{2})} = \frac{-1-i}{\sqrt{2}} = e^{-i\frac{3}{4}\pi}\,, \quad \text{thus } \; h\Big(\frac{1}{2}\Big) = -\frac{3}{4}\pi\,.$$

If condition (4.2) were to hold at $\mathbf{z}_\lambda$ for any $0 < \lambda < 1/2$ then it would follow that

$$\ell_{1,\lambda}\Big|\ell_{1,\lambda}\cos(2h(0) - \theta_{1,\lambda}) + \ell_{2,\lambda}\cos(2h(1) - \theta_{2,\lambda}) - \ell_{3,\lambda}\cos(2h(\lambda + iA(\lambda)) + \theta_{2,\lambda} - \theta_{1,\lambda})\Big| \le$$

$$\le\; C\ell_{2,\lambda}\sin^2\theta_{1,\lambda} \quad \text{for every } 0 < \lambda < 1/2\,,$$

but this is not possible because with our choice of h the lefthand side of this inequality tends to

$$\frac{1}{2}\left(\frac{1}{2}\cos 2\pi \;+\; \frac{1}{2}\cos 2\pi \;-\; \cos\left(-\frac{3}{2}\pi\right)\right) = \frac{1}{2}$$

as $\lambda \to 1/2$, whereas the righthand side tends to 0.

4.2. **Failure of global negativity of $\mathtt{S}[\mathrm{Im}K_\Gamma](\mathbf{z})$ for Γ with fixed concavity.** Let $A(x) = x^2$ and set $\Gamma = \{x + iA(x),\ x \in \mathbb{R}\}$. Consider three-tuples of the form

$$\mathbf{z}_\lambda = \Big(-\lambda + i\lambda^2\,;\ 0\,;\ \lambda + i\lambda^2 \Big), \quad \lambda > 0.$$

Lemma 2.1 gives

$$\mathtt{S}\,[\mathrm{Im}\,K_h\,](\mathbf{z}_\lambda) = \frac{32}{\lambda^2(4+\lambda^2)}\left(\frac{2+\lambda^2}{8(1+\lambda^2)} - \frac{1}{4+a^2}\right)$$

and it is clear from the above that $\mathtt{S}\,[\mathrm{Im}\,K_h\,](\mathbf{z}_\lambda) > 0$ for $\lambda \gg 1$.

4.3. **Failure of global positivity of $\mathtt{S}[\mathrm{Re}K_\Gamma](\mathbf{z})$ in the absence of fixed concavity of Γ.** Let $A(x) = x^3$ and set $\Gamma = \{x + iA(x),\ x \in \mathbb{R}\}$. Fix $a > 0$ and let $\lambda > 0$. Consider three-tuples of the form

$$\mathbf{z}_\lambda = \Big(-a - ia^3\,;\ 0\,;\ \lambda + i\lambda^3 \Big), \quad \lambda > 0.$$

We claim that

$$\mathtt{S}\,[\mathrm{Re}K_h\,](\mathbf{z}_\lambda)\ <\ 0 \quad \text{whenever } \lambda \gg a.$$

To prove the claim we express $\mathtt{S}\,[\mathrm{Re}K_h\,](\mathbf{z}_\lambda)$ using (2.1) and (3.8), and obtain

$$\mathtt{S}\,[\mathrm{Re}K_h\,](\mathbf{z}_\lambda)\ =\ I_\lambda\ +\ II_\lambda\ +\ III_\lambda \quad \text{where}$$

$$\mathtt{I}_\lambda = \frac{2\,a\,(2a^2 - \lambda^2)}{(\lambda+a)(1+a^2)\Big[1 + a^2 + \lambda\,(\lambda - a)\Big](1+9a^2)}\,;$$

$$\mathtt{II}_\lambda = -\frac{a\lambda}{(1+a^2)(1+\lambda^2)}\ <\ 0 \quad \text{for any } a > 0, \lambda > 0\,;$$

$$\mathtt{III}_\lambda = \frac{2\,\lambda\,(2\lambda^2 - a^2 + a\lambda)}{(\lambda+a)(1+\lambda^2)\Big[1 + \lambda^2 + a\,(a - \lambda)\Big](1+9\lambda^2)}\,.$$

Note that

$$\mathtt{I}_\lambda < 0 \quad \text{if} \quad \lambda \gg a\,, \quad \text{and} \quad \mathtt{III}_\lambda\ =\ O(\lambda^{-4}).$$

Thus $\mathtt{I}_\lambda + \mathtt{II}_\lambda + \mathtt{III}_\lambda < 0$ whenever $\lambda \gg a$. The claim is proved.

5. Appendix

As a point of comparison with the results of this article, it is interesting to note that the boundedness and positivity of $\mathtt{S}[\mathrm{Re}K_h]$ and $\mathtt{S}[\mathrm{Im}K_h]$ with K_h as in (1.19) fail globally on $\mathbb{C}$ for *any* continuous, non-constant, globally defined $h : \mathbb{C} \to \mathbb{R}$. This appendix summarizes the main results in this direction. The methods of proof bear some similarities to techniques recently developed in a body of work joint in part by Chousionis, Chunaev, Mateu, Prat and Tolsa [4, 5, 6, 7, 8, 9]. For example, we use representation formulæ for symmetrized forms that rely upon a certain labeling scheme of the vertices of a triangle, analogous to [6, Proposition 3.1] and [7, Lemma 6], as well as the computations accompanying the diagrams Figures 3 and 4 in [7, p. 2738]. Hence, for brevity we limit the exposition here to the statements of the pertinent results and defer all proofs to the auxiliary note [14].

5.1. **Preliminaries.** First note that putting $h \equiv 0$ (or a constant) in (1.19) yields the original kernel K_0 (or a constant multiple of it). Also it is immediate to see that

$$\mathtt{S}\,[K_h](\mathbf{z}) \;=\; \mathtt{S}[K_0](\mathbf{z}) = c^2(\mathbf{z}) \tag{5.1}$$

for any three-tuple of distinct points and for any $h : \mathbb{C} \to \mathbb{R}$ (no continuity assumption needed here).

Definition 5.1. We say that an ordered three-tuple of non-collinear points (a, b, c) is arranged in *admissible order* (or is *admissible*, for short) if `(i)` the orthogonal projection of c onto the line determined by a and b falls in the interior of the line segment joining a and b, and `(ii)` the triangle with vertices a, b and c, henceforth denoted $\Delta(a, b, c)$, has positive counterclockwise orientation.

Proposition 5.2. *For any non-constant $h : \mathbb{C} \to \mathbb{R}$ and for any three-tuple $\mathbf{z}$ of non-collinear points in $\mathbb{C}$ we have*

$$\mathtt{S}\,[\mathrm{Re}K_h](\mathbf{z}) = \; c^2(\mathbf{z})\left(\frac{1}{2} + \mathcal{R}_h(\mathbf{z})\right) \tag{5.2}$$

$$\mathtt{S}\,[\mathrm{Im}K_h](\mathbf{z}) = \; c^2(\mathbf{z})\left(\frac{1}{2} - \mathcal{R}_h(\mathbf{z})\right) \tag{5.3}$$

where $\mathcal{R}_h$ is non-constant and invariant under the permutations of the elements of $\mathbf{z}$. If $\mathbf{z} = (z_1, z_2, z_3)$ is admissible then $\mathcal{R}_h(\mathbf{z})$ is represented as follows:

$$\begin{aligned}\mathcal{R}_h(\mathbf{z}) = \frac{2\,\ell_1\ell_2\ell_3}{(4\mathrm{Area}\,\Delta(\mathbf{z}))^2} \times \Big[&\ell_1 \cos(\mathfrak{h}_{z_1,z_2}(z_1) - \theta_1) + \\ &+ \ell_2 \cos(\mathfrak{h}_{z_1,z_2}(z_2) + \theta_2) - \ell_3 \cos(\mathfrak{h}_{z_1,z_2}(z_3) + \theta_2 - \theta_1)\Big].\end{aligned} \tag{5.4}$$

Here θ_j denotes the angle at z_j, and ℓ_j denotes the length of the side opposite to z_j in $\Delta(\mathbf{z})$. Also, we have set

$$\mathfrak{h}_{z_1,z_2}(z) := 2h(z) - 2\alpha_{21}\,, \quad z \in \mathbb{C},$$

where α_{21} is the principal argument of $z_2 - z_1$ (in an arbitrarily fixed coordinate system for $\mathbb{R}^2$).

5.2. **Failure of universal boundedness and positivity for $\mathtt{S}\,[\mathrm{Re}K_h](\mathbf{z})$ and $\mathtt{S}\,[\mathrm{Im}K_h](\mathbf{z})$.** On account of Proposition 5.2, the behavior of the symmetrized forms of $\mathrm{Re}K_h$ and $\mathrm{Im}K_h$ are reduced to the analysis of the reminder $\mathcal{R}_h$.

Theorem 5.3. *Suppose that $h : \mathbb{C} \to \mathbb{R}$ is continuous. The following are equivalent:*

`(i)` *There is a constant $C < \infty$, possibly depending on h, such that*

$$|\mathcal{R}_h(\mathbf{z})| \le C$$

for any three-tuple $\mathbf{z} = \{z_1, z_2, z_3\}$ of non-collinear points in $\mathbb{C}$.

`(ii)` $\mathcal{R}_h(\mathbf{z}) \;=\; 0$ *for any three-tuple of non-collinear points in $\mathbb{C}$.*

`(iii)` *h is constant.*

Corollary 5.4. *Suppose that $h:\mathbb{C}\to\mathbb{R}$ is continuous. Then*

$$h \text{ is constant} \quad \Longleftrightarrow \quad \left(\frac{1}{2}-\mathcal{R}_h(\mathbf{z})\right)\left(\frac{1}{2}+\mathcal{R}_h(\mathbf{z})\right)>0$$

for any three-tuple of non-collinear points in $\mathbb{C}$.

In fact more is true.

Theorem 5.5. *Suppose that $h:\mathbb{C}\to\mathbb{R}$ is continuous.*

(a) *If h is constant, then*

$$\frac{1}{2}+\mathcal{R}_h(\mathbf{z})>0$$

for any three-tuple of non-collinear points.

(b) *If h is not constant, then the function*

$$\mathbf{z} \;\mapsto \frac{1}{2}+\mathcal{R}_h(\mathbf{z})$$

changes sign. That is, there exist two three-tuples of non-collinear points $\mathbf{z}$ and $\mathbf{z}'$ such that

$$\frac{1}{2}+\mathcal{R}_h(\mathbf{z})\;>\;0 \quad \text{and} \quad \frac{1}{2}+\mathcal{R}_h(\mathbf{z}')\;<\;0.$$

Furthermore, (a) *and* (b) *are also true with $\dfrac{1}{2}-\mathcal{R}_h$ in place of $\dfrac{1}{2}+\mathcal{R}_h$.*

Corollary 5.6. *Suppose that $h:\mathbb{C}\to\mathbb{R}$ is continuous. Then*

(a) *h is constant* $\quad\Longleftrightarrow\quad$ $\frac{1}{2}+\mathcal{R}_h(\mathbf{z})>0$
for all three-tuples of non-collinear points.

(b) *h is constant* $\quad\Longleftrightarrow\quad$ $\frac{1}{2}-\mathcal{R}_h(\mathbf{z})>0$
for all three-tuples of non-collinear points.

5.3. **Further results: the dual kernel of K_h.** We define the dual kernel of $K_h(w,z)$ as

$$K_h^*(w,z)=\overline{K_h(z,w)}. \tag{5.5}$$

Thus

$$K_h^*(w,z)=\frac{e^{-ih(z)}}{\overline{z}-\overline{w}}.$$

In particular

$$K_0^*(w,z)=-\overline{K_0(w,z)},$$

so that $\mathtt{S}\,[K_0^*](\mathbf{z})\;=\;\mathtt{S}\,[K_0](\mathbf{z})\;=\;c^2(\mathbf{z})$. On the other hand, for non-constant h we have

Proposition 5.7. *For any non-constant $h:\mathbb{C}\to\mathbb{R}$ and for any three-tuple $\mathbf{z}$ of non-collinear points in $\mathbb{C}$ we have*

$$\mathtt{S}\,[K_h^*](\mathbf{z})\;=\;c^2(\mathbf{z})\,\mathcal{H}(\mathbf{z}) \tag{5.6}$$

where $\mathcal{H}(\mathbf{z})$ is a non-constant function of $\mathbf{z}$ that is invariant under the permutations of the elements of $\mathbf{z}$. In particular, if $\mathbf{z} = (z_1, z_2, z_3)$ is admissible then $\mathcal{H}(\mathbf{z})$ has the following representation.

$$\mathcal{H}(\mathbf{z}) = \frac{2\,\ell_1\ell_2\ell_3}{(4\text{Area}\,\Delta(\mathbf{z}))^2} \times \Big[\ell_1 \cos(h(z_2) - h(z_3) + \theta_1) + \\ + \ell_2 \cos(h(z_1) - h(z_3) - \theta_2) + \ell_3 \cos(h(z_1) - h(z_2) + \theta_3)\Big]. \tag{5.7}$$

Here θ_j and ℓ_j are as in the statement of Proposition 5.2.

Theorem 5.8. *Suppose that $h : \mathbb{C} \to \mathbb{R}$ is continuous. The following are equivalent:*

(i) *There is a constant $C < \infty$, possibly depending on h, such that*

$$|\mathcal{H}(\mathbf{z})| \leq C$$

for any three-tuple $\mathbf{z} = \{z_1, z_2, z_3\}$ of non-collinear points in $\mathbb{C}$.

(ii) $\mathcal{H}(\mathbf{z}) = 1$ *for any three-tuple of non-collinear points in $\mathbb{C}$.*

(iii) *h is constant.*

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