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# Disturbance decoupling by state feedback for uncertain impulsive linear systems

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**Abstract**

The problem of making the output of a linear system with polytopic uncertainties and discontinuities in the state evolution totally insensitive to an unknown disturbance input by state feedback is investigated. Suitable geometric notions are introduced and used to provide a structural, constructive solvability condition. The requirement of achieving global robust asymptotic stability of the compensated dynamics is then added and further solvability conditions are provided by requiring that the time instants at which discontinuities in the state evolution, or jumps, occur are sufficiently far from each other.

**KEYWORDS**

disturbance decoupling, stability, state feedback, structural methods, uncertain impulsive systems

## 1 | INTRODUCTION

Dynamical systems whose state behavior presents discontinuities (or jumps), called impulsive systems, are useful to model impulsive phenomena that may arise in physical systems, like, for example, anelastic collisions of mechanical parts, the activation of switches or abrupt failures of components (see, e.g., References 1-3 and the references therein). For this reason, impulsive linear systems have recently attracted the interest of many researchers in relation to a number of control problems. By extending to the framework of impulsive linear systems approaches and techniques previously developed for linear systems, solvability conditions have been obtained for stabilization problems,[4-6] linear quadratic control problems,[7,8] disturbance decoupling problems,[9-12] output regulation problems,[13-21] and observation problems.[22-25] Structural geometric methods obtained by extending the classical geometric approach of Basile and Marro[26] and of Wonham[27] have been shown to be particularly effective to deal with the mentioned problems for this class of systems.

It has however to be remarked that in all the previously mentioned papers the parameters of the considered systems are assumed to be completely known. This hypothesis is rarely verified in practice, where the knowledge of the parameters of the system at issue is generally affected by uncertainty, and this reduces the applicability of the available results. In order to overcome this limitation, it is interesting to extend the study to impulsive linear systems whose mathematical models are characterized by uncertainty.

[Corrections added on 15 April 2021, after first online publication: In the first sentence of the Introduction section: the word 'in' in front of 'anelastic', 'activation', and 'abrupt' has been deleted.]

A convenient, general way to include uncertainty in mathematical models avoiding conservatism is that of letting the coefficients of the model equations vary in a given polytope. This results in the introduction of the so-called polytopic systems, which are widely used in dealing with robust control problems (see, e.g., References 28-30). In particular, disturbance decoupling problems for polytopic linear systems have been studied in References 31-35, where the existence of solutions that are robust with respect to variations of the parameters was investigated. Also in that case, structural geometric methods derived from the geometric approach of Basile and Marro[26] and of Wonham[27] have been extended and then profitably employed to deal with the considered problems.

With the above motivations and starting from the above mentioned results, in this article, we consider impulsive linear systems with polytopic uncertainties. These are hybrid objects whose state dynamics consists of a linear component and of a sequence of linear jumps which occur at given time instants (jump instants). In considering control problems for this class of system, we look for solutions that are robust with respect to the variations of the parameters in the polytopes of uncertainty and that hold for all possible sets of jump instants in a given class.

In particular, here we concentrate on the study of the disturbance decoupling problem for impulsive linear systems with polytopic uncertainties. Our aim is to investigate the problem from a methodological point of view in order to derive structural, practically checkable solvability conditions.

The problem of decoupling the output of a given system from a disturbance input is one of the fundamental problems in the control of multivariable dynamical systems[36] and a cornerstone of the development of the structural geometric approach[26,27] to control theory. The problem has been investigated by extending the methods described in References 26 and 27 for several classes of systems, including nonlinear systems, systems over rings, infinite dimensional systems, 2D systems and others. Examples of practical applications in which the problem is efficiently solved by structural methods range from the control of simple mechanical systems,[37] to that of active suspensions for road vehicles,[38] big ships,[39] distillation columns,[40] manufacturing systems,[41] multi-agent systems,[42] and strategies in dynamic games.[43]

Our approach consists first in developing a suitable, novel notion of controlled invariance, which takes into account the occurrence of discontinuities in the state behavior and which is robust with respect to the variations of the system's parameters in the polytope of uncertainty. This is the main geometric structural tool that makes it possible to state a sufficient structural condition for the existence of a solution to the disturbance decoupling problem, namely of a state feedback that robustly decouples the output from an unknown disturbance input in presence of finitely many jump instants in any finite interval. An algorithmic procedure to check the solvability condition and to construct the feedback is provided.

Then, we add the requirement of stability of the compensated system to the decoupling requirement. Stability depends on the interaction between the so-called flow dynamics and the jump behavior and, in case the first is stable, on the length of the time interval that separates consecutive jump instants. The existence of a common Lyapunov function for the flow dynamics of the compensated systems corresponding to the vertices of the polytope of uncertainty is the additional condition that, together with the structural one, assures solvability of the disturbance decoupling problem with stability. If both conditions hold, there exists a state feedback that robustly decouples the output from an unknown disturbance input and that stabilizes the compensated dynamics for all the sequences of jump instants in which consecutive elements are sufficiently distant from each other. The conditions concerning the existence of a common Lyapunov function can be checked using LMIs and an algorithmic procedure to construct the feedback is given.

This article is organized as follows. In Section 2, we introduce the class of impulsive dynamical systems with polytopic uncertainties we will deal with and we state formally the disturbance decoupling problem we want to study. In Section 3, we introduce the geometric notions of robust hybrid controlled invariance and we characterize the robust hybrid controlled invariant subspaces of the state space. An algorithmic procedure to compute the maximum robust hybrid controlled invariant subspace contained in a given subspace is given. The maximum robust hybrid controlled invariant subspace contained in the kernel of the output map for all values of the uncertain parameters is then used to state a sufficient structural solvability condition for the considered problem in Theorem 1. The stability requirement is discussed in Section 4, where Theorem 2 provides a sufficient condition for the solution of the disturbance decoupling problem with stability. An example is presented in Section 5. Appendix A contains the details of the computations performed on the example of Section 5.

*Notations.* The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ are used for the sets of natural numbers, real numbers, non negative real numbers, respectively. Matrices and linear maps between vector spaces are denoted by slanted capital letters like $A$. Sets, vector spaces and subspaces are denoted by calligraphic capital letters like $\mathcal{X}$. The quotient space of a vector space $\mathcal{X}$ over a subspace $\mathcal{V} \subseteq \mathcal{X}$ is denoted by $\mathcal{X}/\mathcal{V}$. The restriction of a linear map $A$ to an $A$-invariant subspace $\mathcal{V}$ is denoted by $A|_{\mathcal{V}}$. For a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes the Euclidean norm and, for a matrix $A$, $\|A\|$ denotes the matrix norm defined by

$\|A\| = sup_{x\neq 0}\frac{\|Ax\|}{\|x\|}$. Analogously, $\|A\|_1$ and $\|A\|_\infty$ denote the 1-norm and the ∞-norm of $A$, which are respectively defined by $\|A\|_1 = \max_{j=1,\dots,n}\sum_{i=1}^{n} a_{ij}$ and by $\|A\|_\infty = \max_{i=1,\dots,n}\sum_{j=1}^{n} a_{ij}$. $I_n$ denotes the identity matrix of order $n$. For a linear map $A : \mathcal{X} \to \mathcal{Y}$ from a vector space $\mathcal{X}$ into a vector space $\mathcal{Y}$, the image and the kernel of $A$ are denoted respectively by Im $A$ ($\subseteq \mathcal{Y}$) and by Ker $A$ ($\subseteq \mathcal{X}$); the inverse image with respect to $A$ of a subspace $\mathcal{V} \subseteq \mathcal{Y}$ is the subspace $A^{-1}\mathcal{V} = \{x \in \mathcal{X}$ such that $Ax \in \mathcal{V} \subseteq \mathcal{Y}\}$. Given a real vector space $\mathcal{X}$ of dimension $n$, we denote by $\mathcal{X}^{\oplus N}$ the external direct sum of $N$ copies of $\mathcal{X}$, namely the $nN$-dimensional vector space consisting of the vectors $(x_1^\top, \dots x_N^\top)^\top$, with $x_i \in \mathcal{X}$, with the obvious addition and scalar multiplication. Given a subspace $\mathcal{V} \subseteq \mathcal{X}$, $\mathcal{V}^{\oplus N} \subseteq \mathcal{X}^{\oplus N}$ denotes the external direct sum of $N$ copies of $\mathcal{V}$, namely $\mathcal{V}^{\oplus N}$ is the subspace consisting of all vectors $(v_1^\top, \dots, v_N^\top)^\top \in \mathcal{X}^{\oplus N}$ with $v_i \in \mathcal{V}$. If $A_i$, for $i = 1, \dots, N$, are $N$ linear maps from $\mathcal{X}$ to $\mathcal{Y}$ (respectively, $N$ matrices of the same dimensions), we denote shortly by $\oplus_{i=1}^{N} A_i$ the linear map from $\mathcal{X}$ to $\mathcal{Y}^{\oplus N}$ given by $\mathcal{X} \ni x \mapsto ((A_1x)^\top, \dots, (A_Nx)^\top)^\top \in \mathcal{Y}^{\oplus N}$ (respectively, the matrix $((A_1)^\top \dots (A_N)^\top)^\top$).

## 2 | PRELIMINARIES AND PROBLEM STATEMENT

Let us denote by $\mathcal{S}$ the set of all maps $\sigma : \mathbb{N} \to \mathbb{R}^+$ which, letting $\tau_\sigma$ be defined by $\tau_\sigma = \inf\{\sigma(0), \sigma(k+1) - \sigma(k); k \in \mathbb{N}, \sigma(k+1) \neq \sigma(k)\}$, satisfy the condition $\tau_\sigma > 0$. This condition implies that $\sigma(k+1)$ is greater than or equal to $\sigma(k)$ for all $k \in \mathbb{N}$. Moreover, it implies that the set of points in the image of $\sigma$, that is, Im $\sigma = \{t \in \mathbb{R}^+, t = \sigma(k)$ for some $k \in \mathbb{N}\}$, is a discrete, finite or countably infinite subset of $\mathbb{R}^+$, whose subsets (including Im $\sigma$ itself) have no accumulation points. We say that $\tau_\sigma$ is the *dwell time* of $\sigma$. In the following, given $\tau \in \mathbb{R}^+$, we will denote by $\mathcal{S}_\tau$ the subset of $\mathcal{S}$ defined by $\mathcal{S}_\tau = \{\sigma \in \mathcal{S}, \tau_\sigma \geq \tau\}$.

The dynamical systems we consider present jump discontinuities, called simply jumps, in the state evolution and coefficients that are affected by polytopic uncertainties. They are described by sets of linear equations of the following form

$$\Sigma_\sigma \equiv \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & \text{for } t \neq \sigma(k),\ k \in \mathbb{N} \\ x(\sigma(k)) = Jx^-(\sigma(k)) & \text{for } t = \sigma(k),\ k \in \mathbb{N} \\ y(t) = Cx(t) & \end{cases} \tag{1}$$

where $t \in \mathbb{R}^+$ is the time variable; $\sigma$ belongs to $\mathcal{S}$; $x \in \mathcal{X} = \mathbb{R}^n$, $u \in \mathcal{U} = \mathbb{R}^m$, and $y \in \mathcal{Y} = \mathbb{R}^p$ denote, respectively, the state, the control input, and the output variables; the matrices $A$, $B$, $C$, and $J$ have, respectively, the form

$$A = A(\mu) = \sum_{i\in I} \mu_i A_i\ ; \quad B = B(\mu) = \sum_{i\in I} \mu_i B_i\ ;$$
$$C = C(\mu) = \sum_{i\in I} \mu_i C_i\ ; \quad J = J(\mu) = \sum_{i\in I} \mu_i J_i$$

where $I = \{1, \dots, N\}$ is a set of indices; the vector of the *polytopic uncertain parameters* $\mu = (\mu_1, \dots, \mu_N)^\top$ belongs to the standard $(N-1)$-dimensional symplex $\Delta^{(N-1)}$ in $\mathbb{R}^N$ (that is: $\mu_i \geq 0$ for $i \in I$ and $\sum_{i\in I}\mu_i = 1$); $A_i$, $B_i$, $C_i$, and $J_i$ are real matrices of suitable dimensions that form the *vertices* of the related *uncertainty polytopes*; $x^-(\sigma(k))$ denotes the limit of $x(t)$ for $t$ which goes to $\sigma(k)$ from the left, that is $x^-(\sigma(k)) = lim_{t\to\sigma(k)^-}x(t)$. In other words, the state $x(t)$ of $\Sigma_\sigma$, starting from an initial condition $x(0) = x_0$ at time $t = 0$, evolves continuously on the time interval $[0, \sigma(0))$ according to the dynamics given by the first block of equations in (1). Then, at time $t = \sigma(0)$, instead of taking the value $x^-(\sigma(0))$, the state jumps to $Jx^-(\sigma(0))$, as stated in the second block of equations in (1). The same behavior repeats on each one of the subsequent time intervals $[\sigma(k), \sigma(k+1))$, with initial condition $x(\sigma(k))$. In practice, we have $x(\sigma(0)) = Je^{A\sigma(0)}x(0)$ and $x(\sigma(k+1)) = Je^{A(\sigma(k+1)-\sigma(k))}x(\sigma(k))$ for $k \in \mathbb{N}$. We say that the equations in the first block in (1) represent the *flow dynamics* of $\Sigma_\sigma$, while the equations in the second block represent the *jump behavior* of $\Sigma_\sigma$. A single jump occurs at each point $\sigma(k) \in$ Im $\sigma$. The points in Im $\sigma$ are called *jump instants* and the time interval between consecutive jumps in the behavior of $\Sigma_\sigma$ is greater than or equal to the dwell time $\tau_\sigma$. The jump behavior depends on $J$ and on the choice of the map $\sigma$ and, in particular, it is not necessarily periodic. Note that we do not assume that the interval between consecutive jumps has an upper bound. In case Im $\sigma$ is a finite set, no jumps occur for $t > \max_{k\in\mathbb{N}}\sigma(k)$ and, on the interval $[\max_{k\in\mathbb{N}}\sigma(k), +\infty)$, the dynamics of $\Sigma_\sigma$ reduces to the flow dynamics.

[Corrections added on 15 April 2021, after first online publication: In the third and fifth line from the bottom: 'consecutive jump' has been changed to 'consecutive jumps'.]

We refer to systems of the above kind, which combine polytopic uncertainty with the discontinuous state behavior of linear impulsive systems,[44,45] as *uncertain impulsive (linear) systems*.

The hybrid control systems $\Sigma_i$ given by

$$\Sigma_i \equiv \begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) \;\; \text{for} \;\; t \neq \sigma(k), \; k \in \mathbb{N} \\ x(\sigma(k)) = J_i x^-(\sigma(k)) \;\; \text{for} \;\; t = \sigma(k), \; k \in \mathbb{N} \\ y(t) = C_i x(t) \end{cases} \tag{2}$$

for $i \in I$ form the family $\{\Sigma_i\}_{i\in I}$ of the *vertex systems*, or simply of the *vertices*, of the uncertain impulsive system $\Sigma_\sigma$.

If we apply a state feedback of the form $u(t) = Fx(t)$ to any element of the family of vertices $\{\Sigma_i\}_{i\in I}$, we have a family $\{\Sigma_i^F\}_{i\in I}$ of closed-loop hybrid systems whose elements are given by

$$\Sigma_i^F \equiv \begin{cases} \dot{x}(t) = (A_i + B_i F) x(t) \;\; \text{for} \;\; t \neq \sigma(k), \; k \in \mathbb{N} \\ x(\sigma(k)) = J_i x^-(\sigma(k)) \;\; \text{for} \;\; t = \sigma(k), \; k \in \mathbb{N} \\ y(t) = C_i x(t) \end{cases} \tag{3}$$

that can be viewed as the family of vertices of the compensated uncertain impulsive system $\Sigma_\sigma^F$ given by

$$\Sigma_\sigma^F \equiv \begin{cases} \dot{x}(t) = (A + BF) x(t) \;\; \text{for} \;\; t \neq \sigma(k), \; k \in \mathbb{N} \\ x(\sigma(k)) = J x^-(\sigma(k)) \;\; \text{for} \;\; t = \sigma(k), \; k \in \mathbb{N} \\ y(t) = C x(t) \end{cases} \tag{4}$$

In case the considered uncertain impulsive system is subject to an additional unknown input $d(t)$, its equations take the form

$$\Sigma_{D\sigma} \equiv \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dd(t) \;\; \text{for} \;\; t \neq \sigma(k), \; k \in \mathbb{N} \\ x(\sigma(k)) = J x^-(\sigma(k)) \;\; \text{for} \;\; t = \sigma(k), \; k \in \mathbb{N} \\ y(t) = C x(t) \end{cases} \tag{5}$$

where $d \in \mathcal{D} = \mathbb{R}^q$ is the disturbance, the matrix $D$ has the form

$$D = D(\mu) = \sum_{i=1}^{N} \mu_i D_i$$

and the other notations are as in Equation (1).

In such situation, one is interested in compensating the system $\Sigma_{D\sigma}$ by a state feedback $u(t) = Fx(t)$ in such a way that the disturbance $d(t)$ does not affect the output $y(t)$ for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathcal{S}$. More formally, this problem is stated as follows.

**Problem 1** (Disturbance Decoupling Problem). Given a disturbed uncertain impulsive system $\Sigma_{D\sigma}$ of the form (5), the Disturbance Decoupling Problem (DDP) by state feedback for $\Sigma_{D\sigma}$ consists in finding a state feedback $F : \mathcal{X} \to \mathcal{U}$, if any exists, such that the disturbance $d(t)$ does not affect the output $y(t)$ of the compensated uncertain impulsive system $\Sigma_{D\sigma}^F$ for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathcal{S}$, or, in other words, such that the output $y(t)$ of $\Sigma_{D\sigma}^F$ initialized at $x(0) = 0$ is identically null (i.e., $y(t) = 0$ for $t \geq 0$), for all $\mu \in \Delta^{N-1}$, all $\sigma \in \mathcal{S}$ and any disturbance input $d(t)$.

Note that the requirement of making the output $y(t)$ insensitive to the disturbance $d(t)$ by means of a single state feedback $F$ for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathcal{S}$ is quite strong, but, since no information is assumed to be available other than the polytopic structure of the uncertainty and the knowledge of the vertex systems, this is the only way to find, if any exists, an implementable solution.

[Corrections added on 15 April 2021, after first online publication: In the first and fourth lines of Problem 1: 'the form' and 'other word' have been changed to 'of the form' and 'other words', respectively.]

## 3 | PROBLEM SOLUTION

In order to find structural conditions for the solvability of the DDP stated in the previous section, we need to introduce suitable geometric tools. Taking into account the uncertainty and the presence of jumps that characterize the systems at issue, we combine the ideas of Conte et al.[10] (where systems with jumps are considered) and of Otsuka[46] (where, in particular, uncertain systems are considered) to give the following novel definitions.

**Definition 1.** Given an uncertain impulsive system $\Sigma_\sigma$ of the form (1), a subspace $\mathcal{V} \subseteq \mathcal{X}$ is said *robust hybrid invariant* for $\Sigma_\sigma$ if and only if

$$A\mathcal{V} = A(\mu)\mathcal{V} \subseteq \mathcal{V} \tag{6}$$

holds together with

$$J\mathcal{V} = J(\mu)\mathcal{V} \subseteq \mathcal{V} \tag{7}$$

for all $\mu \in \Delta^{(N-1)}$.

**Proposition 1.** *Given an uncertain impulsive system $\Sigma_\sigma$ of the form (1), a subspace $\mathcal{V} \subseteq \mathcal{X}$ is robust hybrid invariant for $\Sigma_\sigma$ if and only if the conditions $A_i\mathcal{V} \subseteq \mathcal{V}$ and $J_i\mathcal{V} \subseteq \mathcal{V}$ hold for all $i \in I$.*

*Proof.* If $A_i\mathcal{V} \subseteq \mathcal{V}$ and $J_i\mathcal{V} \subseteq \mathcal{V}$ hold for all $i \in I$, then obviously the conditions (6) and (7) are satisfied for all $\mu \in \Delta^{(N-1)}$. The converse is proved by taking $\mu_i = 1$ in (6) and (7) for $i \in I$. ∎

**Definition 2.** Given an uncertain impulsive system $\Sigma_\sigma$ of the form (1), a subspace $\mathcal{V} \subseteq \mathcal{X}$ is said *robust hybrid controlled invariant* for $\Sigma_\sigma$ if and only if

$$(\oplus_{i=1}^{N} A_i)\mathcal{V} \subseteq \mathcal{V}^{\oplus N} + \mathrm{Im}(\oplus_{i=1}^{N} B_i) \tag{8}$$

holds together with

$$J_i\mathcal{V} \subseteq \mathcal{V} \tag{9}$$

for all $i \in I$.

The following proposition can be obtained from Basile and Marro[26] and from Otsuka.[46]

**Proposition 2.** *Given an uncertain impulsive system $\Sigma_\sigma$ of the form (1) and a subspace $\mathcal{V} \subseteq \mathcal{X}$, condition (8) is equivalent to the existence of an $m \times n$ matrix $F$ such that*

$$(A_i + B_iF)\mathcal{V} \subseteq \mathcal{V} \tag{10}$$

*for all $i \in I$.*

Given a subspace $\mathcal{V} \subseteq \mathcal{X}$, any matrix $F$ for which (10) holds for all $i \in I$ is called *friend* of the subspace $\mathcal{V}$. An obvious consequence of Proposition 2 is that if $\mathcal{V}$ is a robust hybrid controlled invariant subspace for the uncertain impulsive system $\Sigma_\sigma$ of the form (1) and $F$ is one of its friends, then $\mathcal{V}$ is a robust invariant subspace for the compensated system $\Sigma_\sigma^F$ given by (4). This remark, together with the elementary decomposition of the linear dynamics $A_i$ with respect to an invariant subspace (Theorem 3.2-1 of Reference 26), proves the following proposition.

**Proposition 3.** *Given an uncertain impulsive system $\Sigma_\sigma$ of the form (1), let $\mathcal{V} \subseteq \mathcal{X}$ be a robust hybrid controlled invariant subspace of dimension $k \leq n$ and let $F$ be a friend of $\mathcal{V}$. Then, $\mathcal{V}$ is robust hybrid invariant for $\Sigma_\sigma^F$. Moreover, by applying the change of basis $x = Tz$ in $\mathcal{X}$ with $T = (V \;\; T_1)$, where $V$ is a $n \times k$ matrix whose columns are a basis of $\mathcal{V}$, the equations of $\Sigma_\sigma^F$ take the form*

$$\Sigma_\sigma^F \equiv \begin{cases} \dot{z}(t) = \hat{A}z(t) \;\; \text{for} \;\; t \neq \sigma(k), \; k \in \mathbb{N} \\ z(\sigma(k)) = \hat{J}z^-(\sigma(k)) \;\; \text{for} \;\; t = \sigma(k), \; k \in \mathbb{N} \\ y(t) = \hat{C}z(t) \end{cases} \tag{11}$$

[Corrections added on 15 April 2021, after first online publication: In Equation 11: the word ‘for’ has been changed to roman font.]

*with*

$$\hat{A} = T^{-1}(A+BF)T = \begin{pmatrix} A_{11} & A_{12} \\ 0_{(n-k)\times k} & A_{22} \end{pmatrix} = \sum_{i\in I} \mu_i T^{-1}(A_i + B_i F)T$$
$$= \sum_{i\in I} \mu_i \begin{pmatrix} A_{11i} & A_{12i} \\ 0_{(n-k)\times k} & A_{22i} \end{pmatrix} \tag{12}$$
$$\hat{J} = T^{-1}JT = \begin{pmatrix} J_{11} & J_{12} \\ 0_{(n-k)\times k} & J_{22} \end{pmatrix} = \sum_{i\in I} \mu_i T^{-1} J_i T = \sum_{i\in I} \mu_i \begin{pmatrix} J_{11i} & J_{12i} \\ 0_{(n-k)\times k} & J_{22i} \end{pmatrix} \tag{13}$$
$$\hat{C} = CT = \begin{pmatrix} C_1 & C_2 \end{pmatrix} = \sum_{i\in I} \mu_i C_i T = \sum_{i\in I} \mu_i \begin{pmatrix} C_{1i} & C_{2i} \end{pmatrix} \tag{14}$$

*Remark* 1. It is important to note that robust hybrid invariance and robust hybrid controlled invariance of a subspace $\mathcal{V}$, as well as the feedback friends of $\mathcal{V}$, are characterized in terms of the vertex systems that define $\Sigma_\sigma$, without requiring any knowledge of the uncertainty parameters. Moreover, the block-triangular structure of $\hat{A}$ and $\hat{J}$ given by Proposition 3 shows that the free motion $x(t)$ of the state of $\Sigma_\sigma^F$ that originates from any $x(0) \in \mathcal{V}$ remains inside $\mathcal{V}$ for all $t \in \mathbb{R}^+$ and for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathcal{S}$. In other terms, this means that in the uncertain system $\Sigma_\sigma$ the motion $x(t)$ of the state that originates from any $x(0) \in \mathcal{V}$ can be kept inside $\mathcal{V}$ by a feedback control input $u(t) = Fx(t)$, which does not depend on the uncertainty parameter $\mu$ and on the time sequence of jumps defined by $\sigma$, on all the time intervals $[\sigma(k), \sigma(k+1))$ in which the system behaves according to its flow dynamics and at each jump instant $\sigma(k)$. For this reason, the notions of robust hybrid invariance and of robust hybrid controlled invariance we have introduced qualify as the right tools for tackling feedback decoupling control problems, as the DDP, for uncertain impulsive systems.

For any subspace $\mathcal{W} \subseteq \mathcal{X}$, we denote by $\mathbf{V_R}(\mathcal{W})$ the family of all the robust hybrid controlled invariant subspaces contained in $\mathcal{W}$, that is,

$$\mathbf{V_R}(\mathcal{W}) = \{\mathcal{V} \subseteq \mathcal{W},\ \ \textit{such that}\ \ \mathcal{V}\ \ \textit{is robust hybrid controlled invariant}\}.$$

Then, we have the following proposition.

**Proposition 4.** *Given an uncertain impulsive system $\Sigma_\sigma$ of the form (1) and a subspace $\mathcal{W} \subseteq \mathcal{X}$, the family $\mathbf{V_R}(\mathcal{W})$ has a unique maximum element that is denoted by $\mathcal{V}_R^*(\mathcal{W})$.*

*Proof.* It is sufficient to note that the considered family of subspaces is closed with respect to the sum of subspaces. ■

The maximum element $\mathcal{V}_R^*(\mathcal{W})$ of $\mathbf{V_R}(\mathcal{W})$ coincides with the limit of the sequence of subspaces $\mathcal{V}_k$ that is recursively generated by the following procedure:

- $\mathcal{V}_0 = \mathcal{W}$
- $\mathcal{V}_k = \mathcal{V}_{k-1} \cap (\oplus_{i=1}^N A_i)^{-1}(\mathcal{V}_{k-1}^{\oplus N} + \mathrm{Im}(\oplus_{i=1}^N B_i)) \cap (\oplus_{j=1}^N J_i)^{-1}\mathcal{V}_{k-1}^{\oplus N}$

If $\dim(\mathcal{W}) = r \le n$, the above sequence converges in a number of steps at most equal to $r+1$ and therefore $\mathcal{V}_R^*(\mathcal{W}) = \mathcal{V}_r$.
*Notations.* In case $\mathcal{W} = \cap_{i\in I}\mathrm{Ker}\ C_i$, we denote $\mathcal{V}_R^*(\mathcal{W})$ simply by $\mathcal{V}_R^*$.
The solvability of the DDP can now be characterized by the following result.

**Theorem 1.** *Given a disturbed uncertain impulsive system $\Sigma_{D\sigma}$ the form (5), let $\mathcal{V}_R^*$ denote the maximum robust hybrid controlled invariant subspace for the undisturbed system $\Sigma_\sigma$ contained in $\cap_{i\in I}\mathrm{Ker}\, C_i$. Then, the DDP for $\Sigma_{D\sigma}$ is solvable if the following condition is satisfied*

$$\mathrm{Im}\ D_i \subseteq \mathcal{V}_R^* \tag{15}$$

*for all $i \in I$.*

*Proof.* By applying the same change of basis as in Proposition 3, the equations of $\Sigma^F_{D\sigma}$ take the form

$$\Sigma^F_{D\sigma} \equiv \begin{cases} \begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + \begin{pmatrix} D_1 \\ 0 \end{pmatrix} d(t) \ \ for \ \ t \neq \sigma(k),\ k \in \mathbb{N} \\ \begin{pmatrix} z_1(\sigma(k)) \\ z_2(\sigma(k)) \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{pmatrix} \begin{pmatrix} z_1^-(\sigma(k)) \\ z_2^-(\sigma(k)) \end{pmatrix} \ \ for \ \ t = \sigma(k),\ k \in \mathbb{N} \\ y(t) = (0 \ \ C_2) \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \end{cases}$$

which shows that, initializing $\Sigma^F_{D\sigma}$ at $z(0)=0$, one has $y(t)=C_2 z_2(t)=0$ for $t \geq 0$ for all $\mu \in \Delta^{N-1}$, all $\sigma \in \mathcal{S}$ and any disturbance input $d(t)$. ■

It is important to remark that the condition of Theorem 1 can be practically checked by using the algorithmic procedure described above to construct $\mathcal{V}^*_R$ and by checking the inclusion of $\mathrm{Im}\, D_i$ in it.

*Remark* 2. Obviously, Theorem 1 applies also to the particular case in which $J_i = I_n$ for all $i \in I$ and, actually, the state evolution of the uncertain system at issue does not present jumps.

Condition (15) in general is not necessary. An example of this fact is provided by the system $\Sigma_{D\sigma}$ of the form (5) with $I = \{1, 2\}$ and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ;\ A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} ;\ B_1 = B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ;\ D_1 = D_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$J_1 = J_2 = I_3 ;\ C_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} ;\ C_2 = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}.$$

In this case, we can analyze the effect of the disturbance on the output by means of the transfer function of $\Sigma_{D\sigma}$ from $\mathcal{D}$ to $\mathcal{Y}$, which, for all $\mu \in \Delta^1$ and all $\sigma \in \mathcal{S}$, is given by

$$(\mu_1 C_1 + \mu_2 C_2)(sI_3 - (\mu_1 A_1 + \mu_2 A_2))^{-1}(\mu_1 D_1 + \mu_2 D_2)$$
$$= \begin{pmatrix} 0 & -\mu_2 & \mu_1 \end{pmatrix} \begin{pmatrix} s^{-1} & 0 & 0 \\ \mu_1 s^{-2} & s^{-1} & 0 \\ \mu_2 s^{-2} & 0 & s^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0.$$

Therefore, the disturbance does not affect the output and the DDP is solved without applying any feedback. However, it is easily seen that $\mathcal{V}^*_R = (0\ 0\ 0)^\top$ and $\mathrm{Im}\ D_1 = \mathrm{Im}\ D_2 = \mathrm{span}\{(1\ 0\ 0)^\top\} \subseteq \mathcal{X}$ is not contained in $\mathcal{V}^*_R$.

*Remark* 3. Solvability of the DDP for each vertex system by means of the same feedback $F$ is clearly a necessary but not sufficient condition for the solvability of the DDP for $\Sigma_{D\sigma}$. This is seen by considering, for instance, the following simple example. Let $\Sigma_{D\sigma}$ be of the form (5) with $I = \{1, 2\}$ and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} ; B_1 = B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ; D_1 = D_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$J_1 = J_2 = I_3 ;\ C_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} ;\ C_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}.$$

Any feedback of the form $F = (f_1\ f_2\ f_3)$, for arbitrary $f_1, f_2, f_3 \in \mathbb{R}$ solves the DDP problem for both the vertex systems, since we have $C_i(sI - (A_i + B_i F))^{-1} D_i = 0$ for $i = 1, 2$. But no feedback solves the problem for all $\mu \in \Delta^1$, since $C(\mu)(sI - (A(\mu) + B(\mu)F))^{-1} D(\mu) = C(\mu)(sI - A(\mu))^{-1} D(\mu) \neq 0$ if, for example, $\mu_1 = \mu_2 = 1/2$ and any $F$.

[Corrections added on 15 April 2021, after first online publication: In the last line: the parenthesis between '–' and 'A' has been removed.]

## 4 | STABILITY REQUIREMENT

Stability of an uncertain impulsive control system $\Sigma_\sigma$ of the form (1) depends on the characteristics of its flow dynamics and of its jump behavior as well as on the interaction between the two.

**Definition 3.** An uncertain impulsive control system $\Sigma_\sigma$ of the form (1) is said robustly asymptotically stable over $\mathcal{S}_\tau$, with $\tau > 0$, if it is asymptotically stable for all $\mu \in \Delta^{N-1}$ and all $\sigma \in \mathcal{S}_\tau$.

Adding the requirement of stability, the disturbance decoupling problem can be stated as follows.

**Problem 2** (Disturbance Decoupling Problems with Stability). Given a disturbed uncertain impulsive system $\Sigma_{D\sigma}$ the form (5), the Disturbance Decoupling Problem with Stability (DDPS) by state feedback for $\Sigma_{D\sigma}$ consists in finding a state feedback $F : \mathcal{X} \to \mathcal{U}$, if any exists, that solves the corresponding DDP and that makes the compensated uncertain impulsive system $\Sigma^F_{D\sigma}$ robustly asymptotically stable over $\mathcal{S}_\tau$ for some $\tau > 0$.

In order to study the solvability of the DDPS, the following result is useful.

**Proposition 5.** *Given an uncertain impulsive control system $\Sigma_\sigma$ of the form (1), assume that there exists a symmetric, positive definite $n \times n$ matrix $P$ such that $A_i^\top P + PA_i < 0$ for all $i \in I$. Then $A = A(\mu) = \sum_{i=1}^N \mu_i A_i$ is an Hurwitz matrix for all $\mu \in \Delta^{N-1}$. Moreover, $V(x) = x^\top Px$ is a common quadratic Lyapunov function for the flow dynamics given by $A = A(\mu)$ for all $\mu \in \Delta^{N-1}$.*

*Proof.* From $A_i^\top P + PA_i < 0$ for all $i \in I$, we have

$$A(\mu)^\top P + PA(\mu) = \left(\sum_{i=1}^N \mu_i A_i^\top\right) P + P\sum_{i=1}^N \mu_i A_i = \sum_{i=1}^N \mu_i (A_i^\top P + PA_i) < 0$$

for $\mu \in \Delta^{N-1}$. Then, $A = A(\mu)$ is Hurwitz and $V(x) = x^\top Px$ is a Lyapunov function for the flow dynamics defined by $A = A(\mu)$ for all $\mu \in \Delta^{N-1}$. ■

Now, we can state the following result.

**Theorem 2.** *Given a disturbed uncertain impulsive control system $\Sigma_{D\sigma}$ of the form (5), assume that there exists a solution $F$ of the related DDP such that*

$$(A_i + B_iF)^\top P + P(A_i + B_iF) < 0 \tag{16}$$

*for some positive definite matrix $P$ of suitable dimension and for all $i \in I$. Then, the related DDPS is solvable.*

*Proof.* Since $F$ solves the DDP, it is enough to show that $\Sigma^F_{D\sigma}$ is robustly asymptotically stable over $\mathcal{S}_\alpha$ for some $\alpha$.

To this aim, we begin by showing that, under the hypotheses of the Theorem, there exists $\eta \in \mathbb{R}$ such that $||e^{(A(\mu)+B(\mu)F)t}|| < 1$ for $t \geq \eta$ and for all $\mu \in \Delta^{N-1}$. In fact, by Proposition 5, $V(x) = x^\top Px$ is a common quadratic Lyapunov function for the flow dynamics given by $A = A(\mu)$ for all $\mu \in \Delta^{N-1}$. Each flow dynamics is therefore asymptotically stable and, in particular, for all $\mu \in \Delta^{N-1}$, there exist two positive values $\alpha(\mu)$ and $\beta(\mu)$ such that $||e^{(A(\mu)+B(\mu)F)t}x|| \leq \alpha(\mu)e^{-\beta(\mu)t}||x||$. Now, given $\overline{\mu} \in \Delta^{N-1}$, let $S(\overline{\mu}, \delta_\epsilon(\overline{\mu})) \subseteq \Delta^{N-1}$ be an open neighborhood of $\overline{\mu}$ such that

$$||e^{(A(\mu)+B(\mu)F)t}x - e^{(A(\overline{\mu})+B(\overline{\mu})F)t}x|| \leq \epsilon < 1/2$$

for all $\mu \in S(\overline{\mu}, \delta_\epsilon(\overline{\mu}))$ and all $t \in \mathbb{R}^+$ and let $\bar{t} \in \mathbb{R}^+$ be such that $\alpha(\overline{\mu})e^{-\beta(\overline{\mu})\bar{t}} \leq 1/2$. Note that, in general, $\bar{t}$ depends on $\overline{\mu}$ and the notation $\bar{t}$ has been preferred to $\bar{t}(\overline{\mu})$ for simplicity. Now, for all $x \in \mathcal{X}$ with $||x|| = 1$ and for all $t \geq \bar{t}$ we get

$$\begin{aligned}
||e^{(A(\overline{\mu})+B(\overline{\mu})F)t}x|| &= ||e^{(A(\overline{\mu})+B(\overline{\mu})F)t}x - e^{(A(\overline{\mu})+B(\overline{\mu})F)t}x + e^{(A(\overline{\mu})+B(\overline{\mu})F)t}x|| \\
&\leq ||e^{(A(\overline{\mu})+B(\overline{\mu})F)t}x - e^{(A(\overline{\mu})+B(\overline{\mu})F)t}x|| + ||e^{(A(\overline{\mu})+B(\overline{\mu})F)t}x|| \\
&\leq \epsilon + \alpha(\overline{\mu})e^{-\beta(\overline{\mu})t}||x|| \leq \epsilon + \alpha(\overline{\mu})e^{-\beta(\overline{\mu})\bar{t}}||x|| \leq \epsilon + 1/2||x|| < 1
\end{aligned} \tag{17}$$

[Corrections added on 15 April 2021, after first online publication: In the second line of Problem 2: the first letters of the words 'Disturbance', 'Decoupling', 'Problem', and 'Stability' have been capitalized.]

for all $\mu \in S(\overline{\mu}, \delta_\epsilon(\overline{\mu}))$. This says that for $t \geq \overline{t}$ the norm of $e^{(A(\mu)+B(\mu)F)t}$ is smaller than 1, that is, $||e^{(A(\mu)+B(\mu)F)t}|| < 1$, for all $\mu \in S(\overline{\mu}, \delta_\epsilon(\overline{\mu}))$. Since $\Delta^{N-1} = \bigcup_{\overline{\mu} \in \Delta^{N-1}} S(\overline{\mu}, \delta_\epsilon(\overline{\mu}))$, the totality of the open sets $(\overline{\mu}, \delta_\epsilon(\overline{\mu}))$ form a cover $C$ of the symplex $\Delta^{N-1}$ and, since the latter is compact, it is possible to extract a finite subcover $C_f$, namely, a finite set $\{S(\overline{\mu}_j, \delta_\epsilon(\overline{\mu}_J))\}_{j=1,\dots,M}$, such that $\Delta^{N-1} = \bigcup_{j=1,\dots,M} S(\overline{\mu}_j, \delta_\epsilon(\overline{\mu}_j))$. Then, considering the set of values $\{\overline{t}(\overline{\mu}_j)\}_{j=1,\dots,M}$ corresponding to the elements of $C_f$ and letting $\eta$ be such that $\eta \geq \max_{j=1,\dots,M} \overline{t}(\overline{\mu}_j)$, it follows from the above results that $||e^{(A(\mu)+B(\mu)F)t}|| < 1$, for $t \geq \eta$ and for all $\mu \in \Delta^{N-1}$ as claimed.

Now, let us take $\gamma \geq \sum_{i \in I} ||J_i||$ and remark that, since $||J|| = ||J(\mu)|| = ||\sum_{i \in I} \mu_i J_i|| \leq \sum_{i \in I} \mu_i ||J_i|| \leq \sum_{i \in I} ||J_i||$, we have $\gamma \geq ||J(\mu)||$ for all $\mu \in \Delta^{N-1}$. Since $||e^{(A(\mu)+B(\mu)F)t}|| < 1$ for all $t \geq \eta$ and all $\mu \in \Delta^{N-1}$, there exists $\overline{n} \in \mathbb{N}$ such that $||e^{(A(\mu)+B(\mu)F)\overline{n}t}|| \leq \frac{1}{2\gamma}$ for all $t \geq \eta$ and all $\mu \in \Delta^{N-1}$. Therefore, for all $t \geq \overline{n}\eta$ and all $\mu \in \Delta^{N-1}$ we have $||e^{(A(\mu)+B(\mu)F)t}x|| \leq \frac{1}{2\gamma}||x||$ for all $x \in \mathcal{X}$.

It follows that, for the system $\Sigma^F_{D\sigma}$, we have

$$||x(\sigma(k+1))|| = ||J(\mu)e^{(A(\mu)+B(\mu)F)(\sigma(k+1)-\sigma(k))}x(\sigma(k))|| \leq 1/2||x(\sigma(k))||$$

for all $x \in \mathcal{X}$, all $\mu \in \Delta^{N-1}$ and all $\sigma \in \mathcal{S}_{\overline{n}\eta}$. Hence, $\Sigma^F_{D\sigma}$ is robustly asymptotically stable over $\mathcal{S}_\tau$ for $\tau = \overline{n}\eta$. ∎

**Theorem 3.** *Given a disturbed uncertain impulsive system $\Sigma_{D\sigma}$ of the form (5), let $\mathcal{V}^*_R$ denote the maximum robust hybrid controlled invariant subspace for the undisturbed system $\Sigma_\sigma$ contained in $\cap_{i \in I} \mathrm{Ker}\, C_i$. Then, the DDPS for $\Sigma_{D\sigma}$ is solvable if condition (15) is satisfied for all $i \in I$ and there exists a friend $F$ of $\mathcal{V}^*_R$ such that condition (16) is satisfied for some positive definite matrix $P$ of suitable dimensions and all $i \in I$.*

*Proof.* It follows from Theorems 1 and 2. ∎

We have already illustrated how to check condition (15). If it is satisfied, condition (16) can be checked by the following procedure. Note, first of all, that if $V$ is an $n \times k$ matrix whose columns are a basis of $\mathcal{V}^*_R$, condition (10) of Proposition 2 is equivalent to the existence of a matrix $F$ and of matrices $L_1, \dots, L_N$ of suitable dimensions such that $(A_i + B_iF)V = VL_i$ for $i \in I$. It follows, reasoning as in Perdon et al.,[47] that $F'$ is a friend of $\mathcal{V}^*_R$ if and only if

$$F'V = FV + K_2H$$

where $F$ is a given friend, $K_2$ is an $m \times k'$ matrix of suitable dimensions such that, for some $kN \times k'$ matrix $K_1$, the columns of the $(kN+m) \times k'$ matrix $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ are a basis of $\mathrm{Ker} \begin{pmatrix} V & 0 & \dots & \dots & 0 & B_1 \\ 0 & V & 0 & \dots & 0 & B_2 \\ & & \dots & & & \dots \\ 0 & 0 & \dots & 0 & V & B_N \end{pmatrix}$, and $H$ is an arbitrary $k' \times k$ matrix. Then, expressing condition (16) in terms of LMIs by the change of variable $FP^{-1} = Y$ as

$$P^{-1}A_i^\top + A_iP^{-1} + YB_i^\top + Y^\top B_i < 0$$

for $i \in I$, one can search, in particular, for solutions $P$ and $Y$ such that $YPV = FV + K_2H$ for some matrix $H$ of suitable dimensions.

*Remark* 4. When the conditions (15) and (16) are satisfied, Theorem 3 guarantees that the feedback $F$ is a solution of the DDPS, namely that it achieves the decoupling and that it stabilizes the compensated system over $\mathcal{S}_\tau$ for some $\tau \geq 0$. However, the theorem does not indicate how to compute $\tau$. Actually, no general procedure to compute $\tau$ is known and one has to use *ad hoc* procedures depending on the characteristics of the system at issue (see the example in Section 5).

## 5 EXAMPLE

Consider the disturbed, uncertain impulsive linear system with jumps and polytopic uncertainties $\Sigma_{D\sigma}$ of the form (5) defined, for $\mu \in \Delta^1$, by the matrices:

$$A = A(\mu) = \mu A_1 + (1-\mu)A_2 = \mu \begin{pmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \\ 1 & 0 & -3 \end{pmatrix} + (1-\mu) \begin{pmatrix} -1/2 & -1 & 2 \\ 1 & -5/2 & 1 \\ 0 & 1 & -5/2 \end{pmatrix}$$

$$B = B(\mu) = \mu B_1 + (1-\mu)B_2 = \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (1-\mu) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$C = C(\mu) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = C_1 = C_2$$

$$D = D(\mu) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = D_1 = D_2$$

$$J = J(\mu) = \mu J_1 + (1-\mu)J_2 = \mu \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} + (1-\mu) \begin{pmatrix} -1 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Computations show that $\text{Ker}\, C = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $\mathcal{V}_R^* = \text{Ker}\, C$. Since $\text{Im}\, D_i \subseteq \mathcal{V}_R^*$ for $i = 1, 2$, we have by Theorem 1 that the related DDP is solvable. Again, computations show that, for all $f_1 \in \mathbb{R}$, the matrix $F = \begin{pmatrix} -1 & f_1 & -1 \end{pmatrix}$ is a friend of $\mathcal{V}_R^*$ and therefore we can, in particular, choose $F = \begin{pmatrix} -1 & 0 & -1 \end{pmatrix}$ as solution of the DDP.

We can also check directly that $F = \begin{pmatrix} -1 & 0 & -1 \end{pmatrix}$ is a solution by applying in $\mathcal{X}$ the change of basis $x = Tz = (V \;\; T_1)z$, where $V$ is a matrix whose columns are a basis of $\mathcal{V}_R^*$. For instance, for $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, we have

$$\hat{A} = T^{-1}(A + BF)T = \begin{pmatrix} -1/2\,\mu - 3/2 & 1-\mu & -1+\mu \\ 1 & -3/2\,\mu - 3/2 & 1-\mu \\ 0 & 0 & 1/2\,\mu - 5/2 \end{pmatrix}$$

$$\hat{J} = T^{-1}JT \begin{pmatrix} 2\,\mu - 1 & -\mu + 2 & -\mu + 2 \\ 1 & -\mu & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{C} = CT = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \qquad \hat{D} = T^{-1}D = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and, hence, $\Sigma_{D\sigma}^F$ in the new basis takes the form

$$\Sigma_{D\sigma}^F \equiv \begin{cases} \dot{z}_1(t) = (-1/2\,\mu - 3/2)z_1(t) + (1-\mu)z_2(t) + (-1+\mu)z_3(t) + d(t) \\ \dot{z}_2(t) = z_1(t) + (-3/2\mu - 3/2)z_2(t) + (1-\mu)z_3(t) \\ \dot{z}_3(t) = (1/2\,\mu - 5/2)z_3(t) \\ z_1(\sigma(k)) = (2\,\mu - 1)z_1^-(\sigma(k)) + (-\mu + 2)z_2^-(\sigma(k)) + (-\mu + 2)z_3^-(\sigma(k)) \\ z_2(\sigma(k)) = z_1^-(\sigma(k)) - \mu\, z_2^-(\sigma(k)) - z_3^-(\sigma(k)) \\ z_3(\sigma(k)) = z_3^-(\sigma(k)) \\ y(t) = z_3(t) \end{cases} \tag{18}$$

Inspection shows that, for all $\mu \in \Delta^1$, the disturbance does not affect the component $z_3$ of the state of the compensated system $\Sigma_{D\sigma}^F$ and therefore it does not affect its output for all $\sigma \in \mathcal{S}$.

[Corrections added on 15 April 2021, after first online publication: In the third line of Equation (18): the first minus symbol has been removed.]

Moreover, taking $P = I_3$, we also have

$$(A_1 + B_1F)^\top P + P(A_1 + B_1F) = (A_1 + B_1F)^\top + (A_1 + B_1F) = \begin{pmatrix} -4 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & -6 \end{pmatrix} < 0$$

$$(A_2 + B_2F)^\top P + P(A_2 + B_2F) = (A_2 + B_2F)^\top + (A_2 + B_2F) = \begin{pmatrix} -3 & -1 & 2 \\ -1 & -5 & 1 \\ 2 & 1 & -3 \end{pmatrix} < 0$$

Hence, by Theorem 3, the related DDPS is solvable and $F = \begin{pmatrix} -1 & 0 & -1 \end{pmatrix}$ is a solution. Beside assuring decoupling of the output from the disturbance input for all $\mu \in \Delta^1$ and all $\sigma \in \mathcal{S}$, this means that $\Sigma_{D\sigma}^F$ is robustly asymptotically stable over $\mathcal{S}_\tau$ for some value $\tau \geq 0$. In order to find a valid $\tau$ in this particular case, we can proceed as follows. First, using the symbolic calculus capability of MAPLE®, we get an explicit expression of $e^{(A(\mu)+B(\mu)F)t}$ (see the Appendix). Then, using the general inequality

$$||e^{(A(\mu)+B(\mu)F)t}|| \leq \sqrt{||e^{(A(\mu)+B(\mu)F)t}||_1 ||e^{(A(\mu)+B(\mu)F)t}||_\infty} \tag{19}$$

we obtain that $||e^{(A(\mu)+B(\mu)F)t}||$ is smaller than 1 for $t = 1$ and for all $\mu \in \Delta^1$. Indeed, plotting the graph of $||e^{(A(\mu)+B(\mu)F)t}||_1$ and of $||e^{(A(\mu)+B(\mu)F)t}||_\infty$ for $t = 1$ by MAPLE® (see the Appendix), it is possible to conclude that $\max_{\mu\in\Delta^1} ||e^{(A(\mu)+B(\mu)F)1}||_1 < 0.6$ and $\max_{\mu\in\Delta^1} ||e^{(A(\mu)+B(\mu)F)1}||_\infty < 0.55$, so that $||e^{(A(\mu)+B(\mu)F)1}|| < 0.57$. Since $P = I_3$, we have $x^\top Px = ||x||^2$ and the level curves of $x^\top Px$ are circumferences with center in the origin of $\mathbb{R}^2$. It follows that for all $\mu \in \Delta^1$ the function $||e^{(A(\mu)+B(\mu)F)t}x||$ is monotonically decreasing and hence we have

$$||e^{(A(\mu)+B(\mu)F)t}x|| \leq ||e^{(A(\mu)+B(\mu)F)1}x|| \leq ||e^{(A(\mu)+B(\mu)F)1}||\,||x|| < 0.57||x||$$

for all $t \geq 1$, that is $||e^{(A(\mu)+B(\mu)F)t}|| < 0.57$ for all $t \geq \eta = 1$ and all $\mu \in \Delta^1$.

By similar computations, we get $||J(\mu)|| < \gamma = 5.4$ for all $\mu \in \Delta^1$ and then for $\overline{n} = 4$ we have

$$||e^{(A(\mu)+B(\mu)F)\overline{n}t}|| = ||e^{(A(\mu)+B(\mu)F)4t}|| < 0.57^4 < 0.091 < \frac{1}{2\gamma}$$

for all $t \geq \eta = 1$ and for all $\mu \in \Delta^1$.

We therefore obtain

$$\begin{aligned} ||x(\sigma(k+1))|| &= ||J(\mu)e^{(A(\mu)+B(\mu)F)(\sigma(k+1)-\sigma(k))}x(\sigma(k))|| \\ &\leq ||J(\mu)||\,||e^{(A(\mu)+B(\mu)F)(\sigma(k+1)-\sigma(k))}||\,||x|| \leq \gamma\frac{1}{2\gamma}||x|| = \frac{1}{2}||x|| \end{aligned}$$

for all $\sigma \in \mathcal{S}_{\overline{n}\eta}$, that implies robust asymptotic stability of $\Sigma_{D\sigma}^F$ over $\mathcal{S}_\tau$ with $\tau = 4$.

## 6 | CONCLUSIONS

The structural solvability of a class of disturbance decoupling problems for hybrid linear systems with polytopic uncertainties which present jumps in the state behavior has been investigated by means of a structural geometric approach. Sufficient, constructive conditions for the solvability of the disturbance decoupling problem by state feedback and for the same problem with the additional requirement of robust stability of the compensated system have been given. Since the requirements of decoupling for all values of the uncertainty parameter and of robust stability are quite strong, the solvability conditions are tight. However, they can be checked by viable algorithmic procedures.

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### DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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[Corrections added on 15 April 2021, after first online publication: In Reference 25: '18th' has been added before 'European'.]

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## APPENDIX A

The symbolic computation routine of MAPLE® gives the following expression, as functions of $t$ and $\mu$, for the columns $E_1(t,\mu)$, $E_2(t,\mu)$, $E_3(t,\mu)$ of the matrix $e^{(A(\mu)+B(\mu)F)t}$ considered in Section 5:

$$E_1(t,\mu)=\begin{pmatrix} \dfrac{e^{-1/2\ (5+\mu)t}\mu-e^{-1/2\ (5+\mu)t}-e^{-1/2\ (1+3\ \mu)t}}{\mu-2} \\ 0 \\ \dfrac{e^{-1/2\ (5+\mu)t}-e^{-1/2\ (1+3\ \mu)t}}{\mu-2} \end{pmatrix}$$

$$E_2(t,\mu)=\begin{pmatrix} -1/2\ \dfrac{-8\ e^{-1/2\ (5+\mu)t}\mu+4\ e^{-1/2\ (5+\mu)t}\mu^2+4\ e^{-1/2\ (5+\mu)t}+8\ \mu\ e^{1/2\ (\mu-5)t}-3\ \mu^2e^{1/2\ (\mu-5)t}-4\ e^{1/2\ (\mu-5)t}-\mu^2e^{-1/2\ (1+3\ \mu)t}}{(\mu-2)\mu} \\ e^{1/2\ (\mu-5)t} \\ -1/2\ \dfrac{4\ e^{-1/2\ (5+\mu)t}\mu-4\ e^{-1/2\ (5+\mu)t}-4\ \mu\ e^{1/2\ (\mu-5)t}+4\ e^{1/2\ (\mu-5)t}+\mu^2e^{1/2\ (\mu-5)t}-\mu^2e^{-1/2\ (1+3\ \mu)t}}{(\mu-2)\mu} \end{pmatrix}$$

$$E_3(t,\mu)=\begin{pmatrix} -\frac{-e^{-1/2\ (5+\mu)t}+e^{-1/2\ (5+\mu)t}\mu+e^{-1/2\ (1+3\ \mu)t}-\mu\ e^{-1/2\ (1+3\ \mu)t}}{\mu-2}\\ 0\\ -\frac{e^{-1/2\ (1+3\ \mu)t}+e^{-1/2\ (5+\mu)t}-\mu\ e^{-1/2\ (1+3\ \mu)t}}{\mu-2}\end{pmatrix}.$$

Note that the expression of $E_2(t,\mu)$ returned by MAPLE® is not defined for $\mu=0$, but we can extend it by continuity by defining

$$E(t,0)=\lim_{\mu\to 0^+}E(t,\mu)=\begin{pmatrix} 1/2\ e^{-5/2\ t}+1/2\ e^{-1/2\ t} & -te^{-5/2\ t} & 1/2\ e^{-1/2\ t}-1/2\ e^{-5/2\ t}\\ 0 & e^{-5/2\ t} & 0\\ 1/2\ e^{-1/2\ t}-1/2\ e^{-5/2\ t} & te^{-5/2\ t} & 1/2\ e^{-5/2\ t}+1/2\ e^{-1/2\ t}\end{pmatrix}.$$

For $t=1$, we have

$$E_1(1,\mu)=\begin{pmatrix} \frac{e^{-1/2\ \mu-5/2}\mu-e^{-1/2\ \mu-5/2}-e^{-1/2-3/2\ \mu}}{\mu-2}\\ 0\\ \frac{e^{-1/2\ \mu-5/2}-e^{-1/2-3/2\ \mu}}{\mu-2}\end{pmatrix}$$

$$E_2(1,\mu)=\begin{pmatrix} -1/2\ \frac{-8\ e^{-1/2\ \mu-5/2}\mu+4\ e^{-1/2\ \mu-5/2}\mu^2+4\ e^{-1/2\ \mu-5/2}+8\ \mu\ e^{1/2\ \mu-5/2}-3\ \mu^2e^{1/2\ \mu-5/2}-4\ e^{1/2\ \mu-5/2}-\mu^2e^{-1/2-3/2\ \mu}}{(\mu-2)\mu}\\ e^{1/2\ \mu-5/2}\\ -1/2\ \frac{4\ e^{-1/2\ \mu-5/2}\mu-4\ e^{-1/2\ \mu-5/2}-4\ \mu\ e^{1/2\ \mu-5/2}+4\ e^{1/2\ \mu-5/2}+\mu^2e^{1/2\ \mu-5/2}-\mu^2e^{-1/2-3/2\ \mu}}{(\mu-2)\mu}\end{pmatrix}$$

$$E_3(1,\mu)=\begin{pmatrix} -\frac{-e^{-1/2\ \mu-5/2}+e^{-1/2\ \mu-5/2}\mu+e^{-1/2-3/2\ \mu}-\mu\ e^{-1/2-3/2\ \mu}}{\mu-2}\\ 0\\ -\frac{e^{-1/2-3/2\ \mu}+e^{-1/2\ \mu-5/2}-\mu\ e^{-1/2-3/2\ \mu}}{\mu-2}\end{pmatrix}.$$

In particular, we have:

$$\begin{aligned}
||e^{(A(\mu)+B(\mu)F)1}||_\infty = \max\Bigg(&\left|\frac{e^{-1/2\ \mu-5/2}\mu-e^{-1/2\ \mu-5/2}-e^{-1/2-3/2\ \mu}}{\mu-2}\right|+\left|\frac{e^{-1/2\ \mu-5/2}-e^{-1/2-3/2\ \mu}}{\mu-2}\right|,\\
&\left|\frac{-e^{-1/2\ \mu-5/2}+e^{-1/2\ \mu-5/2}\mu+e^{-1/2-3/2\ \mu}-\mu\ e^{-1/2-3/2\ \mu}}{\mu-2}\right|+\left|\frac{e^{-1/2-3/2\ \mu}+e^{-1/2\ \mu-5/2}-\mu\ e^{-1/2-3/2\ \mu}}{\mu-2}\right|,\\
&1/2\ \left|\frac{-8\ e^{-1/2\ \mu-5/2}\mu+4\ e^{-1/2\ \mu-5/2}\mu^2+4\ e^{-1/2\ \mu-5/2}+8\ \mu\ e^{1/2\ \mu-5/2}-3\ \mu^2e^{1/2\ \mu-5/2}-4\ e^{1/2\ \mu-5/2}-\mu^2e^{-1/2-3/2\ \mu}}{(\mu-2)\mu}\right|\\
&+e^{-5/2+1/2\ \mu}+1/2\ \left(\left|\frac{4\ e^{-1/2\ \mu-5/2}\mu-4\ e^{-1/2\ \mu-5/2}-4\ \mu\ e^{1/2\ \mu-5/2}+4\ e^{1/2\ \mu-5/2}+\mu^2e^{1/2\ \mu-5/2}-\mu^2e^{-1/2-3/2\ \mu}}{(\mu-2)\mu}\right|\right)\Bigg)
\end{aligned}$$

$$\begin{aligned}
||e^{(A(\mu)+B(\mu)F)1}||_1 = \max\Bigg(&\left|\frac{e^{-1/2\ \mu-5/2}-e^{-1/2-3/2\ \mu}}{\mu-2}\right|\\
&+1/2\ \left|\frac{4\ e^{-1/2\ \mu-5/2}\mu-4\ e^{-1/2\ \mu-5/2}-4\ \mu\ e^{1/2\ \mu-5/2}+4\ e^{1/2\ \mu-5/2}+\mu^2e^{1/2\ \mu-5/2}-\mu^2e^{-1/2-3/2\ \mu}}{(\mu-2)\mu}\right|\\
&+\left|\frac{e^{-1/2-3/2\ \mu}+e^{-1/2\ \mu-5/2}-\mu\ e^{-1/2-3/2\ \mu}}{\mu-2}\right|,\left|\frac{e^{-1/2\ \mu-5/2}\mu-e^{-1/2\ \mu-5/2}-e^{-1/2-3/2\ \mu}}{\mu-2}\right|\\
&+1/2\ \left|\frac{-8\ e^{-1/2\ \mu-5/2}\mu+4\ e^{-1/2\ \mu-5/2}\mu^2+4\ e^{-1/2\ \mu-5/2}+8\ \mu\ e^{1/2\ \mu-5/2}-3\ \mu^2e^{1/2\ \mu-5/2}-4\ e^{1/2\ \mu-5/2}-\mu^2e^{-1/2-3/2\ \mu}}{(\mu-2)\mu}\right|\\
&+\left(\left|\frac{-e^{-1/2\ \mu-5/2}+e^{-1/2\ \mu-5/2}\mu+e^{-1/2-3/2\ \mu}-\mu\ e^{-1/2-3/2\ \mu}}{\mu-2}\right|,e^{-5/2+1/2\ \mu}\right)
\end{aligned}$$

and, using again MAPLE®, we can plot the graphs of $||e^{(A(\mu)+B(\mu)F)1}||_1$ and of $||e^{(A(\mu)+B(\mu)F)1}||_\infty$ as functions of $\mu$ for $\mu\in\Delta^1$ (see Figure A1). As said in Section 5, we get $\max_{\mu\in\Delta^1}||e^{(A(\mu)+B(\mu)F)1}||_1\le 0.55$ and $\max_{\mu\in\Delta^1}||e^{(A(\mu)+B(\mu)F)1}||_\infty\le 0.6$.

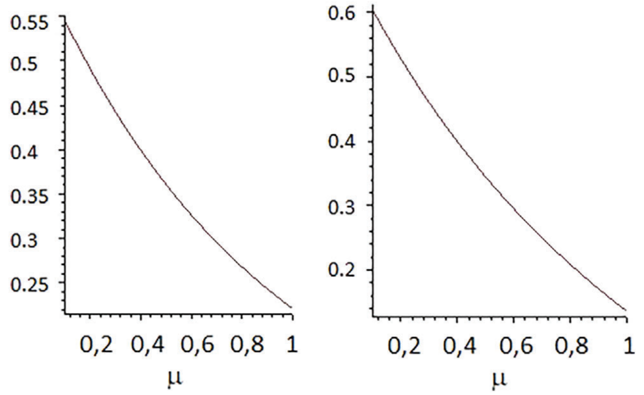


**FIGURE A1** Graph of $||e^{(A(\mu)+B(\mu)F)1}||_1$ and of $||e^{(A(\mu)+B(\mu)F)1}||_\infty$ for $\mu \in \Delta^1$ [Colour figure can be viewed at wileyonlinelibrary.com]