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Production delays, technology choice and cyclical cobweb dynamics*

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Abstract

We develop a cobweb model in which firms, facing a two-period production delay, have access to a flexible (costly) and an inflexible (cheap) production technology. Moreover, firms select between production technologies depending on their evolutionary fitness, measured in terms of past realized profits. The dynamics of our cobweb model is driven by a four-dimensional nonlinear map. We analytically show that its unique steady state may become unstable due to a Neimark-Sacker bifurcation, a scenario that gives rise to cyclical price dynamics, as observed in actual commodity markets. Simulations furthermore reveal that our cobweb model may also produce chaotic motion.

Keywords

Cobweb models; production technology; cyclical price dynamics; bounded rationality and learning; stability and bifurcation analysis; chaos.

JEL classification

D24; E32; Q11.

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1 Introduction

One of the many lessons we were fortunate to learn from Tonu Puu is not to forget what our former scholars accomplished in the past. In many of his inspiring papers, he re-explored classical economic models using new mathematical tools, frequently involving younger scholars, giving them the opportunity to enter the world of academia at the highest possible international level. Although his research interests spanned many fields, he was primarily interested in oligopoly dynamics (Puu 1991, Puu and Sushko 2002), business cycle theory (Puu et al. 2005, Puu and Sushko 2006) and spatial economics (Beckman and Puu 1985, Sushko et al. 2003). In this paper, we endeavor to follow his approach.

In fact, our paper is concerned with the dynamics of the cobweb model, one of the oldest and most studied contributions to the theory of mathematical economic dynamics. Cobweb models describe the price dynamics of nonstorable commodities whose production takes time, thus requiring firms to form price expectations. Pioneering contributions in this line of research were made by Ricci (1930), Tinbergen (1930) and Schultz (1930). On the one hand, these authors aimed at developing dynamic models that allow us to explain the up and down movements of certain agricultural prices, following, for instance, the empirical observations by Moore (1917) on the behavior of cotton prices. On the other hand, and presumably more importantly, these authors were clearly dissatisfied with a pure equilibrium (static) analysis, seeing the need for a dynamic economic theory, e.g. by considering production delays. A particularly elegant contribution in this respect was made by Leontief (1934), who formally established that the stability of the steady state of a classical cobweb model depends on the slopes of its supply and demand schedules, i.e. that their relation has to be smaller than one in modulus at the steady state. Apparently, the term “cobweb model” was first used by Kaldor (1934) to capture the model’s characteristic “zigzag” dynamics in a price-quantity diagram. A further classical reference in this area is the famous paper by Ezekiel (1938), who synthesized the aforementioned works into a general theory of the cobweb phenomenon, paving the way for its entry into economic textbooks. See Waugh (1964) for an early survey of the pertinent literature and Hommes (2018) for a more recent one.

Over the last 100 years, cobweb models have served as a workhorse to study a number of important economic ideas. The papers by Goodwin (1947), Nerlove (1958) and Muth (1961), for instance, introduced and popularized the concepts of extrapolative, adaptive and rational expectation rules, respectively, starting a revolution in economic theory with far-reaching consequences for the real world and academia. With the advent of chaos theory (Lorenz 1963, Li and Yorke 1975, May 1976), economists such as Artstein (1983), Jensen and Urban (1984), Chiarella (1988) and Hommes (1991) showed that nonlinear cobweb models may give rise to quite irregular price dynamics, triggering a wave of research in nonlinear economic dynamics.

A particularly relevant contribution was made by Brock and Hommes (1997), who showed that chaotic price dynamics may arise in cobweb models when firms switch between naive and rational expectation rules subject to their past performance. Since models that deviate from the assumption of fully rational agents face the well-known wilderness of the bounded rationality critique, it should be noted that setups in which agents rely on a set of plausible rules to form expectations and select among them based on their past performance provide a reasonable alternative to the fully rational paradigm. See Hommes (2013) and references therein for a deeper discussion. Other papers that introduce learning in the cobweb model include Goeree and Hommes (2000), Lasselle et al. (2005) and Schmitt and Westerhoff (2015).

A typical property of cobweb models is that their steady states become unstable due to a flip bifurcation, resulting in a period-two cycle or, using the words of Leontief (1934), producing zigzag dynamics. However, even Moore (1917), Ezekiel (1938) and Larson (1964) stressed the fact that actual commodity prices have a cyclical nature. Against this background, a Neimark-Sacker bifurcation, turning fixed-point dynamics into cyclical dynamics, appears to be quite appealing from an empirical perspective.[1] Dieci and Westerhoff (2010) provide such a framework by coupling the dynamics of two cobweb markets via the supply side.

[1] To prevent confusion, we remark that by cyclical price dynamics we mean cycles of higher period and not the zigzag behavior of a period-two cycle.

Hommes (1998) and Cavalli et al. (2021) are just some of the few cobweb models that also manage to produce cyclical price dynamics.

The goal of our paper is to add to this line of research by proposing a cobweb model that is able to produce cyclical commodity price dynamics. As already demonstrated by Ezekiel (1938), cobweb models with a one-period production delay and naïve expectations may create a stable period-two cycle, while cobweb models with a two-period production delay and naïve expectations are able to generate a stable period-four cycle.[2] The latter observation forms our starting point. We assume in our cobweb model that all firms face a production delay of two periods. Moreover, firms have access to two different production technologies: an inflexible production technology, requiring them to start the production process immediately, and a flexible production technology, allowing them to delay the actual production process by one period. While the flexible production technology invokes higher fixed costs than the inflexible production technology, it offers firms an informational advantage in the sense that they can track the commodity price one period closer to the actual trading period. Importantly, firms endogenously select between the two production technologies with respect to their evolutionary fitness, measured in terms of past realized profits. The dynamics of our cobweb model is due to a four-dimensional nonlinear map. We show that its unique steady state may become unstable via a Neimark-Sacker bifurcation, a scenario that gives rise to cyclical price dynamics, as observed in many actual commodity markets. Simulations furthermore reveal that our cobweb model may produce chaotic motion, too.

We remark that our paper is related to the following studies. Gori et al. (2014) and Matsumoto and Szidarovszky (2015) study nonlinear continuous-time models with production delays that yield endogenous cyclical dynamics, though

[2] Many economists seem to have forgotten this part of Ezekiel's (1938) paper – in any case, there are virtually no references to his discussion of larger production delays and the herewith connected possibility of cycles. The same is true for us: we only rediscovered this part of his paper by chance. In a sense, one may regard our attempt to revive this aspect as a Tonu Puu moment, and, hopefully, as a general incentive to all of us to embrace classical economic (and noneconomic) papers.

their models do not consider the fact that firms may switch between different production technologies. Interestingly, their line of reasoning may be traced back to the work of Haldane (1934) and Larson (1964). The continuous-time disequilibrium commodity market model by Mackey (1989) with state-dependent production and storage delays may produce cyclical dynamics, too. Two interesting economic examples for discrete-time models in which firms have access to different production technologies and switch between them with respect to their evolutionary fitness are Hommes and Zeppini (2014) and Lamantia et al. (2018).

We continue as follows. In Section 2, we introduce a cobweb model in which firms, facing a production delay of two periods, can switch between two different production technologies. In Section 3, we present our main analytical and numerical results. In Section 4, we modify our cobweb model by considering that firms have to cope with a production delay of three periods and three production technologies. In Section 5, we conclude our paper and discuss potential avenues for future research.

2 A cobweb model with two different production technologies

Cobweb models describe a dynamic price adjustment process on a competitive market for a single nonstorable commodity with a fixed supply response lag. Cobweb models usually assume that firms face a production delay of one period. Motivated by Ezekiel (1938), we assume here that all firms face a production delay of two periods. However, firms have access to two different production technologies, offering them different degrees of flexibility with respect to the period at which they have to finalize their decisions about their actual production quantities. One production technology, called the slow production technology, is rather inflexible. Firms relying on that production technology have to determine their production quantities and to initiate the production process for period t immediately in period $t-2$. The other production technology, called the fast production technology, is more flexible and allows firms to delay their final production decisions by one period, i.e. they produce the commodity for period t in period $t-1$.

Access to the more flexible production technology invokes higher fixed costs than the slow production technology. Clearly, firms that adopt the fast production technology need to spend these additional fixed costs in period $t-2$ for more efficient machinery and equipment that reduces for them the time to produce the commodity and to bring it to the market by one period. A crucial advantage of spending higher fixed costs obviously is that it allows firms to condition their final production decisions on information they receive up to period $t-1$. In the spirit of Brock and Hommes (1997), firms select production technologies based on their past performance, as observable in period $t-2$, and display a boundedly rational learning behavior in the sense that more of them will use the production technology that was more profitable in the recent past. We also assume that firms are committed to the production technology they choose in period $t-2$ for the commodity market that operates in period t, i.e. they cannot reverse their production technology choices in period $t-1$. To make the implications of our setup as clear as possible, we specify all other model parts as is done in classical linear cobweb models (see, e.g. Gandolfo 2009). Since firms thus have naïve price expectations, firms opting for the slow production technology condition their production decisions on the commodity price observed in period $t-2$, while firms that decided in favor of the fast production technology can monitor the commodity price up to period $t-1$. Due to firms' endogenous technology choice, the dynamics of our cobweb model is driven by the iteration of a four-dimensional nonlinear map.

Let us turn to the details of our model. We consider a fixed number N of firms, indexed by S for slow and by F for fast production technology adopters. For notational convenience, we denote the market shares of firms that have decided in period $t-2$ to use the slow and the fast production technology for providing their supply in period t by W_t^S and W_t^F, respectively. Since an individual firm either supplies quantities S_t^S or S_t^F, the total supply of the commodity can be expressed by

$$S_t = N(W_t^S S_t^S + W_t^F S_t^F). \tag{1}$$

Consumers' commodity demand depends negatively on the current commodity price P_t. Using a linear demand relation, we formulate consumers' commodity demand by

$$D_t = a - bP_t, \tag{2}$$

with $a, b > 0$. Since the commodity is nonstorable, the market clearing condition

$$D_t = S_t \tag{3}$$

holds in every period, implying that the commodity price obeys

$$P_t = \frac{a-(W_t^S S_t^S + W_t^F S_t^F)}{b}, \tag{4}$$

where, for ease of exposition, we have normalized the mass of firms to $N = 1$.

Each firm maximizes its expected profits subject to a quadratic cost function. Let $P_t^{e,i}$ stand for firm i's commodity price expectation and let $C_t^i = \frac{1}{2c}(S_t^i)^2 + d^i$ reflect its cost function, with $c > 0$ and $d^i > 0$. Firm i's fixed costs depend on its choice of production technology, specified in the sequel, and are represented by $d^F > d^S \geq 0$. Hence, we can write firm i's expected profits as

$$\pi_t^{e,i} = P_t^{e,i} S_t^i - \frac{1}{2c}(S_t^i)^2 - d^i, \tag{5}$$

yielding its optimal supply decision

$$S_t^i = cP_t^{e,i}. \tag{6}$$

Accordingly, firm i's commodity supply depends positively on its price expectations. Since firms employ naïve expectations, the optimal supply of a firm relying on the slow production technology, requiring it to fix its output in period $t-2$, is

$$S_t^S = cP_{t-2}, \tag{7}$$

while the optimal supply of a firm applying the fast production technology, providing it the flexibility to determine its output in period $t-1$, amounts to

$$S_t^F = cP_{t-1}. \tag{8}$$

Importantly, firm i's willingness to invest in higher fixed costs allows it to track the commodity price one period closer to the actual trading period. In this sense, we may also regard the difference in fixed costs, i.e. $d = d^F - d^S$, as information costs.[3]

[3] As pointed out by an anonymous referee, one may interpret our setup also in the following way. The difference in fixed costs may reflect costs that provide those firms that are willing to spend them with an unbiased forecast in period $t-2$ about the commodity price in period $t-1$, which

We model firms' production technology choices and their willingness to invest in higher fixed costs via the discrete choice approach. Following Brock and Hommes (1997), the market shares of firms that choose the slow and fast production technology are expressed by

$$W_t^S = \frac{exp[\beta\pi_{t-2}^S]}{exp[\beta\pi_{t-2}^S]+exp[\beta\pi_{t-2}^F]} = \frac{1}{1+exp[\beta(\pi_{t-2}^F-\pi_{t-2}^S)]} \tag{9}$$

and

$$W_t^F = \frac{exp[\beta\pi_{t-2}^F]}{exp[\beta\pi_{t-2}^S]+exp[\beta\pi_{t-2}^F]} = 1 - W_t^S, \tag{10}$$

respectively, where, given the model's underlying time structure, the production technologies' realized profits, observed in period $t-2$, are determined by

$$\pi_{t-2}^S = P_{t-2}S_{t-2}^S - C_{t-2}^S = \frac{c}{2}P_{t-4}(2P_{t-2} - P_{t-4}) - d^S \tag{11}$$

and

$$\pi_{t-2}^F = P_{t-2}S_{t-2}^F - C_{t-2}^F = \frac{c}{2}P_{t-3}(2P_{t-2} - P_{t-3}) - d^F, \tag{12}$$

respectively. Note that the higher the profitability of a production technology, the more firms will select it. In particular, the intensity of choice parameter $\beta > 0$ controls how sensitive the mass of firms is to selecting the most profitable production technology. For $\beta \to 0$, for instance, firms do not observe profit differentials between the two production technologies and, consequently, are divided equally among them, i.e. $W_t^S = W_t^F = 0.5$ for all t. However, the higher the intensity of choice parameter β, the more firms will select the more profitable production technology. In the extreme case of $\beta \to \infty$, the so-called neoclassical limit, all firms will opt for the more profitable production technology.[4]

they can use as a predictor for the commodity price in period t. In fact, firms may be willing to pay money for better predictions.

[4] Hommes (1998) develops a nonlinear cobweb model with linear backward-looking expectations that may yield cyclical commodity price dynamics. In particular, he studies the case in which firms form their price expectations for period t as a weighted average of prices they observe in periods $t-1$ and $t-2$. In a sense, our model resembles his model when we interpret the market shares of firms that opt for the slow and the fast production technology that we model in our paper as time-varying weights that enter the aforementioned weighted average price in his model.

3 Main results

For analytical and notational convenience, and similar to Brock and Hommes (1997), we introduce the difference in market shares of the two production technologies with a one-period lag in the time subscript, i.e. $m_{t-1} = W_t^S - W_t^F = tanh\left[\frac{\beta}{2}(\pi_{t-2}^S - \pi_{t-2}^F)\right]$. Noting that $W_t^S = \frac{1+m_{t-1}}{2}$ and $W_t^F = \frac{1-m_{t-1}}{2}$ then allows us to express our model in the form of a four-dimensional nonlinear map, given by

$$\begin{cases} P_t = \frac{a - \left(\frac{1+m_{t-1}}{2}cX_{t-1} + \frac{1-m_{t-1}}{2}cP_{t-1}\right)}{b} \\ X_t = P_{t-1} \\ Y_t = X_{t-1} \\ m_t = tanh\left[\frac{\beta}{2}\left(\left(\frac{c}{2}Y_{t-1}(2P_{t-1} - Y_{t-1}) - d^S\right) - \left(\frac{c}{2}X_{t-1}(2P_{t-1} - X_{t-1}) - d^F\right)\right)\right] \end{cases}, \quad (13)$$

where $X_t = P_{t-1}$ and $Y_t = X_{t-1}$ are auxiliary variables. In order to find the model's steady state(s), we set $\bar{P} = P_t = P_{t-1} = X_{t-1} = Y_{t-1}$ and $\bar{m} = m_t = m_{t-1}$. Straightforward computations reveal that map (13) possesses the unique steady state

$$FSS = (\bar{P}, \bar{X}, \bar{Y}, \bar{m}) = \left(\frac{a}{b+c}, \frac{a}{b+c}, \frac{a}{b+c}, tanh\left[\frac{\beta}{2}d\right]\right). \quad (14)$$

Since steady-state profit differences between the slow and the fast production technology amount to $\overline{\pi^S} - \overline{\pi^F} = d^F - d^S = d$, their steady-state market shares can be expressed by $\overline{W^S} = \frac{1}{1+exp[-\beta d]}$ and $\overline{W^F} = \frac{1}{1+exp[\beta d]}$, respectively. Moreover, to study the local asymptotic stability properties of the model's steady state, we have to evaluate the Jacobian matrix of (13). At the FSS, we obtain

$$J(FSS) = \begin{bmatrix} -\frac{c\left(1-tanh\left[\frac{\beta}{2}d\right]\right)}{2b} & -\frac{c\left(1+tanh\left[\frac{\beta}{2}d\right]\right)}{2b} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (15)$$

yielding the characteristic polynomial

$$P(\lambda) = \lambda^2\left(\lambda^2 + \frac{c\left(1-tanh\left[\frac{\beta}{2}d\right]\right)}{2b}\lambda + \frac{c\left(1+tanh\left[\frac{\beta}{2}d\right]\right)}{2b}\right). \quad (16)$$

Since two eigenvalues of (16) are obviously always equal to zero, say $\lambda_1 = 0$ and $\lambda_2 = 0$, the local stability of the model's steady state hinges on the remaining two eigenvalues, say λ_3 and λ_4, determined by the term in brackets on the right-hand side of (16). Let us rewrite this term as

$\lambda^2 + a_1\lambda + a_2$, (17)

where $a_1 = \frac{c\left(1-tanh\left[\frac{\beta}{2}d\right]\right)}{2b} = \frac{c}{b}\overline{W^F}$ and $a_2 = \frac{c\left(1+tanh\left[\frac{\beta}{2}d\right]\right)}{2b} = \frac{c}{b}\overline{W^S}$. As is well known (see, e.g. Medio and Lines 2001), necessary and sufficient conditions assuring that the remaining two eigenvalues of (16) are less than one in modulus are given by (i) $1 + a_1 + a_2 > 0$, (ii) $1 - a_1 + a_2 > 0$ and (iii) $1 - a_2 > 0$. Conditions (i) and (ii) are always satisfied. However, condition (iii) requires that

$\overline{W^S}\frac{c}{b} < 1$. (18)

Importantly, $0.5 < \overline{W^S} < 1$ increases in line with β. For $\beta \to 0$, we have that $\overline{W^S} \to 0.5$. Hence, the FSS is always unstable for $c > 2b$. For $\beta \to \infty$, we have that $\overline{W^S} \to 1$, implying that the FSS is locally stable for $c < b$. Since $\overline{W^S} = \frac{1+tanh\left[\frac{\beta}{2}d\right]}{2}$, condition (18), relevant for $b < c < 2b$, is equivalent to

$\beta < \frac{2arctanh\left[\frac{2b-c}{c}\right]}{d}$. (19)

Finally, a violation of (18) and (19), respectively, is associated with a Neimark-Sacker bifurcation. [5] We have thus proven the following proposition.

Proposition: *The dynamics of our cobweb model is driven by a four-dimensional nonlinear map. Its unique steady state, implying that* $\bar{P} = \frac{a}{b+c}$ *and* $\bar{m} = tanh\left[\frac{\beta}{2}d\right]$, *is locally stable for* $c < b$ *and unstable for* $c > 2b$. *For* $b < c < 2b$, *the model's steady state is locally stable if* $\beta < \frac{2arctanh\left[\frac{2b-c}{c}\right]}{d}$, *where a violation of the later condition is associated with a Neimark-Sacker bifurcation.*

[5] Interestingly, firms' evolutionary switching between the slow and the fast production technology stabilizes the model's dynamics relative to the case in which only one production technology is available, whether that is the slow or the fast production technology. As stated above, the local stability of the FSS requires for $\overline{W^S} \to 1$ that $c < b$. Furthermore, condition (ii) reads $(\overline{W^F} - \overline{W^S})\frac{c}{b} < 1$. Since $\overline{W^F} < \overline{W^S}$, condition (ii) is always satisfied. However, consider, contrary to our assumptions, that $d < 0$, and assume that $\beta \to \infty$. Then $\overline{W^F} \to 1$, and the local stability of the FSS again requires that $c < b$. In the latter case, commodity prices become unstable in the form of a flip bifurcation, as in classical cobweb models with naïve expectations and a one-period production delay. See Leontief (1934) or Gandolfo (2009).

Let us discuss the economic implications of our proposition. At the steady state, the commodity price is given by $\bar{P} = \frac{a}{b+c}$. Since $\bar{P}$ only depends on the model's real parameters, we call it the model's fundamental steady state.[6] However, the model's behavioral parameter β, i.e. firms' intensity of choice, influences the steady-state fractions of firms opting for the slow and fast production technology, given by $\overline{W^S} = \frac{1}{1+exp[-\beta d]}$ and $\overline{W^F} = \frac{1}{1+exp[\beta d]}$, respectively. Obviously, the slow production technology receives a larger market share when firms' intensity of choice and/or its fixed costs advantage increases. Needless to say, the same can be concluded from the difference in market shares, i.e. from $\bar{m} = \ tanh\left[\frac{\beta}{2}d\right]$. For completeness, we remark that the steady-state profits of the two production technologies amount to $\overline{\pi^S} = \frac{c}{2}\bar{P}^2 - d^S = \frac{ca^2}{2(b+c)^2} - d^S$ and $\overline{\pi^F} = \frac{c}{2}\bar{P}^2 - d^F = \frac{ca^2}{2(b+c)^2} - d^F$. Hence, firms make profits at the steady state as long as their production technologies' fixed costs are not too high.

With respect to the steady state's stability properties, we have that the FSS is locally stable for $c < b$, while it is unstable for $c > 2b$. For $b < c < 2b$, stability condition $\overline{W^S}\frac{c}{b} < 1$ applies, being equivalent to $\beta < \frac{2arctanh\left[\frac{2b-c}{c}\right]}{d}$. Hence, the local stability of the model's FSS does not only depend on the slopes of the model's demand and supply schedules, but also on firms' intensity of choice and on the fixed costs advantage of the slow production technology. In particular, an increase in parameter b is beneficial for market stability, while an increase in parameters c, d and β is harmful to market stability. Furthermore, the stability of the model's FSS is independent of parameter a, and, consequently, the same is true for the steady-state commodity price and the steady-state profits of the two production technologies. In more general terms, we may conclude that an increase in the market impact of firms relying on the slow production technology is destabilizing. In fact, we may interpret the product of $\overline{W^S}$ and c as the slope

[6] We remark that Ezekiel (1938) called this steady state the normal steady state. Today, the use of the term "fundamental steady state" is more common.

parameter of the aggregate supply function of firms relying on the slow production technology, growing, of course, with the market share of the slow production technology and with the individual slope parameter of a slow production technology adapter.

To illustrate the dynamics of our model, let us assume the following base parameter setting: $a = 25$, $b = 1$, $c = 1.5$ and $d = 1$. Figure 1 shows bifurcation diagrams of the commodity price (top panel) and the market share of firms opting for the slow production technology (bottom panel) versus firms' intensity of choice, where parameter β is varied between 0 and 2. As predicted by our analytical results, the steady-state commodity price $\bar{P} = 10$ becomes unstable due to a Neimark-Sacker bifurcation as parameter β exceeds $\beta_{crit} \approx 0.7$. Since the amplitude of commodity price cycles increases in line with parameter β, we can conclude that our cobweb model does not only become unstable, but also increasingly volatile when firms switch faster between their two production technologies. While $\bar{P}$ is independent of parameter β, $\overline{W^S}$ increases in line with it. Note that $\overline{W^S} = 2/3$ at $\beta_{crit} \approx 0.7$, so that $\overline{W^S}\frac{c}{b} = 1$. Defining $SC = \overline{W^S}c$ as the slope parameter of the aggregate supply schedule of firms adhering to the slow production technology at the steady state, the stability condition $\frac{SC}{b} < 1$ is reminiscent of the one reported in Leontief (1934), except that a violation of our stability condition leads to cyclical dynamics, while a violation of his stability condition creates a period-two cycle.

Figures 2 depicts the time evolution of the commodity price, the time evolution of the market share of firms opting for the slow production technology, the market share of firms opting for the slow production technology in period t versus the commodity price in period t and the commodity price in period $t + 1$ versus the commodity price in period t, assuming that $\beta = 1$. As long as firms' intensity of choice parameter remains near its Neimark-Sacker bifurcation value, our model gives rise to more or less regular oscillatory commodity price dynamics. However, our model is also able to produce complex dynamics, as evidenced by the simulations presented in Figure 3, resting on $\beta = 3.5$. We remark that the emergence of a strange attractor, visible in the bottom two panels of Figure 3, is an indicator of chaotic dynamics.

Despite the high dimension of our cobweb model, it is still possible to understand its basic functioning. Note first that the amplitude of the commodity price cycles depicted in Figure 3 increases when firms stick to the slow production technology for longer periods, as is, for instance, the case around periods, 20, 40 and 60. In contrast, the dynamics of our cobweb model becomes (temporarily) stabilized when sufficiently many firms switch to the fast production technology. In fact, stability condition (18) suggests that for the underlying parameter setting of the dynamics depicted in Figure 3, this is the case when $W_t^S < \frac{2}{3}$. Together, this leads us to the following insights. Near the fundamental steady state, firms prefer the slow production technology: as long as the commodity price is relatively stable, the slow production technology's fixed costs advantage outweighs the errors that arise from having to predict the commodity price for period t on the basis of the commodity price observed in period $t-2$. The cobweb market is then subject to a centrifugal force. As the cobweb market enters a more unstable phase, however, prediction errors of firms using the slow production technology increase, and it eventually pays off for firms to switch to the fast production technology, despite facing higher fixed costs.[7] This allows them to monitor the commodity price for one period closer to the trading period. Now the commodity market is subject to a centripetal force. Unfortunately, this is not the end of the story. Firms return to the slow production technology when the commodity market has sufficiently calmed down and, consequently, the centripetal force gives way to the centrifugal force, keeping the dynamics alive.[8]

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[7] See Appendix A for a more rigorous line of reasoning.

[8] In the cobweb model by Brock and Hommes (1997), firms switch between naïve and rational expectations, implying, essentially, that they face a production lag of one period or no production lag. Similar to our model, the dynamics of their model becomes unstable when naïve (slow) firms dominate the commodity market. A difference between their model and ours is that our model entails a further production lag, turning, roughly speaking, their flip bifurcation scenario into our Neimark-Sacker bifurcation scenario.

4 A cobweb model with three different production technologies

In this section, we modify our cobweb model by assuming that the production process takes three periods and that firms have access to three different production technologies, referred to as the fast, the slow and the lame production technology, effectively necessitating them to fix their production decisions (i.e. the quantity to be produced) one, two or three periods ahead. Once a firm has decided in period $t-3$ for a production technology it wants to use to produce the commodity for period t, its choice cannot be reversed. The three production technologies' fixed costs parameters satisfy the relation $d^F > d^S > d^L \geq 0$, where the index L refers to the new (lame) production technology. Consequently, the total supply of the commodity is $S_t = N(W_t^L S_t^L + W_t^S S_t^S + W_t^F S_t^F)$, where W_t^L stands for the market share of firms opting for the lame production technology and $S_t^L = cP_{t-3}$ indicates the optimal supply of one of those firms. Fixing the mass of firms to $N = 1$, the commodity price is due to $P_t = \frac{a-(W_t^L S_t^L + W_t^S S_t^S + W_t^F S_t^F)}{b}$. Moreover, the market shares of firms choosing the lame, the slow and the fast production technology are given by $W_t^L = \frac{exp[\beta\pi_{t-3}^L]}{exp[\beta\pi_{t-3}^L]+exp[\beta\pi_{t-3}^S]+exp[\beta\pi_{t-3}^F]}$, $W_t^S = \frac{exp[\beta\pi_{t-3}^S]}{exp[\beta\pi_{t-3}^L]+exp[\beta\pi_{t-3}^S]+exp[\beta\pi_{t-3}^F]}$ and $W_t^F = \frac{exp[\beta\pi_{t-3}^F]}{exp[\beta\pi_{t-3}^L]+exp[\beta\pi_{t-3}^S]+exp[\beta\pi_{t-3}^F]}$, respectively. Importantly, firms' choice of production technology now depends on realized profits observed in period $t-3$, namely $\pi_{t-3}^L = P_{t-3}S_{t-3}^L - C_{t-3}^L = \frac{c}{2} P_{t-6}(2P_{t-3} - P_{t-6}) - d^L$, $\pi_{t-3}^S = P_{t-3}S_{t-3}^S - C_{t-3}^S = \frac{c}{2} P_{t-5}(2P_{t-3} - P_{t-5}) - d^S$ and $\pi_{t-3}^F = P_{t-3}S_{t-3}^F - C_{t-3}^F = \frac{c}{2} P_{t-4}(2P_{t-3} - P_{t-4}) - d^F$. The dynamics of this model is due to a six-dimensional nonlinear map. See Appendix B for a brief analytical characterization.

The top panel of Figure 4 shows the time evolution of the commodity price, assuming that $a = 25$, $b = 1$, $c = 1.5$, $d^F = 1$, $d^S = 0.1$, $d^L = 0$ and $\beta = 1.8$. Comparing this panel with the top panel of Figure 2 reveals that allowing for larger production delays may increase the length of commodity price cycles. Indeed, Ezekiel (1938) already demonstrated that a cobweb model with a production delay of two periods and naïve expectations may give rise to a stable period-4 cycle, while a cobweb model with a production lag of three periods and naïve expectations may give rise to a stable period-6 cycle. Further simulations (not depicted) reveal that the average cycle length of the commodity price further increases when even larger production delays are considered. The bottom panel of Figure 4 presents the commodity price versus firms' intensity of choice parameter using the same parameter setting, except that parameter β is varied between $0 < \beta < 2$. As can be seen, the cobweb market becomes unstable in the form of a Neimark-Sacker bifurcation as firms' intensity of choice parameter exceeds $\beta_{crit} \approx 1.418$.

5 Conclusions

A crucial goal of our paper is to explain the fact that commodity markets around the world display significant cyclical price dynamics. To achieve this goal, we follow Ezekiel (1938) and develop a cobweb model in which all firms face a production delay of two periods. However, firms have access to two different production technologies: a slow (cheap) production technology, requiring them to start the production process immediately, and a fast (costly) production technology, allowing them to delay the production process by one period. While the fast production technology invokes higher fixed costs than the slow production technology, it offers firms an informational advantage in the sense that they can monitor the evolution of the commodity market one period closer to the actual trading date. Inspired by Brock and Hommes (1997), we furthermore assume that firms select between production technologies based on their past performance. As it turns out, a four-dimensional nonlinear map, possessing a unique steady state, reigns the dynamics of our cobweb model. We analytically show that its steady state may become unstable in the form of a Neimark-Sacker bifurcation when sufficiently many firms opt for the slow production technology. Our cobweb model then generates regular or, depending on the parameter setting, irregular oscillatory price dynamics. Interestingly, allowing for larger production lags tends to increase the dynamics' average cycle length.

To make the implications of our setup as stark as possible, we assume that firms have naïve expectations. A natural extension of our cobweb model would thus be to consider alternative expectation formation schemes, e.g. by assuming that firms rely on extrapolative, regressive or adaptive expectation rules. It would, of course, be beneficial to build a model in which firms switch between production technologies and expectation rules. The same is true for approaches that postulate nonlinear demand and supply schedules. Finally, we remark that Tonu Puu was always interested in the global behavior of economic models. Hence, it might also be worthwhile to explore our model's out-of-equilibrium dynamics in more detail. In any case, we hope that our paper stimulates more work in these directions.

Appendix A

Note that (11) and (12) allow us to rewrite the profit differential between the fast and the slow production technology as

$$\begin{aligned}\pi_t^F - \pi_t^S &= \frac{c}{2}\, P_{t-1}(2P_t - P_{t-1}) - d^F - \frac{c}{2}\, P_{t-2}(2P_t - P_{t-2}) + d^S \\ &= c\, P_t(P_{t-1} - P_{t-2}) - \frac{c}{2}\,(P_{t-1}^2 - P_{t-2}^2) - (d^F - d^S) \\ &= c\, P_t(P_{t-1} - P_{t-2}) - \frac{c}{2}\,(P_{t-1} - P_{t-2})(P_{t-1} + P_{t-2}) - d \\ &= c\,(P_{t-1} - P_{t-2})\left(P_t - \tilde{P}_{t-1}\right) - d,\end{aligned} \tag{A1}$$

where $\tilde{P}_{t-1} := \frac{(P_{t-1}+P_{t-2})}{2}$ is the average price of the previous two periods. Accordingly, the fast production technology is more profitable in period t if and only if

$$c\,(P_{t-1} - P_{t-2})\left(P_t - \tilde{P}_{t-1}\right) > d. \tag{A2}$$

Since $d > 0$, a *necessary* condition for the fast production technology to be more profitable in period t is that $(P_{t-1} - P_{t-2})$ and $\left(P_t - \tilde{P}_{t-1}\right)$ have the same sign, that is there must be some persistence in the price trend over the last two periods. Conversely, a *sufficient* condition for the slow production technology to be more profitable in period t is that the product between these two quantities is negative, e.g. due to a price reversal over the last two periods. Clearly, a commodity market with pronounced price trends is beneficial for the fast production technology while a more stable commodity market is beneficial for the slow production technology.

Appendix B

The model with three different production delays presented in Section 4 may be expressed by the following six-dimensional nonlinear map:

$$\begin{cases} P_t = \frac{a-(W_t^F cP_{t-1}+W_t^S cV_{t-1}+W_t^L cW_{t-1})}{b} \\ V_t = P_{t-1} \\ W_t = V_{t-1} \\ X_t = W_{t-1} \\ Y_t = X_{t-1} \\ Z_t = Y_{t-1} \end{cases}, \tag{B1}$$

where $W_t^F = \frac{exp[\beta\left(\frac{c}{2}X_{t-1}(2W_{t-1}-X_{t-1})-d^F\right)]}{\Omega_t}$, $W_t^S = \frac{exp[\beta\left(\frac{c}{2}Y_{t-1}(2W_{t-1}-Y_{t-1})-d^S\right)]}{\Omega_t}$, $W_t^L = \frac{exp[\beta\left(\frac{c}{2}Z_{t-1}(2W_{t-1}-Z_{t-1})-d^L\right)]}{\Omega_t}$ and $\Omega_t = exp[\beta\left(\frac{c}{2}X_{t-1}(2W_{t-1}-X_{t-1})-d^F\right)] + exp[\beta\left(\frac{c}{2}Y_{t-1}(2W_{t-1}-Y_{t-1})-d^S\right)] + exp[\beta\left(\frac{c}{2}Z_{t-1}(2W_{t-1}-Z_{t-1})-d^L\right)]$.[9] Now, straightforward computations reveal that the model's unique steady state, given by

$$\bar{P} = \bar{V} = \bar{W} = \bar{X} = \bar{Y} = \bar{Z} = \frac{a}{b+c}, \tag{B2}$$

yields, via its Jacobian matrix, the following characteristic polynomial:

$$P(\lambda) = \lambda^3\left(\lambda^3 + \frac{c}{b}\overline{W^F}\lambda^2 + \frac{c}{b}\overline{W^S}\lambda + \frac{c}{b}\overline{W^L}\right), \tag{B3}$$

where $\overline{W^F} = \frac{1}{1+\exp[\beta(d^F-d^S)]+\exp[\beta(d^F-d^L)]}$, $\overline{W^S} = \frac{1}{1+\exp[\beta(d^S-d^F)]+\exp[\beta(d^S-d^L)]}$ and $\overline{W^L} = \frac{1}{1+\exp[\beta(d^L-d^F)]+\exp[\beta(d^L-d^S)]}$. Using the stability results derived in Lines et al. (2020), we can conclude that the model's steady state loses its local stability when one of the three inequalities becomes broken: (i) $1 + a_1 + a_2 + a_3 > 0$, (ii) $1 - a_1 + a_2 - a_3 > 0$ and (iii) $1 - a_2 + a_1 a_3 - {a_3}^2 > 0$, where $a_1 = \frac{c}{b}\overline{W^F}$, $a_2 = \frac{c}{b}\overline{W^S}$ and $a_3 = \frac{c}{b}\overline{W^L}$. Before we study these inequalities in more detail, a few observations are in order. For $\beta > 0$, the relation $0 < \overline{W^F} < \overline{W^S} < \overline{W^L} < 1$ holds. Moreover, as β moves from 0 to ∞, $\overline{W^F}$ decreases from $\frac{1}{3}$ to 0, while $\overline{W^L}$ increases from $\frac{1}{3}$ to 1. Although $\overline{W^S} = \frac{1}{3}$ for $\beta = 0$ and $\overline{W^S} \to 0$ for $\beta \to \infty$, $\overline{W^S}$ reacts non-

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[9] The auxiliary variable W_t should not be confused with the market shares W_t^L, W_t^S and W_t^F.

monotonically with respect to β. However, it is clear that $\overline{W^S} < \frac{1}{2}$ for any $\beta \geq 0$. Having these observations in mind, condition (i) is obviously always satisfied. Condition (ii), requiring that

$$\frac{c}{b}\left(1 - 2\overline{W^S}\right) < 1, \tag{B4}$$

is certainly satisfied when $c \leq 2b$. A violation of (B4) takes place if the ratio between c and b is sufficiently larger than 1 and a large enough increase in β renders $\overline{W^S}$ sufficiently small. Condition (iii) necessitates that

$$\frac{c}{b}\left(\overline{W^S} + \frac{c}{b}\overline{W^L}\left(\overline{W^L} - \overline{W^F}\right)\right) < 1. \tag{B5}$$

Since the left-hand side of (B5) is equal to $\frac{c}{3b}$ for $\beta = 0$, our previous remarks imply that (B5) is violated if $c > 3b$ and β is not too large. Similarly, since the left-hand side of (B5) tends to $\left(\frac{c}{b}\right)^2$ as $\beta \to \infty$, (B5) holds if $c < b$ and β is large enough. Although no general conclusions can be drawn from intermediate values of the ratio between c and b, the simple and tractable form of conditions (B4) and (B5) allows us to easily identify bifurcation values for β, given specific values for b and c. For instance, the bifurcation diagram in Figure 4 is based on $b = 1$, $c = 1.5$, $d^F = 1$, $d^S = 0.1$ and $d^L = 0$. For $\beta_{crit} \approx 1.418$, we obtain $\overline{W^F} \approx 0.115$, $\overline{W^S} \approx 0.411$ and $\overline{W^L} \approx 0.474$, and (B5) reveals that the steady state is at the border of stability. As demonstrated by Gardini et al. (2021), we can furthermore conclude that a violation of (B5) is associated with a Neimark-Sacker bifurcation. In fact, a slight increase in β elevates $\overline{W^L}$ and $\overline{W^S}$ at the expense of $\overline{W^F}$, and the steady state loses its stability, giving way to oscillatory dynamics.

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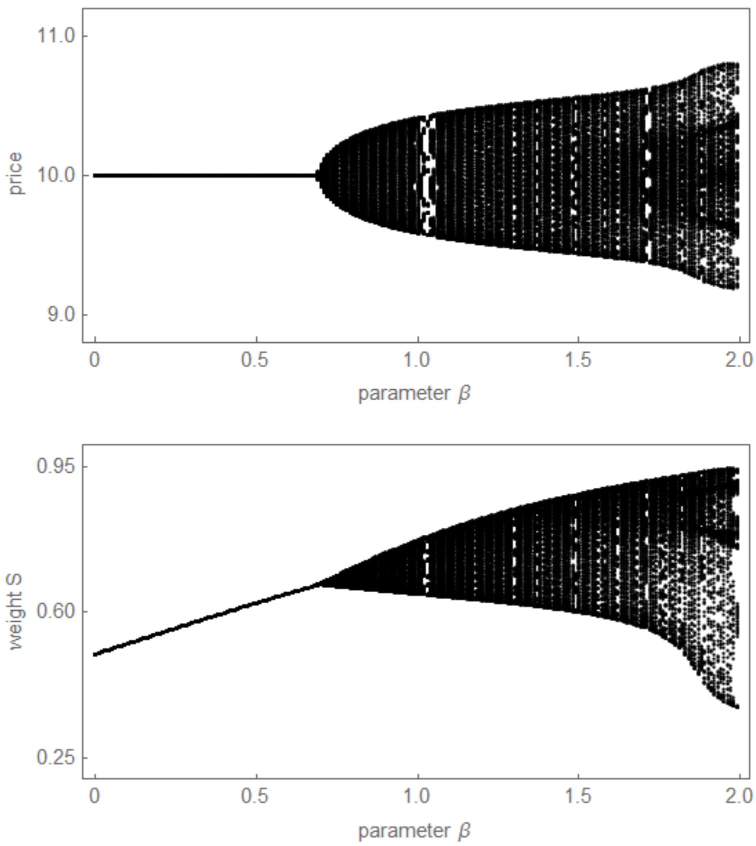

Figure 1: Bifurcation diagrams. The panels show the commodity price and the market share of firms opting for the slow production technology versus firms' intensity of choice parameter β. Remaining parameters: $a = 25$, $b = 1$, $c = 1.5$ and $d = 1$.

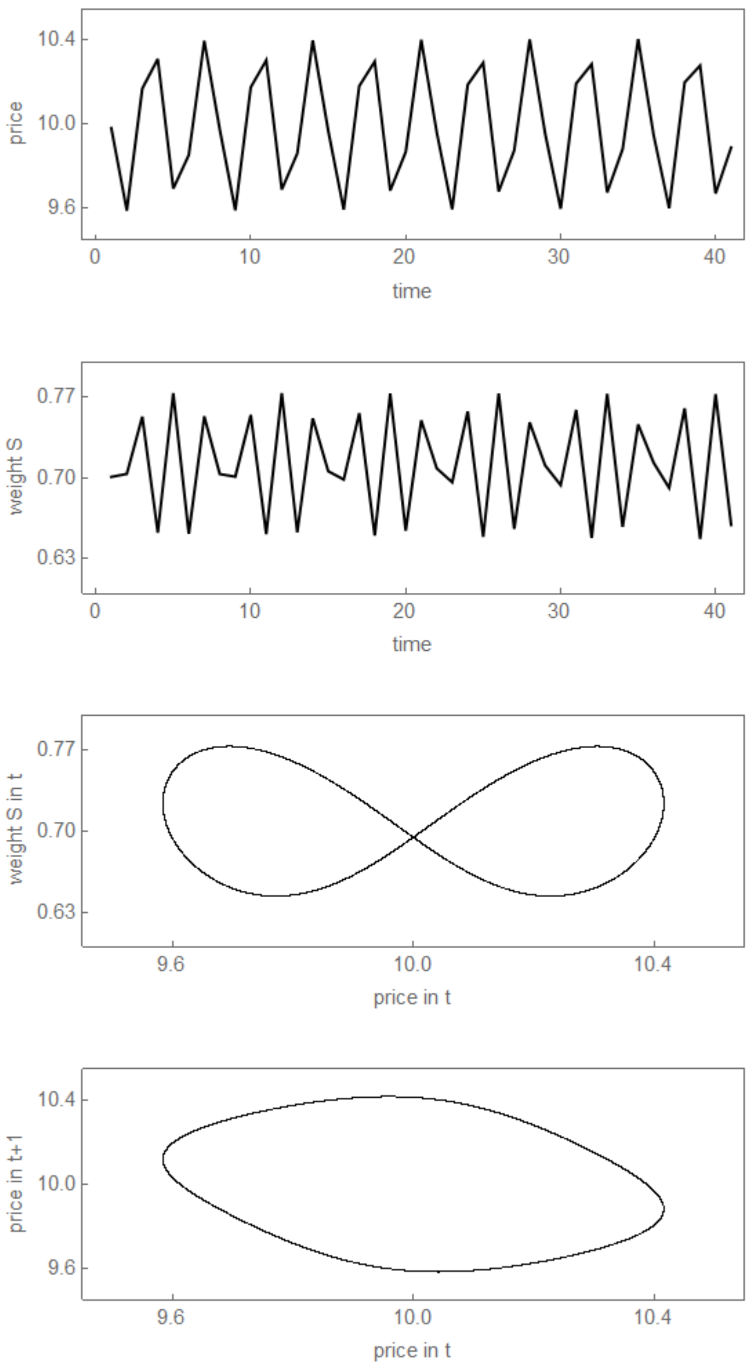

Figure 2: Oscillatory dynamics. The panels show from top to bottom the evolution of the commodity price, the evolution of the market share of firms opting for the slow production technology, the market share of firms opting for the slow production technology in period t versus the commodity price in period t and the commodity price in period $t+1$ versus the commodity price in period t. Parameter setting: $a = 25$, $b = 1$, $c = 1.5$, $d = 1$ and $\beta = 1$.

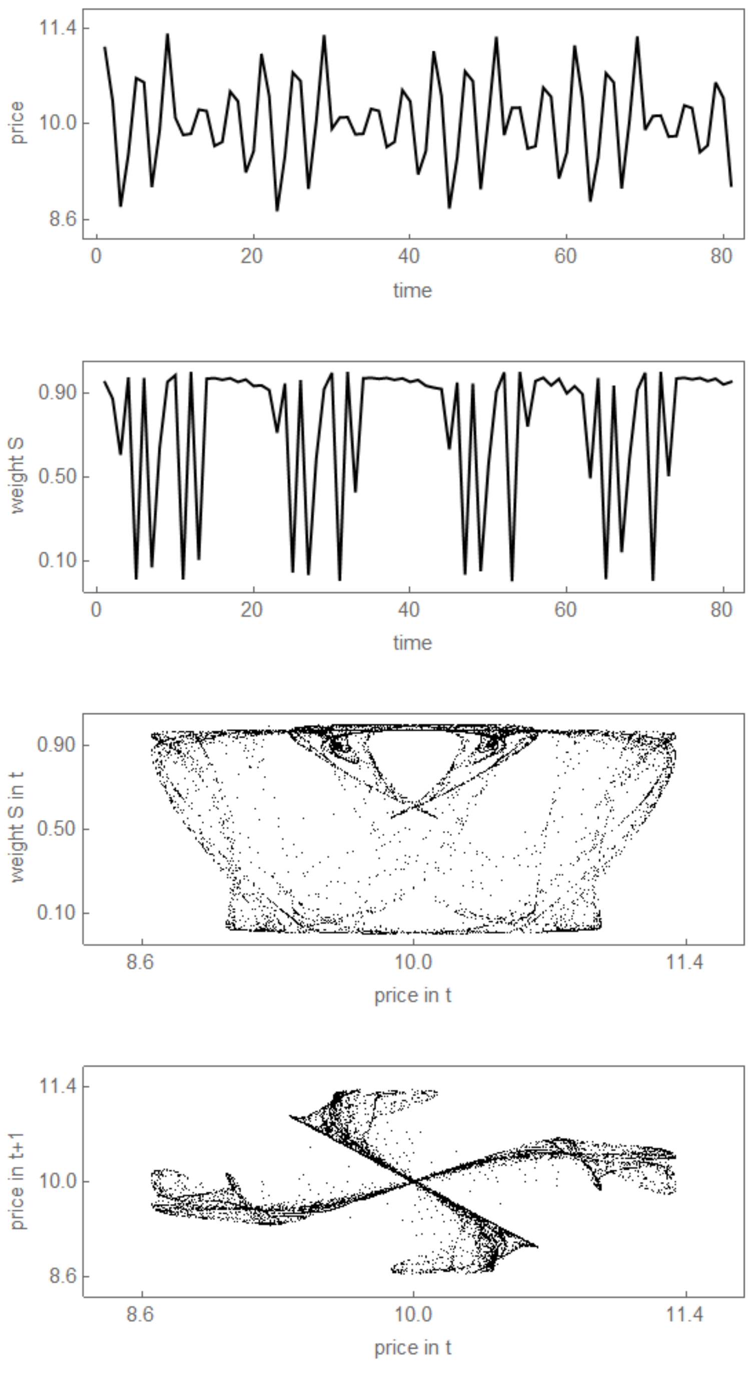

Figure 3: Chaotic dynamics. The panels show from top to bottom the evolution of the commodity price, the evolution of the market share of firms opting for the slow production technology, the market share of firms opting for the slow production technology in period t versus the commodity price in period t and the commodity price in period $t+1$ versus the commodity price in period t. Parameter setting: $a = 25$, $b = 1$, $c = 1.5$, $d = 1$ and $\beta = 3.5$.

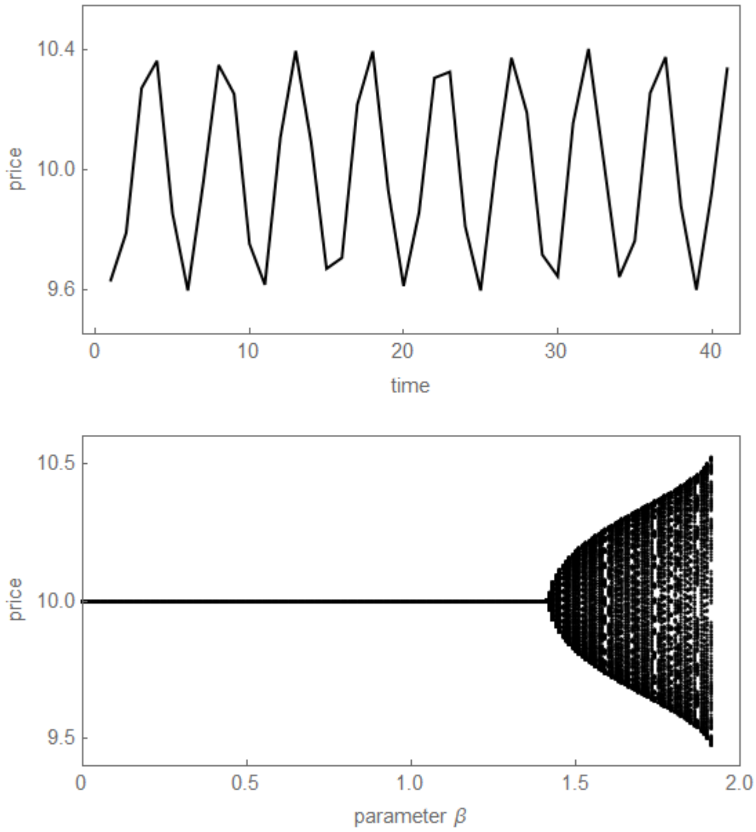

Figure 4: Three production technologies. The top panel shows the time evolution of the commodity price, assuming that $a = 25$, $b = 1$, $c = 1.5$, $d^F = 1$, $d^S = 0.1$, $d^L = 0$ and $\beta = 1.8$. The bottom panel shows the commodity price versus firms' intensity of choice parameter using the same parameter setting, except that the parameter is varied between $0 < \beta < 2$.