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ON K-STABILITY OF SOME DEL PEZZO SURFACES OF FANO INDEX 2

YUCHEN LIU AND ANDREA PETRACCI

Abstract. For every integer $a \geq 2$, we relate the K-stability of hypersurfaces in the weighted projective space $\mathbb{P}(1,1,a,a)$ of degree $2a$ with the GIT stability of binary forms of degree $2a$. Moreover, we prove that such a hypersurface is K-polystable and not K-stable if it is quasi-smooth.

1. Introduction

It is an important problem in algebraic geometry and in differential geometry to decide if a given Fano variety X admits a Kähler–Einstein (KE) metric. The Yau–Tian–Donaldson (YTD) Conjecture predicts that the existence of a KE metric on X is equivalent to the K-polystability of X. Using Cheeger–Colding–Tian theory, the YTD Conjecture was first proved when X is smooth [CDS15, Tia15, Ber16], when X is $\mathbb{Q}$-Gorenstein smoothable [LWX19, SSY16], or when X has dimension 2 [LTW21]. Later, a different method, namely the variational approach, was introduced in [BBJ21]. The analytic side of the variational approach was completed in [LTW21b, Li19] which shows that a $\mathbb{Q}$-Fano variety X, that is, a Fano variety with klt singularities, admits a KE metric if and only if X is reduced uniformly K-stable, a concept introduced in [His16] as an equivariant version of uniform K-stability (see also [XZ20]). Recently, using purely algebro-geometric methods, the work [LXZ21] establishes the equivalence between K-polystability and reduced uniform K-stability. This work, combining with the variational approach, proves the YTD Conjecture for all $\mathbb{Q}$-Fano varieties.

K-stability of del Pezzo surfaces which are quasi-smooth hypersurfaces in weighted projective 3-spaces has been studied extensively. Johnson and Kollár [JK01] classified those which are anticanonically polarised (i.e. have Fano index 1) and decided the existence of a KE metric on many of these, by using Tian's criterion which relates KE metrics to global log canonical thresholds (also called α-invariants) [Tia87, Nad90, DK01, Che08, OS12, Fuj19]. This method was applied to most of these del Pezzo surfaces by Araujo [Ara02], Boyer–Galicki–Nakamaye [BGN03], and Cheltsov–Park–Shramov [CPS10]. One case was missing and was finally solved in [CPS21] by using delta invariants (see [FO18, BJ20]).

The (non-)existence of KE metrics on many del Pezzo surfaces which are quasi-smooth hypersurfaces in weighted projective 3-spaces with Fano index ≥ 2 has been studied in [CPS10, CPS21, CS13, KW21].

In this paper, we study K-polystability of quasi-smooth degree $2a$ hypersurfaces in the weighted projective space $\mathbb{P}(1,1,a,a)$. When $a \in \{2,4\}$, such del Pezzo surfaces are $\mathbb{Q}$-Gorenstein smoothable, and their K-polystability was determined by Mabuchi–Mukai [MM93] and Odaka–Spotti–Sun [OSS16] (see Remark 5). To the authors' knowledge it is not known if they are K-polystable for an integer $a = 3$

or $a \geq 5$. In [KW21] Kim and Won conjecture that these surfaces are K-polystable and not K-stable.

Our main result relates the K-polystability (resp. K-semistability) of degree $2a$ hypersurfaces in $\mathbb{P}(1,1,a,a)$ to GIT polystability (resp. GIT semistability) of degree $2a$ binary forms (see [MFK94, Chapter 4]).

Theorem 1. *Let $a \geq 2$ be an integer and let $\mathbb{P}(1,1,a,a)$ be the weighted projective space with coordinates $[x,y,z,w]$ with weights $\deg x = \deg y = 1$ and $\deg z = \deg w = a$. Let X be a hypersurface of degree $2a$ in $\mathbb{P}(1,1,a,a)$.*

Then X is K-semistable (resp. K-polystable) if and only if, after an automorphism of $\mathbb{P}(1,1,a,a)$, the equation of X is given by $z^2+w^2+g(x,y)=0$ where $g \neq 0$ is GIT semistable (resp. GIT polystable) as a degree $2a$ binary form. Moreover, X is not K-stable.

As a consequence we prove the K-polystability of quasi-smooth hypersurfaces in $\mathbb{P}(1,1,a,a)$ of degree $2a$, hence partially confirming [KW21, Conjecture 1.3].

Corollary 2. *Let $a \geq 2$ be an integer and let X be a degree $2a$ quasi-smooth hypersurface in $\mathbb{P}(1,1,a,a)$. Then X is K-polystable and not K-stable. Moreover, X admits a KE metric.*

Recently the result of this corollary has been independently announced by Viswanathan using different methods.

It is possible to give a proof of K-polystability for a general hypersurface in $\mathbb{P}(1,1,a,a)$ of degree $2a$, when a is odd, by analysing the deformation theory of the toric surface appearing in Proposition 3 similarly to [KP21] and without using Theorem 1.

Notation and conventions. We always work over $\mathbb{C}$. A *del Pezzo surface* is a normal projective surface whose anticanonical divisor is $\mathbb{Q}$-Cartier and ample. Every toric variety we consider is normal. We do not even try to write down the definitions of K-(poly/semi)stability of Fano varieties and of log Fano pairs: we refer the reader to the excellent survey [Xu20], the paper [ADL19], and to the references therein.

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2. Proofs

In what follows a is a fixed integer greater than 1. We consider the weighted projective space $\mathbb{P}(1,1,a,a)$ with coordinates $[x,y,z,w]$ with weights $\deg x = \deg y = 1$ and $\deg z = \deg w = a$.

Proposition 3. *If Y is the hypersurface in $\mathbb{P}(1,1,a,a)$ defined by the equation $zw - x^a y^a = 0$, then Y is a K-polystable toric del Pezzo surface.*

Proof. We fix the lattice $N = \mathbb{Z}^2$ and its dual $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Elements of N will be columns and elements of M will be rows.

Let Q be the convex hull of the points

$$(0,0),\ (0,1),\ (a^{-1},0),\ (-a^{-1},1)$$

in $M_{\mathbb{R}}$. Let Σ be the inner normal fan of Q; thus Σ is the complete normal fan in N whose rays are generated by the vectors

$$\begin{pmatrix} a \\ 1 \end{pmatrix},\ \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} -a \\ -1 \end{pmatrix},\ \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{1}$$

We want to show that Y is the toric variety associated to the fan Σ.

Provisionally, let $\mathrm{TV}(\Sigma)$ denote the toric variety associated to Σ. Consider the cone τ in $M \oplus \mathbb{Z}$ spanned by $Q \times \{1\}$. Consider the finitely generated monoid $\tau \cap (M \oplus \mathbb{Z})$ and the semigroup algebra $\mathbb{C}[\tau \cap (M \oplus \mathbb{Z})]$, which is $\mathbb{N}$-graded via the projection $M \oplus \mathbb{Z} \twoheadrightarrow \mathbb{Z}$. Toric geometry says that $\mathrm{TV}(\Sigma) = \operatorname{Proj} \mathbb{C}[\tau \cap (M \oplus \mathbb{Z})]$. One can see that the minimal set of generators of the semigroup $\tau \cap (M \oplus \mathbb{Z})$ is made up of the vectors

$$(0,0,1),\ (0,1,1),\ (1,0,a),\ (-1,a,a);$$

these vectors satisfy a unique relation:

$$a(0,0,1) + a(0,1,1) = (1,0,a) + (-1,a,a).$$

Hence the $\mathbb{N}$-graded ring $\mathbb{C}[\tau \cap (M \oplus \mathbb{Z})]$ coincides with $\mathbb{C}[x,y,z,w]/(zw - x^a y^a)$, where $\deg x = \deg y = 1$ and $\deg z = \deg w = a$. Therefore $Y = \mathrm{TV}(\Sigma)$.

The vectors in (1) are the vertices of a polytope P in N. This implies that Y is a del Pezzo surface, i.e. $-K_Y$ is $\mathbb{Q}$-Cartier and ample.

Let P° be the polar of P; thus P° is the convex hull of $(0,\pm 1)$ and $\pm(\frac{2}{a}, -1)$ in $M_{\mathbb{R}}$. The polygon P° is the moment polytope of the toric boundary of Y, which is an anticanonical divisor. Since P is centrally symmetric, also P° is centrally symmetric, thus the barycentre of P° is the origin. By [Ber16] Y is K-polystable. $\square$

Remark 4. (1) Another way to show K-polystability of Y is by realising $Y \cong (\mathbb{P}^1 \times \mathbb{P}^1)/(\mathbb{Z}/a\mathbb{Z})$, where the $\mathbb{Z}/a\mathbb{Z}$-action on $\mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$\zeta \cdot ([u_0,u_1],[v_0,v_1]) := ([\zeta u_0, u_1],[\zeta^{-1} v_0, v_1]) \quad \text{with } \zeta = e^{\frac{2\pi i}{a}}.$$

Since the above action is free away from finitely many points, and it preserves the product of Fubini-Study metrics on $\mathbb{P}^1 \times \mathbb{P}^1$, we know that Y admits a KE metric and hence is K-polystable by [Ber16].

(2) A degree $2a$ hypersurface in $\mathbb{P}(1,1,a,a)$ is defined by an equation

$$q(z,w) + f(x,y)z + h(x,y)w + g(x,y) = 0$$

where q is a quadratic form, f and h are forms of degree a, and g is a form of degree $2a$. With an automorphism of $\mathbb{P}(1,1,a,a)$ which is induced by a linear change of the coordinates z,w, we can diagonalise the quadratic form q, so that the term zw disappears. Furthermore, if q has full rank, with an automorphism of $\mathbb{P}(1,1,a,a)$ induced by $z \mapsto z + \frac{f}{2}$ and $w \mapsto w + \frac{h}{2}$, the equation becomes

$$z^2 + w^2 + g(x,y) = 0.$$

Proof of Theorem 1. We start from the "if" part. Suppose $X \subset \mathbb{P}(1,1,a,a)$ is defined by the equation $z^2+w^2+g(x,y)=0$ with $g \neq 0$. Then the "if" part states that X is K-semistable (resp. K-polystable) if g is GIT semistable (resp. GIT polystable).

By forgetting the w-coordinate, we obtain a double cover $\pi : X \to \mathbb{P}(1,1,a)$ with branch locus $D=(z^2+g(x,y)=0)$. Thus by [LZ20, Zhu21] we know that X is K-semistable (resp. K-polystable) if and only if $(\mathbb{P}(1,1,a), \frac{1}{2}D)$ is K-semistable (resp. K-polystable).

Let us assume for the moment that g is an arbitrary degree $2a$ binary form. Denote by $D_0 := (z^2=0)$ as a divisor on $\mathbb{P}(1,1,a)$. It is clear that $\mathbb{P}(1,1,a)$ is the projective cone over $\mathbb{P}^1$ with polarization $\mathcal{O}_{\mathbb{P}^1}(a)$, and $\frac{1}{2}D_0$ is the section at infinity. Since $\mathbb{P}^1$ is Kähler–Einstein, [LL19, Proposition 3.3] shows that $(\mathbb{P}(1,1,a), (1-\frac{r}{2})\frac{1}{2}D_0)$ admits a conical KE metric, where $r \in \mathbb{Q}_{>0}$ satisfies $\mathcal{O}_{\mathbb{P}^1}(a) \sim_{\mathbb{Q}} -r^{-1}K_{\mathbb{P}^1}$, i.e. $r=\frac{2}{a}$. By computation, $(1-\frac{r}{2})\frac{1}{2} = \frac{a-1}{2a}$. Thus $(\mathbb{P}(1,1,a), \frac{a-1}{2a}D_0)$ admits a conical KE metric and hence is K-polystable. It is clear that under the $\mathbb{G}_m$-action σ on $\mathbb{P}(1,1,a)$ given by $\sigma(t)\cdot[x,y,z]=[x,y,tz]$, the log Fano pair $(\mathbb{P}(1,1,a), \frac{a-1}{2a}D)$ specially degenerates to $(\mathbb{P}(1,1,a), \frac{a-1}{2a}D_0)$ as $t \to 0$. Thus by openness of K-semistability [BLX19, Xu20b] we know that $(\mathbb{P}(1,1,a), \frac{a-1}{2a}D)$ is K-semistable.

Next, we assume that $g \neq 0$ is GIT semistable. By GIT of binary forms, we know that each linear factor in $g(x,y)$ has multiplicity at most a. In other words, the curve D has only A_{k-1}-singularities (i.e. locally analytically given by $x^2+y^k=0$) where $k \le a$. Thus we have that $\mathrm{lct}(\mathbb{P}(1,1,a); D) \ge \frac{1}{2}+\frac{1}{a} = \frac{a+2}{2a}$. This implies that $(\mathbb{P}(1,1,a), \frac{a+2}{2a}D)$ is a log canonical log Calabi–Yau pair. Thus interpolation for K-stability [ADL19, Proposition 2.13] implies that $(\mathbb{P}(1,1,a), \frac{1}{2}D)$ is K-semistable.

Next, we assume that $g \neq 0$ is GIT polystable. There are two cases: g is strictly GIT polystable (i.e. GIT polystable but not GIT stable), or g is GIT stable. In the first case, under a suitable coordinate we may write $g(x,y)=x^a y^a$. Thus the double cover X is toric, and as shown in Proposition 3 X is K-polystable. In the second case, we know that each linear factor in $g(x,y)$ has multiplicity at most $a-1$. Thus the curve D has only A_{k-1}-singularities where $k \le a-1$. Thus we have that $\mathrm{lct}(\mathbb{P}(1,1,a); D) \ge \frac{1}{2}+\frac{1}{a-1} > \frac{a+2}{2a}$, which implies that $(\mathbb{P}(1,1,a), \frac{a+2}{2a}D)$ is a klt log Calabi–Yau pair. Thus interpolation for K-stability [ADL19, Proposition 2.13] implies that $(\mathbb{P}(1,1,a), \frac{1}{2}D)$ is K-stable. This finishes the proof of the "if" part.

Next, we treat the "only if" part. In fact, this follows from moduli comparison arguments as in [ADL19]. Let $\mathbf{A} := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a))$ be the affine space parametrizing degree $2a$ binary forms. Let $\mathbf{A}^{\mathrm{ss}} \subset \mathbf{A}\setminus\{0\}$ be the open subset of GIT semistable binary forms. Consider the universal family of weighted hypersurfaces $\mathcal{X} \to \mathbf{A}^{\mathrm{ss}}$ where $\mathcal{X} \subset \mathbb{P}(1,1,a,a)\times \mathbf{A}^{\mathrm{ss}}$ has fibre $(z^2+w^2+g(x,y)=0)$ over each $g \in \mathbf{A}^{\mathrm{ss}}$. By the "if" part we know that each fibre of $\mathcal{X} \to \mathbf{A}^{\mathrm{ss}}$ is K-semistable. Consider the $(\mathbb{G}_m \times \mathrm{SL}_2)$-action λ on $\mathbf{A}$ given by $\lambda(t,A)\cdot g(x,y) = t^2 g(A^{-1}(x,y))$. It is clear that $\mathbf{A}^{\mathrm{ss}}$ is a $(\mathbb{G}_m \times \mathrm{SL}_2)$-invariant open subset. Then there is a $(\mathbb{G}_m \times \mathrm{SL}_2)$-action $\tilde{\lambda}$ on $\mathcal{X}$ as a lifting of λ given by

$$\tilde{\lambda}(t,A)\cdot([x,y,z,w],g) := ([A(x,y),tz,tw], \lambda(t,A)\cdot g).$$

Denote by $\mathcal{M}^{\mathrm{GIT}} := [\mathbf{A}^{\mathrm{ss}}/(\mathbb{G}_m \times \mathrm{SL}_2)]$ and $M^{\mathrm{GIT}} := \mathbf{P} /\!\!/ \mathrm{SL}_2$ where $\mathbf{P} := \mathbb{P}(\mathbf{A})$. It is clear that M^{GIT} is the good moduli space of $\mathcal{M}^{\mathrm{GIT}}$. Taking quotient of the family $\mathcal{X} \to \mathbf{A}^{\mathrm{ss}}$ by $\tilde{\lambda}$, we obtain a $\mathbb{Q}$-Gorenstein flat family of K-semistable $\mathbb{Q}$-Fano varieties over $\mathcal{M}^{\mathrm{GIT}}$, where fibres over closed points are precisely K-polystable fibres.

From a series of important recent works [Jia20, LWX21, CP21, BX19, ABHLX20, Xu20b, BLX19, XZ20, XZ21, BHLLX21, LXZ21], we know that there exists an Artin stack of finite type $\mathcal{M}^{\mathrm{Kss}}_{2,8/a}$ parametrizing K-semistable (possibly singular) del Pezzo surfaces of degree $8/a$. Moreover, $\mathcal{M}^{\mathrm{Kss}}_{2,8/a}$ admits a projective good moduli space $M^{\mathrm{Kps}}_{2,8/a}$ parametrizing K-polystable ones. Let $\mathcal{M}^{\mathrm{K}}$ be the Zariski closure (with reduced structure) of the locally closed substack in $\mathcal{M}^{\mathrm{Kss}}_{2,8/a}$ parametrizing K-semistable degree $2a$ weighted hypersurfaces $X \subset \mathbb{P}(1,1,a,a)$. Let M^{K} be the good moduli space of $\mathcal{M}^{\mathrm{K}}$ as a closed algebraic subspace of $M^{\mathrm{Kps}}_{2,8/a}$. Then the above construction and the "if" part produces a morphism $\Phi : \mathcal{M}^{\mathrm{GIT}} \to \mathcal{M}^{\mathrm{K}}$ which descends to a morphism $\phi : M^{\mathrm{GIT}} \to M^{\mathrm{K}}$. Since a general weighted hypersurface X has the form $z^2 + w^2 + g(x,y) = 0$ in a suitable coordinate where $g \neq 0$ has no multiple linear factors, we know that Φ is dominant. The "if" part shows that Φ sends closed points to closed points. Since M^{GIT} is projective, we know that ϕ is proper and dominant, which implies that ϕ is surjective. Moreover, since SL_2 has no non-trivial characters, we have injections

$$\mathrm{Pic}(M^{\mathrm{GIT}}) = \mathrm{Pic}(\mathbf{P} /\!\!/ \mathrm{SL}_2) \hookrightarrow \mathrm{Pic}_{\mathrm{SL}_2}(\mathbf{P}^{\mathrm{ss}}) \hookrightarrow \mathrm{Pic}(\mathbf{P}^{\mathrm{ss}})$$

by [KKV89, Proposition 4.2 and Section 2.1]. It is clear that $\mathbf{P} \backslash \mathbf{P}^{\mathrm{ss}}$ has codimension at least 2 in $\mathbf{P}$. Thus we have $\mathrm{Pic}(\mathbf{P}^{\mathrm{ss}}) \cong \mathrm{Pic}(\mathbf{P}) \cong \mathbb{Z}$. In particular, the GIT quotient M^{GIT} has Picard rank 1. It is clear that M^{K} is not a single point. Thus $\phi : M^{\mathrm{GIT}} \to M^{\mathrm{K}}$ is a finite surjective morphism by Zariski's main theorem.

Next, we show that K-poly/semistability implies GIT poly/semistability. Since ϕ is surjective, a K-polystable hypersurface $X \subset \mathbb{P}(1,1,a,a)$ satisfies that $[X] = \phi([g]) \in M^{\mathrm{K}}$ for some GIT polystable binary form $g \in \mathbf{A} \setminus \{0\}$. Thus X has the form $z^2 + w^2 + g(x,y) = 0$ with $g \neq 0$ being GIT polystable. If $X \subset \mathbb{P}(1,1,a,a)$ is K-semistable, then it specially degenerates to a K-polystable point $[X_0] \in M^{\mathrm{K}}$ by [LWX21]. Clearly X_0 has the form $z^2 + w^2 + g_0(x,y) = 0$ with $g_0 \neq 0$ being GIT polystable. Since the rank of quadratic forms cannot jump up under degeneration, the quadratic terms in (z,w) of the equation of X has rank 2, which implies that $X = (z^2 + w^2 + g(x,y) = 0)$ for some g. By [Fuj19b, Corollary 1.7], we know that $(\mathbb{P}(1,1,a), \frac{1}{2}D)$ is K-semistable where $D = (z^2 + g(x,y) = 0)$. Since X carries a $\mathbb{Z}/2\mathbb{Z}$-action given by $w \mapsto -w$, we may assume that the special degeneration from X to X_0 is $\mathbb{Z}/2\mathbb{Z}$-equivariant by [LZ20, Zhu21]. In particular, this shows that $(\mathbb{P}(1,1,a), \frac{1}{2}D)$ specially degenerates to $(\mathbb{P}(1,1,a), \frac{1}{2}D_0)$ where $D_0 = (z^2 + g_0(x,y) = 0)$. By the lower semi-continuity of lct (see e.g. [DK01]), we know that $\mathrm{lct}(\mathbb{P}(1,1,a); D) \geq \mathrm{lct}(\mathbb{P}(1,1,a); D_0) \geq \frac{a+2}{2a}$ where the latter inequality was proven in the "if" part due to the fact that g_0 is GIT polystable. Thus this shows that $g \neq 0$, and each linear factor in $g(x,y)$ has multiplicity at most a. Thus we obtain the GIT semistability of g. The proof of the "only if" part is finished.

Finally, we show that any hypersurface $X \subset \mathbb{P}(1,1,a,a)$ of degree $2a$ is not K-stable. If X were K-stable, then it would have equation $z^2 + w^2 + g(x,y) = 0$, or equivalently the equation $zw + g(x,y) = 0$. It is clear that $t \cdot (z,w) = (tz, t^{-1}w)$ defines an effective action of $\mathbb{G}_{\mathrm{m}}$ on X. Thus X is not K-stable by definition. $\square$

Proof of Corollary 2. It is clear that X is quasi-smooth if and only if, up to an automorphism of $\mathbb{P}(1,1,a,a)$, X has the equation $z^2+w^2+g(x,y)=0$ where g has no multiple linear factors. Thus by Theorem 1 we conclude that X is K-polystable and not K-stable. The existence of KE metrics on X follows from [LTW21]. □

Remark 5. For $a=2$, the del Pezzo surface X admits an embedding into $\mathbb{P}^4$ as a complete intersection of two hyperquadrics. This is induced by the linear system $|-K_X|$ which is very ample.

For $a=4$, X (as a double cover of $\mathbb{P}(1,1,4)$) appeared in [OSS16] where it lies in the exceptional divisor of Kirwan blow-up of the GIT moduli space. Hence X admits a $\mathbb{Q}$-Gorenstein smoothing to degree 2 smooth del Pezzo surfaces.

Therefore, in both cases ($a=2$ or $a=4$) our K-moduli space M^{K}, introduced in the proof of Theorem 1, form a divisor in the K-moduli spaces of $\mathbb{Q}$-Gorenstein smoothable del Pezzo surfaces of degree $\frac{8}{a}$ studied in [MM93, OSS16]. We will see in Proposition 6 what happens for $a=3$ or $a\geq 5$.

Proposition 6. *If $a=3$ or $a\geq 5$, then the locus of K-polystable degree $2a$ hypersurfaces in $\mathbb{P}(1,1,a,a)$ is a connected component of $M^{\mathrm{Kps}}_{2,8/a}$.*

Proof. We denote by Γ the connected component of $M^{\mathrm{Kps}}_{2,8/a}$ containing K-polystable degree $2a$ hypersurfaces in $\mathbb{P}(1,1,a,a)$. In the proof of Theorem 1 we showed that the locus of K-polystable degree $2a$ hypersurfaces in $\mathbb{P}(1,1,a,a)$ is closed in Γ; this locus is denoted by M^{K}. We need to prove that M^{K} coincides with Γ. We will achieve this by a dimension count. Using the notation of the proof of Theorem 1, there is a finite surjective morphism $\phi\colon M^{\mathrm{GIT}}\to M^{\mathrm{K}}$. Thus we have

$$\dim M^{\mathrm{K}} = \dim M^{\mathrm{GIT}} = \dim \mathbf{P} - \dim \mathrm{SL}_2 = 2a-3.$$

Let us now compute the dimension of Γ by analysing the deformation theory of the K-polystable toric del Pezzo surface Y introduced in Proposition 3. Note that a similar study was discussed in [MGS21].

Let $\mathscr{T}^0_Y$ denote the sheaf of derivations on Y, i.e. the dual of Ω^1_Y. Let $\mathscr{T}^{\mathrm{qG},1}_Y$ denote the sheaf of 1st order $\mathbb{Q}$-Gorenstein deformations of Y. The singular locus of Y, which consists of 4 points, contains the set-theoretic support of $\mathscr{T}^{\mathrm{qG},1}_Y$.

Since Y is a toric Fano, we have $H^1(\mathscr{T}^0_Y)=H^2(\mathscr{T}^0_Y)=0$ by [Pet19, §4.3]. Via a standard argument about the local-to-global spectral sequence for Ext, we deduce that the tangent space of the $\mathbb{Q}$-Gorenstein deformation functor of Y is $H^0(\mathscr{T}^{\mathrm{qG},1}_Y)$. The $\mathbb{Q}$-Gorenstein deformation functor of Y is unobstructed because Y is a del Pezzo surface with cyclic quotient singularities [ACC+16, Lemma 6]. Therefore the germ at the origin of the vector space $H^0(\mathscr{T}^{\mathrm{qG},1}_Y)$ is the base of the miniversal (Kuranishi) $\mathbb{Q}$-Gorenstein deformation of Y.

Consider the torus $T_N = N\otimes_{\mathbb{Z}}\mathbb{G}_{\mathrm{m}}$ acting on the toric variety Y. There is an action of T_N on the vector space $H^0(\mathscr{T}^{\mathrm{qG},1}_Y)$, hence $H^0(\mathscr{T}^{\mathrm{qG},1}_Y)$ splits into the direct sum of irreducible representations (characters) of the torus T_N.

We observe that the singularities of Y are:

- 2 points of type $\frac{1}{a}(1,-1)=A_{a-1}$, which correspond to the cones in Σ spanned by

$$\pm\begin{pmatrix}a\\1\end{pmatrix},\ \pm\begin{pmatrix}0\\1\end{pmatrix},$$

- 2 points of type $\frac{1}{a}(1,1)$, which correspond to the cones in Σ spanned by
$$\pm\begin{pmatrix}a\\1\end{pmatrix},\ \mp\begin{pmatrix}0\\1\end{pmatrix}.$$

Since $a = 3$ or $a \geq 5$, the surface singularity $\frac{1}{a}(1,1)$ is $\mathbb{Q}$-Gorenstein rigid, so it does not contribute to $H^0(\mathscr{T}_Y^{\mathrm{qG},1})$. One can see that the T_N-representation $H^0(\mathscr{T}_Y^{\mathrm{qG},1})$ is the direct sum of the 1-dimensional representation of T_N associated to the characters

$$(0,\pm 2),(0,\pm 3),\dots,(0,\pm a)\in M. \tag{2}$$

In particular $\dim H^0(\mathscr{T}_Y^{\mathrm{qG},1}) = 2a-2$, so the base of the miniversal $\mathbb{Q}$-Gorenstein deformation of Y is a smooth germ of dimension $2a-2$.

Since the weights in (2) are contained in a rank 1 sublattice of M, there exists a 1-dimensional subtorus of T_N which acts trivially on $H^0(\mathscr{T}_Y^{\mathrm{qG},1})$. More precisely one can prove that the affine quotient $H^0(\mathscr{T}_Y^{\mathrm{qG},1})/T_N$ has dimension $2a-3$.

Since every facet of the polytope P° has no interior lattice points, by [KP21, Proposition 2.6] the automorphism group of Y is $T_N \rtimes \mathrm{Aut}(P)$, where $\mathrm{Aut}(P) \subseteq \mathrm{GL}(N)$ is the finite group consisting of the lattice automorphisms which keep the polytope P invariant. Since the difference between T_N and $\mathrm{Aut}(Y)$ is just a finite group, we deduce that the affine quotient the affine quotient $H^0(\mathscr{T}_Y^{\mathrm{qG},1})/\mathrm{Aut}(Y)$ has dimension $2a-3$. By the local structure of the K-moduli space [ABHLX20, AHR20] we know that the completion of the local ring of Γ at $[Y]$ coincides with the completion at the origin of $H^0(\mathscr{T}_Y^{\mathrm{qG},1})/\mathrm{Aut}(Y)$. This proves that Γ has dimension $2a-3$ at $[Y]$. Since $\dim M^{\mathrm{K}} = 2a-3$, we know that M^{K} is an irreducible component of Γ.

Moreover, since all K-polystable del Pezzo surfaces in M^{K} have cyclic quotient singularities by Theorem 1, they have unobstructed $\mathbb{Q}$-Gorenstein deformations by [ACC+16, Lemma 6]. Thus the stack $\mathcal{M}^{\mathrm{Kss}}_{2,\frac{8}{a}}$ is smooth in an open neighbourhood of $\mathcal{M}^{\mathrm{K}}$. In particular, this implies that Γ is normal in an open neighbourhood of M^{K}. Since M^{K} is an irreducible component of Γ, we have $M^{\mathrm{K}} = \Gamma$. □

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