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BELLMAN FUNCTION SITTING ON A TREE

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ABSTRACT. In this note we give a proof-by-formula of certain important embedding inequalities on a tree. We also consider the case of a bi-tree, where a different approach is explained.

0.1. **Hardy operator on a tree.** Let I^0 be a unit interval. Let us associate the dyadic lattice $\mathcal{D}(I^0)$ and the uniform directed dyadic tree T in a usual way. First we define the Hardy operator, the dual Hardy operator and the logarithmic potential: given a function $\varphi : T \to \mathbb{R}_+$ we let

$$(I\varphi)(\alpha) = \sum_{\beta \geq \alpha} \varphi(\beta), \quad \alpha \in T;$$

$$(I^*\varphi)(\alpha) = \sum_{\beta \leq \alpha} \varphi(\beta), \quad \beta \in T;$$

$$V^{\varphi}(\gamma) = (II^*\varphi)(\gamma), \quad \gamma \in T,$$

where $\leq$ is the natural order relation on T.

We always may think that the tree T is finite (albeit very large). By the boundary ∂T we understand the vertices that are not connected to smaller vertices.

Each dyadic interval Q in $\mathcal{D}(I^0)$ corresponds naturally to a vertex α_Q.

Let μ be a measure on the tree T, so just the collection of non-negative numbers $\{\mu_P\}_{P\in T}$. Assuming μ to be a measure on T, we have

$$(I\mu)(\alpha_R) = \sum_{Q \supset R} \mu_Q, \quad Q, R \in \mathcal{D}(I^0);$$

$$(I^*\mu)(\alpha_Q) = \mu(Q) = \sum_{P \subset Q, \alpha_P \in \partial T} \mu_P, \quad Q \in \mathcal{D}(I^0);$$

$$V^{\mu}(\alpha_P) = (II^*\mu)(\alpha_P), \quad P \in \mathcal{D}(I^0),$$

the second equality is valid under the assumption of $\operatorname{supp}\mu \subset \partial T$.

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We will answer the question when $I : \ell^2(T) \to \ell^2(T,\mu)$. Passing to the adjoint operator we see that this is equivalent to the following inequality

$$\sum_{Q\in T} \Big(\sum_{P\le Q} \varphi(P)\mu_P \Big)^2 \lesssim \Big(\sum_{R\in T} \varphi(R)^2 \mu_R \Big)\,. \tag{0.1}$$

Theorem 0.1. *Operator I is a bounded operator $I : \ell^2(T) \to \ell^2(T,\mu)$ if and only if*

$$\sum_{Q\in T,\, Q\le R} \Big(\sum_{P\le Q} \mu_P \Big)^2 \lesssim \Big(\sum_{Q\le R} \mu_Q \Big) \quad \forall R\in T\,. \tag{0.2}$$

This is proved in Theorem 1.3 below by the use of Bellman function.

1. Bellman function on a tree

Theorem 1.1. *Let dw be a positive measure on $I_0 := [0,1]$. Let $\langle w\rangle_I$ denote $w(I)/|I|$. Let φ be a measurable test function. Then if*

$$\frac{1}{|J|} \sum_{I\in\mathcal{D}(J)} \langle w\rangle_I^2 |I|^2 \le \langle w\rangle_J \quad \forall J\in\mathcal{D}(I_0), \tag{1.1}$$

then

$$\sum_{I\in\mathcal{D}(I_0)} \langle \varphi w\rangle_I^2 |I|^2 \lesssim \langle \varphi^2 w\rangle_{I_0} |I_0|, \tag{1.2}$$

This can be obtained as a direct consequence of the weighted Carleson embedding theorem [4]:

Theorem 1.2. *Let $\mathcal{D}$ be a dyadic lattice, w be any weight, and $\{\alpha_I\}_{I\in\mathcal{D}}$ be a sequence of non-negative numbers. Then, if*

$$\frac{1}{|J|} \sum_{I\subset J} \alpha_I \langle w\rangle_I^2 \le \langle w\rangle_J \quad \forall J\in\mathcal{D}, \tag{1.3}$$

then

$$\sum_{I\in\mathcal{D}} \alpha_I \langle \varphi\sqrt{w}\rangle_I^2 \lesssim \|\varphi\|_{L^2}^2, \tag{1.4}$$

for all $\varphi\in L^2$.

Clearly, the conclusion of (1.4) may be rewritten as

$$\frac{1}{|I_0|} \sum_{I\subset I_0} \alpha_I \langle \varphi w\rangle_I^2 \lesssim \langle \varphi^2 w\rangle_{I_0}.$$

Letting $\alpha_I = |I|^2$ in (1.3), we obtain exactly Theorem 1.1.

We recall here that the proof of Theorem 1.2 in [4] was based upon the Bellman function

$$\mathcal{B}(F,f,A,v) := 4\Big(F - \frac{f^2}{v+A}\Big), \tag{1.5}$$

and three main properties this function satisfies are:

(1) $\mathcal{B}$ is defined on:

$$f^2 \le Fv; A \le v;$$

(2) $0 \le \mathcal{B} \le CF$, in this case with $C = 4$;

(3) Main Inequality:

$$\mathcal{B}(F, f, A, v) - \frac{1}{2}\Big(\mathcal{B}(F_-, f_-, A_-, v_-) + \mathcal{B}(F_+, f_+, A_+, v_+)\Big) \ge \frac{f^2}{v^2}m, \tag{1.6}$$

for all points in the domain such that

$$F = \frac{F_- + F_+}{2}; \;\; f = \frac{f_- + f_+}{2}; \;\; v = \frac{v_- + v_+}{2}, \tag{1.7}$$

and

$$A = m + \frac{A_- + A_+}{2},$$

for some $m \ge 0$.

In particular, we have that the function $\mathcal{B}$ is *concave.*

1.1. **Carleson embedding theorem on a dyadic tree.** Now we wish to prove a version of Theorem 1.1 on a dyadic tree. Specifically, suppose we have a dyadic tree originating at some $I_0 \in \mathcal{D}$. Define a measure Λ on the tree as follows: to each node $I \in \mathcal{D}(I_0)$ we associate a non-negative number $\lambda_I \ge 0$. We may think of $I \in \mathcal{D}(I_0)$ as an interval *in the dyadic tree* by considering $\{K \in \mathcal{D}(I_0) : K \subset I\}$. Then we define

$$\Lambda(I) := \sum_{K \subset I} \lambda_K,$$

and the averaging operator

$$(\Lambda)_I := \frac{1}{|I|}\Lambda(I).$$

Given a function $\varphi = \{\varphi(I)\}_{I \in \mathcal{D}(I_0)}$ on the dyadic tree, we have

$$\int_I \varphi \, d\Lambda = \sum_{K \subset I} \varphi(K)\lambda_K,$$

and

$$(\varphi\Lambda)_I := \frac{1}{|I|}\int_I \varphi \, d\Lambda.$$

Theorem 1.3 (Carleson embedding theorem for a dyadic tree)**.** *Let $I_0 \in \mathcal{D}$, the dyadic tree originating at I_0 with notations as above, and $\{\alpha_I\}_{I \subset I_0}$ be a sequence of non-negative numbers. Then, if*

$$\frac{1}{|I|}\sum_{K \subset I} \alpha_K (\Lambda)_K^2 \le (\Lambda)_I, \quad \forall I \in \mathcal{D}(I_0), \tag{1.8}$$

then

$$\frac{1}{|I_0|}\sum_{I \subset I_0} \alpha_I (\varphi\sqrt{\Lambda})_I^2 \le 4(\varphi^2)_{I_0}, \tag{1.9}$$

where

$$(\varphi\sqrt{\Lambda})_I := \frac{1}{|I|}\sum_{K\subset I}\varphi(K)\sqrt{\lambda_K} \quad \textit{and} \quad (\varphi^2)_{I_0} := \frac{1}{|I_0|}\sum_{I\subset I_0}\varphi(I)^2.$$

Note that the conclusion of (1.9) may be rewritten as

$$\sum_{I\subset I_0}\alpha_I(\varphi\Lambda)_I^2 \le 4(\varphi^2\Lambda)_{I_0}.$$

Letting $\alpha_I = |I|^2$ in (1.8), we obtain:

Corollary 1.4. *Let $I_0 \in \mathcal{D}$, the dyadic tree originating at I_0 with notations as above. Then, if*

$$\frac{1}{|I|}\sum_{K\subset I}|K|^2(\Lambda)_K^2 \le (\Lambda)_I, \quad \forall I \in \mathcal{D}(I_0),$$

then

$$\frac{1}{|I_0|}\sum_{I\subset I_0}|I|^2(\varphi\Lambda)_I^2 \le 4(\varphi^2\Lambda)_{I_0}.$$

The proof of Theorem 1.3 is based also on the function $\mathcal{B}$ in (1.5), and on proving a more involved version of (1.6) – this will be Lemma 1.5.

For now, let us create a Bellman function for the dyadic tree. Below, we have in the left column the setup for the original Bellman function of [4], and on the right we construct the Bellman function for our Carleson embedding theorem on the dyadic tree.

Classic Weighted CET

$$\mathbb{B}_1(F, f, A, v) := \sup_{\varphi,w,\alpha}\frac{1}{|I_0|}\sum_{I\subset I_0}\alpha_I\langle\varphi\sqrt{w}\rangle_I^2$$

where the supremum is over all functions φ on I_0, weights w on I_0, and w-Carleson sequences $\alpha = (\alpha_I)_{I\subset I_0}$ such that:

- $\langle\varphi^2\rangle_{I_0} = F$
- $\langle\varphi\sqrt{w}\rangle_{I_0} = f$
- $\frac{1}{|I_0|}\sum_{I\subset I_0}\alpha_I\langle w\rangle_I^2 = A$
- $\langle w\rangle_{I_0} = v.$

CET on Dyadic Tree

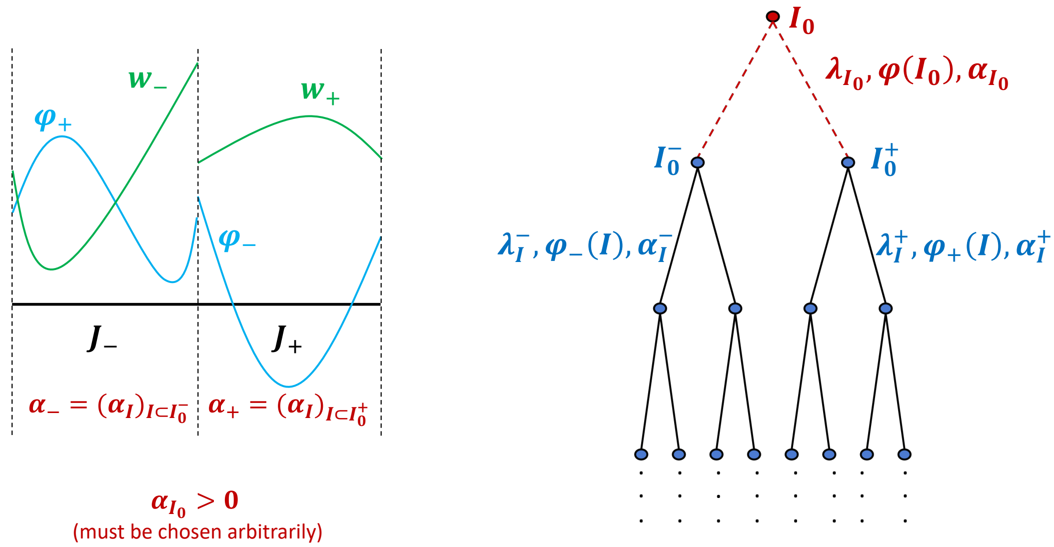
$$\mathbb{B}_2(F, f, A, v) := \sup_{T,\varphi,\Lambda,\alpha}\frac{1}{|I_0|}\sum_{I\subset I_0}\alpha_I(\varphi\sqrt{\Lambda})_I^2$$

where the supremum is over all dyadic trees T originating at I_0, measures Λ on T, functions φ on T and non-negative sequences $\alpha = (\alpha_I)_{I\subset I_0}$ such that:

- $(\varphi^2)_{I_0} = F$
- $(\varphi\sqrt{\Lambda})_{I_0} = f$
- $\frac{1}{|I_0|}\sum_{I\subset I_0}\alpha_I(\Lambda)_I^2 = A$
- $(\Lambda)_{I_0} = v.$

Both will be defined on $\{f^2 \le Fv; A \le v\}$, and both will satisfy some Main Inequality – which will turn out to be the fundamental distinction between the two. Let us also mention here that the function $\mathcal{B}$ in (1.5) is not the "true" Bellman function $\mathbb{B}_1$ above, but a *supersolution*. This refers in this case to any function which satisfies properties (1) – (3) in Section 1. The true Bellman function $\mathbb{B}_1$ was found in [5].

Now let us discuss the Main Inequalities for these functions. For $\mathbb{B}_1$, (1.6) was obtained in the usual way, by running the Bellman machine separately on each half of an interval I_0 – see



(A) Main Inequality for $\mathbb{B}_1$ – classic case (B) Main Inequality for $\mathbb{B}_2$ – dyadic tree case

Figure 1. Deducing the Main Inequalities for $\mathbb{B}$

Figure 1(A): choose weights $w_\pm$ and functions $\varphi_\pm$ supported on $I_0^\pm$, and $w_\pm$-Carleson sequences $\alpha_\pm = (\alpha_I)_{I\subset I_0^\pm}$, in such a way that they give the supremum for $\mathbb{B}_1(F_\pm, f_\pm, A_\pm, v_\pm)$ up to some ϵ.

One then easily obtains a weight w and a function φ on I_0 by concatenation. In the case of the sequence α, one must choose an $\alpha_J > 0$ arbitrarily though, as long as the resulting sequence remains w-Carleson. This is what produces the m term in (1.6).

Now let us turn to $\mathbb{B}_2$ and proceed similarly: take two dyadic trees $T_\pm$ originating at $I_0^\pm$, each equipped with measures $\Lambda_\pm = \{\lambda_I^\pm\}_{I\subset I_0^\pm}$, two function $\varphi_\pm = \{\varphi_\pm(I)\}_{I\subset I_0^\pm}$ on the trees, and non-negative sequences $\alpha_\pm = \{\alpha_I\}_{I\subset I_0^\pm}$ such that:

$$F_\pm = (\varphi_\pm^2)_{I_0^\pm};\;\; f_\pm = (\varphi_\pm\sqrt{\Lambda_\pm})_{I_0^\pm};\;\; A_\pm = \frac{1}{|I_0^\pm|}\sum_{K\subset I_0^\pm} \alpha_K^\pm (\Lambda_\pm)_K^2;\;\; v_\pm = (\Lambda_\pm)_{I_0^\pm},$$

and such that:

$$\mathbb{B}_2(F_\pm, f_\pm, A_\pm, v_\pm) - \epsilon < \frac{1}{|I_0^\pm|}\sum_{I\subset I_0^\pm} \alpha_I^\pm (\varphi_\pm\sqrt{\Lambda_\pm})_I^2.$$

We can "concatenate" the two trees into a new dyadic tree T centered at I_0 – see Figure 1(B) – but, here is the major difference from the usual dyadic situation: λ_{I_0}, $\varphi(I_0)$ and α_{I_0} must all be assigned to I_0, they do not pre-exist. So let some arbitrary $\lambda_{I_0} \geq 0$, $\alpha_{I_0} \geq 0$ and $\varphi(I_0) \in \mathbb{R}$. Now we have a new tree T, a measure Λ, a function φ and a sequence α. Next step is to figure out what F, f, A, and v must be, through straightforward calculations.

$$F = (\varphi^2)_{I_0} = b_{I_0}^2 + \frac{1}{2}(F_- + F_+), \text{ where } b_{I_0} = \frac{\varphi(I_0)}{\sqrt{|I_0|}}$$
$$f = (\varphi\sqrt{\Lambda})_{I_0} = a_{I_0} b_{I_0} + \frac{1}{2}(f_- + f_+), \text{ where } a_{I_0} = \frac{\sqrt{\lambda_{I_0}}}{\sqrt{|I_0|}}$$
$$A = \frac{1}{|I_0|}\sum_{K\subset I_0} \alpha_K(\Lambda)_K^2 = c_{I_0} + \frac{1}{2}(A_- + A_+), \text{ where } c_{I_0} = \frac{1}{|I_0|}\alpha_{I_0}(\Lambda)_{I_0}^2$$
$$v = (\Lambda)_{I_0} = a_{I_0}^2 + \frac{1}{2}(v_- + v_+)$$

The tree T, along with Λ, φ, and α are then admissible for $\mathbb{B}_2(F, f, A, v)$, and:

$$\begin{aligned}
\mathbb{B}_2(F,f,A,v) &\geq \frac{1}{|I_0|}\sum_{I\subset I_0} \alpha_I(\varphi\sqrt{\Lambda})_I^2 \\
&= \frac{c_{I_0} f^2}{v^2} + \frac{1}{2}\Bigg(\frac{1}{|I_0^-|}\sum_{I\subset I_0^-}\alpha_I^-(\varphi_-\sqrt{\Lambda_-})_I^2 + \frac{1}{|I_0^+|}\sum_{I\subset I_0^+}\alpha_I^+(\varphi_+\sqrt{\Lambda_+})_I^2\Bigg) \\
&> \frac{c_{I_0} f^2}{v^2} + \frac{1}{2}\Bigg(\mathbb{B}_2(F_-, f_-, A_-, v_-) + \mathbb{B}_2(F_+, f_+, A_+, v_+)\Bigg) - \epsilon.
\end{aligned}$$

Therefore, we have the Main Inequality for $\mathbb{B}_2$:

$$c\frac{f^2}{v^2} \leq \mathbb{B}_2(F,f,A,v) - \frac{1}{2}\Bigg(\mathbb{B}_2(F_-, f_-, A_-, v_-) + \mathbb{B}_2(F_+, f_+, A_+, v_+)\Bigg), \tag{1.10}$$

for all quadruplets in the domain of $\mathbb{B}_2$ such that

$$F = \widetilde{F} + b^2;\;\; f = \widetilde{f} + ab;\;\; A = \widetilde{A} + c;\;\; v = \widetilde{v} + a^2, \tag{1.11}$$

and

$$\widetilde{F} := \frac{F_- + F_+}{2};\;\; \widetilde{f} := \frac{f_- + f_+}{2};\;\; \widetilde{A} := \frac{A_- + A_+}{2};\;\; \widetilde{v} := \frac{v_- + v_+}{2},$$

and $a \geq 0$, $b \in \mathbb{R}$, $c \geq 0$ are some real numbers.

Lemma 1.5. *The function $\mathcal{B}$ in* (1.5) *satisfies the Main Inequality above in* (1.10)*.*

Before we prove this lemma, let us see how it proves Theorem 1.3.

Proof of Theorem 1.3. For every $I \in \mathcal{D}(I_0)$ define:

$$\begin{aligned}
v_I &:= (\Lambda)_I = \frac{1}{|I|}\lambda_I + \frac{1}{2}(v_{I_-} + v_{I_+}) = a_I^2 + \widetilde{v}_I, \text{ where } a_I := \sqrt{\frac{\lambda_I}{|I|}};\\
F_I &:= (\varphi^2)_I = \frac{1}{|I|}\varphi(I)^2 + \frac{1}{2}(F_{I_-} + F_{I_+}) = b_I^2 + \widetilde{F}_I, \text{ where } b_I := \frac{\varphi(I)}{\sqrt{|I|}};\\
f_I &:= (\varphi\sqrt{\Lambda})_I = \frac{\varphi(I)\sqrt{\lambda_I}}{|I|} + \frac{1}{2}(f_{I_-} + f_{I_+}) = a_I b_I + \widetilde{f}_I;\\
A_I &:= \frac{1}{|I|}\sum_{K\subset I}\alpha_K(\Lambda)_K^2 = \frac{\alpha_I(\Lambda)_I^2}{|I|} + \frac{1}{2}(A_{I_-} + A_{I_+}) = c_I + \widetilde{A}_I, \text{ where } c_I := \frac{\alpha_I(\Lambda)_I^2}{|I|}.
\end{aligned}$$

Note then that

$$\mathcal{B}(F_I, f_I, A_I, v_I) = \mathcal{B}(b_I^2 + \widetilde{F}_I, a_I b_I + \widetilde{f}_I, c_I + \widetilde{A}_I, a_I^2 + \widetilde{v}_I),$$

so we may apply Lemma 1.5 and obtain

$$\alpha_I f_I^2 \le |I|\mathcal{B}(x_I) - |I_-|\mathcal{B}(x_{I_-}) - |I_+|\mathcal{B}(x_{I_+}).$$

Summing over $I \in \mathcal{D}(I_0)$ and using the telescoping nature of the sum, we have

$$\sum_{I\in I_0}\alpha_I f_I^2 \le |I_0|\mathcal{B}(F_{I_0}, f_{I_0}, A_{I_0}, v_{I_0}) \le 4|I_0|F_{I_0},$$

which is exactly (1.9). □

Remark 1.6. Before we proceed with the proof of Lemma 1.5 , let us note that the big and essential difference with Theorem 1.1 now is that in the proof of Theorem 1.1 $\{v_I\}_{I\in\mathcal{D}}$, $\{f_I\}_{I\in\mathcal{D}}$, $\{F_I\}_{I\in\mathcal{D}}$ are all martingales. This is the standard situation, and it is pictured in Figure 2 (A).

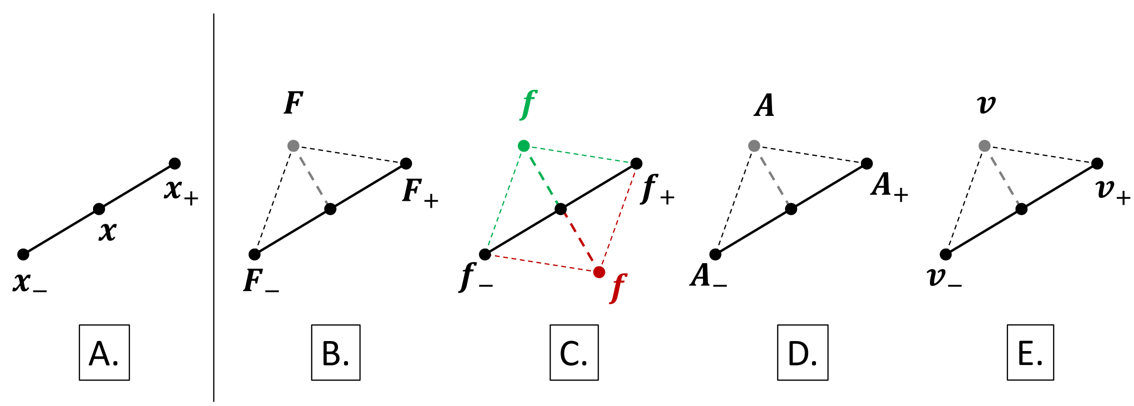


Figure 2. Deducing the Main Inequalities for $\mathbb{B}$

Now looking at (1.11), they are only *supermartingales* in the case of the F, A and v variables, and even worse, in the case of f we can have either a supermartingale or a *submartingale*! In other words, we do not have the property (1.7) anymore. Instead, as pictured in Figure 2 (B) – (E),

$$F_I \geq \frac{F_{I_-} + F_{I_+}}{2}; \;\; A_I \geq \frac{A_{I_-} + A_{I_+}}{2}; \;\; v_I \geq \frac{v_{I_-} + v_{I_+}}{2},$$

and

$$f_I \geq \text{ or } \leq \frac{f_{I_-} + f_{I_+}}{2}.$$

This is an essential difference, because there is less cancellation, and, indeed, the fact that $\{f_I\}_{I\in\mathcal{D}}$ can be both a super or a submartingale can destruct the whole proof. As a small miracle the "good" supermartingale properties of $\{v_I\}_{I\in\mathcal{D}}$, $\{F_I\}_{I\in\mathcal{D}}$ and $\{A_I\}_{I\in\mathcal{D}}$ allow us to neutralize the "bad" sub/supermartingale property of $\{f_I\}_{I\in\mathcal{D}}$. The exact calculation of this "good"–"bad" interplay will be in (1.14). We wish to explain now why some variables, the supermartingales F_I, A_I, v_I, are good and some, namely, f_I is bad. The explanation is simple: the good ones are those that give positive partial derivative of $\mathcal{B}$, the bad is the one that gives a negative partial derivative of $\mathcal{B}$. In fact,

$$\begin{aligned}&\frac{\partial \mathcal{B}}{\partial A} \geq 0, \;\; \frac{\partial \mathcal{B}}{\partial v} \geq 0, \;\; \frac{\partial \mathcal{B}}{\partial F} = 4,\\ &\frac{\partial \mathcal{B}}{\partial f} \leq 0\,.\end{aligned}$$

Proof of Lemma 1.5. Recall that if g is a concave, differentiable function on a convex domain $S \subset \mathbb{R}^n$, then

$$g(x) - g(x^*) \leq \sum_{i=1}^n \frac{\partial g}{\partial x_i}(x^*) \cdot (x_i - x_i^*),$$

for all $x, x^* \in S$. Denoting $x := (F, f, A, v)$, for the function $\mathcal{B}$, this takes the particular form:

$$\begin{aligned}\frac{1}{4}\big(\mathcal{B}(x) - \mathcal{B}(x^*)\big) \;\; &\leq \;\; (F - F^*) - \frac{2f^*}{v^* + A^*}(f - f^*)\\ &\quad + \frac{(f^*)^2}{(v^* + A^*)^2}(A - A^*) + \frac{(f^*)^2}{(v^* + A^*)^2}(v - v^*).\end{aligned} \tag{1.12}$$

In particular:

$$\frac{1}{4}\Big(\mathcal{B}(F, f, A, v) - \mathcal{B}(F, f, A - c, v)\Big) \geq c\frac{f^2}{(v + A)^2} \geq c\frac{f^2}{4v^2}, \tag{1.13}$$

where the last inequality follows because $0 \leq A \leq v$.

By (1.13):

$$c\frac{f^2}{v^2} \leq \Big(\mathcal{B}(F, f, A, v) - \mathcal{B}(\widetilde{F}, \widetilde{f}, A - c, \widetilde{v})\Big) + \Big(\mathcal{B}(\widetilde{F}, \widetilde{f}, A - c, \widetilde{v}) - \mathcal{B}(F, f, A - c, v)\Big).$$

We claim that the term in the second parenthesis is negative: apply (1.12) to obtain

$$\frac{1}{4}\big(\mathcal{B}(\widetilde{F}, \widetilde{f}, A - c, \widetilde{v}) - \mathcal{B}(F, f, A - c, v)\big)$$

$$\leq -b^2 - \frac{2f}{v+A-c}(-ab) + \frac{f^2}{(v+A-c)^2}(-a^2) \tag{1.14}$$
$$- \big(b - \frac{af}{v+A-c}\big)^2 \leq 0.$$

Then

$$\begin{aligned} c\frac{f^2}{v^2} &\leq \left(\mathcal{B}(F,f,A,v) - \mathcal{B}(\widetilde{F},\widetilde{f},A-c,\widetilde{v})\right) \\ &\leq \mathcal{B}(F,f,A,v) - \frac{1}{2}\left(\mathcal{B}(F_-,f_-,A_-,v_-) + \mathcal{B}(F_+,f_+,A_+,v_+)\right), \end{aligned}$$

where $(A_- + A_+)/2 = A - c$, and the last inequality follows by concavity of $\mathcal{B}$. This proves the lemma. □

2. Maximal theorem on a tree

Now we are going to prove the result slightly more general than Corollary 1.4 from the previous section.

Theorem 2.1. *Let $I_0 \in \mathcal{D}$, the dyadic tree originating at I_0 with notations as above. Then, if*

$$\frac{1}{|I|}\sum_{K\subset I} |K|^2 (\Lambda)_K^2 \leq (\Lambda)_I, \quad \forall I \in \mathcal{D}(I_0),$$

then

$$\frac{1}{|I_0|}\sum_{I\subset I_0} |I|^2 (\Lambda)_I^2 \sup_{K:\, I\subset K}\left(\frac{(\varphi\Lambda)_K}{(\Lambda)_K}\right)^2 \lesssim (\varphi^2\Lambda)_{I_0}.$$

The proof – for a change – is a stopping time proof and not a Bellman proof.

Proof. For every vertex H of the tree, let us introduce the set of vertices E_H. Namely, let J be the first successor of H such that

$$\frac{(\varphi\Lambda)_J}{(\Lambda)_J} \geq 2\frac{(\varphi\Lambda)_H}{(\Lambda)_H}\,.$$

It may happen of course that J is not alone, and there are several first successors with this property. We call by E_H all vertices that are successors of all these J's and also all such $J's$.

Now we introduce another set of vertices associated with H. Consider all successors of H which are not in E_H. All of them plus H itself form the collection that is called O_H. This set in never empty (it contains H) and can include all successors of H.

Now we first assign $H = I_0$ and let $\{J\}$ be the first successors of H with the property above. We call this family stopping vertices of first generation, and denote it by $\mathcal{S}_1$. Then for any $H \in \mathcal{S}_1$ we repeat the procedure thus having stopping vertices of the second generation: $\mathcal{S}_2$.

For each j and each $H \in \mathcal{S}_j$, we have E_H and O_H. Notice that all such O_H are disjoint. We call I_0 the stopping vertex of 0 generation, and let $\mathcal{S} = \cup_{j=0}^{\infty}\mathcal{S}_j$.

Then

$$\sum_{I\subset I_0}|I|^2(\Lambda)_I^2\sup_{K:\,I\subset K}\left(\frac{(\varphi\Lambda)_K}{(\Lambda)_K}\right)^2$$
$$=\sum_{H\in\mathcal{S}}\sum_{I\in O_H}|I|^2(\Lambda)_I^2\sup_{K:\,I\subset K}\left(\frac{(\varphi\Lambda)_K}{(\Lambda)_K}\right)^2$$
$$\le 4\sum_{H\in\mathcal{S}}\left(\frac{(\varphi\Lambda)_H}{(\Lambda)_H}\right)^2\sum_{I\in O_H}|I|^2(\Lambda)_I^2$$
$$\le 4\sum_{H\in\mathcal{S}}\left(\frac{(\varphi\Lambda)_H}{(\Lambda)_H}\right)^2\Lambda(H)\,.$$

The last inequality uses the assumptions of the theorem. But notice that by definition of E_H we easily get

$$\Lambda(E_H)\le\frac{1}{2}\Lambda(H)\Rightarrow\ \Lambda(H)\le 2\Lambda(O_H)\,.$$

Hence

$$\sum_{I\subset I_0}|I|^2(\Lambda)_I^2\sup_{H:\,I\subset H}\left(\frac{(\varphi\Lambda)_I}{(\Lambda)_I}\right)^2\le 8\sum_{H\in\mathcal{S}}\left(\frac{(\varphi\Lambda)_H}{(\Lambda)_H}\right)^2\Lambda(O_H). \tag{2.1}$$

Now, define β_H for $H\subset I_0$ by $\beta_H:=\Lambda(O_H)$ if $H\in\mathcal{S}$, and $\beta_H:=0$ otherwise. Note that, by disjointness of O_H, we have

$$\sum_{H\subset K}\beta_H\le\Lambda(K),\ \ \forall K\subset I_0.$$

Therefore, if we let

$$\alpha_H:=\frac{\beta_H}{(\Lambda)_H^2}$$

then the sequence α_H satisfies the requirements of the Carleson Embedding Theorem 1.3 for the dyadic tree. So we may rewrite the right hand side of (2.1) in terms of β_H and apply Theorem 1.3:

$$\frac{1}{|I_0|}\sum_{I\subset I_0}|I|^2(\Lambda)_I^2\sup_{H:\,I\subset H}\left(\frac{(\varphi\Lambda)_I}{(\Lambda)_I}\right)^2$$
$$\le 8\frac{1}{|I_0|}\sum_{H\subset I_0}\alpha_H(\varphi\Lambda)_H^2$$
$$\le 32(\varphi^2\Lambda)_{I_0},$$

completing the proof of Theorem 2.1.

$\square$

3. Two-dimensional version of Theorem 1.1 and dyadic rectangles

Theorem 3.1. *Let μ be a positive measure on $R^0 = [0,1)^2$. Let $\langle \mu \rangle_R$ denote $\frac{\mu(R)}{|R|}$. Let φ be a measurable test function. Then, if*

$$\sum_{Q \subset E,\, Q \in \mathcal{D}(R^0)} \langle \mu \rangle_Q^2 |Q|^2 \le \mu(E), \quad \forall\, E \subset \partial T^2, \tag{3.1}$$

then

$$\sum_{Q \in \mathcal{D}(R^0)} \langle \varphi\mu \rangle_Q^2 |Q|^2 \lesssim \langle \varphi^2 \mu \rangle_{R^0} |R^0|. \tag{3.2}$$

Remark 3.2. There are now two proofs of this theorem, one in [3] and one in [2]. The paper [3] uses capacity and strong capacitary inequalities on the bi-tree, while the proof in [2] avoids the notion of capacity and strong capacitary estimates completely. Note that neither the claim nor the conclusion of Theorem 3.1 uses any kind of capacity.

Remark 3.3. We believe that it would be not enough to check (3.1) only for single rectangles. Let us see anyway what we can achieve assuming this one box condition, through a Bellman argument.

3.1. **One box condition and its corollary.**

In the next theorem it is essential to think that $\mu = \{\lambda_\beta\}$ is the measure on ∂T^2. Every dyadic rectangle R corresponds to a node of T^2, and we will use this in the notations below.

Theorem 3.4. *Let μ be a positive measure on $R^0 = [0,1)^2$. Let $\langle \mu \rangle_R$ denote $\frac{\mu(R)}{|R|}$. Let φ be a measurable test function. Then, if*

$$\sum_{Q \subset R,\, Q \in \mathcal{D}(R^0)} \langle \mu \rangle_Q^2 |Q|^2 \le \mu(R), \quad \forall \text{ rectangle } R, \tag{3.3}$$

then

$$\sum_{Q \in \mathcal{D}(R^0)} \langle \varphi\mu \rangle_Q^2 |Q|^3 \lesssim \langle \varphi^2 \mu \rangle_{R^0} |R^0|. \tag{3.4}$$

Proof. We consider exactly the same function $B(x)$, $x = (F, f, A, v)$,

$$B(x) = F - \frac{f^2}{v + A}\,.$$

Given a rectangle R we consider

$$F_R = \frac{1}{|R|} \sum_{\beta \le R} \phi_\beta^2 \lambda_\beta = \frac{1}{|R|} \int_R \phi^2 d\mu,\; f_R = \frac{1}{|R|} \sum_{\beta \le R} \phi_\beta \lambda_\beta = \frac{1}{|R|} \int_R \phi d\mu,$$

$$v_R = \frac{1}{|R|} \sum_{\beta \le R} \lambda_\beta = \frac{\mu(R)}{|R|},\; A_R = \frac{1}{|R|} \sum_{\beta \le R} v_\beta^2 |R_\beta|^2,\; x_R = (F_R, \dots, v_R)\,.$$

Let R_+, R_- be right and left half-rectangles of R, and R^t, R^b be top and bottom half-rectangles of R (so, e.g., if $R = I\times J$, the $R^t = I\times J_+$). Now let us estimate from below

$$B(x_R) - \frac14\bigg(B(x_{R_-}) + B(x_{R_+}) + B(x_{R^t}) + B(x_{R^b})\bigg)\,.$$

As μ is concentrated on the boundary, we see immediately, that

$$F_R = \frac14\bigg(F_{R_-} + F_{R_+} + F_{R^t} + F_{R^b}\bigg),\ f_R = \frac14\bigg(f_{R_-} + f_{R_+} + f_{R^t} + f_{R^b}\bigg)\,.$$

At the same time,

$$A_R - \frac14\bigg(A_{R_-} + A_{R_+} + A_{R^t} + A_{R^b}\bigg) \ge \frac{1}{|R|}\mu(R)^2,\ v_R = \frac14\bigg(v_{R_-} + v_{R_+} + v_{R^t} + v_{R^b}\bigg)\,.$$

The second equality here is just because $v_R = \frac12\bigg(v_{R_-} + v_{R_+}\bigg)$. And $v_R = \frac12\bigg(v_{R^t} + v_{R^b}\bigg)$. The first one because any β-term in A_R such that this term happens to be in two rectangles, e.g. in A_{R_-} and A_{R^t}, will be cancelled in the difference. The terms that happen only in one rectangle (this is the case for R_{--} as an example) will be in coefficient $\frac{1}{|R|}$ in A_R, and only with coefficient $\frac{1}{2|R|}$ in $\frac14\bigg(A_{R_-} + A_{R_+} + A_{R^t} + A_{R^b}\bigg)$, so it gives a partial (positive) contribution to $A_R - \frac14\bigg(A_{R_-} + A_{R_+} + A_{R^t} + A_{R^b}\bigg)$. And of course, $\frac{1}{|R|}\mu(R)^2$ is in A_R and in none of $A_{R_-}, A_{R_+}, A_{R^t}, A_{R^b}$, so it also the part of the contribution.

So we see that three variables F, f, v split in a "martingale" way, and for A_R we have the above "super-martingale" inequality.

Thus, considering

$$x_R^* = (F_R; f_R; \frac14(A_{R_-} + A_{R_+} + A_{R^t} + A_{R^b}); v_R)$$

we can write

$$\begin{aligned} &B(x_R) - B(x_R^*) \ge \frac{\partial B}{\partial A}(x_R)(x_R - x_R^*);\\ &B(x_R^*) \ge \frac14\bigg(B(x_{R_-}) + B(x_{R_+}) + B(x_{R^t}) + B(x_{R^b})\bigg)\,. \end{aligned}$$

Here both inequalities are corollaries of the concavity of B, in the first one we used that all coordinates of x_R, x_R^* coincide except the A-coordinate. Theerfore, now we get

$$B(x_R) - \frac14\bigg(B(x_{R_-}) + B(x_{R_+}) + B(x_{R^t}) + B(x_{R^b})\bigg) \ge c\frac{f_R^2}{v_R^2}\frac{\mu(R)^2}{|R|}$$

$$\ge c\frac{\frac{1}{|R|^2}\big(\int_R\phi d\mu\big)^2}{\mu(R)^2/|R|^2}\frac{\mu(R)^2}{|R|} = (\int_R \phi d\mu)^2/|R|\,.$$

Multiply this by $|R|^2$. We get a term of telescopic sum on the left:

$$|R|^2B(x_R)-\Big(|R_-|^2B(x_{R_-})+|R_+|^2B(x_{R_+})+|R^t|^2B(x_{R^t})+|R^b|^2B(x_{R^b})\Big)$$
$$\geq c|R|\big(\int_R \phi d\mu\big)^2\,.$$

Notice that on the next step we pick up all terms $|R|\big(\int_R \phi d\mu\big)^2$ with $R:=R_-,R_+,R^t,R^b$. Theorem is proved. □

References

[1] Adams, David R.; Hedberg, Lars Inge Function spaces and potential theory. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 314. Springer-Verlag, Berlin, 1996. xii+366 pp. ISBN: 3-540-57060-8

[2] Nicola Arcozzi, Irina Holmes, Pavel Mozolyako, Alexander Volberg, *Bi-parameter embedding and measures with restriction energy condition*, preprint, arXiv:1811.00978; 2018.

[3] Nicola Arcozzi, Pavel Mozolyako, Karl-Mikael Perfekt, Giulia Sarfatti, *Carleson measures for the Dirichlet space on the bidisc*, preprint, arXiv:1811.04990; 2018.

[4] F. Nazarov, S. Treil, A. Volberg, *The Bellman functions and two-weight inequalities for Haar multipliers*, Journal of the AMS, Vol. 12, Number 4, 1999.

[5] V. Vasyunin, A. Volberg, *Monge-Ampére equation and Bellman optimization of Carleson Embedding Theorems*, Amer. Math. Soc. Transl. (2) 226, 2009.

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