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SHARP QUANTITATIVE STABILITY FOR ISOPERIMETRIC INEQUALITIES WITH HOMOGENEOUS WEIGHTS

E. CINTI, F. GLAUDO, A. PRATELLI, X. ROS-OTON, AND J. SERRA

ABSTRACT. We prove the sharp quantitative stability for a wide class of weighted isoperimetric inequalities. More precisely, we consider isoperimetric inequalities in convex cones with homogeneous weights.

Inspired by the proof of such isoperimetric inequalities through the ABP method (see [CRS16]), we construct a new convex coupling (i.e., a map that is the gradient of a convex function) between a generic set E and the minimizer of the inequality (as in Gromov's proof of the isoperimetric inequality). Even if this map does not come from optimal transport, and even if there is a weight in the inequality, we adapt the methods of [FMP10] and prove that if E is almost optimal for the inequality then it is quantitatively close to a minimizer *up to translations*. Then, a delicate analysis is necessary to rule out the possibility of translations.

As a step of our proof, we establish a sharp regularity result for *restricted* convex envelopes of a function that might be of independent interest.

1. INTRODUCTION

1.1. **Background.** The quantitative stability of functional/geometric inequalities has been an increasingly active field in recent years. The interest lies in understanding whether *almost* minimizers of a certain inequality (i.e., the isoperimetric inequality or the Sobolev inequality) are *quantitatively* close to a minimizer. Let us mention the works [FMP08; CL12; FMP10; FI13; BDS15; BBJ17; IN15] regarding the stability of the isoperimetric inequality, [BE91; FMP07; Cia+09; Ngu19; FG20; FN19; Neu20; FZ20] about the stability of Sobolev inequalities and [MPP14; Bör10; FMM18; FJ17] for the stability of various other inequalities.

On the other hand, many different kinds of weighted isoperimetric inequalities have been established in the literature, let us mention [MP13; Mil15; BL09; Bet+99; BBJ17; Bor75; Cia+11; Ros+08; Cha19; CRS16; CR14; MR14].

The papers that lie at the intersection of the two topics, i.e., quantitative weighted isoperimetric inequalities, are rather rare. Among them, let us mention [Cia+11; BBJ17], which prove the quantitative stability of the Gaussian isoperimetric inequality, and [BDR12], which shows the stability of the weighted isoperimetric inequality for certain radial weights.

In this paper we establish for the first time sharp quantitative stability for a wide class of weighted isoperimetric inequalities. More precisely, we do so for the class of weighted isoperimetric inequalities in convex cones considered in [CRS16]. The analogous stability result without weights is proven with different methods in [FI13].

Let us emphasize that, in our setting (i.e., in a convex cone with a weight), even the characterization of minimizers of the isoperimetric inequality is nontrivial. In [CRS16], minimizers are not characterized. In [CR14, Theorem 5.11] it is proven, classifying stable hypersurfaces, that, in the class of compact sets with smooth boundary, minimizers of the isoperimetric inequality are balls centered at the origin intersected with the cone. The analogous result in the class of convex sets is established, via the Brunn-Minkowsky inequality, in [MR14, Corollary 2.17] (see [MR17]). The characterization of minimizers in the class of measurable sets is a byproduct of our work.

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1.2. **Results.** Given an open convex cone $\Sigma \subseteq \mathbb{R}^n$ and a weight $w : \Sigma \to [0, \infty)$, let us define the weighted volume and perimeter of a set $E \subseteq \Sigma$ with smooth boundary as

$$w(E) := \int_E w(x)\,\mathrm{d}x\,, \qquad \mathrm{Per}_w(E) := \int_{\partial E \cap \Sigma} w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,.$$

See Section 2 for the definition of the perimeter $\mathrm{Per}_w(E)$ for nonsmooth sets. Notice that only the boundary of E that is *inside* the cone matters when computing the perimeter $\mathrm{Per}_w(E)$.

Let us recall the weighted isoperimetric inequality in a convex cone. The ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$ is denoted by $B_r(x)$; when the center is the origin we use the notation $B_r := B_r(0_{\mathbb{R}^n})$.

Theorem 1.1 ([CRS16])**.** *Let $\Sigma \subseteq \mathbb{R}^n$ be an open convex cone with vertex at 0, and let $\alpha > 0$. Let w be a nonnegative continuous function in $\overline{\Sigma}$ (not constantly 0) such that*

$$w \text{ is } \alpha\text{-homogeneous}^1, \text{ and } w^{1/\alpha} \text{ is concave in } \Sigma.$$

Then, for all measurable sets $E \subseteq \Sigma$ with $w(E) < \infty$,

$$\frac{\mathrm{Per}_w(E)}{w(E)^{\frac{D-1}{D}}} \geq \frac{\mathrm{Per}_w(B_1 \cap \Sigma)}{w(B_1 \cap \Sigma)^{\frac{D-1}{D}}}, \tag{1.1}$$

where $D := n + \alpha$. By homogeneity, B_1 can be replaced by B_r for any $r > 0$.

See [CRS16, Remark 1.4] for a geometric justification of the concavity condition on the weight and [CRS16, Section 2] for a number of examples of admissible weights. An important example to have in mind is given by monomial weights:

$$w(x) = x_1^{A_1} \cdots x_n^{A_n} \quad \text{in} \quad \Sigma = \{x_1 > 0, \dots, x_n > 0\}, \quad \text{with } A_i > 0\,.$$

The authors of [CRS16] do not give a characterization of the sets that achieve the equality in (1.1) (see [CRS16, p. 2979]).

Our first result is the following characterization of the optimal sets.

Proposition 1.2. *Let n, α, Σ and w be as in Theorem 1.1 and assume*[2] *that $\Sigma = \mathbb{R}^k \times \widetilde{\Sigma}$, where $0 \leq k < n$ and $\widetilde{\Sigma} \subseteq \mathbb{R}^{n-k}$ is an open convex cone* containing no lines*.*

Then, a measurable set $E \subseteq \Sigma$, with $w(E) < \infty$, achieves the equality in (1.1) if and only if $E = B_r(x_0)$ for some $r > 0$ and $x_0 \in \mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}$.

From now on we assume, without loss of generality, that $\Sigma = \mathbb{R}^k \times \widetilde{\Sigma}$, where $0 \leq k < n$ and $\widetilde{\Sigma} \subseteq \mathbb{R}^{n-k}$ is an open convex cone *containing no lines*. Notice that, since $w^{\frac{1}{\alpha}}$ is concave and nonnegative, the weight w is invariant along the first k-coordinates, that is $\partial_i w = 0$ for each $i = 1, 2, \dots, k$.

Let us measure the distance between a set $E \subseteq \Sigma$ and the minimizers of the weighted isoperimetric inequality with the quantity

$$A_w(E) := \inf_{x_0 \in \mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}} \frac{w(E \Delta (B_r(x_0) \cap \Sigma))}{w(E)}\,,$$

where $r > 0$ is such that $w(E) = w(B_r \cap \Sigma)$.

We define the weighted isoperimetric deficit of a set $E \subseteq \Sigma$ as

$$\delta_w(E) = \frac{\mathrm{Per}_w(E)}{c_* w(E)^{\frac{D-1}{D}}} - 1,$$

where $c_* := \mathrm{Per}_w(B_1 \cap \Sigma)/w(B_1 \cap \Sigma)^{\frac{D-1}{D}} = D w(B_1 \cap \Sigma)^{\frac{1}{D}}$ is the isoperimetric constant that comes from (1.1), and $D := n + \alpha$. Notice that the identity $\mathrm{Per}_w(B_1 \cap \Sigma) = D w(B_1 \cap \Sigma)$ follows from the homogeneity of the weight w.

[1] Here, and everywhere in this paper, we say that a function $w : \Sigma \to \mathbb{R}$ is α-homogeneous if $w(tx) = t^\alpha w(x)$ for any $x \in \overline{\Sigma}$ and any $t > 0$.

[2] This assumption is always satisfied up to a rotation.

The main result of the present paper is the following quantitative version of the weighted isoperimetric inequality.

Theorem 1.3. *Let n, α, Σ and w be as in Theorem 1.1 and assume that $\Sigma = \mathbb{R}^k \times \widetilde{\Sigma}$, where $0 \le k < n$ and $\widetilde{\Sigma} \subseteq \mathbb{R}^{n-k}$ is an open convex cone containing no lines.*

Then, for all measurable sets $E \subseteq \Sigma$ with $w(E) < \infty$, it holds

$$A_w(E) \le C\sqrt{\delta_w(E)}, \tag{1.2}$$

where C is a constant that depends only on n, α, Σ and w.

Notice that the *stability constant* C contained in Theorem 1.3 cannot depend only on n and α. Indeed, if $n = 2$, $\Sigma = \{x_2 > \varepsilon|x_1|\}$ and $w = x_2$, then, as $\varepsilon \to 0$, the constant C of the statement must go to infinity (as any ball with center on $\partial\Sigma$ becomes *almost* optimal).

Let us also point out that the exponent $\frac{1}{2}$ is sharp, as often happens in quantitative stability estimates. We will prove this fact in Remark 8.2.

Remark 1.4. Our proof can be easily adapted (see Theorem 4.1) to recover the stability result of [FMP10] for the anisotropic isoperimetric inequality in $\mathbb{R}^n$.

1.3. **Sketch of the proof.** Our general plan to prove Theorem 1.3 is to make quantitative the proof of the weighted isoperimetric inequality contained in [CRS16]. However, several new difficulties (both conceptual and technical) arise. In this description of the proof we will ignore all technical issues; some estimates are stated in a simplified form that is not exactly what we prove (but is morally equivalent). Only in this section[3], the notation $A \lesssim B$ is equivalent to $A \le CB$, where C is a constant that depends on n, α, Σ, w.

Let us begin by briefly describing the proof of Theorem 1.1 found in [CRS16]. Given a smooth bounded connected set $E \subseteq \Sigma$ with $w(E) = w(B_1 \cap \Sigma) = 1$, consider the elliptic problem[4]

$$\begin{cases} \operatorname{div}(w\nabla u) = w\,\frac{\mathrm{Per}_w(E)}{w(E)} & \text{in } E \\ \partial_\nu u = 1 & \text{on } \partial E \cap \Sigma \\ \partial_\nu u = 0 & \text{on } \partial E \cap \partial\Sigma\,. \end{cases} \tag{1.3}$$

A contact argument implies that, for any $\xi \in B_1 \cap \Sigma$, there is $x \in E$ such that $\nabla u(x) = \xi$ and $\nabla^2 u(x) \ge 0$. Hence, denoting by E' the set $\{x \in E :\ \nabla^2 u \ge 0,\ \nabla u \in B_1 \cap \Sigma\}$, the area formula implies

$$\begin{aligned} w(B_1 \cap \Sigma) &\le \int_{E'} w(\nabla u)\det(\nabla^2 u) = \int_{E'} \frac{w(\nabla u)}{w}\det(\nabla^2 u)\, w \\ &\le \int_{E'} \Big(\frac{\operatorname{tr}(\nabla^2 u) + \alpha\big(\frac{w(\nabla u)}{w}\big)^{\frac{1}{\alpha}}}{D}\Big)^D w \le \int_{E'} \Big(\frac{\Delta u + \frac{\nabla w}{w}\cdot\nabla u}{D}\Big)^D w \\ &= \Big(\frac{\mathrm{Per}_w(E)}{D\,w(E)}\Big)^D w(E') \le \Big(\frac{\mathrm{Per}_w(E)}{D\,w(E)}\Big)^D w(E)\,, \end{aligned}$$

where we have applied the weighted arithmetic-geometric mean inequality (recall that $D = n + \alpha$), then [CRS16, Lemma 5.1] (which assumes only the concavity of $w^{\frac{1}{\alpha}}$) and finally the fact that u satisfies (1.3). Notice that the proven inequality is, up to rearrangement of the terms, the weighted isoperimetric inequality (see [CRS16] for the details). What we have just sketched is the ABP method in a nutshell (in the context of isoperimetric inequalities, the method was introduced in [Cab00; Cab08]; see [Bre21] for a recent striking application of the method[5]). However, it is very hard to exploit directly the function u to

[3]Later on, we will not consider the cone Σ and the weight w as fixed and so a constant that depends on them will not be absorbed by $\lesssim$ (see Section 2).

[4]Note that if $E = B_1 \cap \Sigma$, then the quadratic function $u(x) := \frac{1}{2}|x|^2$ solves (1.3). In this case the gradient map ∇u is the identity.

[5]The author proves a sharp isoperimetric inequality for minimal surfaces (up to codimension 2) embedded in the Euclidean space. We believe that our methods might be applied to show a quantitative version of Brendle's result.

obtain a stability result. The main obstructions being that it is impossible to control any derivative of u (as everything depends wildly on ∂E) and that the proof *sees* only E' and not the whole set E.

Hence we take an appropriate *restricted* convex envelope of u. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be the convex function

$$\varphi(x) := \sup\left\{a + \xi \cdot x : \ \xi \in \overline{B_1 \cap \Sigma},\, a + \xi \cdot y \le u(y)\ \forall y \in E\right\}. \tag{1.4}$$

Notice that φ is simply the supremum of all affine functions with slope in $\overline{B_1 \cap \Sigma}$ that are below u. We go on to prove that φ is much more well-behaved compared to u itself (mainly because φ is convex and controlling the Laplacian of a convex function is sufficient to control the whole Hessian). Precisely, we prove that φ is $C^{1,1}$ (with bounds independent of the regularity of ∂E) and it holds $\nabla\varphi(\overline{E}) = \overline{B_1 \cap \Sigma}$. Moreover, φ retains (in a distilled form) the fact that u satisfies (1.3):

$$\Delta\varphi + \alpha\left(\frac{w(\nabla\varphi)}{w}\right)^{\frac{1}{\alpha}} \le \frac{\mathrm{Per}_w(E)}{w(E)}. \tag{1.5}$$

To understand the meaning of the last inequality, let us remark that, if $\nabla u \in \Sigma$, it holds

$$\Delta u + \alpha\left(\frac{w(\nabla u)}{w}\right)^{\frac{1}{\alpha}} \le \Delta u + \frac{\nabla w}{w} \cdot \nabla u = w^{-1}\,\mathrm{div}(w\nabla u) = \frac{\mathrm{Per}_w(E)}{w(E)},$$

where in the first inequality we used the concavity of $w^{\frac{1}{\alpha}}$ (see Lemma 2.1 below). It turns out that (1.5) is enough for our purposes. Namely, that the properties of the coupling φ are still sufficient to prove the weighted isoperimetric inequality (it is sufficient to replace u with φ in the proof). Moreover, as simple byproducts of the proof, we obtain the following estimates:

$$\int_E |\nabla^2\varphi - 1|\, w \lesssim \delta_w(E)^{\frac{1}{2}}, \tag{1.6}$$

$$\int_{\partial E \cap \Sigma} (1 - |\nabla\varphi|)\, w\, \mathrm{d}\mathscr{H}^{n-1} \lesssim \delta_w(E), \tag{1.7}$$

$$\int_E \left| w(\nabla\varphi)^{\frac{1}{\alpha}} - w^{\frac{1}{\alpha}} \right| \lesssim \delta_w(E)^{\frac{1}{2}}. \tag{1.8}$$

The estimate (1.6) controls the L^1-norm of the differential of $\nabla\varphi$, (1.7) implies that $\nabla\varphi(x)$ almost belongs to $\partial B_1 \cap \Sigma$ when $x \in \partial E$. It is a bit harder to grasp the point of (1.8), but it will be clear later on.

In [FMP10], the authors have shown that if a set has a small isoperimetric deficit then, up to a small modification, it must enjoy a nontrivial Poincaré inequality and a nontrivial trace inequality. Surprisingly, their ideas can be easily adapted to our weighted setting and therefore we can assume that E has nontrivial weighted Poincaré and trace inequalities. The adjective *nontrivial* has to be understood as the fact that the constants of the inequalities do not depend on the set E itself and can be bounded a priori. With these considerations, it is not hard to see that (1.6) and (1.7) imply the existence of $x_0 \in \mathbb{R}^n$ such that

$$\int_E |\nabla\varphi(x) - (x - x_0)|\, w(x)\, \mathrm{d}x \lesssim \delta_w(E)^{\frac{1}{2}}, \tag{1.9}$$

$$\int_{\partial E \cap \Sigma} ||x - x_0| - 1|\, w(x)\, \mathrm{d}\mathscr{H}^{n-1}(x) \lesssim \delta_w(E)^{\frac{1}{2}}. \tag{1.10}$$

The estimate (1.10) tells us that $\partial E \cap \Sigma$ is *almost* contained in the boundary of $B_1(x_0)$, hence it is natural that from (1.10) we are able to deduce

$$w(E \triangle (B_1(x_0) \cap \Sigma)) \lesssim \delta_w(E)^{\frac{1}{2}}. \tag{1.11}$$

Notice that a similar deduction is present also in [FMP10] and [FI13], but their method to prove it does not work in our setting.

It remains to show that in (1.11) we can choose $x_0 = (z, 0_{\mathbb{R}^{n-k}})$ (recall that both the cone and the weight are invariant on the first k coordinates). This final step is far from being straightforward. Indeed, the interplay between the boundary of the cone and the weight makes it cumbersome to rule out that E is close to a *translated* ball, i.e., a ball $B_r(x_0)$ with x_0 not lying in $\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}$. Let us mention that in [FI13]

the authors face an analogous difficulty. Though, in their setting it is easier to rule out translations for two reasons: they work without a weight and they have the stronger assumption $w(E\triangle((B_1\cap\Sigma)+x_0))\lesssim\delta_w(E)^{\frac12}$ (compare with (1.11)).

Without loss of generality we can assume that $x_0=(0_{\mathbb{R}^k},\widetilde{x}_0)$. We will show that $|\widetilde{x}_0|\lesssim\delta_w(E)^{\frac12}$, which is sufficient to conclude the proof. Instead of giving the details of our strategy, we describe it in three different settings in order to show all the ideas without being lost in the technicalities. In all the three cases it holds $k=0$, $n=2$ (i.e., no lines are contained in Σ). We will denote the coordinates with (x,y).

The first method works if x_0 belongs to $\overline{\Sigma}$ or $-\overline{\Sigma}$ and w is constant along the direction of x_0. The second method works if x_0 is not aligned with a constant direction of w. The third method works if x_0 does not belong to Σ nor $-\Sigma$ and w is constant along the direction of x_0. Since we have (morally) covered all possible cases, the proof is concluded.

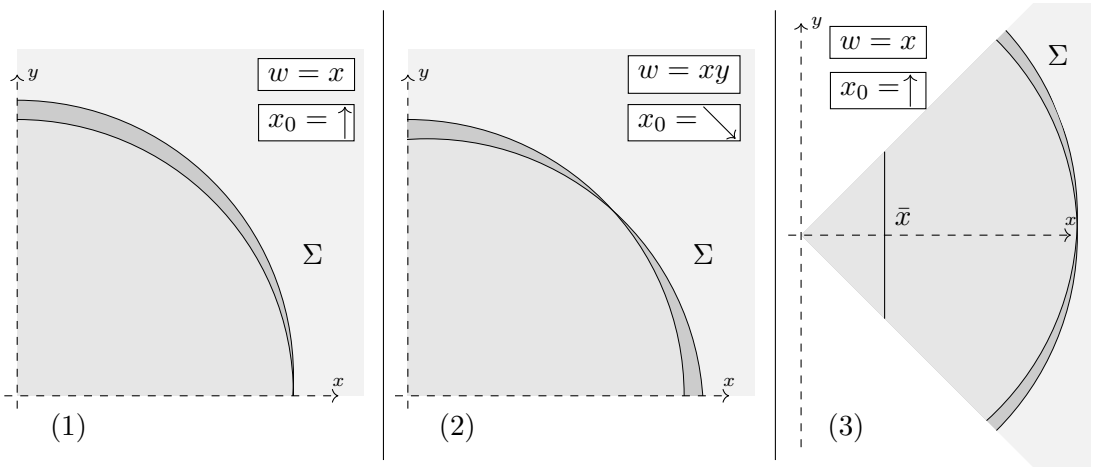

FIGURE 1. Visual description of the three fundamental cases that have to be handled to prove $|\widetilde{x}_0|\lesssim\delta_w(E)^{\frac12}$. The darker region is the symmetric difference between $B_1\cap\Sigma$ and $B_1(x_0)\cap\Sigma$, while the lighter region is their intersection.

(1) Let $\Sigma:=\{x>0,\,y>0\}$, $w:=x$ and assume that $x_0=(0,t)$ for some $t\in\mathbb{R}$. It holds

$$|w(B_1(x_0)\cap\Sigma)-w(B_1\cap\Sigma)|\gtrsim|t|\,, \tag{1.12}$$

that, together with (1.11), implies $t\lesssim\delta_w(E)^{\frac12}$ as desired (recall that $w(E)=w(B_1\cap\Sigma)$).

(2) Let $\Sigma:=\{x>0,\,y>0\}$, $w:=xy$ and assume that $x_0=(t,-t)$ for some $t\in\mathbb{R}$. Joining (1.8) and (1.9), we can prove

$$\int_E|w^{\frac1\alpha}(x-x_0)-w^{\frac1\alpha}|\lesssim\delta_w(E)^{\frac12}\,.$$

Since the weight w is nonconstant in the direction $(1,-1)$, it holds

$$\int_E|w^{\frac1\alpha}(x-x_0)-w^{\frac1\alpha}|\gtrsim t\,. \tag{1.13}$$

The last two inequalities imply $t\lesssim\delta_w(E)^{\frac12}$.

(3) Let $\Sigma:=\{x>|y|\}$, $w:=x$ and assume that $x_0=(0,t)$ for some $t\in\mathbb{R}$. This is the hardest situation: the weight is constant along x_0 (hence (1.8) is useless) and the value of $w(B_1(x_0)\cap\Sigma)$ can be very close to $w(B_1\cap\Sigma)$ (so we cannot use their difference to control t as we did in the first case). Thanks to (1.6) and (1.9), applying the fundamental theorem of calculus we can find $\frac14\le\overline{x}\le\frac12$ such that

$$|\nabla\varphi(\overline{x},y)-((\overline{x},y)-x_0)|\lesssim\delta_w(E)^{\frac12}$$

for any $y \in \mathbb{R}$ such that $(\overline{x}, y) \in \Sigma$ (restricting our attention to a 1-dimensional segment we have improved an L^1 estimate to an L^∞ estimate and this is fundamental). Since $\nabla\varphi \in \overline{B_1 \cap \Sigma}$, we deduce

$$\operatorname{dist}(\Sigma, (\overline{x}, y) - x_0) \lesssim \delta_w(E)^{\frac{1}{2}} .$$

Choosing $y = \overline{x}$ or $y = -\overline{x}$ in the latter estimate (depending on the sign of t), we readily deduce $|t| \lesssim \delta_w(E)^{\frac{1}{2}}$.

1.4. **Comparison between our coupling and the optimal transport map.** Let $E \subseteq \mathbb{R}^n$ be a bounded connected set with smooth boundary. If we set $\Sigma = \mathbb{R}^n$ and $w = 1$, the sketch above[6] provides a *convex* function $\varphi : E \to \mathbb{R}$ such that $\nabla\varphi(E) = B_1$ (up to negligible sets) and

$$\operatorname{tr}(\nabla^2\varphi) \leq \frac{\operatorname{Per}(E)}{|E|} .$$

On the other hand, let $\nabla\psi : E \to B_1$ be the optimal transport map between the two probability measures $|E|^{-1}\mathscr{L}^n|_E$ and $|B_1|^{-1}\mathscr{L}^n|_{B_1}$ with respect to the quadratic cost (see [FMP10] or [Vil09, p. 23] for the missing details). The function $\psi : E \to \mathbb{R}$ is *convex* ([Bre91]) and, by definition of transport map, it holds $\nabla\psi(E) = B_1$ (up to negligible sets) and

$$\det(\nabla^2\psi) = \frac{|B_1|}{|E|} .$$

Summing up, both $\varphi, \psi : E \to \mathbb{R}$ are convex functions such that $\nabla\varphi(E) = \nabla\psi(E) = B_1$ (up to negligible sets) and they both satisfy a condition on the Hessian. Notice also that both φ and ψ encode some nontrivial global information about the set E, indeed both yield a one-line proof of the isoperimetric inequality:

$$|B_1| = |\nabla\varphi(E)| \leq \int_E \det(\nabla^2\varphi) \leq \int_E \left(\frac{\operatorname{tr}(\nabla^2\varphi)}{n}\right)^n \leq \frac{\operatorname{Per}(E)^n}{n^n |E|^{n-1}} , \tag{1.14}$$

$$n|E| \left(\frac{|B_1|}{|E|}\right)^{\frac{1}{n}} = n \int_E \det(\nabla^2\psi)^{\frac{1}{n}} \leq \int_E \operatorname{div}(\nabla\psi) = \int_{\partial E} \nabla\psi \cdot \nu_{\partial E} \, \mathrm{d}\mathscr{H}^{n-1} \leq \operatorname{Per}(E) .$$

Given their many similarities, it is natural to wonder whether the two functions φ, ψ are two instances of the same phenomenon. We believe it would be very interesting to find a unifying framework that allows to treat the two functions (and perhaps other functions with in-between conditions on the Hessian) together.

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1.6. **Organization of the paper.** While describing the content of the various sections of the work we refer to the sketch of the proof given above.

After a section of notation and preliminaries, in Section 3 we study the K-envelope of a function (where K is a compact convex set) obtaining a precise $C^{1,1}$-regularity result that might be of independent interest (and that we will use later on in the proof). When $K = \overline{B_1 \cap \Sigma}$, the K-envelope of a generic function u boils down to (1.4) (thus the coupling φ is a $\overline{B_1 \cap \Sigma}$-envelope). The construction of the coupling φ and the proof of its properties are contained in Section 4. We construct the coupling also in the case of the anisotropic (unweighted) perimeter, as it could be of independent interest. The strategies adopted to deduce $|\widetilde{x}_0| \leq C\delta_w(E)^{\frac{1}{2}}$ are implemented in Section 5. The implication (1.10) $\Longrightarrow$ (1.11) is proven in

[6] It is sufficient to repeat the sketch, ignoring the weight and the cone, until (1.5). Alternatively, see Theorem 4.1.

Section 6. In Section 7, we adapt [FMP10, Section 3] to the weighted setting. Namely, we show that if a set has a small weighted isoperimetric deficit, then it enjoys nontrivial trace and Poincaré inequalities. The proofs of Proposition 1.2 and Theorem 1.3 are contained in Section 8.

This work has three appendices. The first one contains a quantitative version of the weighted inequality of arithmetic and geometric means; the second one is a collection of simple facts regarding 1-homogeneous concave functions in a cone. In the third and final appendix we prove that, in $\mathbb{R}^2$, an indecomposable set (see Definition C.1) can be approximated with *connected* open sets.

2. Notation

Since the statement of Theorem 1.3 is invariant under rescaling of the weight, we can assume that $w(B_1 \cap \Sigma) = 1$. Notice that, under this additional constraint, the isoperimetric constant is simply $c_* = D$ (recall that $D \coloneqq n + \alpha$).

Any constant that depends only on n, α is considered universal and can be hidden in the notation $\lesssim$. Precisely, the notation $A \lesssim B$ is equivalent to $A \le CB$ for a suitable constant $C = C(n, \alpha)$ that depends only on n and α. On the other hand, we will write explicitly constants that depend on the cone Σ and the weight w.

We denote with $B_r(x_0)$ the open ball with center $x_0 \in \mathbb{R}^n$ and radius $r > 0$; when the center is the origin ($x_0 = 0_{\mathbb{R}^n}$) we simply write B_r. The $(n-1)$-dimensional sphere is denoted by ∂B_1 or $\mathbb{S}^{n-1}$. The n-dimensional Lebesgue measure is denoted by $\mathscr{L}^n$, the $(n-1)$-dimensional Hausdorff measure is denoted by $\mathscr{H}^{n-1}$. The Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$. The identity matrix (whose size will always be $n \times n$) is denoted by $\mathbb{1}$. The convex-hull of a set E, that is the smallest closed convex set that contains E, is denoted by $\operatorname{conv}(E)$.

2.1. Assumptions.

Almost all the statements of this work require the same assumptions on the cone and the weight. For notational simplicity we state them here and reference them instead of repeating them in every statement. We assume that n, α, Σ, w satisfy

$$\begin{cases} n \in \mathbb{N} \text{ and } \alpha \in (0, \infty)\,; \\ \Sigma \subseteq \mathbb{R}^n \text{ is an open convex cone with vertex at } 0; \\ \Sigma = \mathbb{R}^k \times \widetilde{\Sigma} \text{ where } 0 \le k < n \text{ and } \widetilde{\Sigma} \subseteq \mathbb{R}^{n-k} \text{ is an open convex cone } \textit{containing no lines}; \\ w : \overline{\Sigma} \to [0, \infty) \text{ is a continuous nonnegative weight such that } w \text{ is } \alpha\text{-homogeneous}, \\ w^{\frac{1}{\alpha}} \text{ is concave, and } w(B_1 \cap \Sigma) = 1. \end{cases} \tag{2.1}$$

Notice that w may be 0 on $\partial\Sigma$, but, since $w^{\frac{1}{\alpha}}$ is concave, it is strictly positive inside Σ. Furthermore, since $w^{\frac{1}{\alpha}}$ is concave and nonnegative, the weight is invariant along the first k coordinates, that is $\partial_i w = 0$ for each $i = 1, 2, \dots, k$.

A simple but useful result from [CRS16] is the following.

Lemma 2.1 ([CRS16])**.** *Let w be a positive homogeneous function of degree $\alpha > 0$ in an open cone $\Sigma \subseteq \mathbb{R}^n$. Then, the following conditions are equivalent:*

- *The function $w^{1/\alpha}$ is concave in Σ.*
- *For each $x, z \in \Sigma$, the following inequality holds:*

$$\alpha \left(\frac{w(z)}{w(x)} \right)^{1/\alpha} \le \frac{\nabla w(x) \cdot z}{w(x)}.$$

We will use such result several times throughout the paper.

2.2. Set of finite (weighted) perimeter and functions of bounded variation.

Let us recall some basic facts about sets of finite perimeter. All the definitions and results we are going to state can be found in the monograph [Mag12].

A measurable set $E \subseteq \mathbb{R}^n$ is a set of finite perimeter if its perimeter

$$\mathrm{Per}(E) := \sup\left\{ \int_E \mathrm{div}(X(x))\,\mathrm{d}x : \ X \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)\,,\ \|X\|_\infty \leq 1 \right\}$$

is finite. There is also a *localized* version of the perimeter; given an open set $\Omega \subseteq \mathbb{R}^n$, the relative perimeter of E inside Ω is

$$\mathrm{Per}(E, \Omega) := \sup\left\{ \int_E \mathrm{div}(X(x))\,\mathrm{d}x : \ X \in C_c^\infty(\Omega, \mathbb{R}^n)\,,\ \|X\|_\infty \leq 1 \right\}.$$

A set of (locally) finite perimeter admits a measure-theoretic notion of boundary (the *reduced boundary*), which is a $(n-1)$-rectifiable set that we denote by $\partial^* E$, and for each point $x \in \partial^* E$ an *outer normal vector* $\nu_{\partial^* E}(x) \in \mathbb{S}^{N-1}$ is defined. For any $X \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$, it holds

$$\int_E \mathrm{div}(X(x))\,\mathrm{d}x = \int_{\partial^* E} X \cdot \nu_{\partial^* E}\,\mathrm{d}\mathscr{H}^{n-1}\,.$$

In addition, $\mathrm{Per}(E) = \mathscr{H}^{n-1}(\partial^* E)$. The reduced boundary is, up to $\mathscr{H}^{n-1}$-negligible sets, the set of points in $\mathbb{R}^n$ where E has density $\frac{1}{2}$. We denote with $E^{(1)}$ the set of points in $\mathbb{R}^n$ where E has density 1.

Let us now define the *weighted* perimeter. Given a measurable set $E \subseteq \Sigma$, its w-perimeter in Σ is defined as

$$\mathrm{Per}_w(E) := \sup\left\{ \int_E \mathrm{div}(X(x)w(x))\,\mathrm{d}x : \ X \in C_c^\infty(\Sigma, \mathbb{R}^n)\,,\ \|X\|_\infty \leq 1 \right\}.$$

It is not hard to prove that if $\mathrm{Per}_w(E) < \infty$, then for any $\Omega \subset\subset \Sigma$ it holds $\mathrm{Per}(E, \Omega) < \infty$. In particular $\partial^* E$ is well-defined whenever $\mathrm{Per}_w(E) < \infty$ and it holds

$$\mathrm{Per}_w(E) = \int_{\partial^* E \cap \Sigma} w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,.$$

Let us define $\mathscr{H}^{n-1}_w := w\mathscr{H}^{n-1}|_\Sigma$, so that $\mathrm{Per}_w(E) = \mathscr{H}^{n-1}_w(\partial^* E)$.

Let us now move our attention to functions of bounded variation. All the definitions and results we are going to state can be found in the monograph [AFP00].

A measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is of bounded variation if its distributional gradient is a vector-valued measure, that we denote with[7] $\widetilde{\nabla} f$. The set of functions of bounded variation is denoted by $BV(\mathbb{R}^n)$.

Given $f \in BV(\mathbb{R}^n)$ and $X \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$, it holds

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{div}(X(x))\,\mathrm{d}x = \int_{\mathbb{R}^n} X(x)\,\mathrm{d}\widetilde{\nabla} f(x)\,. \tag{2.2}$$

Notice that a measurable set $E \subseteq \mathbb{R}^n$ is a set of finite perimeter if and only if its characteristic function χ_E has bounded variation. Moreover, the following relation between the reduced boundary of E and the distributional gradient of the characteristic function,

$$\widetilde{\nabla}\chi_E = -\nu_{\partial^* E}\,\mathscr{H}^{n-1}|_{\partial^* E}\,,$$

holds.

3. Regularity of the K-envelope

Let us start with the definition of the main character of this section: the K-envelope of a function.

Definition 3.1. Let $K \subseteq \mathbb{R}^n$ be a compact, convex set, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. We define the *$K$-envelope of u* as the function $\overline{u}^K : \mathbb{R}^n \to \mathbb{R}$ given by

$$\overline{u}^K(x) := \sup\left\{ a + \xi \cdot x : \ \xi \in K,\ a + \xi \cdot y \leq u(y)\ \forall y \in \overline{\Omega} \right\}.$$

[7] The distributional gradient of a function of bounded variation is usually denoted with Df, and ∇f is used to identify the absolutely continuous part of Df. We use the notation $\widetilde{\nabla} f$ for the distributional gradient to avoid confusion, indeed the letter D is the *effective dimension* $D = n + \alpha$.

Remark 3.2. The K-envelope is, by definition, the supremum of all affine functions with slope in K that are below u. Notice that $\overline{u}^K(x) < \infty$ because K is compact.

Remark 3.3. For our purposes, only the case $K = \overline{B_1 \cap \Sigma}$ is important. Nonetheless, since it is not much easier to handle only that case and because the results in this section might be of independent interest, we decided to drop the assumption $K = \overline{B_1 \cap \Sigma}$ and work with a generic compact convex set.

The goal of this section is to obtain some precise $C^{1,1}$ bounds on $\overline{u}^K$. We are interested in showing that the Hessian of $\overline{u}^K$ is controlled (as a symmetric matrix) by a suitable combination of Hessians (in different points) of the original function u. Similar results are well-known for the classical convex envelope (see for instance [DF15] and the references therein). Nonetheless, we could not find any work on the K-envelope.

The fundamental difficulty arising when considering the K-envelope (compared to the convex envelope) is that at many points it holds $\nabla \overline{u}^K \in \partial K$ and there the standard approaches fail. This shortcoming can be solved neatly introducing the notion of normal cone. Let us define the normal cone and the subdifferential (see [Roc70]) and obtain the first basic results about the K-envelope.

Definition 3.4 (Normal cone)**.** Given a compact, convex set $K \subseteq \mathbb{R}^n$, for any $\xi \in K$, the normal cone $N(\xi, K)$ of K at ξ is defined as

$$N(\xi, K) := \left\{ v \in \mathbb{R}^n : \ v \cdot (\xi' - \xi) \leq 0 \ \text{ for all } \ \xi' \in K \right\} .$$

Notice that the normal cone is most interesting for boundary points $\xi \in \partial K$; in the interior one simply has $N(\xi, K) = \{0\}$.

Definition 3.5 (Subdifferential)**.** Given a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$, its subdifferential $\partial\varphi(x)$ at the point $x \in \mathbb{R}^n$ is defined as

$$\partial\varphi(x) := \{ \xi \in \mathbb{R}^n : \ \varphi(y) \geq \varphi(x) + \xi \cdot (y - x) \ \forall y \in \mathbb{R}^n \} .$$

Lemma 3.6. *Let $K \subseteq \mathbb{R}^n$ be a compact, convex set, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. The function $\overline{u}^K : \mathbb{R}^n \to \mathbb{R}$ is convex and at any point $x \in \mathbb{R}^n$ it holds $\partial \overline{u}^K(x) \cap K \neq \emptyset$.*

Proof. The convexity follows directly from the definition. For the second part of the statement, fix $(a_i)_{i \in \mathbb{N}} \subseteq \mathbb{R}$ and $(\xi_i)_{i \in \mathbb{N}} \subseteq K$ such that $\overline{u}^K(x) = \lim_{i \to \infty} a_i + \xi_i \cdot x$ and $a_i + \xi_i \cdot y \leq u(y)$ for any $y \in \overline{\Omega}$. Since K is compact we can assume that $\xi_i \to \xi \in K$ and as a consequence it must hold $a_i \to a \in \mathbb{R}$. Hence $\overline{u}^K(x) = a + \xi \cdot x$ and $a + \xi \cdot y \leq u(y)$ for any $y \in \overline{\Omega}$; in particular this implies that $\overline{u}^K(y) \geq a + \xi \cdot y$ for any $y \in \mathbb{R}^n$ and we deduce $\xi \in \partial \overline{u}^K(x)$. □

Lemma 3.7. *Let $A, B \subseteq \mathbb{R}^n$ be two nonempty closed convex sets such that for any $b \in B$ there is $a \in A$ such that $a \cdot b \leq 0$. Then there is $\overline{a} \in A$ such that $\overline{a} \cdot b \leq 0$ for any $b \in B$.*

Proof. Given a subset $S \subseteq \mathbb{R}^n$, let S° be its polar cone, that is

$$S^\circ := \{ x \in \mathbb{R}^n : \ x \cdot s \leq 0 \text{ for any } s \in S \} .$$

We want to prove that $A \cap B^\circ \neq \emptyset$. Let us assume by contradiction that $A \cap B^\circ = \emptyset$. Then, since A is a closed convex set and B° is a closed convex cone, we can find $v \in B^{\circ\circ}$ such that $a \cdot v > 0$ for any $a \in A$. Since $B^{\circ\circ}$ is the cone $\{\lambda b : \ b \in B, \lambda \geq 0\}$ generated by B (see [Roc70, Theorem 14.1]), up to rescaling we can assume that $v \in B$. Thus we have reached a contradiction as we are assuming the existence of $a \in A$ such that $a \cdot v \leq 0$. □

Lemma 3.8. *Let $K \subseteq \mathbb{R}^n$ be a compact, convex set, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Given $\xi \in K$, let $S_\xi := \arg\min_{x \in \overline{\Omega}} \{u(x) - \xi \cdot x\}$. Then we have:*

(a) For any $x_\xi \in S_\xi$, it holds $\overline{u}^K(x_\xi) = u(x_\xi)$ and $\xi \in \partial \overline{u}^K(x_\xi)$.
(b) Given $x \in \mathbb{R}^n$, if $\xi \in \partial \overline{u}^K(x)$ then there is $v \in N(\xi, K)$ such that $x - v \in \operatorname{conv}(S_\xi)$.

Proof. Let us start proving (a). Since x_ξ minimizes $u(x) - \xi \cdot x$, for any $y \in \overline{\Omega}$ it holds

$$u(y) \geq u(x_\xi) + \xi \cdot (y - x_\xi)$$

and thus, by definition of K-envelope, we deduce

$$u(y) \geq \overline{u}^K(y) \geq u(x_\xi) + \xi \cdot (y - x_\xi)\,. \tag{3.1}$$

Setting $y = x_\xi$, (3.1) implies $\overline{u}^K(x_\xi) = u(x_\xi)$, thus $\xi \in \partial \overline{u}^K(x_\xi)$.

Let us now move to the proof of (b). Fix $\xi' \in K$ and, for any $0 < \varepsilon < 1$, choose $y_{\xi',\varepsilon} \in S_{\xi+\varepsilon(\xi'-\xi)}$. Thanks to (a), we know that $\xi + \varepsilon \cdot (\xi' - \xi) \in \partial \overline{u}^K(y_{\xi',\varepsilon})$ and, since the subdifferential is a monotone operator (see [Roc70, §24]), this implies

$$(x - y_{\xi',\varepsilon}) \cdot (\xi' - \xi) \leq 0\,. \tag{3.2}$$

Thanks to the compactness of $\overline{\Omega}$, up to subsequence, it holds $y_{\xi',\varepsilon} \to y_{\xi'} \in S_\xi$ as $\varepsilon \to 0$ (we are using that $\xi + \varepsilon(\xi' - \xi) \to \xi$ as $\varepsilon \to 0$). Thus, passing to the limit in (3.2), we deduce

$$(x - y_{\xi'}) \cdot (\xi' - \xi) \leq 0\,. \tag{3.3}$$

We have shown that for any $\xi' \in K$ there exists $y_{\xi'} \in S_\xi$ such that (3.3) holds. The conclusion follows from Lemma 3.7 with $A = x - \mathrm{conv}(S_\xi)$ and $B = K - \xi$. □

We now have all the tools to prove the central result of this section. The idea is the following. Fix $x \in \mathbb{R}^n$ and consider a hyperplane touching $\overline{u}^K$ from below at x. This hyperplane touches u from below at, say, $x_1, \dots, x_m$. We prove that (in a rather strong sense) $\overline{u}^K$ admits an Hessian at x and this Hessian is controlled by a convex combination of the Hessian of u at $x_1, \dots, x_m$. More precisely, $\nabla^2 \overline{u}^K(x)$ belongs to $H(x, \nabla \overline{u}^K(x), K)$, which is defined as follows:

Definition 3.9. Let $K \subseteq \mathbb{R}^n$ be a compact, convex set, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Given $x \in \mathbb{R}^n$ and $\xi \in K$, let us define the family of matrices (the definition of S_ξ is contained in the statement of Lemma 3.8)

$$H(x, \xi, K) := \left\{ \sum_{i=1}^m \lambda_i \nabla^2 u(s_i) : \begin{array}{l} 1 \leq m \leq n+1 \\ \lambda_i \geq 0,\ \sum \lambda_i = 1 \\ s_i \in S_\xi \cap \Omega \\ x - \sum \lambda_i s_i \in N(\xi, K) \end{array} \right\}.$$

Notice that $H(x, \xi, K)$ is convex by definition and it is also closed if $S_\xi \subseteq \Omega$ (since S_ξ is closed by definition).

Proposition 3.10. *Let $K \subseteq \mathbb{R}^n$ be a compact, convex set, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Assume that for any $\xi \in K$ it holds $S_\xi \subseteq \Omega$ (where S_ξ is defined as in Lemma 3.8). Then $\overline{u}^K : \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,1}$ convex function such that*

$$\nabla \overline{u}^K(\mathbb{R}^n) = \nabla \overline{u}^K(\Omega) = K.$$

Moreover, for any $x \in \mathbb{R}^n$ and any $H_x \in H(x, \nabla \overline{u}^K(x), K) \neq \emptyset$, it holds $\nabla^2 \overline{u}^K(x) \leq H_x$, in the sense that, for x' converging to x,

$$\overline{u}^K(x') \leq \overline{u}^K(x) + \nabla \overline{u}^K(x) \cdot (x' - x) + \frac{1}{2}(x' - x) H_x (x' - x) + \mathrm{o}(|x' - x|^2)\,. \tag{3.4}$$

Proof. Fix $x \in \mathbb{R}^n$ and $\xi \in \partial \overline{u}^K(x) \cap K$ (the intersection is nonempty by Lemma 3.6). Thanks to Lemma 3.8 (and Carathéodory's theorem[8] for convex hulls) we can find $v \in N(\xi, K)$, $s_1, \dots, s_m \in S_\xi \subseteq \Omega$, with $1 \leq m \leq n+1$, and $\lambda_i \geq 0$, $\lambda_1 + \dots + \lambda_m = 1$, such that

$$x - v = \lambda_1 s_1 + \dots + \lambda_m s_m\,.$$

In particular, $H(x, \nabla \overline{u}^K(x), K)$ is nonempty. Now, let us fix any such v, $(s_i)_{i=1,\dots,m}$ and $(\lambda_i)_{i=1,\dots,m}$.

Take any $x' \in \mathbb{R}^n$ close enough to x, so that $s_i + x' - x \in \Omega$ for any $i = 1, \dots, m$. Given $\xi' \in \partial \overline{u}^K(x') \cap K$, for any $y \in \Omega$, it holds

$$u(y) \geq \overline{u}^K(y) \geq \overline{u}^K(x') + \xi' \cdot (y - x')$$

[8] Recall that Carathéodory's theorem [Car07] states that if a point $x_\circ \in \mathbb{R}^n$ lies in the convex hull of a set S, then $x_\circ$ can be written as the convex combination of at most $n+1$ points in S.

and thus, plugging $y = s_i + x' - x$, we deduce

$$\overline{u}^K(x') \le u(s_i + x' - x) + \xi' \cdot (x - s_i)\,. \tag{3.5}$$

Since $s_i \in S_\xi$, it holds $\nabla u(s_i) = \xi$ and therefore we have

$$u(s_i + x' - x) = u(s_i) + \xi \cdot (x' - x) + \frac{1}{2}(x' - x)\nabla^2 u(s_i)(x' - x) + \mathrm{o}(|x' - x|^2)\,. \tag{3.6}$$

Moreover, recalling that $\xi \in \partial\overline{u}^K(s_i) \cap \partial\overline{u}^K(x)$, we also have

$$\overline{u}^K(s_i) - \overline{u}^K(x) = \xi \cdot (s_i - x)\,. \tag{3.7}$$

Joining (3.5), (3.6) and (3.7) we obtain

$$\overline{u}^K(x') \le \overline{u}^K(x) + \xi \cdot (x' - x) + \frac{1}{2}(x' - x)\nabla^2 u(s_i)(x' - x) + \mathrm{o}(|x' - x|^2) + (\xi' - \xi) \cdot (x - s_i)\,.$$

Multiplying this latter inequality with λ_i and summing over $i = 1, \dots, m$ we get

$$\overline{u}^K(x') \le \overline{u}^K(x) + \xi \cdot (x' - x) + \frac{1}{2}(x' - x)\left(\sum_{i=1}^m \lambda_i \nabla^2 u(s_i)\right)(x' - x) + \mathrm{o}(|x' - x|^2) + (\xi' - \xi) \cdot v$$

and since $v \in N(\xi, K)$, we have shown (3.4) (notice that $\nabla^2 u(s_i) \ge 0$ as $s_i \in S_\xi$).

As a consequence we get that $\nabla\overline{u}^K(x) = \xi$ and thus, recalling Lemma 3.8, we deduce also $\nabla\overline{u}^K(\mathbb{R}^n) = \nabla\overline{u}^K(\Omega) = K$. To conclude that $\overline{u}^K \in C^{1,1}$, let us observe that, for any $x \in \mathbb{R}^n$, the matrix H_x of the statement can be found in

$$\operatorname{conv}\left(\nabla^2 u\Big(\bigcup_{\xi \in K} S_\xi\Big)\right)$$

and the assumption $S_\xi \subseteq \Omega$ together with the compactness of K implies that $\bigcup_{\xi\in K} S_\xi \subset\subset \Omega$. Therefore there is a constant $C = C(\Omega, u)$ such that for any $x \in \mathbb{R}^n$ it holds $H_x \le C\,\mathbb{1}$. Now it is standard to obtain $\overline{u}^K \in C^{1,1}(\mathbb{R}^n)$ (see, for example, [ACM18, Proposition 5.29]). □

Remark 3.11. The proof of Proposition 3.10 yields also a more *local* result. Namely, if $S_\xi \subseteq \Omega$ for all $\xi \in U \cap K$ (where U is an open set), then for any $x \in \mathbb{R}^n$ such that $\nabla\overline{u}^K(x) \in U$ there is a neighborhood of x where $\overline{u}^K$ is $C^{1,1}$ and (3.4) holds (for some $H_x \in H(x, \nabla\overline{u}^K(x), K)$).

4. Construction of the coupling

To better illustrate the ideas, before describing the construction of the coupling in the weighted setting, we construct the analogous coupling in the anisotropic case (without a weight). Even if the methods are the same, the construction in the anisotropic setting is less technical. In order to do so, we have to define the anisotropic norm and the anisotropic perimeter (to get some further context about the anisotropic perimeter, see [FMP10, Introduction]).

Given a compact convex set $K \subseteq \mathbb{R}^n$ such that $0_{\mathbb{R}^n} \in \mathring{K}$, let us define the anisotropic norm $|\cdot|_{K^*} : \mathbb{R}^n \to [0, \infty)$ as

$$|v|_{K^*} = \sup\{v \cdot x :\ x \in K\}\,.$$

Then, the anisotropic perimeter of a set of finite perimeter $E \subseteq \mathbb{R}^n$ is defined as

$$\mathrm{Per}_K(E) = \int_{\partial^* E} |\nu_{\partial^* E}|_{K^*}\, \mathrm{d}\mathscr{H}^{n-1}\,.$$

We have all the necessary definitions to construct the coupling for the anisotropic perimeter. Before going on, let us remark that, in all the statements of this section, we need an additional assumption in dimension 2 (namely, the indecomposability of the set E). This is because we need to approximate E with *connected* open sets and this is not always possible in dimension 2.

Theorem 4.1 (Anisotropic coupling). *Let $E \subseteq \mathbb{R}^n$ be a set of finite perimeter (with $0 < |E| < \infty$) and let $K \subseteq \mathbb{R}^n$ be a compact convex set whose interior contains the origin. If $n = 2$, we assume also that E is indecomposable (see Definition C.1).*

Then, there exists a $C^{1,1}$ convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ that satisfies $\nabla\varphi(\mathbb{R}^n) = \nabla\varphi(E) = K$ (up to negligible sets) and $\Delta\varphi \leq \frac{\mathrm{Per}_K(E)}{|E|}$.

Proof. First, we prove the statement when E is a bounded connected open set with smooth boundary and then we remove the assumption by approximation.

Let $\nu : \partial E \to \mathbb{R}^n$ be the outer normal to the boundary. As in [CRS16], let us consider the Neumann problem for $u : E \to \mathbb{R}$

$$\begin{cases} \Delta u = \frac{\mathrm{Per}_K(E)}{|E|} & \text{in } E \\ \partial_\nu u = |\nu|_{K^*} & \text{on } \partial E\,. \end{cases}$$

Since the compatibility condition $\int_E \Delta u = \int_{\partial E} \partial_\nu u$ is satisfied and E is connected, the problem admits a solution u, which is smooth up to the boundary.

For any $\xi \in \mathring{K}$ the minimum of the function $u(x) - \xi \cdot x$ cannot be attained on the boundary of E, indeed $\nabla(u(x) - \xi \cdot x) \cdot \nu = |\nu|_{K^*} - \xi \cdot \nu > 0$.

Take a sequence of compact convex sets $(K_i)_{i\in\mathbb{N}}$ such that $K_i \subset\subset \mathring{K}_{i+1}$ and $\cup_{i\in\mathbb{N}} K_i = \mathring{K}$. It is not hard to see that $\overline{u}^{K_i} \nearrow \overline{u}^K$ locally uniformly. Moreover, at any point $x \in E$ such that $\nabla^2 u(x) \geq 0$, it holds

$$0 \leq \nabla^2 u(x) \leq \Delta u(x)\, \mathbb{1} = \frac{\mathrm{Per}_K(E)}{|E|}\, \mathbb{1}$$

and thus, thanks to Proposition 3.10, the family $(\overline{u}^{K_i})_{i\in\mathbb{N}}$ is uniformly bounded in $C^{1,1}$. Hence, $\overline{u}^K \in C^{1,1}$, and the convergence to $\overline{u}^K$ is (up to subsequence) in $C^1_{loc}(\mathbb{R}^n)$. Therefore, $\nabla\overline{u}^K(\mathbb{R}^n) = \nabla\overline{u}^K(\overline{E}) = K$.

Now, for any $i \in \mathbb{N}$ and almost any $x \in \mathbb{R}^n$, the Hessian $\nabla^2\overline{u}^{K_i}$ exists and is controlled by $H_x \in H(x, \nabla\overline{u}^{K_i}, K_i)$. It follows that $\Delta\overline{u}^{K_i} \leq \frac{\mathrm{Per}_K(E)}{|E|}$ (since such an estimate holds for u). Sending $i \to \infty$ we deduce that the same holds for $\overline{u}^K$, thus $\varphi := \overline{u}^K$ satisfies all the requirements.

It remains to drop the regularity assumption on E. Let $(E_i)_{i\in\mathbb{N}} \subseteq \mathbb{R}^n$ be a sequence of *connected* bounded open sets with smooth boundary such that $\mathrm{Per}_K(E_i) \to \mathrm{Per}_K(E)$ and $|E_i \triangle E| \to 0$. In dimension 2, the existence of this sequence is guaranteed by Proposition C.2; in higher dimension we can apply [Mag12, Theorem 13.8] and then notice that the additional requirement of connectedness can be fulfilled adding a finite number of thin pipes between the connected components. For each E_i, let $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ be a function that satisfies the assumptions of the statement.

Without loss of generality, thanks to Arzelà–Ascoli theorem, we can assume that $\varphi_i \to \varphi$ locally in C^1, where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,1}$ convex function such that $\nabla\varphi(\mathbb{R}^n) \subseteq K$ and $\Delta\varphi \leq \frac{\mathrm{Per}_K(E)}{|E|}$. To conclude it is sufficient to prove $|\nabla\varphi(E)| \geq |K|$.

Let $C \subseteq E$ be a compact set. It holds

$$|\nabla\varphi_i(E_i \setminus C)| \leq \mathrm{Lip}(\nabla\varphi_i)^n |E_i \setminus C| \leq \gamma \cdot (|E_i \triangle E| + |E \setminus C|)\,, \tag{4.1}$$

where $\gamma = \gamma(n, K, E) = \left(2n\frac{\mathrm{Per}_K(E)}{|E|}\right)^n$. Since $\nabla\varphi_i(E_i) = K$, it holds $K \setminus \nabla\varphi_i(E_i \setminus C) \subseteq \nabla\varphi_i(C)$ and therefore (4.1) implies

$$|\nabla\varphi_i(C)| \geq |K| - \gamma \cdot (|E_i \triangle E| + |E \setminus C|)\,. \tag{4.2}$$

Take any $y \in \limsup \nabla\varphi_i(C)$ and let $x_i \in C$ be such that $\nabla\varphi_i(x_i) = y$ (we avoid passing to a subsequence for notational simplicity). By compactness we know that $x_i \to x \in C$ and since the convergence $\varphi_i \to \varphi$ is locally in C^1, it follows that $\nabla\varphi(x) = y$. Hence, applying (4.2), we deduce

$$|\nabla\varphi(E)| \geq |\nabla\varphi(C)| \geq |\limsup \nabla\varphi_i(C)| \geq \limsup |\nabla\varphi_i(C)| \geq |K| - \gamma \cdot |E \setminus C|\,.$$

The conclusion now follows as $|E \setminus C|$ can be chosen arbitrarily small. □

Remark 4.2. Notice that the existence of the coupling implies the anisotropic isoperimetric inequality, exactly as in (1.14).

It is now time to construct the convex coupling in the weighted setting. As anticipated, the main idea (taking the convex envelope of the solution of an elliptic problem) is unchanged with respect to Theorem 4.1. On the other hand, the proof that the coupling works for smooth sets/weights and the approximation arguments needed to handle any set/weight are more demanding. Notice that at this level we require $w \equiv 0$ on $\partial\Sigma$; later on we remove this assumption.

Theorem 4.3 (Weighted coupling)**.** *Consider n, α, Σ, w satisfying (2.1) and assume moreover $w \equiv 0$ on $\partial\Sigma$. Let $E \subseteq \Sigma$ be a set of finite w-perimeter with $0 < w(E) < \infty$. If $n = 2$, we assume also that E is w-indecomposable (see Definition C.3). Then, there exists a $C^{1,1}$ convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ that satisfies $\nabla\varphi(\mathbb{R}^n) = \nabla\varphi(E) = B_1 \cap \Sigma$ (up to negligible sets) and*

$$\Delta\varphi + \alpha \left(\frac{w(\nabla\varphi)}{w}\right)^{\frac{1}{\alpha}} \leq \frac{\mathrm{Per}_w(E)}{w(E)}\,. \tag{4.3}$$

Proof. We start by showing that the result holds if E is a bounded open set with smooth boundary such that $\overline{E} \subset\subset \Sigma$ and the weight w is smooth in E. Let $b_E = \mathrm{Per}_w(E)/w(E)$ and let $K := \overline{B_1 \cap \Sigma}$.

Let $\nu : \partial E \to \mathbb{R}^n$ be the outer normal to the boundary. As in [CRS16], let us consider the following Neumann problem for $u : E \to \mathbb{R}$

$$\begin{cases} \mathrm{div}(w\nabla u) = w\, b_E & \text{in } E \\ \partial_\nu u = 1 & \text{on } \partial E\,. \end{cases} \tag{4.4}$$

Since the compatibility condition $\int_E \mathrm{div}(w\nabla u) = \int_{\partial E} w\partial_\nu u$ is satisfied, the problem admits a solution.

With the exact same reasoning adopted in the proof of Theorem 4.1, we deduce that for any $\xi \in \mathring{K}$, the minimum of $u(x) - \xi \cdot x$ cannot be attained on the boundary of E. Thence $\overline{u}^K \in C^{1,1}$ and $\nabla\overline{u}^K(\mathbb{R}^n) = \nabla\overline{u}^K(\overline{E}) = K$.

Let $K_r := \overline{\Sigma \cap B_r}$. Since $K_r \nearrow K$ as $r \nearrow 1$, then it holds $\overline{u}^{K_r} \nearrow \overline{u}^K$ locally uniformly as $r \nearrow 1$. Notice that $K_r \not\subseteq \mathring{K}$, so we cannot apply directly Proposition 3.10. Our goal is to prove (4.3) with $\varphi = \overline{u}^{K_r}$ for any $0 < r < 1$. We consider two cases, depending on whether $\nabla\overline{u}^{K_r}$ belongs to the boundary or to the interior of Σ.

If $\nabla\overline{u}^{K_r}(x) \in \partial\Sigma$, since $w(\partial\Sigma) = 0$, it is sufficient to show that $\Delta\overline{u}^{K_r}(x) \leq b_E$. Applying (4.4) and Lemma 2.1, we have $\Delta u(x) \leq b_E$ at all points $x \in E$ such that $\nabla u(x) \in \Sigma$. Then the same inequality for $\overline{u}^{K_r}$ follows repeating the approximation argument contained at the beginning of the proof of Theorem 4.1 (recall that we cannot apply directly Proposition 3.10).

On the other hand, if $\xi := \nabla\overline{u}^{K_r}(x) \notin \partial\Sigma$, then it belongs to $\mathring{K}$ and we can apply Proposition 3.10 (see Remark 3.11). Thus $\nabla^2\overline{u}^{K_r}(x) \leq H_x \in H(x, \xi, K_r)$. Let $H_x = \sum_{i=1}^m \lambda_i \nabla^2 u(s_i)$ with $s_i \in S_\xi$ and $x - \sum \lambda_i s_i = v \in N(\xi, K_r)$. Notice that, since Σ is convex, $x - v \in \Sigma$. Applying Lemma 2.1 we obtain

$$\begin{aligned} \alpha \left(\frac{w(\xi)}{w(x-v)}\right)^{\frac{1}{\alpha}} &\leq \alpha w(\xi)^{\frac{1}{\alpha}} \sum_{i=1}^m \lambda_i w(s_i)^{-\frac{1}{\alpha}} = \sum_{i=1}^m \lambda_i \alpha \left(\frac{w(\xi)}{w(s_i)}\right)^{\frac{1}{\alpha}} \\ &\leq \sum_{i=1}^m \lambda_i \frac{\nabla w(s_i) \cdot \xi}{w(s_i)} = \sum_{i=1}^m \lambda_i \frac{\nabla w(s_i) \cdot \nabla u(s_i)}{w(s_i)}\,. \end{aligned} \tag{4.5}$$

Since u is a solution of (4.4), it holds

$$\Delta u(s_i) + \frac{\nabla w(s_i) \cdot \nabla u(s_i)}{w(s_i)} = b_E\,. \tag{4.6}$$

Combining (4.5) and (4.6), we deduce

$$\sum_{i=1}^m \lambda_i \Delta u(s_i) + \alpha \left(\frac{w(\xi)}{w(x-v)}\right)^{\frac{1}{\alpha}} \leq b_E\,,$$

and thus

$$\nabla^2 \overline{u}^{K_r}(x) + \alpha \left(\frac{w(\nabla \overline{u}^{K_r})}{w(x-v)} \right)^{\frac{1}{\alpha}} \leq b_E \, .$$

This latter inequality is *almost* the desired one, but for the presence of $w(x-v)$ instead of $w(x)$. To fix this issue we notice that if $\xi \in \mathring{K}_r$ then $v = 0$. Otherwise $\xi \in \partial K_r \cap \mathring{K}$ and therefore $\xi \in \Sigma \cap \partial B_r$ and $N(\xi, K_r) = \{\xi\}$. Hence $v = \xi \in \Sigma$ and, recalling once again Lemma 2.1, it holds $w(x-v) \leq w(x)$.

We managed to prove (4.3) for $\varphi = \overline{u}^{K_r}$ and since $\overline{u}^{K_r} \to \overline{u}^{K}$ locally in C^1 as $r \nearrow 1$, it follows that the same is true also for $\overline{u}^{K}$.

It remains to drop the regularity assumption on E and w. First, we drop the regularity assumption on w. Thanks to Lemma B.2 we can find a sequence of admissible weights $w_i \to w$ that are smooth in E and that converge to w locally uniformly. For any such w_i we can find $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ such that the statement holds. In particular the family φ_i is bounded in $C^{1,1}$ and therefore, up to subsequence, converges locally in C^1 to $\varphi : \mathbb{R}^n \to \mathbb{R}$. It is not hard to check that φ satisfies all the desired properties.

Finally, the method used in the last part of the proof of Theorem 4.1 can be applied also here to remove the assumption that E is compactly contained in Σ and has a smooth boundary (applying Proposition C.5 instead of Proposition C.2). Notice that in this final step we need $\mathrm{Per}_w(E_i) \to \mathrm{Per}_w(E)$, with $E_i \subset\subset \Sigma$, which holds since $w \equiv 0$ on $\partial\Sigma$. □

The next proposition is fundamental for the proof of the stability of the isoperimetric inequality; it contains all the properties of the coupling that we will need. In this proposition we drop the additional assumption $w \equiv 0$ on $\partial\Sigma$ that was necessary for Theorem 4.3.

Proposition 4.4. *Consider n, α, Σ, w satisfying (2.1) and let $E \subseteq \Sigma$ be a set of finite w-perimeter with $w(E) = 1$ and $\delta_w(E) \leq 1$. If $n = 2$, we assume also that E is w-indecomposable (see Definition C.3). There is a $C^{1,1}$ convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\|\nabla^2 \varphi\|_\infty \lesssim 1$, $\nabla\varphi(\mathbb{R}^n) = \nabla\varphi(E) = \Sigma \cap B_1$ (up to negligible sets), and*

$$\int_E |\nabla^2 \varphi - \mathbb{1}| \, w \lesssim \delta_w(E)^{\frac{1}{2}} \, , \tag{4.7}$$

$$\int_{\partial^* E \cap \Sigma} \big(1 - |\nabla\varphi|\big) \, w \, \mathrm{d}\mathscr{H}^{n-1} \lesssim \delta_w(E) \, . \tag{4.8}$$

Moreover, for any $Q \subset\subset \Sigma$, we have

$$\int_{E \cap Q} \left| w(\nabla\varphi)^{\frac{1}{\alpha}} - w^{\frac{1}{\alpha}} \right| \leq C \delta_w(E)^{\frac{1}{2}} \, , \tag{4.9}$$

where C is a constant depending only on n, α, Σ, and Q.

Proof. First, assuming that $w \equiv 0$ on $\partial\Sigma$, we prove that the map built in Theorem 4.3 satisfies all the constraints. Then we remove the additional assumption by approximation.

Let us assume $w(\partial\Sigma) = 0$ and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be the map built in Theorem 4.3.

Applying the area formula, the arithmetic-geometric mean inequality, and the properties of φ described in Theorem 4.3, we get

$$\begin{aligned} 1 = w(B_1 \cap \Sigma) = w(\nabla\varphi(E)) &\leq \int_E \det(\nabla^2\varphi) w(\nabla\varphi) = \int_E \det(\nabla^2\varphi) \Big(\frac{w(\nabla\varphi)}{w}\Big) w \\ &\leq \int_E \Big(\frac{\Delta\varphi}{n}\Big)^n \Big[\Big(\frac{w(\nabla\varphi)}{w}\Big)^{\frac{1}{\alpha}}\Big]^\alpha w \leq \int_E \left(\frac{\Delta\varphi + \alpha\big(\frac{w(\nabla\varphi)}{w}\big)^{\frac{1}{\alpha}}}{D}\right)^D w \leq \int_E \Big(\frac{\mathrm{Per}_w(E)}{D}\Big)^D w \\ &= \big(1 + \delta_w(E)\big)^D \, , \end{aligned} \tag{4.10}$$

where we used $s^n t^\alpha \leq \big(\frac{ns+\alpha t}{n+\alpha}\big)^{n+\alpha}$. Thus

$$\int_E \left(\Big(\frac{\mathrm{Per}_w(E)}{D}\Big)^D - \det(\nabla^2\varphi)\Big(\frac{w(\nabla\varphi)}{w}\Big) \right) w \leq \big(1 + \delta_w(E)\big)^D - 1 \lesssim \delta_w(E) \, . \tag{4.11}$$

Applying Lemma A.1 with $\lambda = (\overbrace{1,\dots,1}^{n},\alpha)$, $(x_1,\dots,x_n)$ equal to the eigenvalues of $\nabla^2\varphi$, $x_{n+1} = (\frac{w(\nabla\varphi)}{w})^{\frac{1}{\alpha}}$, and $c = \frac{\mathrm{Per}_w(E)}{D}$, we obtain

$$|\nabla^2\varphi - \mathbb{1}|^2 + \left|\Big(\frac{w(\nabla\varphi)}{w}\Big)^{\frac{1}{\alpha}} - 1\right|^2 \lesssim \Big(\frac{\mathrm{Per}_w(E)}{D}\Big)^D - \det(\nabla^2\varphi)\Big(\frac{w(\nabla\varphi)}{w}\Big)\,. \tag{4.12}$$

Joining (4.11) and (4.12) we obtain

$$\int_E |\nabla^2\varphi - \mathbb{1}|^2 w + \int_E \left|\Big(\frac{w(\nabla\varphi)}{w}\Big)^{\frac{1}{\alpha}} - 1\right|^2 w \lesssim \delta_w(E)\,.$$

This implies (using Cauchy-Schwarz) (4.7) and

$$\int_E \left|\Big(\frac{w(\nabla\varphi)}{w}\Big)^{\frac{1}{\alpha}} - 1\right| w \lesssim \delta_w(E)^{\frac12}\,.$$

The weight w is bounded in Q from below and above by constants that depend only on n,α,Σ,Q because $w^{\frac{1}{\alpha}}$ is concave and $w(B_1\cap\Sigma)=1$. Thus, the last estimate implies (4.9).

Proceeding as in (4.10), applying Lemma 2.1 and the divergence theorem, we obtain

$$\begin{aligned}
1 &\le \int_E \left(\frac{\Delta\varphi + \alpha\big(\frac{w(\nabla\varphi)}{w}\big)^{\frac{1}{\alpha}}}{D}\right)^D w \le \frac{\mathrm{Per}_w(E)^{D-1}}{D^D}\int_E \left(\Delta\varphi + \alpha\Big(\frac{w(\nabla\varphi)}{w}\Big)^{\frac{1}{\alpha}}\right) w \\
&\le \frac{\mathrm{Per}_w(E)^{D-1}}{D^D}\int_E \mathrm{div}(w\nabla\varphi) = \frac{\mathrm{Per}_w(E)^{D-1}}{D^D}\int_{\partial^* E} (\nabla\varphi\cdot\nu_{\partial^*E}) w\,\mathrm{d}\mathscr{H}^{n-1} \\
&\le \frac{\mathrm{Per}_w(E)^{D}}{D^D} - \frac{\mathrm{Per}_w(E)^{D-1}}{D^D}\int_{\partial^* E} (1-|\nabla\varphi|) w\,\mathrm{d}\mathscr{H}^{n-1}\,,
\end{aligned}$$

where in the last step we applied $\nabla\varphi\cdot\nu \le |\nabla\varphi| \le 1$ (recall that $\nabla\varphi\in B_1\cap\Sigma$). The estimate (4.8) follows rearranging the terms.

It remains to drop the assumption $w(\partial\Sigma)=0$. Thanks to Lemma B.3 we can find a sequence $(w_i)_{i\in\mathbb{N}}$ of α-homogeneous weights such that $w_i\equiv 0$ on $\partial\Sigma$, and $w_i\to w$ uniformly on compact subsets of Σ. Let $\varphi_i:\mathbb{R}^n\to\mathbb{R}$ be the map built in Theorem 4.3 when the weight is w_i.

Notice that $\delta_{w_i}\to\delta_w$ as $i\to\infty$ and therefore the family $(\varphi_i)_{i\in\mathbb{N}}$ is bounded in $C^{1,1}$. Up to subsequence, we can assume that $\varphi_i\to\varphi$ locally in C^1, where $\varphi:\mathbb{R}^n\to\mathbb{R}$ is a $C^{1,1}$-function. Following the method adopted in the proof of Theorem 4.3, we can establish $\nabla\varphi(\mathbb{R}^n) = \nabla\varphi(E) = \Sigma\cap B_1$. The properties (4.7) and (4.8) follow directly from the convergence of weights and maps. On the other hand, passing to the limit in (4.9), we need to avoid $\nabla\varphi\in\partial\Sigma$ because in that case $w_i(\nabla\varphi_i)$ may not converge to $w(\nabla\varphi)$. Hence, we obtain

$$\int_{E\cap Q\cap\{\nabla\varphi\notin\partial\Sigma\}} \left|w(\nabla\varphi)^{\frac{1}{\alpha}} - w^{\frac{1}{\alpha}}\right| \le C\delta_w(E)^{\frac12}\,. \tag{4.13}$$

The area formula for Lipschitz functions implies $\nabla^2\varphi$ is singular almost everywhere in $\{\nabla\varphi\in\partial\Sigma\}$, in particular $|\nabla^2\varphi-\mathbb{1}|\gtrsim 1$. Therefore (4.7) implies

$$w\big(Q\cap\{\nabla\varphi\in\partial\Sigma\}\big)\lesssim\delta_w(E)^{\frac12}\,,$$

and this is sufficient to deduce (4.9) from (4.13). □

5. Ruling out translational freedom

Recall that we assume $\Sigma = \mathbb{R}^k\times\widetilde{\Sigma}$ where $\widetilde{\Sigma}\subseteq\mathbb{R}^{n-k}$ is an open convex cone containing no lines. Moreover, we can split $\mathbb{R}^{n-k}$ as $\mathbb{R}^h\times\mathbb{R}^{n-k-h}$, where $\mathbb{R}^h$ identifies the directions along which w is constant, namely, the vectors ξ such that $w(x+t\xi)=w(x)$ whenever both sides make sense ($x\in\Sigma$, $t\in\mathbb{R}$).

Let

$$\begin{aligned}
&\mathcal{L} := \mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\} \text{ be the subspace of } \mathcal{L}\text{ines contained in } \Sigma,\\
&\mathcal{C} := \{0_{\mathbb{R}^k}\} \times \mathbb{R}^h \times \{0_{\mathbb{R}^{n-k-h}}\} \text{ be the subspace of direction of } \mathcal{C}\text{onstancy for } w,\\
&\mathcal{E} := \{0_{\mathbb{R}^{h+k}}\} \times \mathbb{R}^{n-k-h} \text{ be everything } \mathcal{E}\text{lse.}
\end{aligned} \tag{5.1}$$

Notice that if $\mathcal{C} \neq \{0_{\mathbb{R}^n}\}$ then $w \not\equiv 0$ on $\partial\Sigma$.

To familiarize the reader with these subspaces, let us describe what they are in the three cases of Fig. 1:

(1) $\mathcal{L} = \{(0,0)\}$, $\mathcal{C} = \{(0,t) : t \in \mathbb{R}\}$, $\mathcal{E} = \{(t,0) : t \in \mathbb{R}\}$, $x_0 \in \mathcal{C} \cap \Sigma$.
(2) $\mathcal{L} = \{(0,0)\}$, $\mathcal{C} = \{(0,0)\}$, $\mathcal{E} = \mathbb{R}^2$, $x_0 \in \mathcal{E}$.
(3) $\mathcal{L} = \{(0,0)\}$, $\mathcal{C} = \{(0,t) : t \in \mathbb{R}\}$, $\mathcal{E} = \{(t,0) : t \in \mathbb{R}\}$, $x_0 \in \mathcal{C} \setminus \Sigma$.

Proposition 5.3 corresponds to the strategy (2) described in the introduction to prove $|\widetilde{x}_0| \leq C\delta_w(E)^{\frac{1}{2}}$. Lemma 5.4 implements the strategy (1). Finally, (1) and (3) are condensed in Proposition 5.5.

Note that in this section all the statements involve only the cone Σ and the weight w, there is nothing about sets with small isoperimetric deficit.

We will need the following two technical lemmas.

Lemma 5.1. *Let $\eta : \mathbb{R} \to \mathbb{R}$ be any function. Then for any $a < b$ and $\varepsilon > 0$ it holds*

$$\int_a^b |\eta(t+\varepsilon) - \eta(t)|\,\mathrm{d}t \geq \varepsilon\Big(\inf_{|t-b|\leq\varepsilon} \eta(t) - \sup_{|t-a|\leq\varepsilon} \eta(t)\Big)\,.$$

Proof. Let $m = \lfloor \frac{|a-b|}{\varepsilon} \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x. The triangle inequality implies

$$\int_a^b |\eta(t+\varepsilon) - \eta(t)|\,\mathrm{d}t \geq \int_0^\varepsilon \sum_{i=1}^m |\eta(a+t+i\varepsilon) - \eta(a+t+(i-1)\varepsilon)|\,\mathrm{d}t \geq \int_0^\varepsilon |\eta(a+t+m\varepsilon) - \eta(a+t)|\,\mathrm{d}t\,,$$

and the desired inequality follows since $|a+t+m\varepsilon - b| \leq \varepsilon$ for any $0 \leq t \leq \varepsilon$. □

The second one reads as follows.

Lemma 5.2. *Given $f \in BV_{loc}(\mathbb{R}^n)$ and $\eta \in L^\infty(\mathbb{R}^n)$ with compact support, the function $\psi : \mathbb{R}^n \to \mathbb{R}$ defined as*

$$\psi(\xi) := \int_{\mathbb{R}^n} \eta(x) f(x+\xi)\,\mathrm{d}x$$

is locally Lipschitz continuous, hence differentiable almost everywhere, and its gradient satisfies (almost everywhere)

$$\nabla\psi(\xi) = \int_{\mathbb{R}^n} \eta(x-\xi)\,\mathrm{d}\widetilde{\nabla} f(x)\,.$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a smooth function with compact support. From multiple applications of Fubini's theorem and the divergence theorem for BV functions (2.2), it follows

$$\int_{\mathbb{R}^n} \nabla\varphi(\xi)\psi(\xi)\,\mathrm{d}\xi = -\int_{\mathbb{R}^n} \varphi(\xi)\Big(\int_{\mathbb{R}^n} \eta(x-\xi)\,\mathrm{d}\widetilde{\nabla} f(x)\Big)\,\mathrm{d}\xi\,.$$

Since $\int_{\mathbb{R}^n} \eta(x-\xi)\,\mathrm{d}\widetilde{\nabla} f(x)$ is locally bounded, ψ is locally Lipschitz continuous and $\int_{\mathbb{R}^n} \eta(x-\xi)\,\mathrm{d}\widetilde{\nabla} f(x)$ is its weak gradient (hence its classical gradient almost everywhere). □

The following proposition proves (a technical version of) (1.13), which is employed to rule out translations in the directions along which w is not constant.

Proposition 5.3. *Consider n, α, Σ, w satisfying (2.1), and let $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{E}$ be as in (5.1). There exists an $\hat{\varepsilon} = \hat{\varepsilon}(n,\alpha,\Sigma,w) > 0$ and a compact set $\hat{Q} \subseteq B_{\frac{1}{2}} \cap \Sigma$ (that depends on n,α,Σ,w) such that*

(1) $d(\hat{Q}, \partial\Sigma) > 2\hat{\varepsilon}$,

(2) For any $\xi \in B_{\hat\varepsilon}$ it holds

$$\int_{\hat Q} |w^{\frac{1}{\alpha}}(x+\xi) - w^{\frac{1}{\alpha}}(x)|\,\mathrm{d}x \geq \hat\varepsilon |\pi_{\mathcal{E}}(\xi)|\,,$$

where $\pi_{\mathcal{E}} : \mathbb{R}^n \to \mathcal{E}$ is the orthogonal projection onto $\mathcal{E}$.

Proof. Given a compact set $Q \subseteq \Sigma$, an $\varepsilon > 0$ and a subset $U \subseteq \partial B_1 \cap \mathcal{E}$ we say that the pair (U, ε) is compatible with the compact set Q if $d(Q, \partial\Sigma) > 2\varepsilon$ and

$$\int_{Q} |w^{\frac{1}{\alpha}}(x+\xi) - w^{\frac{1}{\alpha}}(x)|\,\mathrm{d}x \geq \varepsilon |\xi|$$

for any $\xi \in B_\varepsilon$ such that $\xi/|\xi| \in U$

Given two points $p_1, p_2 \in B_{\frac{1}{2}} \cap \Sigma$ and $r > 0$, let us define the compact set

$$Q(p_1, p_2, r) := \left\{ x \in \Sigma \cap \overline{B}_{\frac{1}{2}} :\ d(x, \partial\Sigma) \geq r,\ d(x, [p_1, p_2]) \leq r \right\},$$

where $[p_1, p_2]$ denotes the segment between p_1 and p_2.

Given a direction $\theta \in \partial B_1 \cap \mathcal{E}$, take two points $p_\theta, p'_\theta \in B_{\frac{1}{2}} \cap \Sigma$ such that $w(p_\theta) \neq w(p'_\theta)$ and $p_\theta - p'_\theta$ is a multiple of θ (the existence of the two points follows from $\theta \in \mathcal{E}$). Applying Lemma 5.1, we can find a small $r_\theta > 0$, a small $\varepsilon_\theta > 0$ and a neighborhood $U_\theta \subseteq \partial B_1 \cap \mathcal{E}$ of θ such that $(\varepsilon_\theta, U_\theta)$ is compatible with $Q(p_\theta, p'_\theta, r_\theta)$. Since $\partial B_1 \cap \mathcal{E}$ is compact, we can find $U_{\theta_1}, \dots, U_{\theta_k}$, with $k \in \mathbb{N}$, that cover $\partial B_1 \cap \mathcal{E}$. Hence, it is not hard to check that $\hat\varepsilon := \min(\varepsilon_{\theta_1}, \dots, \varepsilon_{\theta_k})$ and

$$\hat Q := \bigcup_{i=1}^{k} Q(p_{\theta_i}, p'_{\theta_i}, r_{\theta_i})$$

satisfy (1) and (2) with the additional constraint $\xi \in \mathcal{E}$. The whole statement follows because the left-hand side of (2) does not change if we replace ξ with $\pi_{\mathcal{E}}(\xi)$. □

The next lemma proves (1.12), which is employed to rule out translations along $\Sigma \cap \mathcal{C}$.

Lemma 5.4. *Consider n, α, Σ, w satisfying (2.1), and let $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{E}$ be as in (5.1). Then, there is a small constant $c_1 = c_1(n, \alpha, \Sigma, w) > 0$ such that*

$$w(B_1(\xi) \cap \Sigma) \geq w(B_1 \cap \Sigma) + c_1 |\xi|\,,$$

for all $\xi \in B_{c_1} \cap \mathcal{C}$ satisfying $d(\frac{\xi}{|\xi|}, \Sigma) < c_1$.

Proof. For any $\xi \in \mathcal{C}$ and any $x \in \Sigma$ with $x + \xi \notin \Sigma$, let us define $w(x+\xi) := w(x)$ (we are extending the domain of w exploiting its constancy in certain directions). The definition is consistent because of the condition $\xi \in \mathcal{C}$.

Notice that, for any $\xi \in \mathcal{C}$, it holds

$$w(B_1(\xi) \cap \Sigma) = \int_{B_1} \chi_\Sigma(x+\xi) w(x+\xi)\,\mathrm{d}x = \int_{B_1} w(x) \chi_\Sigma(x+\xi)\,\mathrm{d}x\,.$$

Applying Lemma 5.2 with $f = \chi_\Sigma$ and $\eta = w\,\chi_{B_1}$, we deduce that for $\xi \in \mathcal{C}$ it holds

$$w(B_1(\xi) \cap \Sigma) - w(B_1 \cap \Sigma) \geq -\overline{\nu} \cdot \xi - \mathrm{o}(|\xi|)\,, \tag{5.2}$$

where

$$\overline{\nu} := \int_{\partial\Sigma \cap B_1} \nu_{\partial\Sigma}\, w\, \mathrm{d}\mathscr{H}^{n-1}$$

and $\nu_{\partial\Sigma}$ is the *outer* normal to $\partial\Sigma$. If $\xi \in \mathcal{C} \cap \Sigma$, it holds $\nu_\Sigma \cdot \xi < 0$ at any point and therefore $\overline{\nu} \cdot \xi < 0$ (recall that if $\mathcal{C} \neq \{0\}$ then $w \not\equiv 0$ on $\partial\Sigma$). The same strict inequality holds also if $\xi \in \mathcal{C} \cap \partial\Sigma \setminus \{0\}$ (because by definition $\mathcal{C} \cap \Sigma$ does not contain lines). Hence, there is a small $c > 0$ such that for any $\xi \in \mathcal{C} \cap \overline{\Sigma}$ it holds

$$\overline{\nu} \cdot \xi \leq -c|\xi|\,. \tag{5.3}$$

The two estimates (5.2) and (5.3), together with a simple continuity argument, yield the desired result. □

Finally, in the following proposition, we prove a dichotomy: either we have (1.12) and so we can perform strategy (1) or we have the tools (namely the set $S \subseteq \langle\xi\rangle^\perp$ built in the statement) to perform strategy (3).

Proposition 5.5. *Consider n, α, Σ, w satisfying (2.1), and let $\mathcal{L}$, $\mathcal{C}$, and $\mathcal{E}$ be as in (5.1). There is a small constant $c = c(n, \alpha, \Sigma, w) > 0$ such that for any $\xi \in B_c \cap \mathcal{C}$ at least one of the following two statements is true:*

(1) It holds $|w(\Sigma \cap B_1(\xi)) - w(\Sigma \cap B_1)| \geq c|\xi|$.

(2) There exists a set $S \subseteq \langle\xi\rangle^\perp$ such that, setting $t_0(s)$ and $t_1(s)$ so that $\Sigma_s := \Sigma \cap \{s + t\xi : t \in \mathbb{R}\} = \{s + t\frac{\xi}{|\xi|} : \ t_0(s) < t < t_1(s)\}$, the following hold.
(a) For any $s \in S$, it holds $-\infty < t_0(s) < t_0(s) + c < t_1(s) < +\infty$.
(b) For any $s \in S$ and any $z \in \Sigma_s$, it holds $w(z) > c$ (notice that w is constant on Σ_s).
(c) It holds $\mathscr{H}^{n-1}(S) > c$.
(d) For any $s \in S$ and any $t < t_0(s)$ it holds $d(s + t\frac{\xi}{|\xi|}, \Sigma) > c(t_0 - t)$.
(e) For any $s \in S$ and any $t > t_1(s)$ it holds $d(s + t\frac{\xi}{|\xi|}, \Sigma) > c(t - t_1)$.
(f) For any $s \in S$, $\Sigma_s \subseteq B_{\frac{1}{2}} \cap \Sigma$.

Proof. Let c_1 be the constant present in the statement of Lemma 5.4. If $d(\xi, \Sigma) < c_1|\xi|$ or $d(\xi, -\Sigma) < c_1|\xi|$, then Lemma 5.4 implies that (1) holds (choosing c smaller then c_1).

We are going to show that if $d(\xi, \Sigma) > c_1|\xi|$ and $d(\xi, -\Sigma) > c_1|\xi|$, then we can find a set S satisfying the requirements of (2). Given that ξ is *far* from Σ and $-\Sigma$, the properties (d) and (e) are satisfied independently of the choice of S.

Fix an element $v \in \Sigma$ and a small real number $l > 0$. Let S be the image of the projection onto $\langle\xi\rangle^\perp$ of $B_{l^2}(lv)$. Up to choosing c and l sufficiently small, the properties (b) and (c) are satisfied.

The property (f) follows from the observation that for any $s \in S$ it holds $\Sigma_s \cap B_{2l} \neq \emptyset$. Similarly, the property (a) follows from the observation that for any $s \in S$ there is a point $z \in \Sigma_s$ such that $d(z, \partial\Sigma) > \varepsilon_0 l$, where $\varepsilon_0 > 0$ is a small constant that depends only on the chosen vector v. □

6. Spherical boundary implies ball

The goal of this section is to prove the following.

Proposition 6.1. *Consider n, α, Σ, w satisfying (2.1) and let $E \subseteq \Sigma$ be a set of finite w-perimeter with $w(E \cap B_{\frac{1}{2}}) \geq \frac{1}{2} w(B_{\frac{1}{2}} \cap \Sigma)$. Then, we have*

$$w\big(E \triangle (B_1(x_0) \cap \Sigma)\big) \lesssim \int_{\partial^* E \cap \Sigma} ||x - x_0| - 1| w(x) \, \mathrm{d}\mathscr{H}^{n-1}(x) \,,$$

for any $x_0 \in \mathbb{R}^n$ so that $|x_0|$ is sufficiently small (the required smallness depends only on n, α, Σ, w and not on the set E).

The strategy is to obtain first a robust analogous result in 1-dimension (that is Lemma 6.2) and deduce the full statement through a polar slicing.

Lemma 6.2. *For any $\gamma \geq 0$, there is a constant $C_\gamma > 0$ such that the following statement holds.*

Let $E \subseteq [0, \infty)$ be a 1-dimensional set of locally finite perimeter with $|E| < \infty$. For any $1 - \frac{1}{4} \leq l \leq 1 + \frac{1}{4}$ it holds

$$\int_{E \triangle [0, l]} t^\gamma \, \mathrm{d}t \leq C_\gamma \left(\int_{[0, \frac{1}{2}] \setminus E} t^\gamma \, \mathrm{d}t + \int_{\partial^* E} t^\gamma |l - t| \, \mathrm{d}\mathscr{H}^0(t) \right) . \tag{6.1}$$

Proof. For simplicity, we prove it only for $l = 1$; the proof works (up to slightly increasing the value of the constant C_γ) also for any other $1 - \frac{1}{4} \leq l \leq 1 + \frac{1}{4}$. It holds $E \triangle [0, 1] = \big((1, \infty) \cap E \big) \cup \big([0, 1] \setminus E \big)$.

We will estimate independently the two terms

$$\int_{[1, \infty) \cap E} t^\gamma \, \mathrm{d}t \quad \text{and} \quad \int_{[\frac{1}{2}, 1] \setminus E} t^\gamma \, \mathrm{d}t \,.$$

Notice that if $[1,\infty)\cap E=\emptyset$, then the first term is trivially estimated. Otherwise $\partial^* E\cap[1,\infty)$ is nonempty and must have a supremum point, that we denote by t_1 (we may assume $t_1<\infty$, so that the right-hand side in (6.1) is finite). It holds

$$\int_{[1,\infty)\cap E} t^\gamma \,\mathrm{d}t \le \int_1^{t_1} t^\gamma \,\mathrm{d}t \le t_1^\gamma |t_1-1| \le \int_{\partial^* E} t^\gamma |1-t|\,\mathrm{d}\mathscr{H}^0(t)\,,$$

hence we have successfully controlled the first term.

Let us now move our attention to the second term. First notice that its value is a priori bounded by 1. If $\partial^* E\cap\left[\frac14,\frac34\right]\neq\emptyset$, then the right-hand side of (6.1) is at least $C_\gamma\left(\frac14\right)^{\gamma+1}$ and therefore we have the desired estimate (choosing C_γ appropriately).

Thus we can assume $\partial^* E\cap\left[\frac14,\frac34\right]=\emptyset$. If $\left[\frac14,\frac34\right]\setminus E\neq\emptyset$, then (since E has no boundary in that interval) it follows $\left[\frac14,\frac34\right]\cap E=\emptyset$ and in particular the right-hand side of (6.1) is larger than

$$C_\gamma\int_{\frac14}^{\frac34} t^\gamma\,\mathrm{d}t\,,$$

which yields the desired estimate.

Thus we can assume $\left[\frac14,\frac34\right]\subseteq E$. If $\partial^* E\cap\left[\frac14,1\right]=\emptyset$, then $\left[\frac12,1\right]\subseteq E$ and therefore there is nothing to prove. Otherwise, let t_0 be the infimum of $\partial^* E\cap\left[\frac14,1\right]$. It holds

$$\int_{\left[\frac12,1\right]\setminus E} t^\gamma\,\mathrm{d}t \le \int_{t_0}^1 t^\gamma\,\mathrm{d}t \le 4^\gamma t_0^\gamma|1-t_0| \le 4^\gamma\int_{\partial^* E} t^\gamma|1-t|\,\mathrm{d}\mathscr{H}^0(t)\,,$$

and this concludes the proof. □

To perform the polar slicing we will need the following technical lemma.

Lemma 6.3. *Let Σ be an open cone and let $E\subseteq\Sigma$ be a measurable set such that $\mathrm{Per}(E,\Omega)<\infty$ for any $\Omega\subset\subset\Sigma$. For any $\theta\in\mathbb{S}^{n-1}\cap\Sigma$, let us define*

$$E_\theta := \{t\ge 0:\ t\theta\in E\}\,.$$

Then, for any $\eta\in L^1(E)$, we have

$$\int_E \eta = \int_{\mathbb{S}^{n-1}\cap\Sigma}\mathrm{d}\mathscr{H}^{n-1}(\theta)\int_{E_\theta} t^{n-1}\eta(t\theta)\,\mathrm{d}t\,.$$

Moreover, for $\mathscr{H}^{n-1}$-almost every $\theta\in\mathbb{S}^{n-1}\cap\Sigma$, $E_\theta\subseteq\mathbb{R}$ is a 1-dimensional set of locally finite perimeter such that the Vol'pert property $\partial^* E_\theta\cap\{t>0\}=\{t>0:\ t\theta\in\partial^* E\}$ holds. Furthermore, for any nonnegative function $\eta\in L^1(\partial^* E,\mathscr{H}^{n-1})$, we have

$$\int_{\partial^* E\cap\Sigma}\eta \ge \int_{\mathbb{S}^{n-1}\cap\Sigma}\mathrm{d}\mathscr{H}^{n-1}(\theta)\int_{\partial^* E_\theta} t^{n-1}\eta(t\theta)\,\mathrm{d}\mathscr{H}^0(t)\,.$$

Proof. The proof is standard and technical, we give only a sketch. The first part of the statement follows from the coarea formula ([Mag12, Theorem 13.1]), whereas the second part can be shown applying [AFP00, Theorem 3.107] and the area formula for rectifiable sets ([Mag12, Theorem 11.6]). □

Using the previous two results, we can now give the:

Proof of Proposition 6.1. The strategy of the proof is to perform a polar slicing of the set E and apply Lemma 6.2 on each slice.

Using Lemma 6.3, we obtain

$$\int_{\partial^* E\cap\Sigma}||x-x_0|-1|w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x) \ge \int_{\mathbb{S}^{n-1}\cap\Sigma} w(\theta)\,\mathrm{d}\mathscr{H}^{n-1}(\theta)\int_{\partial^* E_\theta} t^{D-1}||t\theta-x_0|-1|\,\mathrm{d}\mathscr{H}^0(t)\,.$$

Keeping in mind that x_0 is a small vector, we have that the set $\{t>0:\ |t\theta-x_0|<1\}$ is an open segment $(0,t(\theta))$, with $t(\theta)$ close to 1, for every $\theta\in\mathbb{S}^{n-1}$. In addition, for any $\theta\in\mathbb{S}^{n-1}$ and $t>0$, we have $||t\theta-x_0|-1|\gtrsim|t-t(\theta)|$, where the hidden constant is purely geometric. Hence we obtain

$$\int_{\mathbb{S}^{n-1}\cap\Sigma} w(\theta)\,\mathrm{d}\mathscr{H}^{n-1}(\theta)\int_{\partial^* E_\theta} t^{D-1}|t-t(\theta)|\,\mathrm{d}\mathscr{H}^0(t)\lesssim\int_{\partial^* E\cap\Sigma}||x-x_0|-1|w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,. \tag{6.2}$$

Applying Lemma 6.3 and the relative isoperimetric inequality, Corollary 7.7, to the set $F:=\big(B_{\frac12}\cap\Sigma\big)\setminus E$, we deduce

$$\begin{aligned}\int_{\mathbb{S}^{n-1}\cap\Sigma} w(\theta)\,\mathrm{d}\mathscr{H}^{n-1}(\theta)\int_{[0,\frac12]\setminus E_\theta} t^{D-1}\,\mathrm{d}t &= w\big(\big(B_{\frac12}\cap\Sigma\big)\setminus E\big)\\ &\lesssim\int_{\partial^* E\cap B_{\frac12}\cap\Sigma} w\,\mathrm{d}\mathscr{H}^{n-1}\lesssim\int_{\partial^* E\cap\Sigma}||x-x_0|-1|w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,.\end{aligned} \tag{6.3}$$

Here we used that $||x-x_0|-1|\gtrsim 1$ in $B_{\frac12}$.

Using Lemma 6.3, the estimates (6.2) and (6.3), and Lemma 6.2, we conclude

$$w\big(E\triangle(B_1(x_0)\cap\Sigma)\big)=\int_{\mathbb{S}^{n-1}\cap\Sigma} w(\theta)\,\mathrm{d}\mathscr{H}^{n-1}(\theta)\int_{E_\theta\triangle[0,t(\theta)]} t^{D-1}\,\mathrm{d}t\lesssim\int_{\partial^* E\cap\Sigma}||x-x_0|-1|w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,.$$

□

7. Weighted trace and Sobolev-Poincaré inequalities

In this section we establish a trace inequality and a Sobolev–Poincaré inequality in the weighted setting and within a cone, as well as a uniform bound on the Cheeger constant for sets that have small weighted isoperimetric deficit. Essentially, we repeat the arguments of [FMP10, Section 3], taking care of the presence of the weight w and of the cone Σ. The only additional result contained here is the Sobolev-Poincaré inequality (7.4). Both the statements and the proofs in this section are simple adaptations of the analogous ones in [FMP10]. Unexpectedly, the constants in the statements of Lemma 7.5 and Theorem 7.6 depend only on $D=n+\alpha$ and not on Σ or w.

Remark 7.1. In this section, we never use directly the homogeneity or the concavity of the weight; all the results remain true for any weight such that the weighted isoperimetric inequality holds.

Given a set of finite w-perimeter $E\subseteq\Sigma$ with $0<w(E)<\infty$, we define the Cheeger constant as

$$\tau(E):=\inf\left\{\frac{\mathrm{Per}_w(F)}{\mathscr{H}^{n-1}_w(\partial^* F\cap\partial^* E)}:\ F\subseteq E,\ 0<w(F)\le\frac{w(E)}{2}\right\}. \tag{7.1}$$

It follows from the definition of $\tau(E)$ that, as long as $\tau(E)>1$, we have the following relative isoperimetric inequality

$$c_* w(F)^{\frac{D-1}{D}}\le\mathrm{Per}_w(F)\le\frac{\tau(E)}{\tau(E)-1}\,\mathscr{H}^{n-1}_w(\partial^* F\cap E) \tag{7.2}$$

for any $F\subseteq E$ with $w(F)\le\frac12 w(E)$, where $c_*:=\mathrm{Per}_w(B_1\cap\Sigma)/w(B_1\cap\Sigma)^{\frac{D-1}{D}}=Dw(B_1\cap\Sigma)^{\frac1D}$ is the isoperimetric constant that comes from (1.1), and $D:=n+\alpha$.

Remark 7.2. Notice that when $\tau(E)-1$ is very small, (7.2) is not very useful (as the constant in the right-hand side explodes). We will see that also the trace inequality (7.3) and the Sobolev-Poincaré inequality (7.4) exhibit a similar behavior. Since we want to apply these inequalities on sets with small weighted isoperimetric deficit, it is crucial to show that, if $\delta_w(E)$ is sufficiently small, then (up to slightly modifying the set E) the value $\tau(E)-1$ is bounded away from 0 by a constant that does not depend on E (see [FMP10, Section 1.6] for an explanation of why it is necessary to modify the set E). This is exactly the statement of Theorem 7.6.

The following trace inequality is the analogue of [FMP10, Lemma 3.1]. In addition to the trace inequality, we prove also a Sobolev-Poincaré inequality. See Section 2 for the definition of $E^{(1)}$.

Lemma 7.3. *Consider n, α, Σ, w satisfying (2.1). For every function $f \in BV_w(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and for every set of finite w-perimeter $E \subseteq \Sigma$ with $w(E) < \infty$, there is a constant[9] $c \in \mathbb{R}$ such that*

$$\int_{E^{(1)}} w(x)\,\mathrm{d}|\widetilde{\nabla} f|(x) \geq (\tau(E)-1)\int_{\partial^* E\cap\Sigma} \mathrm{tr}_E(|f-c|)w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,, \tag{7.3}$$

and also

$$\int_{E^{(1)}} w(x)\,\mathrm{d}|\widetilde{\nabla} f|(x) \geq D\big(1-\tau(E)^{-1}\big)\left(\int_E |f-c|^{\frac{D}{D-1}} w(x)\,\mathrm{d}x\right)^{\frac{D-1}{D}}\,. \tag{7.4}$$

Proof. For every $t \in \mathbb{R}$ we consider the set $F_t := E\cap\{f > t\}$. There exists $c \in \mathbb{R}$ such that $w(F_t) \leq w(E)/2$ for every $t \geq c$, and $w(E \setminus F_t) \leq w(E)/2$ for every $t < c$. We set moreover $g := (f-c)_+$, where $(A)_+$ denotes the positive part of A, and $G_s := E \cap \{g > s\}$.

Using the coarea formula [Mag12, Theorem 13.1], we have that

$$\int_{E^{(1)}} w(x)\,\mathrm{d}|\widetilde{\nabla} g|(x) = \int_0^\infty ds \int_{E^{(1)}\cap\partial^*\{g>s\}} w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,. \tag{7.5}$$

Moreover, by the definition of $\tau(E)$, we have

$$\begin{aligned}\int_{E^{(1)}\cap\partial^*\{g>s\}} w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x) &= \int_{E^{(1)}\cap\partial^* G_s} w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\\ &= \int_{\partial^* G_s\cap\Sigma} w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x) - \int_{\partial^* E\cap\partial^* G_s\cap\Sigma} w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\\ &\geq (\tau(E)-1)\int_{\partial^* E\cap\partial^* G_s\cap\Sigma} w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,.\end{aligned} \tag{7.6}$$

Now, by Fubini, we have

$$\int_{\partial^* E\cap\Sigma} \mathrm{tr}_E(g)w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x) = \int_0^\infty \mathrm{d}s \int_{\partial^* E\cap\{\mathrm{tr}_E(g)>s\}\cap\Sigma} w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x).$$

Using that, up to $\mathscr{H}^{n-1}$-null sets, $\partial^* E\cap\{\mathrm{tr}_E(g) > s\} \subseteq \partial^* E\cap\partial^* G_s$ (the proof of this fact is contained in the proof of [FMP10, Lemma 3.1]), we deduce that

$$\int_{\partial^* E\cap\Sigma} \mathrm{tr}_E(g)w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x) \leq \int_0^\infty \mathrm{d}s \int_{\partial^* E\cap\partial^* G_s\cap\Sigma} w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,. \tag{7.7}$$

Combining together (7.6) and (7.7) into (7.5), we get

$$\begin{aligned}\int_{E^{(1)}} w(x)\,\mathrm{d}|\widetilde{\nabla} g|(x) &\geq (\tau(E)-1)\int_0^\infty \mathrm{d}s \int_{\partial^* E\cap\partial^* G_s\cap\Sigma} w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\\ &\geq (\tau(E)-1)\int_{\partial^* E\cap\Sigma} \mathrm{tr}_E(g)\,w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,.\end{aligned}$$

Now, we repeat the above argument with $(f-c)_-$ in place of $(f-c)_+$ and, using the linearity of the trace operator and the fact that $(f-c)_+ + (f-c)_- = |f-c|$, we deduce that

$$\begin{aligned}&\int_{E^{(1)}} w(x)\,\mathrm{d}|\widetilde{\nabla}((f-c)_+)|(x) + \int_{E^{(1)}} w(x)\,\mathrm{d}|\widetilde{\nabla}((f-c)_-)|(x)\\ &\qquad\geq (\tau(E)-1)\int_{\partial^* E\cap\Sigma} \mathrm{tr}_E(|f-c|)w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\,.\end{aligned}$$

To conclude the proof of (7.3) it is enough to show that, for any open set Ω in $\mathbb{R}^n$

$$\int_\Omega w(x)\,\mathrm{d}|\widetilde{\nabla} f|(x) \geq \int_\Omega w(x)\,\mathrm{d}|\widetilde{\nabla}((f-c)_+)|(x) + \int_\Omega w(x)\,\mathrm{d}|\widetilde{\nabla}((f-c)_-)|(x)\,.$$

[9] The constant c, as it is clear from the proof, can be chosen to be the *median* value of f.

This fact can be seen exactly as in [FMP10], by approximating f with smooth functions in the weighted BV-norm, and using the lower semicontinuity of the weighted total variation.

Let us now move our attention to the Sobolev-Poincaré inequality. Repeating the argument of (7.5) and (7.6) and applying the weighted isoperimetric inequality, we can show

$$\int_{E^{(1)}} w(x)\,\mathrm{d}|\widetilde{\nabla} g|(x) \geq (1-\tau(E)^{-1}) \int_0^\infty \mathrm{d}s \int_{\partial^* G_s \cap \Sigma} w(x)\,\mathrm{d}\mathscr{H}^{n-1}(x) \geq D(1-\tau(E)^{-1}) \int_0^\infty w(G_s)^{\frac{D-1}{D}}\,\mathrm{d}s\,.$$

The estimate (7.4) follows from the last one as described in [ACM18, Theorem A.25]. □

We define now the family of sets (that depends on a set E that will always be clear from the context)

$$\Gamma_\lambda := \left\{ F \subseteq E :\ 0 < w(F) \leq \frac{w(E)}{2},\ \mathrm{Per}_w(F) \leq \lambda \mathscr{H}^{n-1}_w(\partial^* F \cap \partial^* E) \right\}.$$

The following lemma is the analogue [FMP10, Lemma 3.2].

Lemma 7.4. *Consider n, α, Σ, w satisfying (2.1). Let $E \subseteq \Sigma$ be a set of finite w-perimeter with $0 < w(E) < \infty$, and let $\lambda > 1$. If the family Γ_λ is not empty, then it admits a maximal element with respect to the order relation given by set inclusion up to sets of measure zero.*

Proof. We define the increasing sequence $(F_i)_{i\in\mathbb{N}}$ of sets in Γ_λ in the following way. Let F_1 be any element of Γ_λ and, once F_i has been defined, we consider

$$\Gamma_\lambda(i) = \{F \in \Gamma_\lambda :\ F_i \subseteq F\}.$$

Now, let F_{i+1} be an element of $\Gamma_\lambda(i)$ which satisfies

$$w(F_{i+1}) \geq \frac{w(F_i)+s_i}{2}, \quad \text{where } s_i = \sup_{F\in\Gamma_\lambda(i)} w(F).$$

Since F_i is an increasing sequence of sets it admits a limit, we call it F_∞. In what follows we show that $F_\infty \in \Gamma_\lambda$ and F_∞ is a maximal element in Γ_λ.

First, we observe that $w(F_\infty) = \sup_{i\in\mathbb{N}} w(F_i) \leq w(E)/2$. Moreover, by lower semicontinuity of the weighted perimeter, we have

$$\mathrm{Per}_w(F_\infty) \leq \liminf_{i\to\infty} \mathrm{Per}_w(F_i) \leq \lambda \liminf_{i\to\infty} \mathscr{H}^{n-1}_w(\partial^* F_i \cap \partial^* E).$$

Since $F_i \subseteq F_{i+1} \subseteq F_\infty \subseteq E$, we have (up to $\mathscr{H}^{n-1}$-negligible sets)

$$(\partial^* F_i \cap \partial^* E) \subseteq (\partial^* F_{i+1} \cap \partial^* E) \subseteq (\partial^* F_\infty \cap \partial^* E)\,,$$

therefore $F_\infty \in \Gamma_\lambda$. In order to show the maximality of F_∞, we consider another subset $H \subseteq E$ such that $H \cap F_\infty = \emptyset$ and $H \cup F_\infty \in \Gamma_\lambda$. By construction $F_\infty \cup H \in \Gamma_\lambda(i)$, so that for every $i \in \mathbb{N}$

$$s_i \geq w(F_\infty \cup H) \geq w(F_{i+1}) + w(H) \geq \frac{w(F_i)+s_i}{2} + w(H),$$

that is, $w(H) \leq (s_i - w(F_i))/2$. Since $s_i - w(F_i) \leq 2w(F_{i+1} \setminus F_i) \to 0$ as $i \to \infty$, we have deduced that $w(H) = 0$, which gives the maximality of F_∞. □

Recall that we denote the isoperimetric deficit by

$$\delta_w(E) := \frac{\mathrm{Per}_w(E)}{D w(E)^{\frac{D-1}{D}}} - 1,$$

where $D = n + \alpha$ is the isoperimetric constant (since we are assuming $w(B_1 \cap \Sigma) = 1$).

We want to show that if E is almost optimal, then any subset F of E which makes $\tau(E)$ small enough has small volume. In order to do that, following [FMP10], we introduce the strictly concave function $\Psi : [0,1] \to [0, 2^{1/D} - 1]$ given by

$$\Psi(t) := t^{\frac{D-1}{D}} + (1-t)^{\frac{D-1}{D}} - 1.$$

We observe that

$$\Psi(t) \geq (2 - 2^{\frac{D-1}{D}})t^{\frac{D-1}{D}}, \quad \text{for} \quad t \in \left[0, \tfrac{1}{2}\right] \tag{7.8}$$

and we set

$$k(D) := \frac{2 - 2^{\frac{D-1}{D}}}{3}, \tag{7.9}$$

so that $\Psi(t) \geq 3k(D)t^{\frac{D-1}{D}}$ for $t \in \left[0, \frac{1}{2}\right]$.

The following lemma is the analogue of [FMP10, Lemma 3.3].

Lemma 7.5. *Consider n, α, Σ, w satisfying (2.1). Let E, F be two sets of finite w-perimeter, with $F \subseteq E \subseteq \Sigma$ such that*

$$0 < w(F) < \frac{w(E)}{2} < \infty \quad \text{and} \quad \mathrm{Per}_w(F) \leq (1 + k(D))\mathcal{H}^{n-1}_w(\partial^* E \cap \partial^* F)\,. \tag{7.10}$$

Then we have:

(i) $w(F) \leq \left(\dfrac{\delta_w(E)}{k(D)}\right)^{\frac{D}{D-1}} w(E)$;

(ii) $\mathrm{Per}_w(E \setminus F) \leq \mathrm{Per}_w(E)$;

(iii) If $\delta_w(E) \leq k(D)$, then $\delta_w(E \setminus F) \leq \dfrac{3}{k(D)}\delta_w(E)$.

Proof. Using the second inequality in (7.10), we have that

$$\begin{aligned}
\mathrm{Per}_w(E) &= \mathrm{Per}_w(E \setminus F) + \mathrm{Per}_w(F) - 2\mathscr{H}^{n-1}_w(\partial^* F \cap E^{(1)}) \\
&= \mathrm{Per}_w(E \setminus F) + \mathrm{Per}_w(F) - 2\left(\mathrm{Per}_w(F) - \mathscr{H}^{n-1}_w(\partial^* F \cap \partial^* E)\right) \\
&\geq \mathrm{Per}_w(E \setminus F) + \mathrm{Per}_w(F) - 2k(D)\mathscr{H}^{n-1}_w(\partial^* F \cap \partial^* E) \\
&\geq \mathrm{Per}_w(E \setminus F) + \mathrm{Per}_w(F)\,(1 - 2k(D)) \qquad (7.11) \\
&\geq D\,w(E \setminus F)^{\frac{D-1}{D}} + (1 - 2k(D))\,D\,w(F)^{\frac{D-1}{D}}\,, \qquad (7.12)
\end{aligned}$$

where in the last estimate we have applied the weighted isoperimetric inequality to the sets $E \setminus F$ and F.

We set now $t := w(F)/w(E)$, so that $w(E \setminus F)/w(E) = 1 - t$ and, by the first assumption in (7.10) we have $t \leq 1/2$. Dividing by $D\,w(E)^{\frac{D-1}{D}}$ in (7.12), we get

$$\delta_w(E) = \frac{\mathrm{Per}_w(E)}{D\,w(E)^{\frac{D-1}{D}}} - 1 \geq (1 - t)^{\frac{D-1}{D}} + (1 - 2k(D))\,t^{\frac{D-1}{D}} - 1.$$

Now we use the definitions of Ψ and $k(D)$ and inequality (7.8), to deduce

$$\delta_w(E) \geq \Psi(t) - 2k(D)t^{\frac{D-1}{D}} \geq k(D)t^{\frac{D-1}{D}} = k(D)\Big(\frac{w(F)}{w(E)}\Big)^{\frac{D-1}{D}}\,, \tag{7.13}$$

that is equivalent to (i).

The estimate (ii) follows from (7.11), using that $1 - 2k(D) \geq 0$.

It remains to show (iii). First we observe that (7.13) and the assumption $\delta_w(E) \leq k(D)$ imply that

$$t \leq \left(\frac{\delta_w(E)}{k(D)}\right)^{\frac{D}{D-1}} \leq \frac{\delta_w(E)}{k(D)}.$$

Therefore we have that

$$\begin{aligned}
\delta_w(E \setminus F) &= \frac{\mathrm{Per}_w(E \setminus F)}{D\,w(E \setminus F)^{\frac{D-1}{D}}} - 1 = \frac{\mathrm{Per}_w(E \setminus F)}{D\,(1 - t)^{\frac{D-1}{D}} w(E)^{\frac{D-1}{D}}} - 1 \\
&\leq \frac{\mathrm{Per}_w(E \setminus F)}{D\,w(E)^{\frac{D-1}{D}}}(1 + 2t) - 1 = \delta_w(E) + 2t\,(\delta_w(E) + 1) \leq \frac{3}{k(D)}\delta_w(E)\,,
\end{aligned}$$

as wanted. □

Finally, the following theorem is the analogue of [FMP10, Theorem 3.4]. It states that if E has small isoperimetric deficit, then there exists a subset G of E which also has small deficit and, more importantly, such that $\tau(G) - 1$ is bounded below away from zero. The idea of the proof consists in cutting away from E the maximal critical set (whose existence is established in Lemma 7.4) and using the estimates of Lemma 7.5.

Theorem 7.6. *Consider n, α, Σ, w satisfying (2.1). Let $E \subseteq \Sigma$ be a set of finite weighted perimeter and suppose that $\delta_w(E) \leq k^2(D)/8$, with $k(D)$ given by (7.9).*

Then, there exists a set $G \subseteq E$ with finite w-perimeter which satisfies the following estimates:

(a) $w(E \setminus G) \leq \dfrac{\delta_w(E)}{k(D)} w(E)$;

(b) $\delta_w(G) \leq \dfrac{3}{k(D)} \delta_w(E)$;

(c) $\tau(G) \geq 1 + k(D)$.

Proof. If $\tau(E) \geq 1 + k(D)$, then we can choose $G := E$ and there is nothing to prove. Otherwise, let F_∞ be the maximal critical set given in Lemma 7.4 with $\lambda = 1 + k(D)$ (notice that $\Gamma_\lambda \neq \emptyset$ because $\tau(E) < 1 + k(D)$). We define the set G as $G := E \setminus F_\infty$. Since $F_\infty \in \Gamma_\lambda$, then we can apply Lemma 7.5 to $F = F_\infty$ and we deduce the estimates (a) and (b).

It remains to prove (c). In order to do that, we argue by contradiction, that is we assume that $\tau(G) < 1 + k(D) = \lambda$. By definition of $\tau(G)$ we have that there exists a set $H \subseteq G$ with $0 \leq w(H) \leq w(G)/2$ and such that

$$\mathrm{Per}_w(H) < \lambda \mathscr{H}^{n-1}_w(\partial^* H \cap \partial^* G).$$

We aim to show that $F_\infty \cup H \in \Gamma_\lambda$, which gives a contradiction to the maximality of F_∞. Thus, we will have completed the proof once we have checked that

$$0 \leq w(F_\infty \cup H) \leq \frac{w(E)}{2} \quad \text{and} \quad \mathrm{Per}_w(F_\infty \cup H) \leq \lambda \mathscr{H}^{n-1}_w(\partial^*(F_\infty \cup H) \cap \partial^* E). \tag{7.14}$$

Applying (b), the first estimate follows from Lemma 7.5. Indeed applying Lemma 7.5 to $H \subseteq G$, we have that

$$w(H) \leq \frac{\delta_w(G)}{k(D)} w(G) \leq 3 \frac{\delta_w(E)}{k^2(D)} w(E)\,.$$

Moreover, using that F_∞ and H are disjoint and applying Lemma 7.5 to $F_\infty \subseteq E$, we deduce

$$w(F_\infty \cup H) = w(F_\infty) + w(H) \leq \frac{\delta_w(E)}{k(D)} w(E) + 3 \frac{\delta_w(E)}{k^2(D)} w(E) \leq 4 \frac{\delta_w(E)}{k^2(D)} w(E) \leq \frac{w(E)}{2}\,,$$

where in the last inequality we have used the assumption $\delta_w(E) \leq k^2(D)/8$.

It remains to prove the second estimate of (7.14). We start by observing that

$$\mathrm{Per}_w(F_\infty \cup H) = \mathscr{H}^{n-1}_w(\partial^*(F_\infty \cup H) \cap E^{(1)}) + \mathscr{H}^{n-1}_w(\partial^*(F_\infty \cup H) \cap \partial^* E).$$

Since $\lambda = k(D) + 1$, in order to conclude we have to show that

$$\mathscr{H}^{n-1}_w(\partial^*(F_\infty \cup H) \cap E^{(1)}) \leq k(D) \mathscr{H}^{n-1}_w(\partial^*(F_\infty \cup H) \cap \partial^* E). \tag{7.15}$$

First, we write

$$\mathscr{H}^{n-1}_w(\partial^*(F_\infty \cup H) \cap E^{(1)}) = \mathscr{H}^{n-1}_w((\partial^* F_\infty \setminus \partial^* H)) \cap E^{(1)}) + \mathscr{H}^{n-1}_w((\partial^* H \setminus \partial^* F_\infty) \cap E^{(1)}) \tag{7.16}$$

We now estimate the second term on the right-hand side of (7.16). Since (up to $\mathscr{H}^{n-1}$-negligible sets) it holds $(\partial^* H \setminus \partial^* F_\infty) \cap E^{(1)} \subseteq G^{(1)}$, we have

$$\begin{aligned} \mathscr{H}^{n-1}_w((\partial^* H \setminus \partial^* F_\infty) \cap E^{(1)}) &\leq \mathscr{H}^{n-1}_w(\partial^* H \cap G^{(1)}) \\ &= \mathrm{Per}_w(H) - \mathscr{H}^{n-1}_w(\partial^* H \cap \partial^* G) \\ &\leq k(D) \mathscr{H}^{n-1}_w(\partial^* H \cap \partial^* G), \end{aligned} \tag{7.17}$$

where in the last estimate we have used $\mathrm{Per}_w(H) \le \lambda \mathscr{H}^{n-1}_w(\partial^* H \cap \partial^* G)$.

Combining together (7.16) and (7.17), and using that $F_\infty \in \Gamma_\lambda$, we deduce that

$$\begin{aligned}
\mathscr{H}^{n-1}_w(\partial^*(F_\infty \cup H) \cap E^{(1)}) &\le \mathscr{H}^{n-1}_w((\partial^* F_\infty \setminus \partial^* H) \cap E^{(1)}) + k(D)\mathscr{H}^{n-1}_w(\partial^* H \cap \partial^* G) \\
&= \mathscr{H}^{n-1}_w((\partial^* F_\infty \setminus \partial^* H) \cap E^{(1)}) \\
&\quad + k(D)\Big[\mathscr{H}^{n-1}_w((\partial^* H \cap \partial^* G) \cap E^{(1)}) + \mathscr{H}^{n-1}_w((\partial^* H \cap \partial^* G) \cap \partial^* E)\Big] \\
&\le \mathscr{H}^{n-1}_w(\partial^* F_\infty \cap E^{(1)}) + k(D)\mathscr{H}^{n-1}_w((\partial^* H \cap \partial^* G) \cap \partial^* E) \\
&\le k(D)\Big[\mathscr{H}^{n-1}_w(\partial^* F_\infty \cap \partial^* E) + \mathscr{H}^{n-1}_w((\partial^* H \cap \partial^* G) \cap \partial^* E)\Big] \\
&\le k(D)\mathscr{H}^{n-1}_w(\partial^*(F_\infty \cup H) \cap \partial^* E)\,.
\end{aligned}$$

We have established (7.15) and therefore the proof is concluded. □

Corollary 7.7 (Relative isoperimetric inequality in the ball)**.** *For any $F \subseteq B_1 \cap \Sigma$ such that $w(F) \le \frac{1}{2} w(B_1 \cap \Sigma)$, it holds*

$$w(F) \lesssim \mathscr{H}^{n-1}_w(\partial^* F \cap B_1 \cap \Sigma)\,.$$

Proof. Since $\delta_w(B_1 \cap \Sigma) = 0$, Theorem 7.6 (with $E = G = B_1 \cap \Sigma$) tells us that $\tau(B_1 \cap \Sigma) \ge 1 + k(D)$. Then, the statement follows from (7.2). □

8. Proof of the main result

We prove three increasingly stronger statements: the characterization of optimal sets, Proposition 1.2, a nonquantitative stability for almost-optimal sets, Lemma 8.1, and finally the quantitative weighted isoperimetric inequality, Theorem 1.3. We prove them separately because we use the characterization of optimal sets in the proof of the nonquantitative stability and we apply the nonquantitative stability in the proof of the quantitative weighted isoperimetric inequality.

Proof of Proposition 1.2. Let $E \subseteq \Sigma$ be an optimal set for the weighted isoperimetric inequality. Without loss of generality we can assume $w(E) = w(B_1 \cap \Sigma) = 1$. Thanks to Theorem 7.6 and Lemma 7.3, we know that E satisfies a weighted Poincaré inequality (since it must hold $G = E$ in the statement of Theorem 7.6 because $\delta_w(E) = 0$). Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be the convex map described in Proposition 4.4 (notice that E is w-indecomposable because $\tau(E) > 1$). From (4.7), it follows that there exists $x_0 \in \mathbb{R}^n$ such that $\nabla\varphi(x) - x = -x_0$ for any $x \in E$. Hence, since $\nabla\varphi(E) = B_1 \cap \Sigma$, it holds $E = x_0 + B_1 \cap \Sigma$. To conclude, notice that $\mathrm{Per}_w(E) = \mathrm{Per}_w(B_1 \cap \Sigma)$ implies $x_0 + \partial\Sigma \subseteq \partial\Sigma$ and therefore $x_0 \in \mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}$. □

Using the characterization of optimal sets, we next show the following.

Lemma 8.1 (Nonquantitative stability)**.** *Consider n, α, Σ, w satisfying (2.1) and let $(E_i)_{i\in\mathbb{N}} \subseteq \Sigma$ be a sequence of sets of finite w-perimeter such that $E_i \subseteq \Sigma$, $w(E_i) = 1$, and $\delta_w(E_i) \to 0$. Then $w(E_i \triangle (B_1(x_i) \cap \Sigma)) \to 0$ for an appropriate choice of $x_i \in \mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}$.*

Proof. In this proof we will denote with C any constant that depends on n, α, Σ, w. The value of the constant can change from line to line.

We assume that Σ contains no lines, that is $k = 0$. It is easy to adapt the proof to handle the case $k > 0$. We call a *minor perturbation* of the sequence, any replacement of E_i with E_i', with $w(E_i \triangle E_i') \to 0$ and $\delta_w(E_i') \to 0$. We will not change the naming of the sets when performing minor perturbations. We say that a sequence of Borel sets $F_i \subseteq \mathbb{R}^n$ converges to a Borel set $F \subseteq \mathbb{R}^n$ if the convergence holds in the weighted L^1 norm, that is $w(F_i \triangle F) \to 0$ as $i \to \infty$.

Applying Theorem 7.6, up to a minor perturbation, we can assume $\tau(E_i) \ge 1 + k(D)$. Hence, thanks to Lemma 7.3, the sets E_i enjoy *nontrivial* Poincaré and trace inequalities. Let us denote with $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ the convex function described in Proposition 4.4 relative to E_i (notice that E_i is w-indecomposable because

$\tau(E_i) > 1$). From the estimate (4.7), thanks to the trace and Poincaré inequalities, it follows that there exists a sequence of points $(x_i)_{i\in\mathbb{N}} \subseteq \mathbb{R}^n$ such that

$$\int_{\partial^* E_i \cap \Sigma} |\nabla\varphi_i - (x - x_i)|\, w(x)\, \mathrm{d}\mathscr{H}^{n-1}(x) \longrightarrow 0 \quad \text{as } i \to \infty\,, \tag{8.1}$$

$$\int_{E_i} |\nabla\varphi_i - (x - x_i)|\, w(x)\, \mathrm{d}x \longrightarrow 0 \quad \text{as } i \to \infty\,. \tag{8.2}$$

From (8.2) we deduce $w(E_i \setminus B_2(x_i)) \to 0$, and therefore there is $2 < r < 3$ such that $E_i \to E_i \cap B_r(x_i)$ is a minor perturbation. Hence we can assume $E_i \subseteq B_3(x_i)$. Repeating the argument that led to (8.1) and (8.2) for the original E_i, we can also assume that (8.1) and (8.2) hold.

Combining (8.1) with (4.8) we deduce

$$\int_{\partial^* E_i \cap \Sigma} d\big(x - x_i, \partial B_1 \cap \Sigma\big)\, w(x)\, \mathrm{d}\mathscr{H}^{n-1}(x) \longrightarrow 0\,.$$

Because of the concavity of $w^{\frac{1}{\alpha}}$, it holds $\inf_{x\in\Sigma} \frac{w(x)}{d(x,\partial\Sigma)^\alpha} > 0$ and therefore the last inequality implies

$$\int_{\partial^*(E_i - x_i)} d\big(x, \partial B_1 \cap \Sigma\big)\, d(x, \Sigma - x_i)^\alpha\, \mathrm{d}\mathscr{H}^{n-1}(x) \longrightarrow 0\,. \tag{8.3}$$

Let us prove that

$$\liminf_{i\to\infty} |E_i| \geq |B_1 \cap \Sigma|\,. \tag{8.4}$$

Choose a sequence of positive numbers $(\varepsilon_i)_{i\in\mathbb{N}} \in \mathbb{R}$ so that $\varepsilon_i \to 0$ and $\delta_w(E_i)^{\frac{1}{2}}\varepsilon_i^{-1-\alpha} \to 0$ as $i \to \infty$. Let us decompose E_i into three subsets $E_i = E_i^{(1)} \sqcup E_i^{(2)} \sqcup E_i^{(3)}$ defined as

$$\begin{aligned}
E_i^{(1)} &:= E_i \cap \{x \in \Sigma :\, d(x, \partial\Sigma) < \varepsilon_i\}\,,\\
E_i^{(2)} &:= \big(E_i \setminus E_i^{(1)}\big) \cap \{x \in E_i :\, |\nabla^2\varphi_i(x) - \mathbb{1}| \geq \varepsilon_i\}\,,\\
E_i^{(3)} &:= E_i \setminus \big(E_i^{(1)} \sqcup E_i^{(2)}\big)\,.
\end{aligned}$$

Keep in mind that $E_i^{(3)}$ is the *good big piece*, while $E_i^{(1)}$ and $E_i^{(2)}$ are the *bad small pieces* (they are bad because $E_i^{(1)}$ is close to the boundary and on $E_i^{(2)}$ the Hessian of φ_i is not close to the identity). Let us control the Lebesgue measure of $\nabla\varphi_i(E_i^{(k)})$ for each $k = 1, 2, 3$:

(1) We have $|E_i^{(1)}| \leq C\varepsilon_i$ and therefore, since $\|\nabla^2\varphi_i\|_\infty$ is uniformly bounded, the area formula implies

$$|\nabla\varphi_i(E_i^{(1)})| \leq C\varepsilon_i\,. \tag{8.5}$$

(2) For any $x \in \Sigma$, we have $Cw(x) \geq d(x, \partial\Sigma)^\alpha$ and therefore $Cw(x) \geq \varepsilon_i^\alpha$ for any $x \in E_i^{(2)}$. Thanks to (4.7), we obtain

$$|E_i^{(2)}| \leq C \int_{E_i^{(2)}} \frac{|\nabla^2\varphi_i(x) - \mathbb{1}| w(x)}{\varepsilon_i^{1+\alpha}}\, \mathrm{d}x \lesssim C\delta_w(E_i)^{\frac{1}{2}}\varepsilon_i^{-1-\alpha}\,.$$

Thus, once again because $\|\nabla^2\varphi_i\|_\infty$ is uniformly bounded, we deduce

$$|\nabla\varphi_i(E_i^{(2)})| \lesssim C\delta_w(E_i)^{\frac{1}{2}}\varepsilon_i^{-1-\alpha}\,. \tag{8.6}$$

(3) On $E_i^{(3)}$ we have $|\nabla^2\varphi_i - \mathbb{1}| < \varepsilon_i$, hence the area formula implies

$$|\nabla\varphi_i(E_i^{(3)})| \leq (1 + C\varepsilon_i)|E_i^{(3)}| \leq (1 + C\varepsilon_i)|E_i|\,. \tag{8.7}$$

Joining (8.5), (8.6) and (8.7) we obtain

$$|B_1 \cap \Sigma| = |\nabla\varphi_i(E_i)| \leq |\nabla\varphi_i(E_i^{(1)})| + |\nabla\varphi_i(E_i^{(2)})| + |\nabla\varphi_i(E_i^{(3)})| \leq (1 + C\varepsilon_i)|E_i| + C\varepsilon_i + C\delta_w(E_i)^{\frac{1}{2}}\varepsilon_i^{-1-\alpha}$$

and (8.4) follows.

In the next two paragraphs we will use repeatedly the compactness of sets of finite perimeter with bounded perimeter ([Mag12, Theorem 12.26]).

If $|x_i|$ stays bounded, then (up to subsequence) the sequence E_i converges to a set E_∞ with $w(E_\infty) = 1$ and $\delta_w(E_\infty) = 0$. Then, since $B_1 \cap \Sigma$ is the unique minimizer of the weighted isoperimetric inequality (see Proposition 1.2), we deduce $E_\infty = B_1 \cap \Sigma$.

On the other hand, let us show that $|x_i| \to \infty$ yields a contradiction. The locally finite perimeter sets $\Sigma - x_i$ subconverge (locally) to $\Sigma' + y_0$, where:

- Σ' is an open convex cone so that $\Sigma \subseteq \Sigma'$ and Σ' contains a line.
- $y_0 \in \mathbb{R}^n$ is an arbitrary point (which is nonzero if the distance $d(x_i, \partial\Sigma)$ does not go to 0 or to ∞ as $i \to \infty$).

Notice that, since Σ' contains a line while Σ does not, the cone Σ' is *strictly* larger than Σ. Moreover the sets $E_i - x_i$ subconverge, locally in $\Sigma' + y_0$, to a set of locally finite[10] perimeter $E_\infty \subseteq B_3 \cap (\Sigma' + y_0)$. Thanks to (8.4), we have $|E_\infty| \geq |B_1 \cap \Sigma|$. Moreover, thanks to the limit (8.3), we deduce

$$\partial^* E_\infty \subseteq (\partial B_1 \cap \Sigma) \cup \partial(\Sigma' + y_0)\,. \tag{8.8}$$

Consider the set $\Omega := (\Sigma' + y_0) \setminus \overline{(B_1 \cap \Sigma)}$. The set Ω is open, connected and unbounded. Moreover, thanks to (8.8), we have $\Omega \cap \partial^* E_\infty = \emptyset$. Thus, up to negligible sets, either $\Omega \subseteq E_\infty$ or $\Omega \cap E_\infty = \emptyset$. Since E_∞ is bounded it must be $\Omega \cap E_\infty = \emptyset$ and therefore $E_\infty \subseteq B_1 \cap \Sigma$. Since $|E_\infty| \geq |B_1 \cap \Sigma|$, we have shown $E_\infty = B_1 \cap \Sigma$ up to negligible sets. Hence (8.8) is equivalent to

$$B_1 \cap \partial\Sigma \subseteq \partial(\Sigma' + y_0)\,,$$

which yields a contradiction as it would imply $y_0 = 0_{\mathbb{R}^n}$ and $\Sigma' = \Sigma$, but we know that Σ' is strictly larger than Σ. □

We can finally give the:

Proof of Theorem 1.3. In this proof we will denote with C any constant that depends on n, α, Σ, w. The value of the constant can change from line to line.

Without loss of generality, we may assume that $\delta_w(E)$ is small, and that $w(E) = w(B_1 \cap \Sigma) = 1$. Thanks to Theorem 7.6 we can also assume that the Cheeger constant $\tau(E)$ defined in (7.1) satisfies $\tau(E) - 1 \gtrsim 1$ (up to replacing E with the set G described in the statement of Theorem 7.6). Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be the convex function associated to E described in Proposition 4.4 (notice that E is w-indecomposable because $\tau(E) > 1$).

Applying the trace inequality (7.3) and the Poincaré inequality (7.4) together with (4.7), we deduce

$$\int_{\partial^* E \cap \Sigma} |\nabla\varphi - (x - x_0)|\, w(x)\, \mathrm{d}\mathscr{H}^{n-1}(x) \lesssim \delta_w(E)^{\frac{1}{2}}\,, \tag{8.9}$$

$$\int_E |\nabla\varphi - (x - x_0)|\, w(x)\, \mathrm{d}x \lesssim \delta_w(E)^{\frac{1}{2}}\,, \tag{8.10}$$

for a suitable $x_0 \in \mathbb{R}^n$. Without loss of generality we can assume that $x_0 = (0_{\mathbb{R}^k}, \widetilde{x}_0)$ with $\widetilde{x}_0 \in \mathbb{R}^{n-k}$. Combining (8.9) with (4.8), it follows

$$\int_{\partial^* E \cap \Sigma} ||x - x_0| - 1|\, w(x)\, \mathrm{d}\mathscr{H}^{n-1}(x) \lesssim \delta_w(E)^{\frac{1}{2}}\,. \tag{8.11}$$

Since $\delta_w(E)$ is assumed to be small, Lemma 8.1 implies the validity of the hypotheses needed by Proposition 6.1. Hence, recalling (8.11), we deduce

$$w\big((B_1(x_0) \cap \Sigma) \triangle E\big) \lesssim \delta_w(E)^{\frac{1}{2}}\,. \tag{8.12}$$

It remains only to establish that $|x_0|$ is controlled by $\delta_w(E)^{\frac{1}{2}}$ (as this allows to replace x_0 with $0_{\mathbb{R}^n}$ in the last inequality). First, we prove that $\pi_{\mathcal{E}}(x_0) \lesssim C\delta_w(E)^{\frac{1}{2}}$ (applying Proposition 5.3) and then we conclude that also the component of x_0 along $\mathcal{C}$ is controlled by $\delta_w(E)^{\frac{1}{2}}$ (applying Proposition 5.5). Let

[10] Here *locally finite* has to be understood in the sense of $\Sigma' + y_0$, that is for any $\Omega \subset\subset (\Sigma' + y_0)$ it holds $\mathrm{Per}(E, \Omega) < \infty$.

us recall the notation introduced in (5.1): $\mathcal{L}$ is the subspace of lines contained in Σ (that is $\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}$), $\mathcal{C}$ is the subspace orthogonal to $\mathcal{L}$ such that w is constant moving along $\mathcal{C}$, $\mathcal{E}$ is the orthogonal subspace to $\mathcal{L} \times \mathcal{C}$.

Notice that (8.12) implies

$$w\big((B_{\frac{1}{2}} \cap \Sigma) \setminus E\big) \lesssim \delta_w(E)^{\frac{1}{2}} . \tag{8.13}$$

Let $\hat{\varepsilon}, \hat{Q}$ be the small value and the compact set described in the statement of Proposition 5.3. Thanks to (4.9), we know

$$\int_{\hat{Q}\cap E} |w^{\frac{1}{\alpha}}(\nabla\varphi) - w^{\frac{1}{\alpha}}(x)| \, \mathrm{d}x \leq C\delta_w(E)^{\frac{1}{2}} . \tag{8.14}$$

To proceed let us assume that $|x_0| < \hat{\varepsilon}$ (notice that $\hat{\varepsilon}$ does not depend on E). Recall that, thanks to Lemma 8.1, we can assume that $|x_0|$ is arbitrarily small (and our goal is to show that it is controlled by $\delta_w(E)^{\frac{1}{2}}$). For any $x \in \hat{Q}$ and any $y \in B_1 \cap \Sigma$, it holds

$$|w^{\frac{1}{\alpha}}(x - x_0) - w^{\frac{1}{\alpha}}(y)| \leq C|y - (x - x_0)| ,$$

where C is a constant that depends on the Lipschitz constant of $w^{\frac{1}{\alpha}}$ in an $\hat{\varepsilon}$-neighborhood of $\hat{Q}$ and on $\hat{\varepsilon}$ itself. Therefore it holds

$$\int_{\hat{Q}\cap E} |w^{\frac{1}{\alpha}}(\nabla\varphi) - w^{\frac{1}{\alpha}}(x - x_0)| w(x) \, \mathrm{d}x \leq C \int_{\hat{Q}\cap E} |\nabla\varphi - (x - x_0)| w(x) \, \mathrm{d}x \lesssim C\delta_w(E)^{\frac{1}{2}} , \tag{8.15}$$

where in the second step we used (8.10).

Since $w \geq C$ on $\hat{Q}$, the inequalities (8.14) and (8.15) imply

$$\int_{\hat{Q}\cap E} |w^{\frac{1}{\alpha}}(x) - w^{\frac{1}{\alpha}}(x - x_0)| \, \mathrm{d}x \leq C\delta_w(E)^{\frac{1}{2}} . \tag{8.16}$$

Finally, notice that thanks to (8.13) and $\hat{Q} \subseteq B_{\frac{1}{2}}$, we can replace $\hat{Q} \cap E$ with $\hat{Q}$ in (8.16).

Hence we can apply Proposition 5.3 and deduce the fundamental bound

$$|\pi_{\mathcal{E}}(x_0)| \leq C\delta_w(E)^{\frac{1}{2}} ,$$

where $\mathcal{E}$ is the subspace of directions orthogonal to the constancy directions of w.

Thanks to the latter control on $\pi_{\mathcal{E}}(x_0)$, changing slightly the value of x_0, we can assume that $x_0 \in \mathcal{C}$.

Applying Proposition 5.5 with $\xi = x_0$, we know that either Proposition 5.5-(1) or Proposition 5.5-(2) holds. If Proposition 5.5-(1) holds, since $w(B_1 \cap \Sigma) = w(E)$, then (8.12) implies $|x_0| \lesssim C\delta_w(E)^{\frac{1}{2}}$, that is exactly the desired estimate. Let us assume that Proposition 5.5-(2) holds. For the ease of the reader, let us state again (4.7) and (8.10):

$$\int_E |\nabla^2\varphi - \mathbb{1}| w \lesssim \delta_w(E)^{\frac{1}{2}} ,$$
$$\int_E |\nabla\varphi - (x - x_0)| w(x) \, \mathrm{d}x \lesssim \delta_w(E)^{\frac{1}{2}} .$$

From (8.13) and Proposition 5.5 (2)-(b) and (2)-(f), we deduce

$$\int_S \mathrm{d}\mathscr{H}^{n-1}(s) \int_{\Sigma_s} |\nabla^2\varphi - \mathbb{1}| \, \mathrm{d}\mathscr{H}^1 \leq C\delta_w(E)^{\frac{1}{2}} ,$$
$$\int_S \mathrm{d}\mathscr{H}^{n-1}(s) \int_{\Sigma_s} |\nabla\varphi - (x - x_0)| \, \mathrm{d}\mathscr{H}^1(x) \leq C\delta_w(E)^{\frac{1}{2}} .$$

Thanks to (2)-(c), there exists $\overline{s} \in S$ such that

$$\int_{\Sigma_{\overline{s}}} |\nabla^2 \varphi - \mathbb{1}| \, \mathrm{d}\mathscr{H}^1 \leq C\delta_w(E)^{\frac{1}{2}} \,,$$
$$\int_{\Sigma_{\overline{s}}} |\nabla \varphi - (x - x_0)| \, \mathrm{d}\mathscr{H}^1(x) \leq C\delta_w(E)^{\frac{1}{2}} \,.$$

From the latter two inequalities, together with (2)-(a), it follows that (here it is crucial that $\Sigma_{\overline{s}}$ is 1-dimensional)

$$\|\nabla \varphi - (x - x_0)\|_{L^\infty(\Sigma_{\overline{s}})} \leq C\delta_w(E)^{\frac{1}{2}} \,.$$

In particular, denoting $z := \overline{s} + t_0(\overline{s})\frac{x_0}{|x_0|}$, it holds

$$|\nabla \varphi(z) - (z - x_0)| \leq C\delta_w(E)^{\frac{1}{2}} \,.$$

On the other hand, (2)-(d) implies

$$|\nabla \varphi(z) - (z - x_0)| \geq d\Big(\Sigma, \overline{s} + (t_0 - |x_0|)\frac{x_0}{|x_0|}\Big) \geq C|x_0| \,.$$

The last two estimates together conclude the proof. □

Remark 8.2. The statement of Theorem 1.3 is sharp with respect to the exponent, i.e., the exponent $\frac{1}{2}$ present in the right-hand side of (1.2) cannot be increased. Let us prove it when Σ does not contain lines; the method can be easily adapted to handle the general case.

Given a smooth positive function $r : \partial B_1 \cap \Sigma \to (0, \infty)$, let

$$E_{(r)} := \big\{ t\theta : \ \theta \in \partial B_1 \cap \Sigma, \ 0 < t < r(\theta) \big\} \,.$$

With some standard computations (see also Lemma 6.3) we obtain

$$w(E_{(r)}) = \frac{1}{D} \int_{\partial B_1 \cap \Sigma} r(\theta)^D \, w(\theta) \, \mathrm{d}\mathscr{H}^{n-1}(\theta) \,,$$
$$\mathrm{Per}_w(E_{(r)}) = \int_{\partial B_1 \cap \Sigma} r(\theta)^{D-1} \sqrt{1 + \frac{|\nabla r|^2}{r^2}} \, w(\theta) \, \mathrm{d}\mathscr{H}^{n-1}(\theta) \,.$$

Fix a smooth function $\eta : \partial B_1 \cap \Sigma \to \mathbb{R}$ and, for any $\varepsilon > 0$, define $E_\varepsilon := E_{(1+\varepsilon\eta)}$. If $\int_{\partial B_1 \cap \Sigma} \eta \, w \, \mathrm{d}\mathscr{H}^{n-1} = 0$, then the previous formulas imply

$$w(E_\varepsilon) = w(B_1 \cap \Sigma) + \mathcal{O}(\varepsilon^2) \,,$$
$$\mathrm{Per}_w(E_\varepsilon) = \mathrm{Per}_w(B_1 \cap \Sigma) + \mathcal{O}(\varepsilon^2) \,.$$

Since it holds $A_w(E_\varepsilon) \geq C(\Sigma, w, \eta)\varepsilon$ (with $C > 0$ provided $\eta \not\equiv 0$), the family $(E_\varepsilon)_{\varepsilon > 0}$ shows the sharpness of the exponent $\frac{1}{2}$.

Appendix A. Quantitative weighted mean inequality

The aim of this appendix is to show the following.

Lemma A.1. *Let $(\lambda_i)_{i=1,\dots,m}$ be positive real numbers with $s := \lambda_1 + \cdots + \lambda_m \geq 1$ and let $(x_i)_{i=1,\dots,m}$ be nonnegative real numbers. If $\sum \lambda_i x_i \leq cs$ for some $c > 0$, then it holds*

$$\sum_{i=1}^m \lambda_i (x_i - c)^2 \leq \frac{8}{3} \frac{c^{2-s} s^3}{\min\limits_{i=1,\dots,m} \lambda_i^2} \Big(c^s - x_1^{\lambda_1} \cdots x_m^{\lambda_m} \Big) \,.$$

Proof. We follow the proof of [FMP10, Lemma 2.5]. Without loss of generality we can assume that $c = 1$. For any $t > 0$ it holds

$$\log(t) \leq t - 1 - \frac{(t-1)^2}{2\max(1, t)^2} \,.$$

Notice that $\max(1, x_i) \leq s\lambda^{-1}$, where λ is the minimum among $\lambda_1, \dots, \lambda_m$. Therefore it holds

$$\log\left(x_1^{\lambda_1} \cdots x_m^{\lambda_m}\right) = \sum_{i=1}^m \lambda_i \log(x_i) \leq \sum_{i=1}^m \lambda_i \Big(x_i - 1 - \frac{\lambda^2 (x_i-1)^2}{2s^2}\Big) \leq -\frac{\lambda^2}{2s^2} \sum_{i=1}^m \lambda_i (x_i-1)^2 \,. \tag{A.1}$$

Since $s \geq 1$, for any $0 \leq t \leq \frac{1}{2}$ we have $e^{-st} \leq e^{-t} \leq 1 - \frac{3}{4}t$. Moreover, the right-hand side of (A.1) has absolute value below $\frac{s}{2}$, thus taking the exponential of both sides we deduce

$$x_1^{\lambda_1} \cdots x_m^{\lambda_m} \leq 1 - \frac{3\lambda^2}{8s^3} \sum_{i=1}^m \lambda_i (x_i-1)^2 \,,$$

that is exactly the desired estimate. □

Appendix B. Concave 1-homogeneous functions

In this appendix we collect some basic facts about concave 1-homogeneous functions on a cone, as well as a couple of approximation results.

Remark B.1. Let $\Sigma \subseteq \mathbb{R}^n$ be a convex cone. Then:

- If $u, v : \Sigma \to \mathbb{R}$ are 1-homogeneous concave functions, then so is $\min(u, v)$.
- If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isometry and $u : \Sigma \to \mathbb{R}$ is 1-homogeneous and concave, then so is $u \circ T : T^{-1}(\Sigma) \to \mathbb{R}$.
- A function $u : \Sigma \to \mathbb{R}$ is 1-homogeneous and concave if and only if for every $x \in \Sigma$ there is $\xi_x \in \mathbb{R}^n$ such that $u(x) = \xi_x \cdot x$ and for any $y \in \mathbb{R}^n$ it holds $u(y) \leq \xi_x \cdot y$.

We first prove the following.

Lemma B.2. *Let $\Sigma', \Sigma \subseteq \mathbb{R}^n$ be two open convex cones such that $\Sigma' \cap \partial B_1 \subset\subset \Sigma \cap \partial B_1$. For any concave 1-homogeneous function $v : \Sigma \to \mathbb{R}$ and any $\varepsilon > 0$, there is a concave 1-homogeneous function $\widetilde{v} : \Sigma \to \mathbb{R}$ such that $\widetilde{v} \geq v$ in Σ, $\widetilde{v} = v$ on $\partial\Sigma$, $\|\widetilde{v} - v\|_{L^\infty(\partial B_1 \cap \Sigma')} < \varepsilon$, and $\widetilde{v}$ is smooth in Σ'.*

Proof. Fix an open convex cone $\Sigma'' \subseteq \mathbb{R}^n$ such that $\Sigma' \cap \partial B_1 \subset\subset \Sigma'' \cap \partial B_1 \subset\subset \Sigma \cap \partial B_1$. During the proof we will require some further properties on the cone Σ''.

To regularize a concave 1-homogeneous function we exploit the convolution with respect to the Haar measure (see [Fed69, p. 2.7]) on $SO(n)$. Let $\mu \in \mathcal{P}(SO(n))$ be the Haar measure and let $\rho : SO(n) \to \mathbb{R}$ be a smooth kernel, that is $\int \rho \, \mathrm{d}\mu = 1$, $\rho \geq 0$ and ρ is supported in a small neighborhood of the identity. Let us define

$$v''(x) \coloneqq \int_{SO(n)} v(T(x)) \rho(T) \, \mathrm{d}\mu(T) \,.$$

If ρ is supported in a sufficiently small region, then v'' is well-defined in Σ'' and smooth in it. Moreover it is concave and 1-homogeneous thanks to Remark B.1. Choosing appropriately the kernel ρ, it is also true that $\|v'' - v\|_{L^\infty(\partial B_1 \cap \Sigma'')} < \varepsilon/8$.

As observed in Remark B.1, for any $x \in \Sigma''$ there is $\xi_x \in \mathbb{R}^n$ such that $v''(x) = \xi_x \cdot x$ and $v''(y) \leq \xi_x \cdot y$ for any $y \in \Sigma''$. Let us define

$$v'(y) = \min_{x \in \partial B_1 \cap \Sigma'} \xi_x \cdot y \,.$$

It holds $v' = v''$ in Σ' and clearly v' is concave and 1-homogeneous in Σ (but could, a priori, take the value $-\infty$). Let us show that v' is *almost* above v in Σ. It holds $v' \geq v''$ in Σ'' and $v''(x) \geq v(x) - \frac{\varepsilon}{2}|x|$ for any $x \in \Sigma''$, thus $v'(x) \geq v(x) - \frac{\varepsilon}{2}|x|$ in Σ''. Fix $x \in \partial B_1 \cap \Sigma'$ and $z \in \Sigma \setminus \Sigma''$ with $|z| = 1$. Up to choosing Σ'' appropriately, we can assume that there is $0 < \lambda < \frac{1}{2}$ such that $y \coloneqq \lambda x + (1-\lambda) z \in \Sigma''$. From the properties of ξ_x and the concavity of v, it follows

$$\begin{aligned} v(z) &\leq \frac{v(y) - \lambda v(x)}{1-\lambda} \leq \frac{v''(y) + \frac{\varepsilon}{8}|y| - \lambda v''(x) + \lambda\frac{\varepsilon}{8}|x|}{1-\lambda} \leq \frac{\xi_x \cdot y + \frac{\varepsilon}{8}|y| - \lambda \xi_x \cdot x + \lambda\frac{\varepsilon}{8}|x|}{1-\lambda} \\ &= \xi_x \cdot z + \frac{\varepsilon}{8} \frac{|y| + \lambda|x|}{1-\lambda} \leq \xi_x \cdot z + \frac{\varepsilon}{2} \,. \end{aligned}$$

Hence, by definition of v', it holds $v(z) \le v'(z) + \frac{\varepsilon}{2}|z|$ for any $z \in \Sigma \setminus \Sigma''$ and, since we have already established the same inequality in Σ'', we deduce $v(x) \le v'(x) + \frac{\varepsilon}{2}|x|$ for any $x \in \Sigma$.

The function $x \mapsto v'(x) + \frac{\varepsilon}{2}|x|$ satisfies all the requirements of the statement apart from $\widetilde{v} = v$ on $\partial\Sigma$. To conclude let $\widetilde{v} : \Sigma \to \mathbb{R}$ be the minimum of all 1-homogeneous and concave functions $h : \Sigma \to \mathbb{R}$ such that $h \ge v$ in Σ and $h \ge v'(x) + \frac{\varepsilon}{2}|x|$ in Σ'. With this final step we obtain $\widetilde{v} = v$ on $\partial\Sigma$ and we do not lose any of the other properties. □

We will also need the following.

Lemma B.3. *Let $\Sigma', \Sigma \subseteq \mathbb{R}^n$ be two convex cones such that $\Sigma' \cap \partial B_1 \subset\subset \Sigma \cap \partial B_1$. For any nonnegative concave 1-homogeneous function $v : \Sigma \to [0, \infty)$ and any $\varepsilon > 0$, there is a concave 1-homogeneous function $\widetilde{v} : \Sigma \to \mathbb{R}$ such that $\widetilde{v} = 0$ on $\partial\Sigma$ and $\widetilde{v} = v$ in Σ'.*

Proof. Let $\widetilde{v} : \Sigma \to [0, \infty)$ be the infimum of all the concave 1-homogeneous functions $h : \Sigma \to \mathbb{R}$ such that $h \ge 0$ in Σ and $h \ge \widetilde{v}$ in Σ'. Thanks to the observation of Remark B.1, the function $\widetilde{v}$ is concave and 1-homogeneous. It is straightforward to check that $\widetilde{v} = v$ in Σ' and $\widetilde{v} = 0$ on the boundary of Σ. □

For completeness, let us conclude observing that a 1-homogeneous function v is concave if and only if its restriction on the sphere is *pseudo-concave* (either in the pointwise or differential sense).

Lemma B.4. *Let $\Sigma \subseteq \mathbb{R}^n$ be a convex cone and let $v : \Sigma \to \mathbb{R}$ be a 1-homogeneous function. The following statements are equivalent.*

(1) The function v is concave.

(2) For any unit-speed geodesic on the sphere $\gamma : [-s, t] \to \partial B_1 \cap \Sigma$ (with $s, t \ge 0$), it holds

$$v(\gamma(0)) \ge \frac{\sin(t)}{\sin(s+t)} v(\gamma(-s)) + \frac{\sin(s)}{\sin(s+t)} v(\gamma(t)) \,.$$

(3) The function v is twice-differentiable almost everywhere and, for any unit-speed geodesic on the sphere $\gamma : (-\varepsilon, \varepsilon) \to \partial B_1 \cap \Sigma$, if v is twice differentiable at $\gamma(0)$ then it holds

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} v(\gamma(0)) + v(\gamma(0)) \le 0 \,.$$

Proof. Since we do not use this result and the proof is standard, we give only a sketch of the proof.

The statements (1) and (3) are equivalent because of the following identity

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} v(\gamma(t)) = \nabla^2 v(\gamma)[\dot\gamma, \dot\gamma] + \nabla v(\gamma) \cdot \ddot\gamma = \nabla^2 v(\gamma)[\dot\gamma, \dot\gamma] - v(\gamma) \,,$$

where in the last step we used that v is 1-homogeneous.

If v_1 and v_2 satisfy (2), then also $\min(v_1, v_2)$ does. Therefore, since linear functions satisfy (2) (with equality), any 1-homogeneous concave function satisfies (2). It remains only to prove that (2) implies (1). Given two points $p, q \in \Sigma$, let $L : \mathbb{R}^n \to \mathbb{R}$ be a linear map such that $L(p) = v(p)$ and $L(q) = v(q)$. Given that L satisfies the equality in (2), for any $\lambda, \mu \ge 0$ with $\lambda + \mu = 1$, it holds

$$\frac{v(\lambda p + \mu q)}{|\lambda p + \mu q|} = v\Big(\frac{\lambda p + \mu q}{|\lambda p + \mu q|}\Big) \ge L\Big(\frac{\lambda p + \mu q}{|\lambda p + \mu q|}\Big) = \frac{\lambda v(p) + \mu v(q)}{|\lambda p + \mu q|}$$

which is the sought concavity of v. □

Appendix C. Indecomposable sets in $\mathbb{R}^2$ are approximated by connected sets

In this appendix we show that, in $\mathbb{R}^2$, an *indecomposable* set of finite perimeter can be approximated by *connected* smooth open sets (we prove an analogous result also in the weighted setting). Notice that, in higher dimension, any set of finite perimeter can be approximated by smooth connected open sets, while in $\mathbb{R}^2$ this is false. As a fundamental technical tool, we exploit the theory devised in [Amb+01]; the interested reader shall refer to that paper for a thorough study of indecomposable sets of finite perimeter.

Definition C.1. A set of finite perimeter $E \subseteq \mathbb{R}^2$ is indecomposable if it cannot be written as $E = E_1 \cup E_2$ with E_1, E_2 disjoint nonnegligibile sets of finite perimeter such that $\mathrm{Per}(E) = \mathrm{Per}(E_1) + \mathrm{Per}(E_2)$.

Proposition C.2. *Let $E \subseteq \mathbb{R}^2$ be an indecomposable set of finite perimeter with $|E| < \infty$. Then, there is a sequence $(\Omega_i)_{i\in\mathbb{N}}$ of bounded connected open sets with smooth boundary such that $|\Omega_i \triangle E| \to 0$ and $\mathrm{Per}(\Omega_i) \to \mathrm{Per}(E)$ as $i \to \infty$.*

Proof. Given a Jordan curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$, we denote with $\mathrm{int}(\gamma)$ the bounded connected component of $\mathbb{R}^2 \setminus \gamma$.

Thanks to [Amb+01, Corollary 1], there are $\gamma, (\gamma_i)_{i\in I}$ Jordan curves (with I at most countable) such that (up to negligible sets)

$$E = \mathrm{int}(\gamma) \setminus \bigcup_{i\in I} \overline{\mathrm{int}(\gamma_i)}\,,$$

and $\mathrm{Per}(E) = \mathscr{H}^1(\gamma) + \sum_{i\in I} \mathscr{H}^1(\gamma_i)$. Moreover $\mathrm{int}(\gamma_i) \subseteq \mathrm{int}(\gamma)$ for any $i \in I$ and $\mathrm{int}(\gamma_i) \cap \mathrm{int}(\gamma_j) = \emptyset$ for any $i \neq j$.

Thus, for any $\varepsilon > 0$, we can find a *finite* subset $I' \subseteq I$ such that $|E' \triangle E| < \varepsilon$ and $|\mathrm{Per}(E') - \mathrm{Per}(E)| < \varepsilon$, where

$$E' \coloneqq \mathrm{int}(\gamma) \setminus \bigcup_{i\in I'} \overline{\mathrm{int}(\gamma_i)}\,.$$

Notice that E' is an open connected set of finite perimeter (the connectedness follows from the indecomposability of E). Since $\varepsilon > 0$ can be chosen arbitrarily, we can directly assume that E is open and connected.

Let us now prove the statement for a connected open set E with finite perimeter. For any $k \in \mathbb{N}$, consider the sequence of open sets $(E_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$ defined as

$$E_k \coloneqq \left\{x \in E : \ d(x, E^{\mathsf{c}}) > \frac{1}{k}\right\}.$$

Let $\widetilde{E}_k$ be the connected component of E_k with the largest measure. Since E is open and $(E_k)_{k\in\mathbb{N}}$ is an increasing chain, it is not hard to check that $|\widetilde{E}_k \triangle E| \to 0$ as $k \to \infty$. Now, let $(\Omega_i)_{i\in\mathbb{N}}$ be a sequence of smooth open sets obtained taking a superlevel set of a convolution of χ_E, more precisely

$$\Omega_i \coloneqq \left\{x \in \mathbb{R}^2 : \chi_E * \left(i^2\eta(i\,\cdot)\right)(x) > t_i\right\},$$

where $\eta : \mathbb{R}^2 \to [0,1]$ is a smooth kernel ($\int \eta = 1$) and $(t_i)_{i\in\mathbb{N}}$ is a sequence of values $0 < t_i < 1$ such that the resulting Ω_i are smooth and $|\Omega_i \triangle E| \to 0$ and $\mathrm{Per}(\Omega_i) \to \mathrm{Per}(E)$. This approximation with smooth open sets is very standard, see [Mag12, Theorem 13.8] for the details.

Given $k \in \mathbb{N}$, it follows from the definition of Ω_i that, for any sufficiently large $i \in \mathbb{N}$, it holds $E_k \subseteq \Omega_i$. Choose an increasing sequence of indices $(i_k)_{k\in\mathbb{N}}$ such that $E_k \subseteq \Omega_{i_k}$ and let $\widetilde{\Omega}_{i_k}$ be the connected component of Ω_{i_k} that contains $\widetilde{E}_k$. Notice that, for any $k \in \mathbb{N}$, it holds

$$\widetilde{E}_k \subseteq \widetilde{\Omega}_{i_k} \subseteq \Omega_{i_k}\,,$$

hence, since $|\widetilde{E}_k \triangle E| \to 0$ and $|\Omega_{i_k} \triangle E| \to 0$, we have $|\widetilde{\Omega}_{i_k} \triangle E| \to 0$. Moreover $\mathrm{Per}(\widetilde{\Omega}_{i_k}) \leq \mathrm{Per}(\Omega_{i_k}) \to \mathrm{Per}(E)$, hence the sequence $(\widetilde{\Omega}_{i_k})_{k\in\mathbb{N}}$ satisfies all the requirements of the statement. □

Let us now give the definition of indecomposable set in the weighted setting and prove a proposition analogous to the latter one.

Definition C.3. Let $\Sigma \subseteq \mathbb{R}^2$ and $w : \Sigma \to [0,\infty)$ be as in (2.1). A set of finite w-perimeter $E \subseteq \Sigma$ is w-indecomposable if it cannot be written as $E = E_1 \cup E_2$ with E_1, E_2 disjoint nonneglegibile sets of finite w-perimeter such that $\mathrm{Per}_w(E) = \mathrm{Per}_w(E_1) + \mathrm{Per}_w(E_2)$.

Remark C.4. For a set $E \subseteq \Sigma$ such that $\max(|E|, w(E), \mathrm{Per}(E), \mathrm{Per}_w(E)) < \infty$, being indecomposable is equivalent to being w-indecomposable.

Proposition C.5. *Let $\Sigma \subseteq \mathbb{R}^2$ and $w : \Sigma \to [0,\infty)$ be as in (2.1), with the additional assumption $w \equiv 0$ on $\partial\Sigma$, and let $E \subseteq \Sigma$ be a w-indecomposable set of finite w-perimeter with $w(E) < \infty$. Then, there is a sequence $(\Omega_i)_{i\in\mathbb{N}}$ of bounded connected open sets with smooth boundary such that $\Omega_i \subset\subset \Sigma$ and $w(\Omega_i \setminus E) \to 0$ and $\mathrm{Per}_w(\Omega_i) \to \mathrm{Per}_w(E)$ as $i \to \infty$.*

Proof. As a first step, we prove that E can be approximated by a sequence $(F_k)_{k\in\mathbb{N}} \subseteq \Sigma$ of sets of finite w-perimeter such that $F_k \subset\subset \Sigma$. Since the proof is standard and technical we will skip some details.

We can find a sequence of radii $r_k \to \infty$ such that $E_k := E \cap B_{r_k}$ is a set of finite w-perimeter and $\mathrm{Per}_w(E_k) \to \mathrm{Per}_w(E)$ as $k \to \infty$. Thanks to [Amb+01, Theorem 1] (adapting their arguments to our setting is straightforward), we can define $\widetilde{E}_k$ as the largest w-indecomposable component of E_k (i.e., $w(\widetilde{E}_k) \geq w(C)$ for any w-indecomposable component of E_k). Since $E_k \nearrow E$ and E is indecomposable, it follows that $w(\widetilde{E}_k \triangle E) \to 0$ and $\mathrm{Per}_w(\widetilde{E}_k) \to \mathrm{Per}_w(E)$ as $k \to \infty$. Now, fix a vector $v \in \Sigma$ and define $F_k := \varepsilon_k v + \widetilde{E}_k$, where $\varepsilon_k > 0$ is such that $w(F_k \triangle \widetilde{E}_k) \to 0$ and $|\mathrm{Per}_w(F_k) - \mathrm{Per}_w(\widetilde{E}_k)| \to 0$ as $k \to \infty$ (here we need the assumption $w \equiv 0$ on $\partial\Sigma$). The sets of finite w-perimeter F_k are compactly contained in Σ and satisfy $w(F_k \triangle E) \to 0$ and $\mathrm{Per}_w(F_k) \to \mathrm{Per}_w(E)$ as $k \to \infty$.

Taking into account the approximation result we have just shown, we can assume, without loss of generality, that $E \subset\subset \Sigma$. To conclude, it is sufficient to repeat the proof of Proposition C.2 (notice that the weight is bounded away from 0 and ∞ in an open set A such that $E \subset\subset A \subset\subset \Sigma$). □

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