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Unstable operations on K-theory for singular schemes

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Abstract

We study the algebraic structures, such as the lambda ring structure, that arise on K-theory seen as an object of some homotopy categories coming from model categories of simplicial presheaves. In particular, we show that if we take the Jardine local injective model category of simplicial presheaves over the category of divisorial, hence possibly singular, schemes with respect to the Zariski topology, these structures are in bijection with the ones we have on K_0 seen as a presheaf of sets. This extends some results of Riou ([Rio10]) from smooth schemes to singular ones and does not require $\mathbb{A}^1$-invariance. We also discuss similar results for symplectic K-theory.

Contents

1 Introduction

Algebraic structures and operations on K-theory have played a very important role since Grothendieck invented K-theory. Indeed, he associated to any scheme X the so called algebraic K-theory group $K_0(X)$ in order to state and prove the most general form of what is nowadays known as the Grothendieck-Riemann-Roch theorem (see [SGA71]). To prove such a theorem, $K_0(X)$ had to be given the structure of a *lambda ring*. This notion was envisioned by Grothendieck and explored for the first time in [SGA71], see [Yau10] for a modern introduction. Loosely speaking, a lambda ring is the datum of a ring R together with a family of functions $\lambda^i : R \to R$ satisfying some formal properties axiomatising the behaviour of the exterior powers in algebra. This formalism and this structure revealed itself to be very important also in other contexts. Given a lambda ring R, we can define the so called Adams operations, namely a family of *ring homomorphisms* $\psi^j : R \to R$ whose properties were famously used by Adams to give a short and elegant solution to the Hopf invariant one problem in topology. After higher algebraic K-theory was defined and explored in seminal works of Quillen ([Qui67]), Waldhausen ([Wal85]) and Thomason ([TT90]) among others, it became meaningful to study which algebraic structure can be given to the higher algebraic K-theory groups (or space) of a scheme (or of a ring). While the K-theory space carries a natural H-group structure and a natural multiplicative structure as well (see [Wal85, page 342] and [Wei81]), the task of defining lambda operations has proved to be more challenging. They have been defined at various levels of generality by several authors, Hiller, Kratzer, Gillet, Soulé, Lecomte, Levine and Grayson among others, using a blend of representation theory and homotopy theory. More recently, Harris, Köck and Taelman in [HKT17] defined lambda operations using the explicit presentation of the higher K-theory groups given by Grayson ([Gra12]). All these works allowed to study lambda operations for the higher K-theory groups of possibly any Noetherian scheme X, leading also to Grothendieck-Riemann-Roch theorems as the one proved in [Sou85] for regular schemes, but in general the constructions are quite involved and difficult to handle. For example, Grayson was not able to verify the lambda ring axioms using his definition (see [Gra89]) and it is difficult to compare all the a priori different operations with each other. After the introduction of $\mathbb{A}^1$-homotopy theory, as developed by Morel and Voevodsky in [MV99], Riou introduced a new powerful tool to explore algebraic structures on K-theory, seen as an object of the unstable motivic homotopy category. For a given regular divisorial (see [SGA71, II 2.2.3] or Definition A.1) Noetherian scheme S, we shall denote by $\mathcal{H}^{\mathrm{Div}}(S)$ the unstable motivic homotopy category over S ([MV99]) built starting from the category DSm_S of smooth and divisorial schemes over S. From now until the end of the introduction, we let K to denote Thomason's K-theory simplicial presheaf. The starting point of this paper is then the following result:

Theorem 1.1. *(Riou, [Rio10, Theorem 1.1.4]). If S is a Noetherian divisorial regular scheme, then for any $n \in \mathbb{N}$ one has the following isomorphism*

$$\mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n, K) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(K_0(-)^n, K_0(-))$$

where $K_0(-)$ is the presheaf of sets associating to every smooth scheme X its algebraic K-theory $K_0(X)$ and $\mathbf{Pre}(\mathrm{DSm}_S)$ is the category of presheaves of sets on DSm_S.

This result allows us to define algebraic structures on K seen as an object of $\mathcal{H}^{\mathrm{Div}}(S)$ lifting the ones we have on K_0. In particular, this gives a lambda ring structure to $K \in \mathcal{H}^{\mathrm{Div}}(S)$ and also allows to see that such structure is uniquely determined by its behaviour on K_0 ([Rio10, Theorem 2.3.1]). The lambda operations obtained can be seen to induce operations on the higher K-theory groups of any scheme in DSm_S. The proof of this theorem relies on the fundamental fact that $K \cong \mathbb{Z} \times \mathrm{Gr}$ in $\mathcal{H}^{\mathrm{Div}}(S)$, together with the fact that for divisorial schemes Thomason's K-theory is equivalent to Quillen's one, Jouanolou's trick is available (A.6), K-theory satisfies Nisnevich descent and is $\mathbb{A}^1$-invariant. The drawback of this approach is that it can only give structures to *regular* Noetherian schemes, while operations have been studied in more general contexts, e.g. [Lev97] or [GS99]. The aim of this work is then to extend the result of Riou to a larger class of possibly singular schemes. Inspired by the seminal work of Thomason and Trobaugh [TT90], it seems natural to consider an extension to the class of divisorial schemes, also called schemes with an ample family of line bundles. Indeed, these schemes are quite general and important: any projective or quasi-projective scheme is divisorial, for example, and they do satisfy the resolution property, see [Tot04] for an extensive discussion of the importance of this property. For such schemes the K-theory of perfect complexes agrees with the one of vector bundles (i.e. with Quillen's K-theory) and we retain desirable descent properties such as Zariski descent. However, these schemes can be singular and therefore we loose $\mathbb{A}^1$-invariance so that we cannot use the argument of Riou any more to show a result analogous to Theorem 1.1. The main goal of this paper is to show that we still have a perfect analogue of Riou's Theorem 1.1 in this setting. Given a Noetherian base scheme S we denote the category of divisorial schemes of finite type over S by DSch_S, the category of Noetherian schemes of finite type over S by Sch_S and we consider the model category $\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S)$ ($\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)$) of simplicial presheaves over DSch_S (Sch_S) with the choice of the Jardine local model structure with respect to the Zariski topology. The main result of this paper is the following:

Theorem 1.2 (3.5, 4.6, 5.11). *If S is a regular quasi-projective scheme over a Noetherian affine scheme R we have, applying π_0, that for any $n \in \mathbb{N}$*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(K^n, K) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(K_0(-)^n, K_0(-))$$

where $\mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{DSch}_S))$ denotes the homotopy category of $\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{DSch}_S)$. Moreover, for any $n \in \mathbb{N}$ we have

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(K^n, K) \cong \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n, K)$$

Finally, for any $n \in \mathbb{N}$ we have

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{Sch}_S))}(K^n, K) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(K^n, K)$$

Similar statements, if $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$, holds for symplectic K-theory KSp, i.e. $\mathrm{GW}^{[2]}$ in the terminology of [Sch17].

This theorem extends the result of Riou from regular schemes to singular ones and states that all the operations on K-theory seen as an object of $\mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{DSch}_S))$ are uniquely determined by their behaviour on the level of Quillen's algebraic K-theory presheaf $K_0 \in \mathbf{Pre}(\mathrm{DSch}_S)$. Moreover, their behaviour on the affine schemes suffices, see 4.12 for a precise statement. The key ingredients in the proof of this theorem are methods inspired from the classical localisation with respect to homology and the verification that K_0^n satisfies the so called property (P) with respect to the fully faithful inclusion $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$ (4.4). The proprerty (P) is introduced and studied in Section 4 (4.4) and it is studied in the context of K-theory and of GW theory in Section 6. For a given small category $\mathcal{C}$, a full subcategory $\mathcal{A} \subseteq \mathcal{C}$ and a fixed presheaf $F \in \mathbf{Pre}(\mathcal{C})$ the property (P) gives a sufficient condition for the restriction map $\mathrm{Hom}_{\mathbf{Pre}(\mathcal{C})}(F, G) \to \mathrm{Hom}_{\mathbf{Pre}(\mathcal{A})}(F, G)$ to be injective for any $G \in \mathbf{Pre}(\mathcal{C})$. We remark that the property (P) has to be satisfied by F, G may be chosen arbitrarily. It is used not only to prove our Main Theorem 1.2 for both K-theory and Symplectic K-theory, denoted KSp, and to prove Theorem 1.5 below concerning the Picard presheaf $\mathrm{Pic}(-)$, but can be used also for Hermitian K-theory, i.e. for the K-theory of symmetric forms, GW, i.e. $\mathrm{GW}^{[0]}$ in the terminology of [Sch17]. In particular we have the following:

Theorem 1.3 (4.5 together with 6.2 and 6.5). *Let S be a quasi-projective Noetherian scheme of finite type over a Noetherian affine scheme R, having 2 invertible if considering K theories of forms*[1]*, and F one of the functors $K_0, \mathrm{KSp}_0, \mathrm{Pic}(-)$ or GW_0 in $\mathbf{Pre}(\mathrm{DSch}_S)$. Then for any $G \in \mathbf{Pre}(\mathrm{DSch}_S)$ and any $n \in \mathbb{N}$ the natural restriction map*

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(F^n, G) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(F^n, G)$$

is injective.

In particular, the previous theorem applies to $F = G = \mathrm{GW}_0$. A result instrumental to the proof of the above theorem for the K-theories of forms is the following result which generalises [Zan20, Theorem 5.5] and allows to see that any form on a divisorial scheme comes as a pullback of a form on a *smooth* scheme:

Theorem 1.4 (6.3). *Assume X is a divisorial scheme of finite type over a scheme S which is quasi-projective over a Noetherian affine scheme R where 2 is invertible. Then given a finite number of ϵ-inner product spaces over X, $V_1 = (\mathcal{E}_1, \varphi_1), ..., V_n = (\mathcal{E}_n, \varphi_n)$, there is a divisorial smooth scheme Y_V over S and ϵ-inner product spaces $V_{1,Y_V}, ..., V_{n,Y_V}$ over it together with a morphism $\psi_V : X \to Y_V$ such that $\psi_V^*(V_{i,Y_V}) \cong V_i$ for every $i = 1, ..., n$. If X and S are affine schemes, then we can take Y_V to be affine.*

The methods used in the proof of Theorem 1.2 can be refined to obtain the following theorem. We denote the presheaf associating to any scheme its Picard group with $\mathrm{Pic} \in \mathbf{Pre}(\mathrm{DSch}_S)$ and $K_i \in \mathbf{Pre}(\mathrm{DSch}_S)$ will denote the ith higher K-theory presheaf.

Theorem 1.5 (7.18, 7.19). *Let be S a regular quasi-projective scheme over a Noetherian affine scheme. Then, for any $i \in \mathbb{N}$ we have the following isomorphisms*

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S, \mathrm{Ab})}(K_0, K_i) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(\mathrm{Pic}, K_i) \cong \lim_n K_i(\mathbb{P}^n)$$

where we have denoted the category of presheaves over DSch_S with values in abelian groups by $\mathbf{Pre}(\mathrm{DSch}_S, \mathrm{Ab})$.

This result generalises to singular schemes [Rio10, Proposition 5.1.1] which is valid only for regular ones and which was instrumental in the definition of both the homotopy invariant K-theory spectrum in the stable motivic homotopy category and in the construction of the Chern character as studied in *op.cit.* Given the simplicial presheaf K we define for every $n \geq 0$ and any $\mathcal{X} \in \mathrm{s}\mathbf{Pre}(\mathrm{Sch}_S)$,

$$K_n(\mathcal{X}) := \mathrm{Hom}_{\mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)}(\mathbb{S}^n \wedge \mathcal{X}_+, K)$$

where $\mathbb{S}^n$ is the standard simplicial nth-sphere. Using Theorem 1.2 we are finally able to construct interesting algebraic structures on the higher K-groups of schemes.

Theorem 1.6 (7.1, 7.14). *Let S be a regular Noetherian scheme. The K-theory simplicial presheaf K on Sch_S has a lambda ring structure in $\mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$. In addition:*

- *for any $\mathcal{X} \in \mathrm{s}\mathbf{Pre}(\mathrm{Sch}_S)$ and for any $n \in \mathbb{N}$, lambda, Adams and γ-operations $K_0(-) \to K_0(-)$ lift to maps in the pointed homotopy category $\mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)$ so that they act, by composition, on the groups $K_n(\mathcal{X})$. The relations that hold at level of K_0 are true even in this setting.*
- *The lambda ring structure on K induces, for any $\mathcal{X} \in \mathrm{s}\mathbf{Pre}(\mathrm{Sch}_S)$, a graded ring structure on the graded $K_0(\mathcal{X})$-module*

$$K_*(\mathcal{X}) := \bigoplus_{n \in \mathbb{N}} K_n(\mathcal{X})$$

In addition, the lambda operations just defined give, for any $X \in \mathrm{Sch}_S$, a well defined functorial lambda ring structure on the $K_0(X)$-module $K_*(X) := \bigoplus_{n \in \mathbb{N}} K_n(X)$ where the product of two elements of degree ≥ 1 is set to be zero.

[1]Note we do not assume R or S to be regular

As an additional consequence, we can prove formally an Adams-Riemann-Roch theorem for projective l.c.i morphisms between divisorial schemes, see Theorem 7.17 below. Finally, we wish to point out that the study of the operations is at the center of many recent developments in the context of GW theory. Indeed, in 2018 Zibrowius ([Zib18]) defined lambda operations for GW_0 of any scheme of finite type over a field (with 2 invertible) and his results have been generalised by Fasel and Haution ([FH20]) in 2020 to divisorial $\mathbb{Z}[1/2]$-schemes. The latter authors, and independently Bachmann and Hopkins ([BH20]) discovered in 2020 how to define Adams operations on the KO-spectrum, a.k.a. homotopy invariant Hermitian K-theory, in the stable motivic homotopy category, although under suitable assumptions (see *op. cit.* for details), thus suggesting that there is still room for many further developments in the area.

We shall now describe the contents of the paper.

In Section 2 we start with recollecting some facts concerning the Bousfield-Kan completion for simplicial sets as in [BK72] and in [GJ09], and we extend the notion of completion and $\mathbb{Z}$-completeness to the context of simplicial presheaves, which is not well known and documented in literature. Using these facts we prove (2.17) that the maps from BGL to K in $\mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$ are in bijection with those from BGL^+ to K. Note we do not need to assume that our schemes are divisorial.

In Section 3, building on the results of Section 2, we show that $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{Sch}_S))}(K^n, K) \cong \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n, K)$ for any $n \in \mathbb{N}$. This can be seen as a consequence of some general machinery we develop and that allows us to handle also symplectic K-theory (3.8).

In Sections 4 and 5 we prove Theorem 1.2 and its analogue for symplectic K-theory. We also prove that these operations only depend on their behaviour on affine schemes (4.10 and 5.12). These proofs rely essentially on showing that K_0 and KSp_0 satisfy the so called property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$ (4.4), which is done in Sections 6, where the reader can also find the proof of Theorem 1.4.

In the last section we discuss the algebraic structures determined by our Main Theorem 1.2 on algebraic K-theory, proving Theorem 1.6 (7.6 and 7.14 in the text). This is done by giving a general description of the lambda ring structure arising on the homotopy groups of a lambda ring in some suitable category of spaces (see 7.10). We also provide an Adams-Riemann-Roch Theorem for projective l.c.i. morphisms between divisorial schemes involving the Adams operations defined (7.17). Finally, we extend [Rio10, Proposition 5.1.1] to singular schemes proving Theorem 1.5.

There are 3 appendices. In Appendix A we collect some useful and not widespread facts about divisorial schemes. In Appendix B we review the argument that Riou used in order to prove his main theorem 1.1.4 in [Rio10] adapting it to our discussion is such a way it is immediately applicable to GW theory. Here we follow more closely Riou's thesis [Rio06], which is available only in French. In Appendix C we collect some facts about lambda rings and we study this notion in the context of an arbitrary category with finite products and a terminal object.

Acknowledgements. The main results contained in this work (essentially) appeared in the author's PhD thesis at the University of Warwick [Zan19], although some of the proofs in *op. cit.* are slightly different (for example, there is no explicit mention of the property (P)). The author is grateful to his advisor Marco Schlichting for suggesting him the problem of extending Riou's results from smooth schemes to possibly singular ones and for sharing many insights during several enlightening conversations. The author is also grateful to John Greenlees, Jens Hornbostel, Bernhard Köck and Heng Xie for many friendly mathematical discussions and for useful comments on previous drafts of this work and to Denis-Charles Cisinski, Christian Dahlhausen, Adeel Khan, Alberto Navarro Garmendia, Charles Weibel and Marcus Zibrowius for useful discussions and for their interest in this project. Finally, the anonymous referee deserves special thanks for providing the author with a plethora of very useful comments and remarks that led to significant improvements of the paper's quality. The author was supported by the EPSRC grant EP/M508184/1.

1.1 Assumptions and notations

All schemes will be always assumed to be Noetherian of finite dimension unless otherwise stated. The underlying category of any Grothendieck site considered is assumed to be small and we will always tacitly assume that we have choosen an universe big enough to avoid any set theoretic related issue. Whenever we will say that a base scheme S is regular, we will mean that its local rings are regular [Sta18, Tag 02IS] and we say that a scheme X is smooth over a base S if its structure map is smooth ([GD67, IV 6.8.6, 17.3.1], [GW10, 6.14], [Sta18, Tag 01V5]) so that we do not require such schemes to be separated as it is sometimes assumed.

Notations 1.7. For a given base scheme S, we shall denote the category of schemes of finite type over S by Sch_S and its full subcategory of smooth schemes over S by Sm_S. We shall denote the full subcategories of divisorial and separated schemes in Sch_S by DSch_S and Sch_S^{Sep}, respectively. We shall use DSm_S and Sm_S^{Sep} when we furthermore ask our schemes to be smooth over S. We will denote the full subcategory of Sch_S of affine schemes (in the absolute sense, i.e. over $\mathrm{Spec}(\mathbb{Z})$) by Aff_S and the full subcategory of Sm_S of smooth affine (over $\mathrm{Spec}(\mathbb{Z})$) schemes by SmAff_S. Throughout this article, given any model category $\mathcal{C}$, we will denote the *homotopy category* of $\mathcal{C}$ by $\mathrm{Ho}(\mathcal{C})$. If we will speak about pointed homotopy categories of a given model category $\mathcal{C}$ we will mean the homotopy category of the model category obtained by considering the pointed category $\mathcal{C}_\bullet$ and giving to it the pointed model structure induced from $\mathcal{C}$ (see [Hov99, Proposition 1.1.8]). We will denote the global projective or injective model structure on a given category of simplicial diagrams by $\mathcal{P}$ or $\mathcal{I}$. Accordingly, we shall denote the model categories of simplicial presheaves over a small category $\mathcal{C}$ endowed with the global projective or the global injective model structure by $\mathbf{sPre}^{\mathcal{P}}(\mathcal{C})$ or $\mathbf{sPre}^{\mathcal{I}}(\mathcal{C})$, respectively. Whenever we handle categories of simplicial presheaves over a Grothendieck site $(\mathcal{C}, \tau)$ we will denote the model categories of simplicial presheaves over $\mathcal{C}$ with the local projective or injective model structure relative to the Grothendieck topology τ by $\mathbf{sPre}_\tau^{\mathcal{P}^l}(\mathcal{C})$ or $\mathbf{sPre}_\tau^{\mathcal{I}^l}(\mathcal{C})$. If moreover A is a class of maps in $\mathcal{C}$ for which the left Bousfield localisation of these model categories at A exists as a model category we shall employ the notation $\mathbf{sPre}_\tau^{\mathcal{I}^l_A}(\mathcal{C})$ and $\mathbf{sPre}_\tau^{\mathcal{P}^l_A}(\mathcal{C})$ for the Bousfield localisation of $\mathbf{sPre}_\tau^{\mathcal{I}^l}(\mathcal{C})$ and $\mathbf{sPre}_\tau^{\mathcal{P}^l}(\mathcal{C})$, respectively. Thus, for example, $\mathbf{sPre}_{\mathrm{Zar}}^{\mathcal{I}^l}(\mathrm{Sch}_S)$ will denote the model category of simplicial presheaves over Sch_S where we consider the injective local model structure with respect to the Zariski topology. If the choice of the injective local structure is assumed, we will use the simpler notation $\mathbf{sPre}_\tau(\mathcal{C})$. In addition, we will use the notation $\mathcal{H}(S)$, $\mathcal{H}^{\mathrm{Div}}(S)$, $\mathcal{H}^{\mathrm{Sep}}(S)$ and $\mathcal{H}^{aff}(S)$ for the unstable motivic homotopy categories over Sm_S, DSm_S, Sm_S^{Sep} and SmAff_S respectively (see [MV99] or [AHW17]).

2 Recollections: completion for simplicial presheaves and consequences

Given a category $\mathcal{C}$ with a terminal object, we shall denote its category under the terminal object by $\mathcal{C}_\bullet$. We denote the category of (pointed, pointed and connected) simplicial sets by $\mathcal{S}$ ($\mathcal{S}_\bullet$, $\mathcal{S}^c_\bullet$), while Top ($\mathrm{Top}_\bullet$) will denote a category of (pointed) topological spaces which is a convenient category for homotopy theory, such as the category of compactly generated Hausdorff spaces (see for examples [Vog71]). We use the term space for an object of either $\mathcal{S}$ or Top. We shall freely use the language of model categories, see [Hir03],[Hov99],[BK72], [GJ09], [Qui67], [DS95]. When dealing with a *simplicial* model category $\mathcal{C}$, we will denote its simplicial mapping space ([Hir03, 9.1.2]) by $\mathfrak{Map}_{\mathcal{C}}(-,-)$.

Definition 2.1. Given a model category $\mathcal{C}$ we will say that a pointed object in it is a H-space (group) if it is a monoid (group) object in the pointed homotopy category $\mathrm{Ho}(\mathcal{C}_\bullet)$.

In this section we will be primarily concerned with H-groups in the category of spaces. For a given commutative ring R, R-nilpotent spaces ([BK72, III 5.2]) are assumed to be path connected and we shall keep this assumption to be consistent with the literature. Recall that Bousfield and Kan defined for every solid ring R, i.e. a commutative unital ring such that the multiplication $R \otimes_{\mathbb{Z}} R \to R$ is an iso, see [BK72, page 20], a functor $R_\infty : \mathcal{S} \to \mathcal{S}$, the so called Bousfield-Kan completion, see [BK72, I 4.2], [BK71] and [GJ09]. The main feature of this functor is that if a simplicial map $f : X \to Y$ induces an isomorphism on $H_*(-, R)$, then $R_\infty f$ is a weak equivalence (see [BK72, I 5.5]). In addition, recall that for any $X \in \mathcal{S}$, $R_\infty X$ is fibrant ([BK72, I, 4.2]).

Definition 2.2. ([BK72, I 5.1]). Consider a solid ring R. A simplicial set $X \in \mathcal{S}$ is called R-complete if the map $i_X : X \to R_\infty X$ is a weak equivalence.

Theorem 2.3. *([BK72, III 5.4] or [BK71, 4.2]). Consider a solid ring R. Every R-nilpotent space $X \in \mathcal{S}^c_\bullet$ is R-complete.*

We define the connected components of a simplicial set as follows (in some literature the definition is slightly different, see [GJ09]). Let $v_\alpha \in X_0$, define X_α as the smallest subcomplex of X such that its zero skeleton consists of vertices w with the property $w \sim v_\alpha$ in $\pi_0|X|$. One can see that $\pi_0(X) = \mathrm{colim}(\ X_1 \overset{d_1}{\underset{d_0}{\rightrightarrows}} X_0\ )$. This definition gives us for every simplicial set X a decomposition (see [Lur19, Subsection 00G5] where this decomposition appears as Proposition 1.1.6.13 Tag 00GJ)

$$X \cong \coprod_{v_\alpha \in \pi_0 X} X_\alpha =: \coprod_\alpha X_\alpha$$

Therefore, using [BK72, I.7.1-7.5] we get:

Proposition 2.4. *Let be X any space such that its connected components are nilpotent or $\mathbb{Z}$-complete. Then X is $\mathbb{Z}$-complete.*

We immediately obtain the following corollary:

Corollary 2.5. *Any H-group is $\mathbb{Z}$-complete. (This is true both in the category of simplicial sets and in the category of topological spaces.)*

Proof. Indeed, if $R = \mathbb{Z}$, $X \in \mathcal{S}^c_\bullet$ is $\mathbb{Z}$-nilpotent if it is nilpotent and if $X \in \mathcal{S}^c_\bullet$ is simple then it is nilpotent, see [MP12, page 49] and [Spa95, page 384] for the definition of a simple space. Moreover, any path connected H-space is simple ([Spa95, Theorem 9 page 384]) and all the path connected components of every H-group are homotopy equivalent ([Hat02, page 291], [Dug66, page 387]). Therefore Proposition 2.4 allows us to conclude. □

We also have the following proposition of independent interest which is proved in [BK72, II.2.7] (see also [GS99]):

Proposition 2.6. *Given a solid ring R, every simplicial R-module is R-complete.*

We shall use now the notion of localisation with respect to homology. We are interested in $h_*^{\mathbb{Z}}$-localisations, i.e. into localisation with respect to integral homology, see [Bou75] or [GJ09, Chapter X]. For $\mathcal{S}$, the $h_*^{\mathbb{Z}}$-local model structure will be the one where an object is fibrant if and only if it is a fibrant simplical set Y which is $h_*^{\mathbb{Z}}$-local ([GJ09, X Corollary 3.3]), i.e. for every map of simplicial sets $f : X \to Z$ such that $H_*(f, \mathbb{Z})$ is an isomorphism, then the induced map $\mathfrak{Map}_{\mathcal{S}}(Z, Y) \to \mathfrak{Map}_{\mathcal{S}}(X, Y)$ is a weak equivalence. We start with the following:

Lemma 2.7. *Let X be a simplicial set. Then $\mathbb{Z}_\infty X$ is $h_*^{\mathbb{Z}}$-local.*

This is proved in [GJ09, X Remark 3.7]. The following lemma then follows from the properties of the simplicial mapping space together with [Hir03, Corollary 9.3.3].

Lemma 2.8. *Any fibrant $\mathbb{Z}$-complete simplicial set X is also $h_*^{\mathbb{Z}}$-local.*

To make use of the notion of completion in this paper, we will need functors valued in simplicial sets. Suppose I is a small category, and consider the category of simplicial presheaves on it, a.k.a. the category of functors $I^{op} \to \mathcal{S}$, denoted $\mathbf{sPre}(I)$. We can put several model structures on this category, general references are [BK72], [Jar87], [Jar04], [Jar15], [Dug01b].

- The Bousfield-Kan projective global model structure $\mathcal{P}$ ([BK72]) where weak equivalences are sectionwise weak equivalences, fibrations are sectionwise fibrations and cofibrations are induced by LLP (Left Lifting Property).
- The injective Heller global model structure $\mathcal{I}$: as before but in this case the cofibrations are defined sectionwise and fibrations by lifting property.

Both of these model structures are simplicial. We now focus on the $\mathcal{P}$-model structure on $\mathbf{sPre}(I)$, unless otherwise stated.

Definition 2.9. Given a small category I, an object X of $\mathbf{sPre}(I)$ is called $\mathbb{Z}$-complete if for every $i \in I$, $X(i)$ is a $\mathbb{Z}$-complete simplicial set.

This definition will turn out to be very useful because of the following variation of a theorem by Levine. Given a small category I, we say that a morphism $f : X \to Y$ in $\mathbf{sPre}(I)$ induces $H_*(-, \mathbb{Z})$-isomorphisms sectionwise if for every $i \in I$ the map $f(i) : X(i) \to Y(i)$ induces an $H_*(-, \mathbb{Z})$-isomorphism.

Theorem 2.10. *Given any small category I, assume that Z is a $\mathcal{P}$-fibrant object of $\mathbf{sPre}(I)$ such that for every $i \in I$, $Z(i)$ is a $h_*^{\mathbb{Z}}$-local simplicial set. Then given a map of $\mathcal{P}$-cofibrant objects $f : X \to Y$ inducing $H_*(-, \mathbb{Z})$-isomorphisms sectionwise, we have that the map*

$$f^* : \mathfrak{Map}_{\mathbf{sPre}(I)}(Y, Z) \to \mathfrak{Map}_{\mathbf{sPre}(I)}(X, Z)$$

is a weak equivalence.

Proof. For every objects i, j of I, since $Z(j)$ is $h_*^{\mathbb{Z}}$-local, we have that

$$f^*(i) : \mathfrak{Map}_{\mathcal{S}}(Y(i), Z(j)) \to \mathfrak{Map}_{\mathcal{S}}(X(i), Z(j))$$

is a weak equivalence. Hence the result follows from Corollary B.4 of [Lev97]. □

We make use of this Theorem thanks to the following lemma:

Lemma 2.11. *Given a $\mathbb{Z}$-complete simplicial presheaf X, there exists a map $\varphi_X : X \to X_{h_*^{\mathbb{Z}} f}$ which is a sectionwise weak equivalence and such that $X_{h_*^{\mathbb{Z}} f}(i)$ is $h_*^{\mathbb{Z}}$-local for any $i \in I$.*

Proof. We define $X_{h_*^{\mathbb{Z}} f}$ by applying sectionwise the $\mathbb{Z}_\infty$-completion functor. $X_{h_*^{\mathbb{Z}} f}$ is then sectionwise $h_*^{\mathbb{Z}}$-local by 2.7. □

We can now start to exploit the usefulness of Theorem 2.10. We start with the following Proposition:

Proposition 2.12. *For a given small category I, assume $X \in \mathbf{sPre}(I)$ is $\mathcal{P}$-fibrant and $\mathbb{Z}$-complete. If $f : Y \to Y'$ is a map inducing $H_*(-, \mathbb{Z})$-isomorphisms sectionwise, one has that*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{P}}(I))}(Y', X) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{P}}(I))}(Y, X)$$

Proof. We can assume that both Y and Y' are $\mathcal{P}$-cofibrant after applying the $\mathcal{P}$-cofibrant replacement functor. The result then follows from the characterization of $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{P}}(I))}(-, -)$ as $\pi_0 \mathfrak{Map}(-, -)$ (if the first entry is cofibrant and the second fibrant) and Theorem 2.10. □

Consider the case where $I = (\mathcal{C}, \tau)$ is a Grothendieck site, i.e. a small category $\mathcal{C}$ together with the choice of a specified Grothendieck topology τ, see [SGA72] or [Jar15] for a discussion in this context. One can put model structures on the category $\mathbf{sPre}(\mathcal{C})$ such that weak equivalences becomes local weak equivalences, see [Jar15, page 64], for example. The most known is the Jardine's injective local model structure (described in [Jar86] or [Jar15]), denote it by $\mathcal{I}^l$, where all presheaves are cofibrant. We shall denote this model category by $\mathbf{sPre}_\tau^{\mathcal{I}^l}(\mathcal{C})$. The second one is the Blander's local projective model structure (described for example in [Dug01b] or [Bla01]), denote it by $\mathcal{P}^l$. We denote this model category by $\mathbf{sPre}_\tau^{\mathcal{P}^l}(\mathcal{C})$. These two structures are homotopy equivalent (see [Dug01b] for an explanation or [DHI04] for a full proof) and $\mathcal{P}^l$-fibrant objects are also $\mathcal{P}$-fibrant, since $\mathcal{P}^l$ is obtained from $\mathcal{P}$ by Bousfield localising at the class of all hypercovers ([DHI04, Corollary 6.3], [Dug01b, Definition 5.4]). We remind that an object is $\mathcal{P}^l$-fibrant if and only if it is $\mathcal{P}$-fibrant and it satisfies descent as it is explained in [DHI04] or [AHW17]. One then get:

Corollary 2.13. *Suppose we are given a Grothendieck site $(\mathcal{C}, \tau)$. Let X be a simplicial presheaf which is $\mathcal{P}^l$-fibrant and $\mathbb{Z}$-complete. If $f : Y \to Y'$ is a map inducing $H_*(-, \mathbb{Z})$-isomorphisms sectionwise, one has*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_\tau^{\mathcal{P}^l}(\mathcal{C}))}(Y', X) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_\tau^{\mathcal{P}^l}(\mathcal{C}))}(Y, X) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_\tau^{\mathcal{I}^l}(\mathcal{C}))}(Y, X) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_\tau^{\mathcal{I}^l}(\mathcal{C}))}(Y', X)$$

Proof. After applying the $\mathcal{P}$-cofibrant replacement functor, we can also assume that both Y and Y' are $\mathcal{P}$-cofibrant. In addition, $\mathcal{P}^l$-fibrant objects are in particular $\mathcal{P}$-fibrant. So $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_\tau^{\mathcal{P}^l}(\mathcal{C}))}(Y, X) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_\tau^{\mathcal{P}}(\mathcal{C}))}(Y, X)$ (same for Y'). Then the first isomorphism follows from 2.12 and the last two isomorphisms follow from the fact that the local injective and the local projective model structures are Quillen equivalent. □

Remark 2.14. If we Bousfield localise with respect to sectionwise $H_*(-, \mathbb{Z})$-isomorphisms the projective global model structure on $\mathbf{sPre}(I)$ (this is possible arguing as in [Jar15, Chapter 7]), then the previous results identify the class of $\mathcal{P}$-fibrant and $\mathbb{Z}$-complete diagrams as a full subcategory of fibrant objects in the localised model structure. Moreover, localisation with respect to sectionwise $H_*(-, \mathbb{Z})$-isomorphisms exists for the global injective model structure because of [GJ98], therefore we get that $\mathcal{I}$-fibrant diagrams that are levelwise $h_*^{\mathbb{Z}}$-local simplicial sets are fibrant objects in the localised model category. This implies that " $\mathbb{Z}$-complete fibrant diagrams satisfy $h_*^{\mathbb{Z}}$-descent" where descent is intended as in [Jar15, page 102].

From now until the end of the section, we will let S to be a Noetherian base scheme. Whenever we will consider categories of divisorial or separated schemes, we will always assume that S is divisorial or separated, respectively. Let us consider the small category Sch_S of schemes of finite type over it and its full subcategory of divisorial schemes DSch_S (see 1.1 and Appendix A). For $X \in \mathrm{Sch}_S$, denote the Waldhausen category of perfect complexes of globally finite Tor-amplitude ([TT90, 2.2.11]) having quasi-isomorphisms as weak equivalences by $\omega\mathrm{Perf}(X)$, and denote the exact category of vector bundles over X by $\mathrm{Vect}(X)$. We define Thomason's and Quillen's K-theory spaces (see [TT90, 1.5.2] and [Qui73]) as the simplicial sets

$$K^T(X) := \Omega \mathrm{Ex}^\infty \omega S_\bullet Perf(X) \qquad K^Q(X) := \Omega \mathrm{Ex}^\infty (Q\mathrm{Vect}(U))$$

where Ex^∞ is the standard fibrant replacement introduced by Kan (see [GJ09, page 182]), $S_\bullet$ is Waldhausen's construction introduced in [Wal85] and Q denote Quillen's Q-construction. For divisorial schemes, those two spaces are homotopy equivalent via a natural zig-zag map. This is spelt out in [TT90, 3.10] and a proof can be given combining [TT90, 1.11.7, 3.8] and [Wal85, 1.9]. The assignment associating to any scheme X its Quillen's or Thomason's K-theory space $K(X)$ can be made (strictly) functorial using the technique of [FS02, Appendix C.4]. Therefore, this gives us simplicial presheaves K^Q and K^T in $\mathbf{sPre}(\mathrm{Sch}_S)$. In addition, there is a local Zariski-weak equivalence $K^Q \simeq \mathbb{Z} \times \mathrm{BGL}^+$ as proved in [GS99, Lemma 18] for example. The same result holds true replacing K^Q with K^T. For these reason, in many statements concerning K-theory as an object is some homotopy category of simplicial presheaves, we are allowed not to make a difference between K^Q and K^T. When we will use the symbol K for the K-theory simplicial presheaf, we will always refer to Thomason's K-theory unless otherwise specified. Consider the Zariski sites $(\mathrm{Sch}_S, \mathrm{Zar})$, $(\mathrm{DSch}_S, \mathrm{Zar})$ and the Nisnevich site $(\mathrm{Sch}_S, \mathrm{Nis})$.

Theorem 2.15. *K^T, $K^Q \in \mathbf{sPre}(\mathrm{Sch}_S)$ are $\mathbb{Z}$-complete and $\mathcal{P}$-fibrant. K^T is also fibrant in $\mathbf{sPre}_{\mathrm{Zar}}^{\mathcal{P}^l}(\mathrm{Sch}_S)$ and $\mathbf{sPre}_{\mathrm{Nis}}^{\mathcal{P}^l}(\mathrm{Sch}_S)$ while K^Q is fibrant in $\mathbf{sPre}_{\mathrm{Zar}}^{\mathcal{P}^l}(\mathrm{DSch}_S)$.*

Proof. By their definition, both K^T and K^Q are presheaves of Kan complexes and they are $\mathbb{Z}$-complete because of 2.5 (sectionwise they are indeed loop spaces). The final sentence comes from the fact that Thomason's K-theory satisfies Zariski and Nisnevich descent ([TT90, 8.1 and 10.8]) and is equivalent to Quillen K-theory for divisorial schemes ([TT90, 3.10]): these results are enough to verify the explicit fibrancy conditions ([AHW17, 3.1.4 and 3.2.5]) in the model categories we are considering because both the Zariski and the Nisnevich topologies are generated by a cd structure. □

For any positive natural number n we can define the general and the symplectic linear algebraic groups GL_n and Sp_{2n}. These are smooth over S: for the general linear group it follows from [GD71, I 9.6.4] while for the symplectic one this can be proven explicitly when S is a field, see [Wat79], and for the general case one can reduce to fields because of [DG80, page 289]. We can define the ind-schemes

$$\mathrm{GL} := \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{GL}_n \qquad \mathrm{Sp} := \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Sp}_{2n}$$

as customary. Taking the classifying spaces of these presheaves (for a detailed account in the context of algebraic geometry, see [Lev98, pages 357-358]) gives the simplicial presheaves

$$\mathrm{BGL} \cong \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{BGL}_n \qquad \mathrm{BSp} \cong \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{BSp}_{2n}$$

Notice that it doesn't matter if we apply the nerve functor to those presheaves after or before taking the colimits: indeed the nerve functor preserves directed colimits since the standard simplices $[n]$ are compact objects in the category of small categories. Thus, BGL and BSp are the simplicial presheaves associating to any scheme $X \in \mathrm{Sch}_S$ the simplicial sets $\mathrm{BGL}(\Gamma(X, \mathcal{O}_X))$ and $\mathrm{BSp}(\Gamma(X, \mathcal{O}_X))$, respectively. We now let G to be either GL or Sp to simplify the notation. Fixed a commutative unital ring R we can apply to $\mathrm{BG}(R)$ Quillen's + construction (references are [Wei13], [Sri08], [Ros94]) to get the simplicial set $\mathrm{BG}(R)^+$ or the Bousfield-Kan $\mathbb{Z}_\infty$ completion functor to get the simplicial set $\mathbb{Z}_\infty(\mathrm{BG}(R))$. These two simplicial sets are homotopy equivalent because of the following argument due to Dror (see [Ger73, 2.16]). Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{BG}(R) & \xrightarrow{\ f\ } & \mathrm{BG}(R)^+ \\ \downarrow & & \downarrow \\ \mathbb{Z}_\infty(\mathrm{BG}(R)) & \xrightarrow{\mathbb{Z}_\infty(f)} & \mathbb{Z}_\infty(\mathrm{BG}(R)^+) \end{array}$$

where f is the canonical map given with Quillen's $+$ construction. Then f induces an isomorphism on integral homology so $\mathbb{Z}_\infty(f)$ is a weak equivalence. As $\mathrm{BG}(R)^+$ is a connected H-space (this can be also seen as a consequence of the general machinery in [Sch17, Appendix A]), Corollary 2.5 applies giving that the right vertical map is a weak equivalence, concluding the argument. This also shows that the canonical map $i_{\mathrm{BG}(R)} : \mathrm{BG}(R) \to \mathbb{Z}_\infty(\mathrm{BG}(R))$ is an $H_*(-,\mathbb{Z})$-isomorphism. As a consequence, we have a simplicial presheaf $\mathbb{Z}_\infty\mathrm{BG} \in \mathbf{sPre}(\mathrm{Sch}_S)$ which is equivalent to the one obtained by applying to BG any functorial contruction of the + construction. We shall therefore denote this presheaf by BG^+.
In this paper, in addition to the categories Sch_S and Sm_S of Noetherian schemes of finite type over S and of smooth schemes over S, we shall be interested in other categories of schemes. Indeed, in many situations these two categories can be considered to be too large or uninteresting for K-theoretic purposes and additional hypotheses as divisoriality or separatedness might be wanted for technical reasons. To the extent of this article, the property of being divisorial will be very important, while the hypothesis of being separated can be forgotten. However, because of their mutual importance in literature, we think that it's worthwhile to discuss what happens when we drop or add one of these hypothesis. Accordingly, we shall focus on the categories Sch_S, DSch_S, Sch_S^{Sep}, Sm_S, DSm_S and Sm_S^{Sep} as introduced in 1.1. We denote the unstable motivic homotopy categories of divisorial and separated smooth schemes over S by $\mathcal{H}^{\mathrm{Div}}(S)$ and $\mathcal{H}^{\mathrm{Sep}}(S)$, respectively.
We remark that for any $n \in \mathbb{N}$, the schemes GL_n and Sp_n are divisorial, smooth and separated over the chosen base S so that GL and Sp are in effect ind-schemes in all the categories of schemes considered.

Proposition 2.16. *Let $\mathcal{C} = (\mathcal{D}, \tau)$ be a Grothendieck site where $\mathcal{D}$ can be either* Sch_S, DSch_S, Sch_S^{Sep}, Sm_S, DSm_S *or* Sm_S^{Sep}. *Let be F a $\mathcal{P}^l$-fibrant simplicial presheaf which is $\mathbb{Z}$-complete. Then* $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_\tau(\mathcal{C}))}((\mathrm{BG}^+)^n, F) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_\tau(\mathcal{C}))}(\mathrm{BG}^n, F)$ *for any $n \in \mathbb{N}$ where G can be either* GL *or* Sp.

Proof. Since F is $\mathcal{P}^l$-fibrant we can apply 2.13. □

Recall that the equivalence $K \simeq \mathbb{Z} \times \mathrm{BGL}^+$ continues to hold even if we consider divisorial, separated or smooth schemes. We can now prove the following:

Proposition 2.17. *Let $\mathcal{C}$ be either* Sch_S, DSch_S, Sch_S^{Sep}, Sm_S, DSm_S *or* Sm_S^{Sep}. *The map* $i : \mathrm{BGL} \to \mathrm{BGL}^+$ *induces an isomorphism*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}((\mathrm{BGL}^+)^n, K) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}(\mathrm{BGL}^n, K)$$

for any $n \in \mathbb{N}$. As a consequence

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}(K^n, K) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}((\mathbb{Z} \times \mathrm{BGL})^n, K)$$

for any $n \in \mathbb{N}$.

Proof. The first statement follows directly from Proposition 2.16 and Theorem 2.15 (notice $\mathbb{Z}_\infty$ commutes with finite products up to homotopy because of [BK72, I 7.3]). The last statement follows from the fact that as simplicial presheaves, $\mathbb{Z} \times \mathrm{BGL}^+ \cong \amalg_{n\in\mathbb{Z}} \mathrm{BGL}^+$ and this coproduct is already an homotopy coproduct for the injective model structure because of [Hov99, Example 1.3.11]. □

We shall need a few recollections on homotopy limits and colimits. Throughout this paper, we shall stick with the definitions and formulas found in [Hir, Chapter 18]: all the model categories we consider in this work are indeed simplicial.

Lemma 2.18. *Consider a small category I. Let $X_\bullet$ be an object in $s\mathbf{Pre}^{\mathcal{I}}(I)$. Then we have $X_\bullet \simeq \operatorname{hocolim}_{[n]\in\Delta^{op}} X_n$ where the X_n are seen as constant simplicial diagrams. The same is true if we Bousfield localise $s\mathbf{Pre}^{\mathcal{I}}(I)$ at some class of morphisms.*

Proof. The argument used to show that a simplicial set is the homotopy colimit of its simplices by considering it as a bisimplicial set and then noticing that its realization is equivalent to its diagonal applies even in this case. More details to be found in [nLa19]. □

For a given category I, we recall that (see [BK72, XII]) $\mathfrak{Map}_{s\mathbf{Pre}(I)}(-,-)$ takes homotopy colimits in the first variable to homotopy limits ([Hir03, 9.2.2]), i.e. $\mathfrak{Map}_{s\mathbf{Pre}(I)}(\operatorname{hocolim}_{i\in J} X_i, Y)$ is weakly equivalent to $\operatorname{holim}_{i\in J^{op}} \mathfrak{Map}_{s\mathbf{Pre}(I)}(X_i, Y)$ and these two simplicial sets are actually isomorphic if one uses the definition of homotopy limits and colimits given in [Hir03] which we to adopt in our work. See [Hir03, 18.1.10] for the proof of this fact and [Hir03, 18.1.11] for a comparison with the definition of [BK72]. (Warning: there is a minor error in [BK72]. See the reference just given for a discussion.) Remember that filtered colimits of simplicial sets are homotopy equivalent to their homotopy colimits via the standard map. This is true for filtered colimits in any combinatorial model category, because of [Dug01a, Proposition 7.3]. Therefore we get

$$\mathrm{BGL} \cong \operatorname{colim}_{n\in\mathbb{N}} \mathrm{BGL}_n \simeq \operatorname{hocolim}_{n\in\mathbb{N}} \mathrm{BGL}_n \qquad \mathrm{BSp} \cong \operatorname{colim}_{n\in\mathbb{N}} \mathrm{BSp}_n \simeq \operatorname{hocolim}_{n\in\mathbb{N}} \mathrm{BSp}_n$$

We conclude this section with two very important propositions:

Proposition 2.19. *Let $(\mathcal{C}, \tau)$ be any Grothendieck site. If F is any $\mathcal{I}$-fibrant simplicial presheaf in $s\mathbf{Pre}(\mathcal{C})$, J a small filtered set, $(X_j)_{j\in J}$ a directed family of simplicial objects of $\mathcal{C}$ and $X \cong \operatorname{colim}_{j\in J} X_j \simeq \operatorname{hocolim}_{j\in J} X_j$ in $s\mathbf{Pre}(\mathcal{C})$, we have*

$$\begin{aligned}\mathfrak{Map}_{s\mathbf{Pre}(\mathcal{C})}(X, F) &\simeq \lim_{j\in J^{op}} \operatorname{holim}_{i\in\Delta} F((X_j)_i) \\ &\simeq \operatorname{holim}_{j\in J^{op}} \operatorname{holim}_{i\in\Delta} F((X_j)_i)\end{aligned}$$

A a consequence, if G is any $\mathcal{P}$-fibrant simplicial presheaf satisfying descent (so that it is $\mathcal{P}^l$-fibrant)

$$\begin{aligned}\operatorname{Hom}_{\mathrm{Ho}(s\mathbf{Pre}_\tau^{\mathcal{I}^l}(\mathcal{C}))}(X, G) &\cong \pi_0 \lim_{j\in J^{op}} \operatorname{holim}_{i\in\Delta} G((X_j)_i) \\ &\cong \pi_0 \operatorname{holim}_{j\in J^{op}} \operatorname{holim}_{i\in\Delta} G((X_j)_i)\end{aligned} \tag{A}$$

Under the same hypothesis on G, let A be a class of maps s.t. we can perform the left Bousfield localization on $s\mathbf{Pre}_\tau^{\mathcal{P}^l}(\mathcal{C})$ and $s\mathbf{Pre}_\tau^{\mathcal{I}^l}(\mathcal{C})$ at it in order to obtain the model categories $s\mathbf{Pre}_\tau^{\mathcal{I}^l_A}(\mathcal{C})$ and $s\mathbf{Pre}_\tau^{\mathcal{P}^l_A}(\mathcal{C})$. Then if a sectionwise weakly equivalent $\mathcal{I}^l$-fibrant replacement of G is also A-local, one has in addition that

$$\operatorname{Hom}_{\mathrm{Ho}(s\mathbf{Pre}_\tau^{\mathcal{I}^l_A}(\mathcal{C}))}(X, G) \cong \operatorname{Hom}_{\mathrm{Ho}(s\mathbf{Pre}_\tau^{\mathcal{I}^l}(\mathcal{C}))}(X, G) \tag{B}$$

Proof. It all boils down to the properties of the simplicial model structure. Indeed

$$\begin{aligned}\mathfrak{Map}_{s\mathbf{Pre}(\mathcal{C})}(X, F) &\cong \mathfrak{Map}_{s\mathbf{Pre}(\mathcal{C})}(\operatorname{colim}_{j\in J} X_j, F) \\ &\cong \lim_{j\in J^{op}} \mathfrak{Map}_{s\mathbf{Pre}(\mathcal{C})}(X_j, F) \\ &\simeq \lim_{j\in J^{op}} \operatorname{holim}_{i\in\Delta} F((X_j)_i)\end{aligned}$$

As before we used in the first isomorphism the definition of X, in the second the fact that $\mathfrak{Map}$ takes colimits to limits and the third weak equivalence comes from the properties of representable presheaves together with Lemma 2.18. The fact that

$$\mathfrak{Map}_{s\mathbf{Pre}(\mathcal{C})}(X, F) \simeq \operatorname{holim}_{j\in J^{op}} \operatorname{holim}_{i\in\Delta} F((X_j)_i)$$

follows similarly. We turn now to the second assertion. By assumption, there is a sectionwise weak equivalence $d : G \to G_f$ with G_f $\mathcal{I}^l$-fibrant. We can write

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_\tau(\mathcal{C}))}(X, G) &\cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_\tau(\mathcal{C}))}(\operatorname*{colim}_{j\in J} X_j, G) \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_\tau(\mathcal{C}))}(\operatorname*{colim}_{j\in J} X_j, G_f) \\
&\cong \pi_0\, \mathfrak{Map}_{\mathbf{sPre}(\mathcal{C})}(\operatorname*{colim}_{j\in J} X_j, G_f) \\
&\cong \pi_0 \lim_{j\in J} \operatorname*{holim}_{i\in\Delta} G_f((X_j)_i) \\
&\cong \pi_0 \lim_{j\in J} \operatorname*{holim}_{i\in\Delta} G((X_j)_i)
\end{aligned}$$

Where we have used the simplicial model structure on $\mathbf{sPre}^{\mathcal{I}^l}_\tau(\mathcal{C})$ together with the result just proved and the fact that d is a sectionwise fibrant replacement (between $\mathcal{P}$-fibrant presheaves). Similarly, we could have replaced the first colimit indexed by J with its homotopy colimit. The final statement, given the validity of (A), is true because of the characterisation of fibrant objects in the model categories arising as a Bousfield localisation of a model category, see [Hir03, Proposition 3.4.1]. Indeed, recall that both the injective local and the projective local model structure are proper ([Jar15, Theorem 5.9], [Bla01, Lemma 1.7]). □

Applying Proposition 2.19 to the Grothendieck sites of schemes we are considering leads to the following:

Proposition 2.20. *Let $\mathcal{D} \subseteq \mathcal{C}$ be one of the following inclusions: $\mathrm{Sm}_S \subseteq \mathrm{Sch}_S$, $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$, $\mathrm{Sm}_S^{Sep} \subseteq \mathrm{Sch}_S^{Sep}$, $\mathrm{DSch}_S \subseteq \mathrm{Sch}_S$, $\mathrm{Sch}_S^{Sep} \subseteq \mathrm{Sch}_S$, $\mathrm{DSm}_S \subseteq \mathrm{Sm}_S$ or $\mathrm{Sm}_S^{Sep} \subseteq \mathrm{Sm}_S$. Let J be a small filtered set and let $(X_j)_{j\in J}$ be a directed family of simplicial objects of $\mathcal{D}$ and $X := \operatorname*{colim}_{j\in J} X_j \in \mathbf{sPre}(\mathcal{C})$. Let G be a $\mathcal{P}$-fibrant simplicial presheaf in $\mathbf{sPre}(\mathcal{C})$ satisfying Zariski descent (so that it is $\mathcal{P}^l_{\mathrm{Zar}}$-fibrant). Denote as $\mathrm{res}(G)$ its restriction to $\mathbf{sPre}(\mathcal{D})$. Then we have*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}(X, G) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{D}))}(X, \mathrm{res}(G))$$

If the Nisnevich topology is well defined on $\mathcal{D}$ and if $\mathrm{res}(G)$ satisfies Nisnevich descent, so that, being $\mathcal{P}$-fibrant, it is both $\mathcal{P}^l_{\mathrm{Zar}}$ and $\mathcal{P}^l_{\mathrm{Nis}}$-fibrant, we have:

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}(X, G) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Nis}}(\mathcal{D}))}(X, \mathrm{res}(G))$$

If moreover G_s has a $\mathcal{I}^l_{\mathrm{Nis}}$-fibrant replacement which is also $\mathbb{A}^1$-local, denoting as $\mathcal{H}^*(S)$ the motivic homotopy category over $\mathcal{D}$, we have

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Nis}}(\mathcal{D}))}(X, \mathrm{res}(G)) \cong \mathrm{Hom}_{\mathcal{H}^*(S)}(X, \mathrm{res}(G))$$

Proof. This is a simple application of the previous Proposition once we notice that for any $j \in J$ and any $i \in \Delta$, $(X_j)_i$ is represented by a scheme that, being in $\mathcal{D}$, is also in $\mathcal{C}$. Indeed, the first and the second isomorphisms follows from equation (A) and the last isomorphism from equation (B). □

Remark 2.21. In the previous Proposition, the bijection $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}(X, G) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{D}))}(X, \mathrm{res}(G))$ can be seen to be induced by the functor

$$\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C})) \to \mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{D}))$$

induced by deriving the restriction functor $\mathrm{res} : \mathbf{sPre}(\mathcal{C}) \to \mathbf{sPre}(\mathcal{D})$.
The bijection $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}(X, G) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Nis}}(\mathcal{D}))}(X, \mathrm{res}(G))$ can be seen to be induced by the functor

$$\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C})) \to \mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{D}))$$

followed by the localization functor

$$\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{D})) \to \mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Nis}}(\mathcal{D}))$$

and finally $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Nis}}(\mathcal{D}))}(X, \mathrm{res}(G)) \cong \mathrm{Hom}_{\mathcal{H}^*(S)}(X, \mathrm{res}(G))$ can be seen to be induced by the localisation functor

$$\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Nis}}(\mathcal{D})) \to \mathcal{H}^*(S)$$

Indeed, the only non trivial step to see this can be to convince yourself that the restriction functor res induces a bijection $\pi_0\mathfrak{Map}_{\mathbf{sPre}(\mathcal{C})}(X, F) \cong \pi_0\mathfrak{Map}_{\mathbf{sPre}(\mathcal{D})}(X, \mathrm{res}(F))$ for any $X \in \mathcal{D}$ and $F \in \mathbf{sPre}(\mathcal{C})$, but this follows from the Yoneda lemma.

3 Unstable operations on K-theory

In this section we stick to the notation introduced in the previous section and recalled in 1.1. We fix a regular Noetherian base scheme S, which we will suppose to be also divisorial or separated every time we consider it as the base scheme of a category of divisorial or separated schemes, respectively. We will denote by K Thomason's K-theory simplicial presheaf. We start with:

Proposition 3.1. *Consider $\mathcal{D} \subseteq \mathcal{C}$ to be one of the following inclusions:* $\mathrm{Sm}_S \subseteq \mathrm{Sch}_S$, $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$, $\mathrm{Sm}_S^{Sep} \subseteq \mathrm{Sch}_S^{Sep}$. *Denote the motivic homotopy category over $\mathcal{D}$ by $\mathcal{H}^*(S)$. We have*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}((\mathrm{BGL}^+)^n, K) \cong \mathrm{Hom}_{\mathcal{H}^*(S)}((\mathrm{BGL}^+)^n, K)$$

Proof. We have

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}((\mathrm{BGL}^+)^n, K) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}((\mathrm{BGL})^n, K)$$

from Proposition 2.17. In addition, the isomorphism

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}((\mathrm{BGL})^n, K) \cong \mathrm{Hom}_{\mathcal{H}^*(S)}((\mathrm{BGL})^n, K)$$

can be deduced from Proposition 2.20 noticing that the simplices of the simplicial object involved in the colimit defining BGL are representable by products of general linear groups. Finally, we get

$$\mathrm{Hom}_{\mathcal{H}^*(S)}((\mathrm{BGL})^n, K) \cong \mathrm{Hom}_{\mathcal{H}^*(S)}((\mathrm{BGL}^+)^n, K)$$

from the fact that $\mathrm{BGL}^+ \cong \mathrm{BGL}$ in $\mathcal{H}^*(S)$. (This was first noticed in [MV99], see [Rio02, Proposition 7.17].) □

We are now ready to prove the following, that should be regarded as the main result in this section.

Theorem 3.2. *Consider $\mathcal{D} \subseteq \mathcal{C}$ to be one of the following inclusions:* $\mathrm{Sm}_S \subseteq \mathrm{Sch}_S$, $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$, $\mathrm{Sm}_S^{Sep} \subseteq \mathrm{Sch}_S^{Sep}$. *Denote the motivic homotopy category over $\mathcal{D}$ by $\mathcal{H}^*(S)$. For any natural number n, we have that*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}(K^n, K) \cong \mathrm{Hom}_{\mathcal{H}^*(S)}(K^n, K)$$

Proof. We already observed that there is a local Zariski (hence Nisnevich) weak equivalence $K \simeq \mathbb{Z} \times \mathrm{BGL}^+$. Moreover, $K \simeq \mathbb{Z} \times \mathrm{BGL}^+ \cong \amalg_{n \in \mathbb{Z}} \mathrm{BGL}^+$ in the homotopy categories considered and disjoint unions (finite products) of cofibrant (fibrant) objects are still coproducts (or finite products) in these homotopy categories so that Hom takes coproducts to products. Therefore, the result follows from Proposition 3.1. □

After this result was proved, the author discovered that part of the previous statement was essentially implied by results contained in unpublished 2013 notes by Cisinski, who sketches a different argument to reach essentially the same conclusion. However our method is different and in some extent, "more explicit". As a corollary, we mention the following result, interesting on its own. Recall that we have a functor $\mathrm{Gr}_{-,-} : \mathbb{N}^2 \to \mathrm{Sch}_S$ associating to any $(n, d) \in \mathbb{N}^2$ the Grassmannians $\mathrm{Gr}_{n,d}$ classifying locally free quotients of rank d of the trivial bundle of rank $n + d$ (see [GD71, I 9.7.3] or [GW10, page 211], notice the different indices) and defined via the canonical closed embeddings on the maps ($\mathbb{N}^2$ is a poset in a natural way). The infinite Grassmannian Gr is defined as customary as the ind-scheme resulting as the colimit of the previous functor (in the category of presheaves).

Corollary 3.3. *Consider $\mathcal{D} \subseteq \mathcal{C}$ to be one of the following inclusions:* $\mathrm{Sm}_S \subseteq \mathrm{Sch}_S$, $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$, $\mathrm{Sm}_S^{Sep} \subseteq \mathrm{Sch}_S^{Sep}$. *Denote the motivic homotopy category over $\mathcal{D}$ by $\mathcal{H}^*(S)$. For any natural number n,* $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}((\mathbb{Z} \times \mathrm{Gr})^n, K) \cong \mathrm{Hom}_{\mathcal{H}^*(S)}((\mathbb{Z} \times \mathrm{Gr})^n, K) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathcal{C}))}(K^n, K)$

Proof. Under our assumptions on S, we have that $K \cong \mathbb{Z} \times \mathrm{Gr}$ in $\mathcal{H}^*(S)$ because of [MV99, 4.3.7], so that, being a colimit of representable smooth schemes, as a direct application of 2.20, we get the first isomorphism. The last isomorphism now follows from Theorem 3.2. □

Remark 3.4. After work of Cisinski and Khan ([Kha16, Proposition 2.4.6]), it seems that under suitable assumptions on S, the unstable motivic homotopy category $\mathcal{H}(S)$ we considered so far is equivalent to the homotopy category of the ∞-category called *spectral* motivic homotopy category as defined in [Kha16, Definition 2.4.1]. This, together with our and Riou's result then provide a way to define and study operations and algebraic structures for a certain class of spectral schemes, although we will not address this point in this paper.

We summarise in a corollary some consequences of the previous theorems and propositions for convenience of the reader.

Corollary 3.5. *For any $n \in \mathbb{N}$, all the arrows in the following commutative diagram are bijective:*

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(K^n, K) & \longleftarrow & \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{Sch}_S))}(K^n, K) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{Sch}^{Sep}_S))}(K^n, K) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n, K) & \longleftarrow & \mathrm{Hom}_{\mathcal{H}(S)}(K^n, K) & \longrightarrow & \mathrm{Hom}_{\mathcal{H}^{\mathrm{Sep}}(S)}(K^n, K)
\end{array}$$

Proof. To start with, because of Propositions 2.17 and 2.20, we have

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{Sch}_S))}((\mathrm{BGL}^+)^n, K) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{DSch}_S))}((\mathrm{BGL}^+)^n, K)$$

As a consequence,

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{Sch}_S))}(K^n, K) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(K^n, K)$$

follows using the fact that $K \simeq \mathbb{Z} \times \mathrm{BGL}^+ \cong \amalg_{n\in\mathbb{Z}}\mathrm{BGL}^+$ in the homotopy categories considered arguing as in the proof of 3.2 and we can use it together with 3.2 to prove that

$$\mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n, K) \cong \mathrm{Hom}_{\mathcal{H}(S)}(K^n, K)$$

The case of separated schemes follows similarly. We conclude using Remark 2.21. □

Remark 3.6. Denoting as Ω^i_f the right derived functor of Ω^i in the simplicial model categories we considered, we could have replaced K with $\Omega^i_f K$ in the second variable of all the Hom sets considered for any $i \in \mathbb{N}$ without changing the final results. Indeed we can still apply 2.16 and 2.20. In addition, since a product of fibrant simplicial presheaves is a product in the homotopy categories considered, this implies that for any $n, m, i \in \mathbb{N}$ we have

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{Sch}_S))}(K^n, \Omega^i_f K^m) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(K^n, \Omega^i_f K^m),$$

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(K^n, \Omega^i_f K^m) \cong \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n, \Omega^i_f K^m)$$

and

$$\mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n, \Omega^i_f K^m) \cong \mathrm{Hom}_{\mathcal{H}(S)}(K^n, \Omega^i_f K^m)$$

The same applies considering separated schemes instead of divisorial ones.

3.1 Symplectic K-theory

In this section we will consider a Noetherian divisorial base scheme S where $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$, although recent progresses indicate that this might be unecessary. We will only consider the category DSch_S of divisorial schemes of finite type over a fixed base scheme S. Indeed, the descent results we need for Hermitian K-theory proved in [Sch17] fall under this assumption and although it seems to be folklore that they can be extended to perfect complexes over general Noetherian schemes, we refrain here from using unpublished results. As in the case of K-theory, we can define a simplicial presheaf over DSch_S representing n-shifted hermitian K-theory. Denote it by $\mathrm{GW}^{[n]}$, $\mathrm{GW}^{[2]}$ being symplectic K-theory. Roughly speaking one start from a presheaf of dg categories with weak equivalences and dualities and then one applies the construction made explicit in [Sch17, 9.1]. What is relevant to our discussion is that we end up with a simplicial presheaf which is $\mathcal{P}$-fibrant and $\mathbb{Z}$-complete since it is an H-group for any n as remarked for example in [Sch10, 2.7 Remark 2]. Moreover, this presheaf also satisfies Zariski and Nisnevich descent and it is $\mathbb{A}^1$-homotopy invariant on regular schemes. This can be found in [Sch17, Theorems 9.7, 9.8 and 9.9], notice that the assumption of separatedness is used in Theorem 9.8 of *op. cit.* only to make [Bal01, Theorem 3.4] to work, but the proof goes through as well replacing separatedness with the weaker assumption of having affine diagonal. So, in particular, $\mathrm{GW}^{[n]}$ is $\mathcal{P}^l$-Zariski fibrant. We then have because of 2.16 the following:

Proposition 3.7. *We have that*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{DSch}_S))}((\mathrm{BSp}^+)^m, \mathrm{GW}^{[n]}) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(\mathrm{BSp}^m, \mathrm{GW}^{[n]})$$

for any $m \in \mathbb{N}$, $n \in \mathbb{Z}$. In particular this holds for $n = 2$.

From now on we shall assume that our base scheme S is regular in addition to being Noetherian and divisorial. We consider the simplicial presheaf $\mathrm{GW}^{[2]} =: \mathrm{KSp}$ over DSch_S. We already recalled its descent properties and there is a local weak equivalence $\mathbb{Z} \times \mathrm{BSp}^+ \simeq \mathrm{KSp}$ in $\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{DSch}_S)$ (argue as in the case of K-theory using [Sch17, Theorem A.1 and Corollary A.2]). in addition, note that that in the paper [ST15], it is shown that $\mathrm{KSp} \simeq \mathbb{Z} \times \mathrm{BSp}$ in $\mathcal{H}^{\mathrm{Div}}(S)$ and we do not need to consider the étale classifying space in this context because of the equivalence between symplectic vector bundles and fppf Sp_{2n}-torsors (the proof of this fact is contained in [AHW18, page 1205], see also [PW10a, page 25]). Then we can repeat the arguments of the previous section used for K-theory and get:

Theorem 3.8. *We have*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(\mathrm{BSp}^+, \mathrm{KSp}) \cong \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(\mathrm{BSp}, \mathrm{KSp})$$

Moreover for any natural number n it holds

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(\mathrm{KSp}^n, \mathrm{KSp}) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}^{\mathcal{I}^l}_{\mathrm{Nis}}(\mathrm{DSm}_S))}((\mathbb{Z}\times \mathrm{BSp})^n, \mathrm{KSp}) \cong \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(\mathrm{KSp}^n, \mathrm{KSp})$$

If S is in addition separated (over $\mathrm{Spec}(\mathbb{Z})$*) then we also have* $\mathrm{Hom}_{\mathcal{H}^{\mathrm{Sep}}(S)}(\mathrm{KSp}^n, \mathrm{KSp}) \cong \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(\mathrm{KSp}^n, \mathrm{KSp})$

Proof. The proof follows the lines of the proofs of 3.2 and 3.5 using 2.20 and 3.7. □

Remark 3.9. Similar considerations as in Remark 3.6 apply.

4 Unstable operations on K-theory depend only on π_0

In this section, we make the blanket assumption that all our schemes are Noetherian and divisorial unless otherwise stated. In addition, our base scheme S will be always assumed to be regular. We denote by K Thomason's K-theory simplicial presheaf. The starting point of this section is the following theorem of Riou ([Rio06, III.31], [Rio10, 1.1.4]):

Theorem 4.1. *(Riou, [Rio10, Theorem 1.1.4]). For any $n \in \mathbb{N}$ one has the following isomorphism*

$$\mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n, K) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(K_0(-)^n, K_0(-))$$

Moreover, for $n = 1$, this becomes

$$\mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K, K) \cong \prod_{i\in\mathbb{Z}} K_0(S)[[c_1, ..., c_n, ...]]$$

with $c_1, ...$ the usual Chern classes. Finally, the pointed analogue of the first isomorphism holds, i.e. we have

$$\mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}_\bullet(S)}(K^n, K) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)_\bullet}(K_0(-)^n, K_0(-))$$

Remark 4.2. The former theorem in both [Rio10] and [Rio06] assumes our schemes to be separated as well, but, replacing separatedness with divisoriality, Riou's argument still goes through. The structure of Riou's argument has been outlined in Appendix B for the convenience of the reader

Remark 4.3. From now on we will always suppose to point $K \in \mathcal{H}^{\mathrm{Div}}(S)$ $(\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S)))$ and $K_0 \in \mathbf{Pre}(\mathrm{DSch}_S)$ with the same element of $K_0(S)$ whenever we consider these objects as pointed. Unless otherwise stated, from now on the default choice will be the one of $0 \in K_0(S)$.

Our aim is to extend as much as we can the result of Riou to divisorial (possibly non regular) schemes. We get for any $n \in \mathbb{N}$ the following commutative diagram, where res is induced by the inclusion $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(K^n, K) & \xrightarrow[\cong]{} & \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n, K) \\
\Big\downarrow{\scriptstyle \pi_0} & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle \pi_0} \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(K_0(-)^n, K_0(-)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(K_0(-)^n, K_0(-))
\end{array} \quad (1)$$

The existence of the commutative diagram is obtained using 2.21 and Theorem 3.2 gives that the top horizontal arrow is an isomorphism, while the Theorem of Riou 4.1 gives that the right vertical map is an isomorphism. Chasing the diagram we then get that the left vertical and bottom horizontal maps are injective and surjective, respectively. To show that all these maps are isomorphisms, it suffices then to show that res is injective. In order to do this, we shall need some generalities about the property (P) that we now introduce.

Definition 4.4 (Property (P)). Consider a small category $\mathcal{C}$ and consider a full subcategory $\mathcal{A} \subseteq \mathcal{C}$. We say that the presheaf $F \in \mathbf{Pre}(\mathcal{C})$ satisfies the property (P) with respect to $\mathcal{A} \subseteq \mathcal{C}$ if for every $X \in \mathrm{Ob}(\mathcal{C})$ and for every $a \in F(X)$ there exist $Y_{X,a} \in \mathrm{Ob}(\mathcal{A})$, $\varphi : X \to Y_{X,a}$ and $b \in F(Y_{X,a})$ so that $\varphi^*_F(b) := F(\varphi)(b) = a$.

The previous definition leads to a simple proof of the following key Proposition:

Proposition 4.5. *Let $\mathcal{A} \subseteq \mathcal{C}$ be a full subcategory of a given small category $\mathcal{C}$ and* $\mathrm{Res} : \mathbf{Pre}(\mathcal{C}) \to \mathbf{Pre}(\mathcal{A})$ *the restriction functor. Consider the map* $\mathrm{res} : \mathrm{Hom}_{\mathbf{Pre}(\mathcal{C})}(F, G) \to \mathrm{Hom}_{\mathbf{Pre}(\mathcal{A})}(F, G)$ *induced by* Res *for two fixed* $F, G \in \mathbf{Pre}(\mathcal{C})$ *and suppose that F satisfies the property (P) with respect to $\mathcal{A} \subseteq \mathcal{C}$. Then* res *is injective.*

Proof. Suppose we have two natural transformations $F \overset{f}{\underset{g}{\rightrightarrows}} G$ such that $\mathrm{res}(f) = \mathrm{res}(g)$. To show that $f = g$ it suffices to show that for any $X \in \mathrm{Ob}(\mathcal{C})$,

$$f_X = g_X : F(X) \rightrightarrows G(X)$$

In order to do that, let us consider $a \in F(X)$, $Y_{X,a} \in \mathcal{A}$, $b \in F(Y_{X,a})$ and $\varphi : X \to Y_{X,a}$ as in the statement. Then we have

$$f_X(a) = f_X(\varphi_F^*(b)) = \varphi_G^*(f_{Y_{X,a}}(b)) = \varphi_G^*(g_{Y_{X,a}}(b)) = g_X(\varphi_F^*(b)) = g_X(a)$$

Iterating this for any $a \in F(X)$ gives the result. □

As a corollary, we get our main result:

Theorem 4.6. *Assume that, for a given $n \in \mathbb{N}$, $K_0(-)^n$ satisfies the property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$. Then all the arrows in diagram (1) are bijections. The same is true if we replace the categories in diagram (1) with their pointed versions $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S)_\bullet)$, $\mathbf{Pre}(\mathrm{DSch}_S)_\bullet$, $\mathcal{H}^{\mathrm{Div}}_\bullet(S)$ and $\mathbf{Pre}(\mathrm{DSm}_S)_\bullet$.*

If S is quasi-projective scheme over a Noetherian affine scheme R, we will prove in Proposition 6.2 that for all $n \in \mathbb{N}$, $K_0(-)^n$ satisfies the property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$.

proof of Theorem 4.6. If $K_0(-)^n$ satisfies the property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$, then all the arrows in diagram (1) are bijections as consequence of Proposition 4.5. The statement about the pointed case follows from Diagram (1) by diagram chase using Lemma B.9. □

Corollary 4.7. *Assume that, for a given $n \in \mathbb{N}$, $K_0(-)^n$ satisfies the property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$. Then the following maps are bijective:*

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}((\mathbb{Z}\times \mathrm{Gr})^n, K_0) \xrightarrow{\mathrm{res}} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}((\mathbb{Z}\times \mathrm{Gr})^n, K_0) \xleftarrow{\tau^*_{(\mathbb{Z}\times\mathrm{Gr})^n}\circ \mathrm{res}} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(K_0(-)^n, K_0(-))$$

where res *is induced by the restriction functor* $\mathrm{res} : \mathbf{Pre}(\mathrm{DSch}_S) \to \mathbf{Pre}(\mathrm{DSm}_S)$ *and* $\tau_{(\mathbb{Z}\times\mathrm{Gr})^n}$ *is defined in B.1.*

Proof. The fact that the first arrow is a bijection can be proved using Yoneda lemma because $(\mathbb{Z}\times \mathrm{Gr})^n$ is a colimit of representables (in both $\mathbf{Pre}(\mathrm{DSch}_S)$ and $\mathbf{Pre}(\mathrm{DSm}_S)$). The second arrow is a bijection because of the previous Theorem and Riou's theorem B.5 (that we can use because of Proposition B.8). □

Remark 4.8. As in Remark 3.6, we could have replaced K with $\Omega^i_f K$ and K_0 with K_i in the second variable of all the Hom sets considered for any $i \in \mathbb{N}$ without changing the final result, using the [Rio06, Theorem III.32] which is proved using the machinery recalled in Appendix B. This implies that for any $n, m, i \in \mathbb{N}$ we have

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(K^n, \Omega^i_f K^m) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(K_0(-)^n, K_i(-)^m) \cong \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n, \Omega^i_f K)$$

4.1 Restriction to affine schemes

In this sections we will see that Theorem 4.6 only relies on what happens to affine schemes, in a sense we will make precise below. We fix a Noetherian regular divisorial base scheme S. Recall that we denote the full subcategory of DSm_S generated by the schemes of DSm_S which are affine (over $\mathrm{Spec}(\mathbb{Z})$) by $\mathrm{SmAff}_S \subseteq$ and by $\mathcal{H}^{aff}(S)$ the homotopy category of the model category $\mathbf{sPre}(\mathrm{SmAff}_S)$ having model structure determined by considering the injective local model structure relative to the affine Nisnevich topology ([AHW17, Example 2.1.2.5]) on it and then by inverting $\mathbb{A}^1$-weak equivalences (see [AHW17]). We have the following adjoint functors arising from the inclusion $\mathrm{SmAff}_S \subset \mathrm{Sm}_S$ (see [SGA72, I Proposition 5.1])

$$i_{\#,s}, i_{*,s} : \mathbf{sPre}(\mathrm{SmAff}_S) \underset{\longrightarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathbf{sPre}(\mathrm{DSm}_S) : i_s^*$$

where $i_{\#,s}$ and $i_{*,s}$ are respectively left and right adjoint of i_s^*. Recall that a weak Quillen adjunction is a pair of adjoint functors such that the left (right) adjoint is only required to preserve cofibrant (fibrant) objects and weak equivalences between them (indeed, this is enough to derive the adjunction). If we give to both categories the Nisnevich injective local model structure and we invert $\mathbb{A}^1$-weak equivalences then these adjunctions becomes Quillen adjunctions (weak in the case of $i_{\#,s} \dashv i_s^*$) and we can derive them. One notices that i_s^* preserves weak equivalences so we do not need to derive it to get a functor $i_s^* : \mathcal{H}(S) \to \mathcal{H}^{aff}(S)$. It then makes sense to study the following commutative diagram, for any $n \in \mathbb{N}$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n, K) & \longrightarrow & \mathrm{Hom}_{\mathcal{H}^{aff}(S)}(K^n, K) \\ \cong \downarrow \pi_0 & & \downarrow \pi_0 \\ \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(K_0^n, K_0) & \xrightarrow[i^*]{} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}_S)}(K_0^n, K_0) \end{array} \tag{2}$$

We have

Proposition 4.9. *The arrow* $\mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K,K) \to \mathrm{Hom}_{\mathcal{H}^{aff}(S)}(K,K)$ *is an isomorphism for any* $n \in \mathbb{N}$*.*

Proof. We notice that the functor i_s^*, because of [AHW17, Theorem 3.3.2 and Lemma 5.1.2] and the fact that K is $\mathbb{A}^1$-invariant over regular schemes, induces an equivalence on $\mathcal{I}^l_{\mathrm{Nis}}$-fibrant simplicial presheaves so that we can see directly that the arrow $\mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n,K) \to \mathrm{Hom}_{\mathcal{H}^{aff}(S)}(K^n,K)$ is an isomorphism. Strictly speaking, in *op.cit.* the authors do not assume their schemes to be divisorial but we can repeat their argument verbatim even in this case or use 3.2. $\square$

We then get

Theorem 4.10. *For any* $n \in \mathbb{N}$*, all the arrows in diagram (2) are bijections.*

Proof. One uses Proposition 4.9 together with Corollary B.12. Notice that we can use B.12 because K_0 is $\mathbb{A}^1$-invariant over regular schemes so that using in addition Nisnevich descent we get that it is also $\mathcal{T}$-invariant[2]. $\square$

Let $\mathrm{Aff}_S \subseteq \mathrm{DSch}_S$ the full subcategory of DSch_S generated by the schemes of DSch_S which are affine (over $\mathrm{Spec}(\mathbb{Z})$). From now on we assume that S is an affine regular scheme R since we need to use the fact that BGL can be seen as a homotopy colimit of affine (in the absolute sense) schemes. We see Aff_S as a site by considering the Zariski affine topology. Even in this case as in the case of the smooth affine schemes we have a functor $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S)) \to \mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Aff}_S))$ arising from the adjunctions

$$i_{\#,s}, i_{*,s} : \mathbf{sPre}(\mathrm{Aff}_S) \rightleftarrows \mathbf{sPre}(\mathrm{DSch}_S) : i_s^*$$

By what we know so far, for any $n \in \mathbb{N}$ we have the following commutative cube

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Aff}_S))}(K^n,K) & \xrightarrow[\cong]{\varphi} & \mathrm{Hom}_{\mathcal{H}^{aff}(S)}(K^n,K) \\
\uparrow{\scriptstyle\cong} \quad \downarrow{\scriptstyle\pi_0} & & \uparrow{\scriptstyle\cong} \quad \downarrow{\scriptstyle\cong\,\pi_0} \\
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(K^n,K) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(K^n,K) \\
\downarrow{\scriptstyle\cong\,\pi_0} \qquad \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Aff}_S)}(K_0^n,K_0) & \twoheadrightarrow & \downarrow{\scriptstyle\cong\,\pi_0} \qquad \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}_S)}(K_0^n,K_0) \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(K_0^n,K_0) \rightarrowtail \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Aff}_S)}(K_0^n,K_0) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(K_0^n,K_0) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}_S)}(K_0^n,K_0)
\end{array} \tag{3}$$

The only thing stated in the diagram that we haven't proven so far is that all the arrows of the upper square are isomorphisms but this is easily solved by the following lemma

Lemma 4.11. *For any* $n \in \mathbb{N}$ *the arrow* $\varphi : Hom_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Aff}_S))}(K^n,K) \to \mathrm{Hom}_{\mathcal{H}^{aff}(S)}(K^n,K)$ *is an isomorphism.*

Proof. The proof follows arguing as Theorem 3.2. Indeed, 2.16 can be seen to apply also to affine schemes and we can still use 2.19 because all the $\mathrm{GL}_{n,R}$ are affine schemes. $\square$

We conclude this section with the following theorem, showing that unstable operations on divisorial schemes are uniquely determined by their behaviour on the K_0 of affine schemes.

Theorem 4.12. *For any* $n \in \mathbb{N}$ *the restriction map*

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Aff}_S)}(K_0^n,K_0) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}_S)}(K_0^n,K_0)$$

is injective. As a consequence, all the arrows in diagram (3) are bijections. This is also true replacing the categories involved with their pointed version as in 4.6.

Proof. The first assertion is a direct application of Proposition 4.5 since K_0^n satisfies the property (P) with respect to $\mathrm{SmAff}_S \subseteq \mathrm{Aff}_S$ because of Proposition 6.2. A diagram chase gives then that all the arrows in diagram (3) are bijections and the statement about the pointed situation now follows from B.10. $\square$

Remark 4.13. Similar considerations to Remark 4.8 apply.

[2] i.e. invariant under vector bundle torsors, see the end of Appendix B for this notion.

4.2 Separated Schemes

We shall now study the behaviour of the unstable operations on K-theory over separated schemes, because of the importance of such schemes in literature. In this section we will then suppose we have a base scheme S which is regular, Noetherian and separated (over $\mathrm{Spec}(\mathbb{Z})$). Recall that because of A.3 we have a fully faithful inclusion $\mathrm{Sm}_S^{Sep} \subseteq \mathrm{DSm}_S$. In the context of separated schemes, we cannot use some property (P) relative to the embedding of the category separated schemes into some larger ambient category to study the behaviour of the operations on K-theory as in the case of divisorial schemes. Indeed, it not (yet) known if we can embed a separated divisorial scheme into a separated smooth one using arguments similar to the ones contained in [Zan20]: handling the homogeneous spectra of [BS03] usually leads only to schemes with affine diagonal, the most natural separation axiom for divisorial schemes. However, we can still prove the following

Proposition 4.14. *Given two presheaves $F, G \in \mathbf{Pre}(\mathrm{DSch}_S)$, if G is $\mathcal{T}$-invariant*[3] *the restriction map*

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(F, G) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}_S^{Sep})}(F, G)$$

is injective. As a consequence, if F satisfies the property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$, the restriction map

$$\mathrm{res} : \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(F, G) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}_S^{Sep})}(F, G)$$

is injective.

Proof. For every $X \in \mathrm{DSm}_S$ we can use the Jouanolou's trick to find an affine vector bundle torsor $\pi : T \to X$. Here T will be affine in the absolute sense, so separated (in the absolute sense), divisorial, smooth over X and as a consequence over S: hence it lies in Sm_S^{Sep}. Moreover because of our hypothesis, π induces an isomorphism on G so we get the following commutative diagram:

$$\begin{array}{ccc} F(T) & \overset{f_T}{\underset{g_T}{\rightrightarrows}} & G(T) \\ \uparrow{\scriptstyle \pi^*} & & \cong \uparrow{\scriptstyle \pi^*} \\ F(X) & \overset{f_X}{\underset{g_X}{\rightrightarrows}} & G(X) \end{array}$$

and since $f_T = g_T$ by assumption, we get $f_X = g_X$. This shows the first assertion in the statement. The last assertion follows from Proposition 4.5 noticing that res factors through $\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(F, G)$ □

Putting everything together we obtain the following theorem:

Theorem 4.15. *Suppose that for some $n \in \mathbb{N}$, K_0^n satisfies the property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$. For example, this holds if S is affine. Then all the arrows in the following commutative diagram are bijections*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(K^n, K) & \xrightarrow[\cong]{} & \mathrm{Hom}_{\mathcal{H}^{\mathrm{Sep}}(S)}(K^n, K) \\ \downarrow{\scriptstyle \pi_0} & & \cong \downarrow{\scriptstyle \pi_0} \\ \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(K_0(-)^n, K_0(-)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}_S^{Sep})}(K_0(-)^n, K_0(-)) \end{array}$$

This holds true even if we replace the categories we are considering with their pointed versions.

Proof. The right vertical map is a bijection because of [Rio10, Theorem 1.1.4]. Indeed in *op. cit.* Riou considers separated schemes, although as we noticed in 4.1 and 4.2 his arguments go through also in the case of divisorial schemes. The upper horizontal one is bijective because of Corollary 3.5. The others maps have the properties depicted in the diagram because of diagram chase. The argument is now concluded by Proposition 4.14, and the assertion concerning the pointed situation follows using B.9. □

5 Unstable operations on symplectic K-theory

In this section we introduce a common ground useful to study both symmetric and symplectic K-theory. The material here is basically contained in [ST15], although our presentation deviates from their. Indeed, they do only consider symmetric forms explicitly, but the generalization is straightforward as they notice. Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. For $\epsilon = \pm 1$, an ϵ-symmetric bilinear form on $\mathcal{F}$ is a map $\varphi : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{O}_X$ of $\mathcal{O}_X$-modules such

[3] i.e. invariant under vector bundle torsors, see the end of Appendix B for this notion.

that $\varphi \circ \tau = \epsilon\varphi$ where $\tau : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$ is the twisting map. If $2 \in \Gamma(S, \mathcal{O}_S)^*$, -1 (skew-)symmetric forms $(\mathcal{F}, \varphi)$ are called symplectic and they are uniquely determined by $\varphi \circ \Delta = 0$ where $\Delta : \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$ $x \mapsto x \otimes x$ is the diagonal map. A form φ is called *non-degenerate* and $(\mathcal{F}, \varphi)$ is called an *ϵ-inner product space* if $\mathcal{F}$ is a vector bundle on X and the adjoint morphism $\widehat{\varphi} : \mathcal{F} \to \mathcal{F}^* = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) : s \mapsto \varphi(- \otimes s)$ is an isomorphism. The form φ is ϵ-symmetric if and only if $\widehat{\varphi} = \epsilon\widehat{\varphi}^* \mathrm{can}_F$ where $\mathrm{can}_F : \mathcal{F} \xrightarrow{\cong} \mathcal{F}^{**}$ is the canonical isomorphism. One can see that if $g : \mathcal{G} \to \mathcal{F}$ is a map of $\mathcal{O}_X$-modules, we can define the restriction $\varphi_{|\mathcal{G}}$ of φ to $\mathcal{G}$ using adjoint map $\widehat{\varphi}_{|\mathcal{G}} = g^* \circ \widehat{\varphi} \circ g : \mathcal{G} \xrightarrow{g} \mathcal{F} \xrightarrow{\widehat{\varphi}} \mathcal{F}^* \xrightarrow{g^*} \mathcal{G}^*$. If $p : X \to S$ is a morphism of schemes and $\mathcal{F}$ is a sheaf on S, we denote $\mathcal{F}_X := p^*\mathcal{F}$. Fix a Noetherian base scheme S with $2 \in \Gamma(S, \mathcal{O}_S)^*$. We assume that this condition holds until the end of this section.

Definition 5.1. For an ϵ-symmetric form $V = (\mathcal{F}, \varphi)$ with $\mathcal{F}$ a quasi-coherent sheaf on S we define the ϵ-bilinear grassmannian of non degenerate locally free subspaces of V to be the presheaf

$$\begin{aligned} \mathrm{GrB}_S(V) :& (\mathrm{Sch}_S)^{op} \to \mathrm{Sets} \\ & (p : X \to S) \mapsto \{E \subset \mathcal{F}_X \mid E \text{ loc.free of finite rank s.t. } \varphi_{|E} \text{ is non degenerate}\} \end{aligned}$$

on the objects, and in the case of morphisms $f : X \to Y$ in Sch_S, $\mathrm{GrB}_S(V)(Y) \to \mathrm{GrB}_S(V)(X)$ is induced by f^*. We define the ϵ-bilinear grassmannian of non degenerate locally free of rank n subspaces of V to as the subpresheaf of $\mathrm{GrB}_S(V)$ of the following form

$$\begin{aligned} \mathrm{GrB}_{n,S}(V) :& (\mathrm{Sch}_S)^{op} \to \mathrm{Sets} \\ & (p : X \to S) \mapsto \{E \subset \mathcal{F}_X \mid E \text{ loc.free of rank n s.t. } \varphi_{|E} \text{ is non degenerate}\} \end{aligned}$$

We then have the following result, that can be proven following verbatim [ST15, Lemma 2.2]:

Theorem 5.2. *Let $V = (\mathcal{F}, \varphi)$ be an ϵ-symmetric inner product space over a Noetherian scheme S. Then for every n we have that $\mathrm{GrB}_{n,S}(V)$ is representable by a scheme which is smooth divisorial and affine over S (notice for $\epsilon = -1$ n has to be even), so in particular it is a sheaf. This is an open subscheme of the Grassmannian $\mathrm{Gr}_{n,S}(\mathcal{F})$ of rank n subbundles of $\mathcal{F}$. We explicitly spell out the universal property of this scheme. For every S-scheme X and every rank n ϵ-inner product space $B = (\mathcal{B}, \alpha)$ which comes as a restriction along a mono $\mathcal{B} \hookrightarrow \mathcal{F}_X$ there exists a unique map $f : X \to \mathrm{GrB}_{n,S}(V)$ over S such that $f^*(\mathcal{T} \subset \mathcal{F}_{\mathrm{Gr}_{n,S}(\mathcal{F})}) \cong \mathcal{B} \subset \mathcal{F}_X$ via the canonical isomorphism $\mathcal{F}_X \cong f^*\mathcal{F}_{\mathrm{Gr}_{n,S}(\mathcal{F})}$ and $B = f^*T$ where $T = (\mathcal{T}, \varphi_{|\mathcal{T}})$ is the ϵ-inner product space induced by $V_{\mathrm{GrB}_{n,S}(V)}$ on $\mathcal{T}$. Here $\mathcal{T}$ is the restriction to $\mathrm{GrB}_{n,S}(V)$ of the tautological rank n vector bundle on $\mathrm{Gr}_{n,S}(\mathcal{F})$.*

There are certain particular forms that play an important role in the theory

Definition 5.3. Suppose $V = (\mathcal{F}, \varphi)$ is an ϵ-symmetric form on a vector bundle over S. We then define the *split metabolic space* $M(\mathcal{F}, \varphi)$ or $M(V)$ as

$$M(V) = M(\mathcal{F}, \varphi) = \left(\mathcal{F} \oplus \mathcal{F}^*, \left(\begin{smallmatrix} \widehat{\varphi} & 1 \\ \epsilon\mathrm{can} & 0 \end{smallmatrix}\right) : F \oplus F^* \to F^* \oplus F^{**}\right)$$

where can is the canonical isomorphism $\mathcal{F} \xrightarrow{\cong} \mathcal{F}^{**}$. For a locally free sheaf $\mathcal{F}$ we define the hyperbolic space $\mathbb{H}_\epsilon(\mathcal{F})$ as $M(\mathcal{F}, 0)$, i.e.

$$\mathbb{H}_\epsilon(\mathcal{F}) = (\mathcal{F} \oplus \mathcal{F}^*, \left(\begin{smallmatrix} 0 & 1 \\ \epsilon\mathrm{can} & 0 \end{smallmatrix}\right))$$

This is an ϵ-inner product space. Notice $\mathbb{H}_\epsilon(\mathcal{O}_X)$ for a scheme X are the hyperbolic spaces considered in the next sections.

We recall the following well known lemma (use [Knu91, page 19]):

Lemma 5.4. *Let X be any quasi-compact scheme such that $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. Then every split metabolic space of the form $M(\mathcal{F}, \varphi)$ is isomorphic to $\mathbb{H}_\epsilon(\mathcal{F})$.*

The following is then immediate:

Corollary 5.5. *Let be X any quasi-compact scheme such that $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. Then for every rank n ϵ-inner product space $V = (\mathcal{F}, \varphi)$ we have a morphism $f : V \hookrightarrow M(V) \cong \mathbb{H}_\epsilon(\mathcal{F}) =: \mathbb{H}(V)$ given by the inclusion $(\mathcal{F}, \varphi) \hookrightarrow M(\mathcal{F}, \varphi) : x \mapsto \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$.*

From now until the end of the section, we shall make the blanket assumption that unless stated otherwise, we fix a regular Noetherian divisorial base scheme S so that $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$, and any scheme we will consider will be always assumed to be in DSch_S. Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. A symmetric bilinear form is an 1-symmetric bilinear form on $\mathcal{F}$ as defined before. Same terminology for symmetric inner product spaces. Analogously, a symplectic bilinear form is a -1-symmetric bilinear form and we use the same terminology for symplectic inner product spaces. We can then define orthogonal and symplectic (or quaternionic) grassmannians following [ST15] and [PW10a], [PW10b]. We shall introduce the notation $H_X := \mathbb{H}_1(\mathcal{O}_X)$ for the so called *hyperbolic plane* and $\mathbb{H} \cong (\mathbb{H}_{-1}(\mathcal{O}_X))$ for the *quaternionic plane*. Their n-fold sums, for $n \in \mathbb{N}$, will be denoted by $H_X^n := H_X^{\perp n}$ and by $\mathbb{H}^n := \mathbb{H}^{\perp n}$, respectively.

Definition 5.6. ([ST15, Definition 2.3], [PW10b]). For $n, d \in \mathbb{N}$ we define the orthogonal grassmannians as $\mathrm{GrO}_{d,n} := \mathrm{GrB}_d(H_S^{n+d})$ and the *infinite orthogonal grassmannian* over S as the ind-scheme

$$\mathrm{GrO} := \operatorname*{colim}_{n \in \mathbb{N}} \mathrm{GrB}_{2n}(H_S^n \perp H_S^n) \cong \operatorname*{colim}_{(d,n)\in\mathbb{N}^2} \mathrm{GrB}_{2d}(H_S^d \perp H_S^n)$$

We define the *infinite symplectic or quaternionic grassmannian* as

$$\mathrm{GrH} := \operatorname*{colim}_{d,n} \mathrm{GrH}_{(d,n)\in\mathbb{N}^2}$$

where we have denoted the ordinary quaternionic grassmannians by $\mathrm{GrH}_{d,n} := \mathrm{GrB}_{2d}(\mathbb{H}^{n+d})$.

The fact that for any $(d, n) \in \mathbb{N}^2$, $\mathrm{GrO}_{d,n}$ and $\mathrm{GrH}_{d,n}$ are indeed smooth schemes follows from 5.2. Moreover ([PW10a, page 22]) there are closed immersions $\mathrm{GrH}_{d,n} \hookrightarrow \mathrm{GrH}_{d,n+1}$ and $\mathrm{GrH}_{d,n} \hookrightarrow \mathrm{GrH}_{d+1,n}$ classified by the inclusions $\mathcal{T}_{d,n} \oplus 0 \subset \mathbb{H}^{n+d} \oplus \mathbb{H}$ and $\mathbb{H} \oplus \mathcal{T}_{d,n} \subset \mathbb{H} \oplus \mathbb{H}^{n+d}$ where $\mathcal{T}_{d,n}$ is the restriction of the universal symplectic bundle on $\mathrm{GrH}_{d,n}$ induced by $\mathbb{H}_S^{n+d}$ on the restriction of the universal rank $2d$ bundle on $\mathrm{Gr}_{2d}(\mathcal{O}_S^{2(n+d)})$ and the same is true for the orthogonal grassmannians, so that we can give precise meaning to the colimits appearing in the previous Definition. As for K-theory, we can form a system $\mathcal{K}\mathrm{Sp}_\bullet$ indexed by $\mathbb{N}$ having $\mathcal{K}\mathrm{Sp}_n := \sqcup_{2n+1}\mathrm{GrH}_{n,n}$ with colimit $\mathbb{Z} \times \mathrm{GrH}$. The same holds true for symmetric hermitian K-theory. This way we have that both $\mathbb{Z} \times \mathrm{GrH}$ and $\mathbb{Z} \times \mathrm{GrO}$ can be seen as filtered colimits of smooth (over S) schemes having a cofinal sequence so that we can try to apply Theorem B.5. First of all we remind that these ind-schemes represent Hermitian K-theory:

Theorem 5.7. *([ST15, Theorem 1.1], [PW10a] or [ST15, Theorem 8.2.]). The Hermitian K-theory* GW *(say* $\mathrm{GW}^{[0]}$ *of [Sch17, Definition 9.1]) as an object of* $\mathcal{H}^{\mathrm{Div}}(S)$ *is representable by* $\mathbb{Z} \times \mathrm{GrO}$ *so that* $\pi_0(\mathbb{Z} \times \mathrm{GrO}) \cong \mathrm{GW}_0(-)$ *as objects of* $\mathbf{Pre}(\mathrm{DSm}_S)$*. Analogously, for the Symplectic K-theory* KSp *(i.e.* $\mathrm{GW}^{[2]}$ *in the terminology of [Sch17]) we have* $\mathbb{Z} \times \mathrm{GrH} \simeq \mathrm{KSp}$ *in* $\mathcal{H}^{\mathrm{Div}}(S)$ *so that* $\pi_0(\mathbb{Z} \times \mathrm{GrH}) \cong \mathrm{KSp}_0(-)$ *as objects of* $\mathbf{Pre}(\mathrm{DSm}_S)$.

We can now prove that $\mathbb{Z} \times \mathrm{GrH}$ and $\mathbb{Z} \times \mathrm{GrO}$ satisfy the property (ii) (Definition B.2). This is an analogue of Proposition B.8.

Proposition 5.8. *The presheaves* $\mathbb{Z} \times \mathrm{GrO}$ *and* $\mathbb{Z} \times \mathrm{GrH}$ *as objects of* $\mathbf{Pre}(\mathrm{DSm}_S)$ *satisfy the property* (ii) *relative to* SmAff_S.

Proof. The proof follows *mutatis mutandis* the one found in [Rio06, Proposition III.14] once we have the Hermitian analogue of [Rio06, Assertion III.4] coming from the representability results contained in 5.7. For symmetric Hermitian K-theory this property is spelt out in [Zib11b, page 38] or in [Zib11a, page 477]. In the symplectic case, it is spelt out in [Ana15, Theorem 6.3]. □

Therefore, as a direct application of B.5 we have:

Proposition 5.9. *Then for any* $n \in \mathbb{N}$ *the map*

$$\pi_0 : \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(\mathrm{GW}^n, \mathrm{GW}) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(\mathrm{GW}_0^n, \mathrm{GW}_0)$$

is surjective.

Unfortunately, we were not capable to go on proving that we indeed have a bijection. This is still an open question. For KSp, however, we can say more:

Theorem 5.10. *Then for every natural number* n *one has the following isomorphisms*

$$\mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(\mathrm{KSp}, \mathrm{KSp}) \cong \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(\mathbb{Z} \times \mathrm{GrH}, \mathbb{Z} \times \mathrm{GrH}) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(\mathrm{KSp}_0(-), \mathrm{KSp}_0(-)) \cong \mathrm{KSp}_0(S)[[b_1, b_2, \ldots]]$$
$$\mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(\mathrm{KSp}^n, \mathrm{KSp}) \cong \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}((\mathbb{Z} \times \mathrm{GrH})^n, \mathbb{Z} \times \mathrm{GrH}) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(\mathrm{KSp}_0^n(-), \mathrm{KSp}_0(-))$$
$$\mathrm{Hom}_{\mathcal{H}_\bullet^{\mathrm{Div}}(S)}(\mathrm{KSp}^n, \mathrm{KSp}) \cong \mathrm{Hom}_{\mathcal{H}_\bullet^{\mathrm{Div}}(S)}((\mathbb{Z} \times \mathrm{GrH})^n, \mathbb{Z} \times \mathrm{GrH}) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)_\bullet}(\mathrm{KSp}_0^n(-), \mathrm{KSp}_0(-))$$

the b_i *being the Borel classes described in [PW10b].*

Proof. For $n = 1$, by B.5 we have to show that

$$R^1 \varprojlim_{m\in\mathbb{N}} \mathrm{KSp}_1(\mathcal{K}\mathrm{Sp}_m) = 0,$$

i.e that KSp satisfies the property (K) (B.3) with respect to the system $\mathcal{K}\mathrm{Sp}_\bullet$ which follows as in the case of K-theory using the explicit calculations of [PW10b, Theorem 11.4] (see indeed [PW10a, Theorems 9.4, 9.5]) to show that the involved tower satisfies the Mittag-Leffler property. The case with n factors follows by considering the system $\mathrm{KSp}_\bullet^n$. Indeed the computations of [PW10a, Theorems 9.4, 9.5] allow us to conclude the argument even in this case since they handle the products of the symplectic grassmannians involved. The statement concerning pointed operations follows from B.9. □

For $n = 1$ the observation that we can apply Riou's machinery also to quaternionic Grassmannians has been independently noted also in the recent work [DF19] (Proposition 4.0.4 in *op. cit.*). We can now argue as in the case of Theorem 4.6 to extend it to symplectic K-theory. To start with we draw the analogue of Diagram (1) in this setting:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(\mathrm{KSp}^n, \mathrm{KSp}) & \xrightarrow[\cong]{} & \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(\mathrm{KSp}^n, \mathrm{KSp}) \\ \downarrow{\scriptstyle \pi_0} & & {\scriptstyle \cong}\downarrow{\scriptstyle \pi_0} \\ \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(\mathrm{KSp}_0(-)^n, \mathrm{KSp}_0(-)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(\mathrm{KSp}_0(-)^n, \mathrm{KSp}_0(-)) \end{array} \tag{4}$$

Theorem 5.11. *Assume that, for a given $n \in \mathbb{N}$, $\mathrm{KSp}_0(-)^n$ satisfies the property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$. Then all the arrows in diagram (4) are bijections. The same is true if we replace the categories in diagram (4) with their pointed versions $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S)_\bullet)$, $\mathbf{Pre}(\mathrm{DSch}_S)_\bullet$, $\mathcal{H}^{\mathrm{Div}}_\bullet(S)$ and $\mathbf{Pre}(\mathrm{DSm}_S)_\bullet$.*

If S is quasi-projective scheme over a Noetherian affine scheme R, we will prove in Proposition 6.5 that for all $n \in \mathbb{N}$, $\mathrm{KSp}_0(-)^n$ satisfies the property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$.

proof of Theorem 5.11. The proof is *mutatis mutandis* the same as the one of Theorem 4.6. □

If S is now assumed to be affine, we can prove the following analogue of 4.12:

Theorem 5.12. *All the arrows in the following commutative cube are isomorphisms for every $n \in \mathbb{N}$*

$$\begin{array}{ccccccc} & & \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Aff}_S))}(\mathrm{KSp}^n, \mathrm{KSp}) & & \longrightarrow & & \mathrm{Hom}_{\mathcal{H}^{aff}(S)}(\mathrm{KSp}^n, \mathrm{KSp}) \\ & \nearrow & & & & \nearrow & \\ \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(\mathrm{KSp}^n, \mathrm{KSp}) & & \downarrow{\scriptstyle \pi_0} & \longrightarrow & \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(\mathrm{KSp}^n, \mathrm{KSp}) & & \downarrow{\scriptstyle \pi_0} \\ \downarrow{\scriptstyle \pi_0} & & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Aff}_S)}(\mathrm{KSp}_0^n, \mathrm{KSp}_0) & & \downarrow{\scriptstyle \pi_0} \longrightarrow & & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}_S)}(\mathrm{KSp}_0^n, \mathrm{KSp}_0) \\ & \nearrow & & & & \nearrow & \\ \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(\mathrm{KSp}_0^n, \mathrm{KSp}_0) & & \longrightarrow & & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(\mathrm{KSp}_0^n, \mathrm{KSp}_0) & & \end{array}$$

The pointed version of this theorem also holds.

Proof. The proof is the same of Theorem 4.12 given Theorems 5.11 and 6.5. □

Remark 5.13. Similar considerations to Remarks 4.8 and 4.13 apply.

As for ordinary K-theory, we shall conclude this section by considering the case of separated schemes. We will then suppose to have a base scheme S which is regular, Noetherian, separated (over $\mathrm{Spec}(\mathbb{Z})$) and such that $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. We remind that because of A.3 we have a fully faithful inclusion $\mathrm{Sm}_S^{Sep} \subseteq \mathrm{DSm}_S$ so that we can state and prove the following:

Theorem 5.14. *Suppose that for some $n \in \mathbb{N}$, $\mathrm{KSp}_0(-)^n$ satisfies the property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$, for example that S is affine. Then all the arrows in the following commutative diagram are bijections*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(\mathrm{KSp}^n, \mathrm{KSp}) & \xrightarrow[\cong]{} & \mathrm{Hom}_{\mathcal{H}^{\mathrm{Sep}}(S)}(\mathrm{KSp}^n, \mathrm{KSp}) \\ \downarrow{\scriptstyle \pi_0} & & {\scriptstyle \cong}\downarrow{\scriptstyle \pi_0} \\ \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(\mathrm{KSp}_0(-)^n, \mathrm{KSp}_0(-)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}_S^{Sep})}(\mathrm{KSp}_0(-)^n, \mathrm{KSp}_0(-)) \end{array}$$

This holds true even if we replace the categories we are considering with their pointed versions.

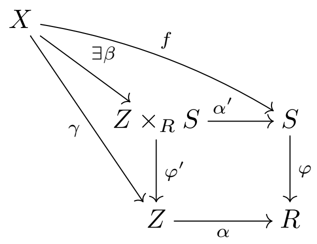
Proof. The right vertical map is a bijection because of Theorem 5.10, proven for separated schemes using Theorem B.5 (see also Remark 4.2), while the upper horizontal one is bijective beacause of Corollary 3.8. We can now finish using Proposition 4.14 and the assertion concerning the pointed situation follows using B.9. □

6 Property (P) for algebraic K-theory, Pic and Hermitian K-theory

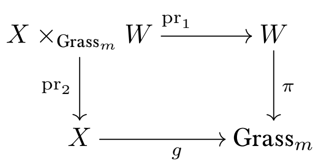
In this section we shall show that some important presheaves satisfy the property (P) (see Definition 4.4) for certain classes of schemes. We fix a base scheme S which is quasi-projective over a Noetherian affine scheme R throughout this section. We start with a mild generalisation of [Zan20, Theorem 5.5].

Proposition 6.1. *Let X be a divisorial scheme over S. Then given a finite number of vector bundles $\mathcal{E}_1, ..., \mathcal{E}_n \in \mathrm{Vect}(X)$ there is a smooth divisorial scheme $Y_{\mathcal{E}}$ over S and vector bundles $\mathcal{E}_{1,Y_{\mathcal{E}}}, ..., \mathcal{E}_{n,Y_{\mathcal{E}}}$ over it together with a morphism $\psi_{\mathcal{E}} : X \to Y_{\mathcal{E}}$ such that $\psi_{\mathcal{E}}^*(\mathcal{E}_{i,Y_{\mathcal{E}}}) \cong \mathcal{E}_i$ for every $i = 1, ..., n$. If moreover S and X are affine, then $Y_{\mathcal{E}}$ can be chosen to be affine.*

Proof. We begin with the non affine case and we consider $n = 1$. Because of the assumptions denoting $f : X \to S$ and $\varphi : S \to R$ the two structure morphisms, we have by [Zan20, Theorem 5.5] that there exists a divisorial scheme Z smooth over R and and an arrow $X \xrightarrow{\gamma} Z \xrightarrow{\alpha} R$ over R such that there exists a vector bundle $\mathcal{G}$ on Z having the property that $\gamma^*\mathcal{G} \cong \mathcal{E}$. We now consider the following diagram



Where the inner square is a pullback, the outer square commutes because of our assumptions, β exists because of the universal property of the pullback and φ' and α' are of finite type and smooth, respectively because of stability under base change of these two properties. If we denote $\psi := \beta$, $Y_{\mathcal{E}} := Z \times_R S$ and $\mathcal{F} := \varphi'^*\mathcal{G}$ the lemma is fully proved: indeed $Y_{\mathcal{E}}$ is divisorial because φ' is quasi-projective (quasi-projective maps are stable under base change) so that we can apply [TT90, 2.1.2 (h)]. This conclude the argument for $n = 1$. For $n > 1$ one repeat the same argument or follows verbatim the proof of Theorem [Zan20, Theorem 5.5]. We now assume that X and S are affine. Consider first the case $n = 1$. Every vector bundle P (say of rank m for simplicity otherwise we can reason on the connected components or we can use the result proved in the first part of the proof directly) on X is generated by global sections so there exists a grassmannian Grass_m over S together with a map $g : X \to \mathrm{Grass}_m$ in Sch_S such that $g^*\mathcal{T} \cong P$ where $\mathcal{T}$ is the universal vector bundle of the grasmannian. Since the Grasmannians are divisorial we can use Jouanolou's device (A.6) to build an affine vector bundle torsor $\pi : W \to \mathrm{Grass}_m$ over the Grassmannian, which is then an object of SmAff_S. Now consider the following pullback



We then have that $\mathrm{pr}_2 : X \times_{\mathrm{Grass}_m} W \to X$ is a torsor under a vector bundle and it is affine (π is affine so it is pr_2) so that it is a vector bundle ([Wei89, page 475]) and there exists an arrow $i : X \to X \times_{\mathrm{Grass}_m} W$ which splits pr_2. If we set $X_P := W$, $Q := \pi^*\mathcal{T}$ and $f := \mathrm{pr}_1 \circ i$ we have a datum as the one wanted in the statement of the Proposition in the case $n = 1$. For $n > 1$ the argument is similar.

□

As a consequence we can establish the property (P) in some notable cases. Recall that given a full subcategory of $\mathcal{C} \subseteq \mathrm{Sch}_S$, we denote the Picard presheaf associating to any scheme $X \in \mathcal{C}$ its Picard group $\mathrm{Pic}(X)$ ([Har77, page 143]) by $\mathrm{Pic} \in \mathbf{Pre}(\mathcal{C})$.

Proposition 6.2. *Then for any $n \in \mathbb{N}$ the presheaves K_0^n and $\mathrm{Pic}(-)^n$ satisfy the property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$. If S is affine, then for any $n \in \mathbb{N}$ the presheaves K_0^n and $\mathrm{Pic}(-)^n$ satisfy the property (P) with respect to $\mathrm{SmAff}_S \subseteq \mathrm{Aff}_S$.*

Proof. We prove only the first statement about K_0, the others can be proved in the same way. One first notices that for any $X \in \mathrm{DSch}_S$ (representatives of) elements $\mathcal{E} \in K_0(X)$ are of the form $\mathcal{E} = [E_0] - [E_1]$ where $E_0, E_1 \in \mathrm{Vect}(X)$. Using Proposition 6.1, we can find for every such $E_0, E_1 \in \mathrm{Vect}(X)$ vector bundles over X, a divisorial smooth scheme $Y_{\mathcal{E}}$ over S and vector bundles E_0', E_1' over it together with a morphism $\psi_{\mathcal{E}} : X \to Y_{\mathcal{E}}$ such that $\psi_{\mathcal{E}}^*(E_i') \cong E_i$ for $i = 1, 0$. One now notices, since pullback is a group homomorphism, that this implies that the element $\mathcal{E}_{Y_{\mathcal{E}}} = ([E_0'] - [E_1']) \in K_0(Y_{\mathcal{E}})$ has the property that $\psi_{\mathcal{E}}^*(\mathcal{E}_{Y_{\mathcal{E}}}) = \mathcal{E}$. This means that for every $\mathcal{E} \in K_0(X)$ we can find a divisorial smooth scheme $Y_{\mathcal{E}}$ over S and $\mathcal{E}_{Y_{\mathcal{E}}} \in K_0(Y_{\mathcal{E}})$ together with a morphism $\psi_{\mathcal{E}} : X \to Y_{\mathcal{E}}$ (over S) such that $\psi_{\mathcal{E}}^*(\mathcal{E}_{Y_{\mathcal{E}}}) = \mathcal{E}$. Now, an element of $K_0(X)^n$ is simply an n tuple of elements of $K_0(X)$ and for any morphism of schemes $f : Y \to Z$, $K_0(f)^n$ is the map $(f^*)^{\times n} K_0(Z)^n \to K_0(Y)^n$ given on each component by the usual pullback $f^* : K_0(Z) \to K_0(Y)$. As a consequence for every element $\mathcal{E} = (\mathcal{E}_1, ..., \mathcal{E}_n) \in K_0(X)^n$, using Proposition 6.1 and arguing as before, we can find a smooth scheme $Y_{\mathcal{E}}$ over S and $\mathcal{E}_{Y_{\mathcal{E}}} = (\mathcal{E}_{1,Y_{\mathcal{E}}}, ..., \mathcal{E}_{1,Y_{\mathcal{E}}}) \in K_0(Y_{\mathcal{E}})^n$ together with a morphism $\psi_{\mathcal{E}} : X \to Y_{\mathcal{E}}$ such that $\psi_{\mathcal{E}}^*(\mathcal{E}_{i,Y_{\mathcal{E}}}) = \mathcal{E}_i$ for every $i = 1, ..., n$. □

We shall study the property (P) for Hermitian K-theory. We begin with the analogue of 6.1 for inner product spaces.

Theorem 6.3. *Assume X is a divisorial scheme over S having 2 invertible. Then given a finite number of ϵ-inner product spaces over X, $V_1 = (\mathcal{E}_1, \varphi_1), ..., V_n = (\mathcal{E}_n, \varphi_n)$, there is a divisorial smooth scheme Y_V over S and ϵ-inner product spaces $V_{1,Y_V}, ..., V_{n,Y_V}$ over it together with a morphism $\psi_V : X \to Y_V$ such that $\psi_V^*(V_{i,Y_V}) \cong V_i$ for every $i = 1, ..., n$. If X and S are affine schemes, then we can take Y_V to be affine.*

Proof. We first assume $n = 1$ and that X is connected so that $\mathcal{E}$ is a vector bundle of rank n. We can use 6.1 to find a scheme W which is divisorial and smooth over S together with a vector bundle $\mathcal{F}$ on it and a map $g : X \to W$ such that $g^*(\mathcal{F}) \cong \mathcal{E}$. If X and S are affine, we remark that we may choose W to be affine. Now, we can consider the bilinear Grassmannian $\mathrm{GrB}_{n,W}(\mathbb{H}_\epsilon(\mathcal{F}))$. This is a divisorial smooth scheme affine over W. In particular, if W is affine, then it is affine in the absolute sense. Now the universal property of the bilinear grassmannians 5.2 together with Corollary 5.5 gives us a map $f : X \to \mathrm{GrB}_{n,W}(\mathbb{H}_\epsilon(\mathcal{F})) =: Y_V$ over W and then over S and an ϵ-inner product space E_V over Y_V such that $f^*(E_V) \cong V$, as desired. Now if X is not connected we can reason componentwise and then glue together the resulting schemes to get the assertion, as in the proof of [Zan20, Theorem 5.5]. The case $n > 1$ is similar. ☐

Remark 6.4. Observe that we do not require all our inner product spaces to have the same value of ϵ.

As a consequence, we have the following property (P) for Hermitian K-theory. We denote the symplectic K-theory presheaf by KSp_0 and by GW_0 the presheaf associating to a scheme X its ordinary Grothendieck-Witt groups of non degenerate symmetric forms ([Kne77, page 138]). GW_0 is $\mathrm{GW}_0^{[0]}(X)$ in the modern terminology introduced by Schlichting in [Sch17, page 74].

Proposition 6.5. *Assume that 2 is invertible in S. Then for any $n \in \mathbb{N}$ the presheaves KSp_0^n and GW_0^n satisfy the property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$. If S is affine, for any $n \in \mathbb{N}$ the presheaves KSp_0^n and GW_0^n satisfy the property (P) with respect to $\mathrm{SmAff}_S \subseteq \mathrm{Aff}_S$.*

Proof. The proof goes mutatis mutandis as the ones of 6.2 using Theorem 6.3. ☐

We then immediately have the following corollary using 4.5:

Corollary 6.6. *For any $n \in \mathbb{N}$ the natural restriction maps*

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(\mathrm{GW}_0(-)^n, \mathrm{GW}_0(-)) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(\mathrm{GW}_0(-)^n, \mathrm{GW}_0(-))$$
$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(\mathrm{KSp}_0(-)^n, \mathrm{KSp}_0(-)) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(\mathrm{KSp}_0(-)^n, \mathrm{KSp}_0(-))$$

are injective if S is as in the previous Proposition.

7 Applications

7.1 Algebraic structures on K-theory

In this section we will study the algebraic structures we can put on K-theory using Corollary 3.5 and Theorem 4.6. We fix S to be a Noetherian regular divisorial base scheme and we denote with K Thomason's K-theory simplicial presheaf. We shall begin with the most general result:

Theorem 7.1. *There exists a structure of lambda ring on K in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$. This structure is the unique structure inducing the standard lambda ring structure on $K_0 \in \mathbf{Pre}(\mathrm{DSm}_S)$.*

Proof. We notice that by [SGA71, VI, 3.2] we have a functorial lambda ring structure on $K_0(X)$ for every $X \in \mathrm{DSm}_S$, since for schemes in DSm_S this is really the algebraic K-theory of vector bundles, so that K_0 becomes a lambda ring object in $\mathbf{Pre}(\mathrm{DSm}_S)$ in the sense of Definition C.8. Therefore we can combine Theorem 4.1, [Rio10, Proposition 2.2.3] and Corollary 3.5 to obtain a commutative ring $(K, +, \times, 0, 1)$ together with a family of unary operations $\lambda^n : K \to K$ in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$ making it a lambda ring in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$ in the sense of Definition C.8. ☐

Remark 7.2. In the ring structure we have just defined on $K \in \mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$, 0 and 1 are defined as the two morphisms $0 : \bullet \to K$ and $1 : \bullet \to K$, $\bullet$ being the terminal object S, associated via the Yoneda lemma to the elements $0, 1 \in K_0(S)$. We also remark that, unless explicitly stated, we will always see K as canonically pointed by 0.

Proposition 7.3. *The endomorphisms $\psi^k : K \to K$ defined by lifting the Adams operations $\psi_0^k \in \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(K_0, K_0)$ are ring morphisms for every $k \geq 1$. Moreover for every $m, n \geq 1$ we have $\psi^m\psi^n = \psi^{nm} = \psi^n\psi^m$. As a consequence, $K \in \mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$ has the structure of a ψ-ring (see Definitions C.6 and C.8).*

Proof. $K_0 \in \mathbf{Pre}(\mathrm{DSm}_S)$ being a ψ-ring, we can argue as in Theorem 7.1 to conclude. □

Remark 7.4. Notice that given the lambda operations defined by Theorem 7.1, one could define Adams operations directly using the Newton formulas (see C.4), but one would get the same operations because of Corollary 3.5 and Theorem 4.1.

Before investigating the behaviour of the lambda operations, we will reassure the reader by showing that the ring structure we have just defined agrees with the one usually considered.

Proposition 7.5. *The additive and multiplicative operations of the ring structure on $K \in \mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$ given by Theorem 7.1 coincide with the ones induced by the usual H-group structure on K-theory and by Waldhausen's product for K-theory (as in [Wal85, page 342]).*

Proof. This is a simple consequence of Corollary 3.5 and the theorem of Riou 4.1 (as observed in [Rio10, Proposition 3.2.1] for smooth schemes). Indeed, the usual additive and multiplicative operations on K are explicitly given by (pointed) maps $\oplus : K \times K \to K$ and $\otimes : K \times K \to K$ in $\mathbf{sPre}(\mathrm{Sch}_S)$ (the functorialities can be checked using the construction found in [Wal85, page 342]) that induce a ring structure on $K \in \mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$. To show that this structure agree with the one of Theorem 7.1 it suffices to check that the structures they induce on $K_0 \in \mathbf{Pre}(\mathrm{DSm}_S)$ agree, which is clear. □

Corollary 7.6. *The lambda ring structure defined by Theorem 7.1 restricts to a lambda ring structure on K in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{DSch}_S))$, whose underlying ring structure agree with the canonical one. Furthermore if for any $n \in \mathbb{N}$, K_0^n satisfies the property (P) with respect to $\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$, this structure induce on $K_0 \in \mathbf{Pre}(\mathrm{DSch}_S)$ the lambda ring structure defined in [SGA71, VI, 3.2].*

Proof. It is a simple application of Theorem 4.6 and Corollary 3.5 given Proposition 7.5. □

In particular, the above corollary applies to the case when S is quasi-projective over an affine Noetherian ring R. To speak about higher algebraic K-theory groups, we need to refine the structures and some of the maps given by Proposition 7.1 to the *pointed* unstable homotopy category $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)$. This can be done using Corollary 3.5 and the pointed part of the statement of Theorem 4.1 which is obtained using B.9 since K-theory is naturally an H-group in the sense of Definition 2.1. We therefore get that $K \in \mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$ has the so called structure of an H-ring, i.e. the ring structure it has in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$ can be refined to a ring structure in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)$ (in this last category the ring structure is not unital[4]). This means that the multiplicative product of K comes from a map $\times : K \times K \to K$ in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)$. Using the argument contained in [Rio06, page 96] or directly using 3.5, 4.6 and [Rio06, Lemme III.33] we get that there is an injective map $\alpha : \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)}(K \wedge K, K) \to \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)}(K \times K, K)$ induced by $K \times K \to K \wedge K$ such that $\times \in \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)}(K \times K, K)$ is the image under α of a map $\times_\bullet \in \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)}(K \wedge K, K)$. In particular, $\times_\bullet$ is the unique morphism which makes the following diagram to commute

$$\begin{array}{ccc} K \times K & & \\ \downarrow & \overset{\times}{\searrow} & \\ K \wedge K & \xrightarrow[\times_\bullet]{} & K \end{array}$$

The fact that $\times$ is commutative implies that $\times_\bullet$ is commutative too, i.e. we have $\times_\bullet = \times_\bullet \circ \tau$ where τ is the usual switch map of $\wedge$. For any simplicial presheaf $\mathcal{X} \in \mathbf{sPre}(\mathrm{Sch}_S)$, we shall be interested to study the following groups

$$K_n(\mathcal{X}) := \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)}(\mathbb{S}^n \wedge \mathcal{X}_+, K)$$

where we have denoted the simplicial n-sphere by $\mathbb{S}^n$ as customary. Explicitly note that in the particular case where $\mathcal{X} = X \in \mathrm{DSch}_S$ is a divisorial scheme, these groups agree with the ordinary Quillen's higher algebraic K-theory groups. We can bundle these groups, letting n to vary, in a ring, as the following proposition makes precise:

Proposition 7.7. *For any $\mathcal{X} \in \mathbf{sPre}(\mathrm{Sch}_S)$ the multiplication law $- \times - : K \times K \to K$ induces a natural graded ring structure on*

$$K_*(\mathcal{X}) := \bigoplus_{n \in \mathbb{N}} K_n(\mathcal{X})$$

Denote this ring together with its multiplication by $(K_*(\mathcal{X}), \cup)$. If $\mathcal{X}$ is represented by a scheme $X \in \mathrm{Sch}_S$, the induced pairings $K_p(X) \times K_q(X) \to K_{p+q}(X)$ induced by this ring structure agree with the ones by Waldhausen, Loday and May as discussed in [Wei81].

[4] the map $1 : \bullet \to K$ in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$ is not pointed and therefore we cannot hope to refine it to a map in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)$.

Proof. By the preceding discussion, we can consider the map $\times_\bullet \in \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)}(K \wedge K, K)$ induced by the multiplicative product $\times$ defined by Theorem 7.1. If we denote the map in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)$ induced by the diagonal map $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ by $\Delta_{\mathcal{X}} : \mathbb{S}^{i+j} \wedge \mathcal{X}_+ \to (\mathbb{S}^i \wedge \mathcal{X}_+) \wedge (\mathbb{S}^j \wedge \mathcal{X}_+)$, for any $\mathcal{X} \in \mathbf{sPre}(\mathrm{Sch}_S)$ we get a multiplication

$$- \cup - : K_i(\mathcal{X}) \times K_j(\mathcal{X}) \to K_{i+j}(\mathcal{X})$$

induced by the map

$$\begin{array}{c}
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)}(\mathbb{S}^i \wedge X_+, K) \times \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)}(\mathbb{S}^j \wedge X_+, K) \\
\downarrow \wedge \\
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)}((\mathbb{S}^i \wedge X_+) \wedge (\mathbb{S}^j \wedge X_+), K^{\wedge 2}) \\
\downarrow \times_\bullet \circ - \circ \Delta_X \\
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)}(\mathbb{S}^{i+j} \wedge X_+, K)
\end{array}$$

This multiplication induces the desired graded ring structure and the naturality is clear. The agreement with the other pairings when $\mathcal{X}$ is a scheme follows from Proposition 7.5. □

To study the lambda ring structure, we explicitly note that for all $j > 0$ the lambda and Adams operations λ^j and ψ^j are pointed, and therefore they refine to maps $K \to K$ in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)$. As a consequence, the following makes sense:

Definition 7.8. For every simplicial presheaf $\mathcal{X} \in \mathbf{sPre}(\mathrm{Sch}_S)$ we define the lambda and the Adams operations $\lambda^r_n, \psi^j_n : K_n(\mathcal{X}) \to K_n(\mathcal{X})$ by postcomposition with the maps $\lambda^r, \psi^j : K \to K$ in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)$ coming from the maps defined in Theorem 7.1 using B.9. In other words, they are defined as

$$\lambda^r_n, \psi^j_n : \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)),\bullet}(\mathbb{S}^n \wedge \mathcal{X}_+, K) \to \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)),\bullet}(\mathbb{S}^n \wedge \mathcal{X}_+, K) \quad f \mapsto \lambda^r, \psi^j \circ f$$

Before going on, we make a digression of more general nature, which is not only of general interest, but might have direct applications in the near future, as it will be explained below. We fix a Grothendieck site $\mathcal{C}$ having a terminal object, and we consider the model category $\mathbf{sPre}(\mathcal{C})$ of simplicial presheaves with the Jardine local model structure localised at some class of maps S. This covers all the situations relevant for this paper. We denote the homotopy category $\mathrm{Ho}(\mathbf{sPre}(\mathcal{C}))$ by $\mathcal{H}$. We let $\mathcal{H}_\bullet$ to be its pointed version. By this we mean that we consider the pointed category of simplicial presheaves, we give to it the pointed model category structure induced by the one we are considering on the unpointed one and we take the homotopy category, as customary (so we are not considering the homotopy category pointed). We suppose to have a lambda ring $(K, +, \bullet, -, 0, 1)$ in $\mathcal{H}$ where all the maps $\lambda^r : K \to K$ are pointed for $r > 0$ so that by B.9 and B.10 they can be promoted to maps in $\mathcal{H}_\bullet$, and where the ring structure comes from a ring structure in $\mathcal{H}_\bullet$ (in this last category we are only looking at the non unital ring structure because we have to take care of the base point) meaning therefore that K is an H-ring. We also assume that the product $\bullet : K \times K \to K$ factors through the smash product i.e. that there exists a map $\wedge_\bullet$ so that the following diagram commutes

$$\begin{array}{ccc}
K \times K & & \\
\downarrow \varphi & \searrow^{\bullet} & \\
K \wedge K & \xrightarrow[\wedge_\bullet]{} & K
\end{array}$$

Remark that K is in particular an H-group, hence for every simplicial presheaf F, the set

$$\pi_n K(F) := \mathrm{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^n \wedge F_+, K) =: K_n(F)$$

has a group structure inherited from the H-group structure of K. Now, we assume that K satisfies descent (in the sense of [Jar15, page 102], i.e. it admits a weakly equivalent sectionwise fibrant replacement) so that for every object $X \in \mathcal{C}$, $K_n(X)$ is really the nth homotopy group of the simplicial set $K(X)$. Moreover we assume that the H-group structure on K is compatible with the homotopy groups, i.e. that the group structure on $K_n(F)$ for $n \geq 0$ induced from the H-group structure coincides with the standard one, i.e. the one defined as in topology using the co-group structure on $\mathbb{S}^1$. Loopspaces are of this form, for example. We could relax these assumptions but we do not have a reason to do that since they allow the discussion to be simpler and all the examples we have in mind fall in this description. Now, for any simplicial presheaf F we immediately notice that by applying the functor π_0 we obtain a lambda ring structure on the set $K_0(F)$. This is true because π_0 preserves finite products so that taking π_0 of the datum of maps and compatibilities we have for K in $\mathcal{H}$ gives us what we want.

Remark 7.9. Notice that the product induced by $\bullet$ on any $K_n(F)$ is trivial if $n \geq 1$ because of [Kra80, Lemme 5.2]. Indeed that lemma says that if we are given an H-space E together with a distributive multiplication over its H-space structure that factors through the smash product and a co-H-space X having the comultiplication factoring through the join (for example any suspension) in $\mathrm{Ho}(\mathrm{Top}_\bullet)$ then the monoid structure induced on $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top}_\bullet)}(X, E)$ by the multiplicative structure of E is trivial and in our case the same argument holds.

Since the multiplication of K factors through the smash product, we can define pairings $K_0(F) \times K_j(F) \to K_j(F)$ as for K-theory in Proposition 7.7. We define the graded group

$$K_*(F) := \oplus_{n\geq 0} K_n(F)$$

where the $K_n(F)$ are $K_0(F)$-modules using the pairings just introduced. Consider the maps $\lambda_n^r : \pi_n(\lambda^r) : K_n(F) \to K_n(F)$ for $r \geq 0$ if $n = 0$ and $r > 0$ if $n \geq 1$ defined as in Definition 7.8. Notice that for $n \geq 1$ these maps are group homomorphisms. Henceforth, $K_0(F)$ being a lambda ring since $K \in \mathcal{H}$ is such, it is possible to give to $K_*(F)$ the structure of pre-lambda ring $(K_*(F), \cdot)$ as in Example C.3 (axioms 1)-3) of the definition of λ ring are satisfied) and we call the lambda operations we have λ_*^r. We want to check that this is indeed a lambda ring.

Proposition 7.10. *For any simplicial presheaf F, the ring $(K_*(F), \cdot, \lambda_*^r)$ is a lambda ring.*

Proof. We have to check that the axioms 4) and 5) of Definition C.1 are satisfied. For elements in $K_0(F)$ this has already been done. Then we notice that because of the definitions and the considerations we made in C.3, we only need to verify that for any $n \geq 1$, the groups $K_n(F)$ are $K_0(F)$-lambda algebras. Since we already know that they are pre-$K_0(F)$-lambda algebras this really amounts to check axioms 4) and 5) for elements $x \in K_n(F)$ and $y \in K_0(F)$ as in Definition C.1. Using our dictionary, $x \in K_n(F)$ is a map in $\mathrm{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^n \wedge F_+, K)$. Now the verification of the axiom can be done in two steps. As a first step one has from the fact that K is a lambda ring that $\lambda^r \circ \lambda^s : K \to K$ and $P_{r,s}^\lambda : K \to K$ in $\mathcal{H}_\bullet$ given from the polynomial $P_{r,s}$ using the techniques of Appendix C are equal. Then one sees that the left hand side of the equality prescribed by axiom 5) equals the map obtained from $\mathrm{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^n \wedge F_+, K)$ by postcomposition with the pointed map $\lambda^r \circ \lambda^s$ with $r, s \geq 1$. As a second step, using Remark 7.9 we see that the polynomial maps $(P_{r,s}^\lambda)_n : K_n(X) \to K_n(X)$ involved in the right hand side of axiom 5) defined using $(K_*(F), \cdot, \lambda_*^r)$ equals the ones obtained from $\mathrm{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^n \wedge F_+, K)$ by postcomposition with the map $P_{r,s}^\lambda : K \to K$ in $\mathcal{H}_\bullet$. So axiom 5) is verified. Notice, because of our definitions, that since $n \geq 1$, many of the products on the RHS are equal so that it will be really a multiple of $\lambda^{rs}(x)$ as noted in the proof of [HKT17, Theorem 8.18]. The verification of axiom 4) can be done in a similar way with the caveat that in the construction of the polynomial maps involved in the verification of axiom 4) (see Appendix C), we build the monomial maps x^J using smash products instead of products, i.e. we use maps $x^J : K \wedge^n K \to K$. We can do this since under our assumptions, the multiplicative product we have factors through the smash. ☐

Remark 7.11. Suppose that $\mathcal{C}$ is a point with the chaotic topology and that we consider only the Jardine injective model structure on $\mathbf{sPre}(\mathcal{C})$. Then the homotopy category we obtain is the classical homotopy category of topological spaces $\mathrm{Ho}(\mathrm{Top})$. This means, as a corollary of the previous definition, that if we have a lambda ring X in $\mathrm{Ho}(\mathrm{Top})$ satisfying the above properties, then to the direct sum of its homotopy groups $\pi_*(X) := \oplus_{n\geq 0}\pi_n(X)$ can be given a structure of lambda ring using the previous Proposition.

Remark 7.12. If a Riou-like theorem will be proved for Hermitian K-theory, then we will be able to use 7.10 to give a lambda ring structure on the direct sum on the (symmetric) higher Grothendieck-Witt groups of schemes.

We can now come back to K-theory to discuss the structure we can put on $K_*(\mathcal{X})$ for every $\mathcal{X} \in \mathbf{sPre}(\mathrm{Sch}_S)$. This abelian group in principle can have two multiplicative structure as a $K_0(\mathcal{X})$-algebra. The first one is the one given in Example C.3, where the product of two homogeneous elements of positive degree is set to be 0. We will refer to this ring simply as $K_*(\mathcal{X})$ or $(K_*(\mathcal{X}), \cdot)$ if confusion might arise. The second one is the noncommutative structure induced on it by the Theorem 7.7. In this case, we will denote the resulting graded-commutative $K_0(\mathcal{X})$-algebra by $(K_*(\mathcal{X}), \cup)$. We defined in Definition 7.8 families of operations $\lambda_n^k : K_n(\mathcal{X}) \to K_n(\mathcal{X})$ and $\psi_n^k : K_n(\mathcal{X}) \to K_n(\mathcal{X})$ which bundle to maps $\lambda^k, \psi^k : K_*(\mathcal{X}) \to K_*(\mathcal{X})$. Notice the following:

Proposition 7.13. *For every $\mathcal{X} \in \mathbf{sPre}(\mathrm{Sch}_S)$ and every $a \in K_n(\mathcal{X})$, $b \in K_m(\mathcal{X})$, we have for every $k \geq 1$ that $\psi_*^k(a \cup b) = \psi_*^k(a) \cup \psi_*^k(b)$ where the product is induced by the pairing defined in Theorem 7.7. This is also trivially true for the product $\cdot$.*

Proof. One follows Riou [Rio06, page 99]. In fact as a consequence of Proposition 7.3 and Theorem 7.1 we have that in $\mathrm{Ho}(\mathbf{sPre}_{\mathrm{Zar}}(\mathrm{Sch}_S)_\bullet)$, for every $k \geq 1$, the equality $\psi^k \circ \times_\bullet = \times_\bullet \circ (\psi^k \wedge \psi^k)$ holds. This concludes the proof. ☐

We then have the following theorem:

Theorem 7.14. *Consider $\mathcal{X} \in \mathbf{sPre}(\mathrm{Sch}_S)$. Then the datum $(K_*(\mathcal{X}), \cdot, \lambda^k)$ is a lambda ring with associated ψ-ring $(K_*(\mathcal{X}), \cdot, \psi^k)$. Moreover, $(K_*(\mathcal{X}), \cup, \psi^k)$ is a noncommutative ψ-ring and the maps $\psi^k : (K_*(\mathcal{X}), \cup) \to (K_*(\mathcal{X}), \cup)$ are morphisms of noncommutative ψ-rings. These structures are functorial.*

Proof. The noncommutative assertions follows simply from Proposition 7.13 and Proposition 7.3. For the first part, to check 4) and 5) (1)-3) follow from the very definition, see C.3) we use Proposition 7.10. To check that the Adams operations we defined before agree with the ones induced by λ^k using the Newton formulas we notice that they are both additive so that we only need to check for elements of the form $x \in K_n(\mathcal{X})$, but this follows from the Newton formulas we have for $K \in \mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$ and the definitions we have given. Indeed as a first step one notices that the Newton formulas we have for $K \in \mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{Sch}_S))$ restrict on $K_n(\mathcal{X})$ to the usual Newton formulas for $K_0(\mathcal{X})$ in the case $n = 0$ and to $\psi_n^k = (-1)^{k+1} k \lambda^k$ for $n \neq 0$ in virtue of Remark 7.9 so they are the same formulas we get starting from $(K_*(X), \cdot, \lambda^k)$ because the product of two positive homogeneous elements is set to be trivial (this comparison only requires the pre-lambda ring structure because of Remark C.5). The fact that $(K_*(\mathcal{X}), \cdot, \psi^k)$ is a ψ-ring follows from this comparison or can be proved independently using 7.13. □

Corollary 7.15. *For every scheme $X \in \mathrm{Sch}_S$, the ring $(K_*(X), \cdot)$ is a lambda ring in the sense of Definition C.1, and this structure is functorial in X. Moreover, $(K_*(X)_{\mathbb{Q}}, \cdot)$, if $(K_*(X), \cdot)$ is $\mathbb{Z}$-torsion free, admits a (unique) lambda ring structure induced from $(K_*(X), \cdot, \lambda^k)$ defined before. Finally, all $K_n(X)$ are $K_0(X)$-lambda algebras, the product of the elements in $K_n(X)$ being trivial for $n \geq 1$.*

Proof. The first two assertions follow from our construction of the lambda operations, Theorem 7.14 and [Yau10, Theorem 3.49]. The last part follows from Theorem 7.14. □

Remark 7.16. One might wonder if changing our base scheme S, we change the structures induced by the operations on $K_n(\mathcal{X})$. Indeed, a scheme can be seen as a scheme over a priori many bases. However, Riou showed ([Rio10, Proposition 2.3.2]) that the operations we get on K-theory in $\mathcal{H}(S)$ do not depend on the choice of S, as long as S is divisorial and regular. Since we can reduce to the smooth schemes, we have that the operations we define for any divisorial scheme of finite type over our allowed bases S are the unique we can define using this method.

We conclude this section by giving an Adams-Riemann-Roch theorem (see[FL85, Chapter V]) that applies to the operations we have defined. Fix S to be a regular quasi-projective scheme over a Noetherian affine scheme R and consider the category DSch_S. Recall that a map $f : X \to Y$ in DSch_S is called a **projective local complete intersection (l.c.i.) morphism** if it factors as $f = \pi \circ i : X \to \mathbb{P}(\mathcal{E}) \to Y$ for some vector bundle $\mathcal{E}$ over Y where i is a regular embedding and $\pi : \mathbb{P}(\mathcal{E}) \to Y$ is the canonical projection. For such maps, the pushforward on Thomason's higher K-theory groups is well defined: indeed because of [TT90, 3.16], if a map is of the form [TT90, 3.16.4-3.16.7], we have that pushforwards are well defined. The results in *op. cit.* also give us a projection formula. This means that, having a well defined lambda ring structure on $K_*(X)$ for any $X \in \mathrm{DSch}_S$ which coincides with the one we have on $K_0(X)$ as studied in [FL85], we can repeat the argument used in [FL85] to prove the Adams-Riemann-Roch theorem ([FL85, Theorem V.7.6]) even in this context almost verbatim as it has been done in [Kö98] for higher equivariant K-theory. Indeed we can use the technique of the deformation of the normal bundle as in [FL85, IV 5] as it doesn't leave the category DSch_S and because of [TT90, Theorem 4.1] we also have a projective bundle theorem for higher K-theory. This means that we get the following, see [Zan19] for an explicit detailed check of the fact that the usual proof goes through:

Theorem 7.17 (Adams-Riemann-Roch). *Assume that S is quasi-projective over a Noetherian affine scheme R. Let be $f : X \to Y$ a a projective local complete intersection morphism in DSch_S. For every scheme Z, denote the direct sum of the higher K-theory groups by $K_*(Z) := \oplus_n K_n(Z)$ and by $\psi_Z^j : K_*(Z) \to K_*(Z)$ the Adams operations induced by Theorem 7.14. Let τ_f be the usual Adams-Riemann-Roch multiplier in $K_0(X)$ (see [FL85, V.6.3]). Then (after inverting $j \in \mathbb{Z}$ if necessary) we have $f_* \circ (\psi_X^j \cdot \tau_f) = \psi_Y^j \circ f_*$.*

This should be in agreement with the result of [Kö98] which however employs Grayson's definition of the Adams operations (that has still to be compared with ours).

7.2 Additive results

In this section we will assume that our base scheme S is divisorial regular and quasi-projective over a Noetherian affine scheme. So far we have studied only the algebraic K-theory K_0 as a presheaf of *sets*. However, it is more naturally a presheaf of *abelian groups*. For a given full subcategory $\mathcal{C}$ of Sch_S let us consider the natural transformation $\delta : \mathrm{Pic} \to K_0$ in $\mathbf{Pre}(\mathcal{C})$, given for any scheme $X \in \mathcal{C}$ by the assignment $[L] \mapsto [L]$ for a any class $[L]$ of a line bundle L on X. Denoting as $(\mathbf{Pre}(\mathcal{C}), \mathrm{Ab})$ the category of presheaves of abelian groups over a given small category $\mathcal{C}$, we remind that Riou was able to show ([Rio10, Proposition 5.1.1]) that the map $\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S, \mathrm{Ab})}(K_0, K_0) \xrightarrow{\delta^*_{\mathrm{Sm}}} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S), \mathrm{Sets}}(\mathrm{Pic}, K_0)$ obtained by composition with δ is a bijection. One should replace the hypothesis of being divisorial with the one of being separated to be consistent with the assumptions of Riou, but Riou's proof goes through also in this case. We would like to remove the hypothesis of smoothness from the results of Riou. First of all, recall that in both $\mathcal{H}^{\mathrm{Div}}(S)$ and $\mathrm{Ho}(\mathrm{s}\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{DSch}_S))$ the functor Pic is represented by $\mathrm{B}\mathbb{G}_m$> for an explicit argument, one can check that the argument of [NSOsr09, Lemma 2.6] goes through. Moreover, in $\mathcal{H}^{\mathrm{Div}}(S)$, Pic is also represented by $\mathbb{P}^\infty$, the standard ind-scheme obtained as colimit of all the projective space along the standard inductive system (see [MV99]). We can then prove the following:

Theorem 7.18. *For any $n \in \mathbb{N}$, all the arrows in the following diagram are isomorphimsms*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Ho}(s\mathbf{Pre}_{\mathrm{Zar}}(\mathrm{DSch}_S))}(B\mathbb{G}_m^n, K) & \longrightarrow & \mathrm{Hom}_{\mathcal{H}^{\mathrm{Div}}(S)}(B\mathbb{G}_m^n, K) \\
\downarrow \pi_0 & & \pi_0 \downarrow \cong \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(\mathrm{Pic}^n, K_0) & \xrightarrow[\mathrm{res}]{} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(\mathrm{Pic}^n, K_0)
\end{array}$$

Moreover, also all the arrows in the following commutative diagram are isomorphisms

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S,\mathrm{Ab})}(K_0, K_0) & \xrightarrow{\delta^*_{\mathrm{Sch}}} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S)}(\mathrm{Pic}, K_0) \\
\beta \downarrow & & \downarrow \cong \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S,\mathrm{Ab})}(K_0, K_0) & \underset{\cong}{\xrightarrow{\delta^*_{\mathrm{Sm}}}} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S)}(\mathrm{Pic}, K_0) \cong \lim_n K_0(\mathbb{P}^n) \cong K_0(S)[[U]]
\end{array}$$

where the maps δ^*_{Sm} and δ^*_{Sch} are induced from the presheaves maps $\delta : \mathrm{Pic} \to K_0$ and $U = [\mathcal{O}(1)] - 1$ is the compatible family in $\lim_n K_0(\mathbb{P}^n)$.

Proof. For the first diagram, we see that the top horizontal map is an isomorphism because of Proposition 2.20, and the lower horizontal map is injective because of Propositions 4.5 and 6.2. This closes the argument since the right vertical π_0 map is an isomorphism because of [Rio10, Proposition 5.1.1]. For the second diagram one notices that the right vertical arrow is an isomorphism because of what we just proved. The isomorphisms $\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}_S)}(\mathrm{Pic}, K_0) \cong \lim_n K_0(\mathbb{P}^n) \cong K_0(S)[[U]]$ are proved in [Rio10, 5.1.1], which proves also that the bottom horizontal line is an isomorphism. The arrow β is injective because of Theorem 4.6 together with the fact that the category of presheaves of abelian groups admits a faithful embedding into the category of presheaves of sets. So also the arrow δ^*_{Sch} is 1-1 by diagram chase. We are then left to prove that β is surjective. Let us study the map

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSch}_S,\mathrm{Ab})}(K_0, K_0) \underset{\beta}{\rightarrowtail} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S,\mathrm{Ab})}(K_0, K_0) \underset{\cong}{\xrightarrow{\varphi}} K_0(S)[[U]]$$

arising from the diagram. Denote the kth Adams operation by $\psi^k : K_0 \to K_0$. Riou shows in [Rio10] that denoting as $x \cdot \psi^k \in \mathrm{Hom}_{\mathbf{Pre}(\mathrm{DSm}_S,\mathrm{Ab})}(K_0, K_0)$ the map given by $y \mapsto x \cdot \psi^k(y)$ for $x \in K_0(S)$, this is mapped via φ to $x(1+U)^k$ in $K_0(S)[[U]]$ and these elements generates the image of φ by [Rio06, IV.15] or [Rio10, page 246]. So if we show that all the $x \cdot \psi^k$ are in the image β, we can conclude. This is true because the Adams operations on K_0 over smooth schemes comes, because of our theorems, as the unique restriction of the operations we have built on K_0 for singular schemes. Hence the theorem is fully proved. □

Remark 7.19. In the statement of Theorem 7.18, as in Remarks 3.6 and 4.8, for any $i \in \mathbb{N}$, denoting as Ω^i_f the right derived functor of Ω^i in the simplicial model categories we considered, we could have replaced K with $\Omega^i_f K$ and K_0 with K_i everywhere in the second variable of the Hom-sets considered because of [Rio10, Proposition 5.1.1]. However, since we do not need this extra generality, we preferred to keep the notation and the assumptions to be simpler.

A Divisorial schemes

We collect some notions about divisorial schemes.

Definition A.1. ([SGA71, II 2.2.3] or [TT90, Definition 2.1]) A quasi-compact and quasi-separated (qcqs for short) scheme X is called *divisorial* (or *has an ample family of line bundles*) if there is a finite family of line bundles $L_1, ..., L_n$ on X together with finitely many global sections $s_i \in \Gamma(X, L_i)$ such that their non vanishing loci X_{s_i} ([GD71, 0, 4.1.9]) form an open affine cover of X.

See [TT90, 2.1.1] for equivalent characterizations of divisorial schemes.

Remark A.2. Recall that a morphism between schemes $f : X \to S$ is said to have affine diagonal if the diagonal embedding $X \to X \times_S X$ ([Sta18, Tag 01KJ]) is affine. A very important property of divisorial schemes is that they have affine diagonal over $\mathrm{Spec}(\mathbb{Z})$. For a simple proof see [BS03, Proposition 1.2]. Also, for a morphism of schemes, the property of having affine diagonal is stable under composition and base change because the same proof used for the property of being separated found for example in [Sta18, Tag 01KH] goes through, affine morphisms being stable under composition and base change because of [Sta18, Tag 01SC] and [Sta18, Tag 01SD]. In addition, if we have morphisms $f : X \to Y$ and $g : Y \to Z$ so that g and $g \circ f$ have affine diagonal, then also f has affine diagonal because we can mimick the proof of [Sta18, Tag 01KV] using the fact that every affine morphism is separated ([Sta18, Tag 01S7]) and Remark 9.11 of [GW10, page 230] (or [Sta18, Tag 01SG]). This implies that any morphism between schemes with affine diagonal has affine diagonal. We remark that instead of the notion of having affine diagonal, one can use the equivalent notion of semi-separated schemes and morphisms detailed in [TT90, Appendix B.7] that we find less explicit, although equivalent. In *op. cit.* one can find observations similar to the ones we just made on the schemes and the morphisms having affine diagonal in terms of semi-separatedness.

Lemma A.3. *([SGA71, II 2.2.7.1]) Every regular (or more generally locally factorial) Noetherian scheme with affine diagonal is divisorial.*

Proof. In [SGA71] the hypothesis of having affine diagonal is replaced by the stronger separated hypothesis. However the separated hypothesis is used in the proof of [SGA71, II 2.2.7] only in order to apply [SGA71, II 2.2.6]; in particular it is required that an open embedding of an affine scheme into our given scheme X is an affine morphism. But this is true if X has affine diagonal ([Sta18, Tag 01SG]), so the proof goes through (see also [BS03, Proposition 1.3]). □

We say that a scheme X is smooth over a base S if its structure map is smooth ([GD67, IV 6.8.6, 17.3.1], [GW10, 6.14], [Sta18, Tag 01V5]). Explicitly note that a smooth morphism is locally of finite presentation ([GD67, IV 1.4.2]) and so locally of finite type. We do not assume that a smooth morphism is separated.

Corollary A.4. *Every quasi-compact scheme X that has affine diagonal and that is smooth over a Noetherian regular base scheme S is divisorial.*

Recall that, fixed a Noetherian base scheme S, we denote the category of schemes of finite type over S by Sch_S and by DSch_S its full subcategory of divisorial schemes. Remark that any morphism of finite type having Noetherian target is of finite presentation ([Sta18, Tag 01TX]). Notice that because S is Noetherian, every scheme in Sch_S is Noetherian as well ([GD71, I 6.2.2]) and therefore quasi-separated ([Sta18, Tag 01OY]). In addition, given $X \in \mathrm{Sch}_S$, any open subscheme $U \subseteq X$ is Noetherian ([GD67, I 6.1.4], [GW10, Corollary 3.22]) and it is canonically embedded in X via a quasi-compact open immersion. Therefore, the Zariski topology is well defined on both Sch_S and DSch_S (because of [TT90, 2.1.2 (e)]). Given a Noetherian regular divisorial base scheme S recall that we shall denote the full subcategory of (divisorial) smooth schemes over S by $\mathrm{Sm}_S \subseteq \mathrm{Sch}_S$ ($\mathrm{DSm}_S \subseteq \mathrm{DSch}_S$). The Zariski topology is well defined on both Sm_S and DSm_S and the Nisnevich topology as well. All the topologies introduced are generated by a cd structure in the sense of Voevodsky as detailed in [AHW17, Section 2].

Remark A.5. To see that the Nisnevich topology is well defined as a Grothendieck topology on DSm_S one uses the facts concerning schemes with affine diagonal that we recollected at the beginning of this Appendix together with A.3 and A.4.

We conclude by recalling a fundamental property of divisorial schemes, the so called Jouanolou's trick. This was proven for projective schemes by Jouanolou and then generalised to the following statement by Weibel, who apparently learnt this fact from Thomason, see [Wei89].

Proposition A.6. *(Jouanolou's Trick, [Wei89, Proposition 4.4]). Let X be a divisorial scheme. Then there exists an affine scheme T and a morphism $T \to X$ which is a torsor under a vector bundle.*

B Riou's methods

In this section we shall review some methods of Riou from [Rio06] and [Rio10]. We fix some Noetherian base scheme S and we let Sch_S to be the category of schemes of finite type over S. We let $\mathcal{C}$ to be some Grothendieck site whose underlying category some full subcategory of Sch_S. We assume that $\mathbf{sPre}(\mathcal{C})$ comes endowed with a simplicial model category structure which comes as a left Bousfield localization of the Jardine injective local model structure on $\mathbf{sPre}(\mathcal{C})$. We denote the homotopy category of $\mathbf{sPre}(\mathcal{C})$ with this model structure by $\mathcal{H}$. We can then give the following definition:

Definition B.1. Let $X \in \mathcal{H}$. Define a presheaf $\pi_0 X : (\mathcal{C})^{op} \to \mathrm{Sets}$ as

$$\pi_0 X(-) := \mathrm{Hom}_{\mathcal{H}}(-, X)$$

If $X \in \mathbf{Pre}(\mathcal{C})$, we define a morphism $\tau_X : X \to \pi_0 X$ in $\mathbf{Pre}(\mathcal{C})$ using the Yoneda lemma in the obvious way. For any $U \in \mathcal{C}$ we denote the function $X(U) \to \pi_0 X(U)$ induced by the natural transformation τ_X with $\tau_{X,U}$.

We now come to a minor generalisation of the property (ii), which was introduced in [Rio06, III.8] and then appeared in a different terminology in [Rio10, Definition 1.2.5].

Definition B.2. Let $X \in \mathbf{Pre}(\mathcal{C})$. Assume we have a full subcategory $\mathcal{A}$ of $\mathcal{C}$. Then we say that X satisfies the property (ii) relative to $\mathcal{A}$ if for every $U \in \mathcal{A}$, the arrow $\tau_{X,U}$ defined above is surjective.

We can now formalise Riou's property (K).

Definition B.3. Let $X_\bullet = (X_i)_{i\in I}$ an inductive system in $\mathcal{C}$ indexed by a directed set I having a cofinal sequence. Denote $X := \mathrm{colim} X_\bullet$ its colimit in $\mathbf{Pre}(\mathcal{C})$. We say that E satisfies the property (K) with respect to the system $X_\bullet$ (or simply with respect to X) if the arrow

$$\alpha : \mathrm{Hom}_{\mathcal{H}}(X, E) \to \mathrm{Hom}_{\mathbf{Pre}(\mathcal{C})}(\pi_0 X, \pi_0 E)$$

induced by taking π_0 is injective.

The following Proposition can be proved exactly as in [Rio06, Proposition III.10] with a diagram chase argument.

Proposition B.4. *Let $X \in \mathbf{Pre}(\mathcal{C})$ and E be an object in $\mathcal{H}$. Assume that X satisfies the property (ii) relative to $\mathcal{A}$ and that for every $B \in \mathcal{C}$ we have an object $A \in \mathcal{A}$ and at least one arrow $A \to B$ in $\mathcal{C}$ which induces an isomorphism in $\mathcal{H}$. Then the map*

$$\tau_X^* : \mathrm{Hom}_{\mathbf{Pre}(\mathcal{C})}(\pi_0 X, \pi_0 E) \to \mathrm{Hom}_{\mathbf{Pre}(\mathcal{C})}(X, \pi_0 E)$$

is injective.

As a corollary we have the following theorem, whose proof follows verbatim the one of [Rio10, Proposition 1.2.9] using Milnor's exact sequence (see [Rio06, pages 72-73]).

Theorem B.5. *Let $\mathcal{A}$ a full subcategory of $\mathcal{C}$ satisfying the hypothesis of the previous Proposition. Let $X_\bullet = (X_i)_{i\in I}$ an inductive system in $\mathcal{C}$ indexed by a directed set I having a cofinal sequence (indexed by the naturals). Set $X = \mathrm{colim} X_\bullet \in$ $\mathbf{Pre}(\mathcal{C})$ and suppose that X satisfy the property (ii) relative to $\mathcal{A}$. For every H-group E we can form the diagram*

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{H}}(X,E) & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathbf{Pre}(\mathcal{C})}(\pi_0 X, \pi_0 E) & & \\ & \searrow^{\gamma} & \downarrow{\scriptstyle \tau_X^*} & & \\ & & \mathrm{Hom}_{\mathbf{Pre}(\mathcal{C})}(X, \pi_0 E) & \xrightarrow{\cong} & \varprojlim_{i\in I^{op}} (\pi_0 E)(X_i) \end{array}$$

where the map α is the one induced by taking π_0. Then α and γ are surjective and τ_X^* is bijective. Moreover $\mathrm{Ker}\,\gamma =$ $\mathrm{Ker}\,\alpha \cong R^1 \varprojlim_{i\in I^{op}} \mathrm{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^1 \wedge X_{i+}, E)$

Remark B.6. Under the hypotheses of the previous theorem, E satisfies the property (K) with respect to the system $X_\bullet$ if $R^1 \varprojlim_{i\in I^{op}} \mathrm{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^1 \wedge X_{i+}, E) = 0$.

Remark B.7. If $\mathcal{C} = \mathrm{DSm}_S$ is the category of smooth divisorial schemes, $\mathcal{A} = \mathrm{SmAff}_S$, S is regular divisorial and Noetherian and we consider the $\mathbb{A}^1$-localised Nisnevich injective local model structures over s$\mathbf{Pre}(\mathcal{C})$, then we get the property (ii) as studied by Riou, modulo the fact that our schemes can be non separated. In addition, the conditions on $\mathcal{A}$ required by the previous two statements are verified in this case. This uses the Jouanolou's trick A.6 and the fact that affine vector bundle torsors are $\mathbb{A}^1$-weak equivalences because of [MV99, Example 2.3 page 106]. Note also that if two presheaves satisfy the property (ii) then also their product does.

Proposition B.8. *Let be S a Noetherian, regular and divisorial base scheme. For any $n \in \mathbb{N}$, the presheaf $(\mathbb{Z} \times \mathrm{Gr})^n$ as an object of $\mathbf{Pre}(\mathrm{DSm}_S)$ satisfies the property (ii) relative to SmAff_S. In addition, K-theory satisfies the property (K) with respect to it.*

Proof. The statement has been proved by Riou in [Rio10, Lemma 1.2.6] and [Rio06, page 94]. This uses the fact that K-theory can be represented in $\mathcal{H}^{\mathrm{Div}}(S)$ by $\mathbb{Z} \times \mathrm{Gr}$ (see [Rio06, III.4]) and the computations of the K-theory of ordinary Grassmannians. These facts don't rely on the hypothesis of separatedness, so the argument of Riou goes through if we drop that assumption keeping the one of divisoriality. □

We also need the following very useful observation of Riou, which we state here together with the proof for the reader's convenience. This is essentially [Rio06, Lemme III.19], we just noticed that the proof applies to a more general situation).

Lemma B.9. *Consider some Grothendieck site of the form $(\mathcal{A}, \tau)$ where $\mathcal{A}$ is a full subcategory of Sch_S and denote $\mathcal{C} :=$ s$\mathbf{Pre}_\tau(\mathcal{A})$. Let be E an H-group in $\mathcal{C}$. Then for every object X of $\mathcal{C}_\bullet$ the evident morphism*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C}_\bullet)}(X, E) \to \{f \in \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, E), f^\star(\bullet) = \bullet \in \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(S, E)\}$$

is a bijection, where we have denoted the composition with f by $f^\star$.

Proof. First one notices that we can assume E to be fibrant. We have a cofiber sequence

$$S_+ \to X_+ \to X$$

in both $\mathcal{C}$ and $\mathcal{C}_\bullet$, obtained as the pushout of the two maps $S_+ \to X_+$ and $S_+ \to *$ (remark that all the objects involved are cofibrant and that $S_+ \to X_+$ is a cofibration so this pushout is actually an homotopy pushout). We can then apply to this cofiber sequence the pointed mapping space $\mathfrak{Map}_{\mathcal{C}_\bullet}(-, E)$ to get a fibration sequence

$$\mathfrak{Map}_{\mathcal{C}_\bullet}(X, E) \to \mathfrak{Map}_{\mathcal{C}_\bullet}(X_+, E) \to \mathfrak{Map}_{\mathcal{C}_\bullet}(S_+, E)$$

which induces a long exact sequence on the homotopy groups. Now, since E is an H-group, we have that the π_0 terms of this sequence are groups, so that, using the fact that the map $S_+ \to X_+$ has a retract induced by the terminal map $X \to S$ we can split the long exact sequence of the homotopy groups in short exact sequences, obtaining for the π_0 terms the following exact sequence

$$1 \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C}_\bullet)}(X, E) \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, E) \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(S, E) \to 1$$

that allows us to conclude the proof. □

Remark B.10. The proof of the above lemma shows that the same result holds when replacing $\mathcal{C}$ with any model category coming from any simplicial model category $\mathcal{C}$ where every object is cofibrant, X with an object of $\mathcal{C}_\bullet$ so that its distinguished point is given by a cofibration and E with a fibrant H-group having the same property of X.

We shall need the following facts, proved by Riou in [Rio06] and [Rio10]. We recall some terminology in *op.cit.* From now until the end of this appendix, we shall fix a Noetherian regular divisorial base scheme S. We denote the collection of maps in DSm_S which are vector bundle torsors by $\mathcal{T}$ and by $\mathcal{T}_{\mathrm{aff}}$ the collection of projection maps of the form $\mathbb{A}^1_X \to X$ in SmAff_S. We have the following result

Proposition B.11. *([Rio06, Proposition II.16]). There is an equivalence of categories* $\Theta : \mathrm{SmAff}_S[\mathcal{T}_{\mathrm{aff}}^{-1}] \xrightarrow{\simeq} \mathrm{DSm}_S[\mathcal{T}^{-1}]$

Using [Rio06, Proposition B.8] or [Rio10, Remark 1.2.8] we then have

Corollary B.12. *The equivalence Θ induces an equivalence between the category of $\mathcal{T}$-invariant presheaves in* $\mathbf{Pre}(\mathrm{DSm}_S)$ *and the category of $\mathbb{A}^1$-invariant presheaves in* $\mathbf{Pre}(\mathrm{SmAff}_S)$

Remark B.13. Notice that the proof of the previous two claims does not require the schemes in SmAff_S and DSm_S to be separated.

C Lambda ring objects in a category

The aim of this appendix is to collect some facts about lambda rings that are scattered through the literature or can be considered folklore. Let us start with the following definition that can be found in [Wei13, Definition I 4.3.1] or [Yau10, Definition 1.10]:

Definition C.1. A lambda ring, is the datum of a commutative unital ring R together with a family of set maps $\lambda^k : R \to R$, $k \geq 0$ such that

1) $\lambda^0(x) = 1$, $\lambda^1(x) = x$ for every $x \in R$.

2) $\lambda^k(x + y) = \lambda^k(x) + \lambda^k(y) + \sum_{i=1}^{k-1} \lambda^i(x)\lambda^{k-i}(y)$ for every $x, y \in R$ for $k > 1$.

3) $\lambda^k(1) = 0$ for $k \geq 2$.

4) $\lambda^k(xy) = P_k(\lambda^1(x), ..., \lambda^k(x); \lambda^1(y), ..., \lambda^k(y))$ for all $x, y \in R$.

5) $\lambda^k(\lambda^l(x)) = P_{k,l}(\lambda^1(x), ..., \lambda^{kl}(x))$ for all $k, l \in \mathbb{N}$ and $x \in R$.

where P_k and $P_{k,l}$ are certain universal polynomial with coefficients in $\mathbb{Z}$ (see [Yau10, Examples 1.7 and 1.9]). A lambda homomorphism between lambda rings $(R, \{\lambda^r_R\})$ and $(S, \{\lambda^r_S\})$ is a ring homomorphism $f : R \to S$ such that $f \circ \lambda^r_R = \lambda^r_S \circ f$ for all $r \geq 0$ ([Yau10, Definition 1.25]).

In literature one can find the name pre-lambda ring or simply lambda ring for a ring satisfying 1)-2) above and the name special lambda ring for rings satisfying 1)-5). Since we will be interested mainly in special lambda rings we will stick to the notation introduced in the previous definition. Finally, note that for a lambda ring, a splitting principle is always satisfied, see [Yau10, Theorem 1.44].

Definition C.2. Suppose R is a lambda ring and A is an R-algebra (not necessarily unital) together with a family of set maps $\lambda^k : A \to A$ for $k \geq 1$, we will say that A is an R-lambda algebra if $R \times A$ with the addition, the multiplication and the operations defined below is a lambda ring (see [Kra80, page 240]).

1) For all $a, b \in R$ and $x, y \in A$ we set $(a, x) + (b, y) := (a + b, x + y)$.

2) For all $a, b \in R$ and $x, y \in A$ we set $(a, x)(b, y) := (ab, ay + bx + xy)$

3) For all $(a, x) \in R \times A$ we set $\lambda^k(a, x) := (\lambda^k(a), \sum_{i=0}^{k-1} \lambda^i(a)\lambda^{k-i}(x))$, $k \geq 1$.

Example C.3. (See also [HKT17, page 436]). Suppose we have an $\mathbb{N}$-graded sum of R-modules $M_* = \oplus_{n\in\mathbb{N}} M_n$ where $M_0 = R$ is a lambda ring. Assume that we give to M_* the following product

$$(a_0, a_1, a_2, ...)(b_0, b_1, b_2, ...) := (a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + b_0a_2, ...)$$

and that we define, for $k \geq 1$ (λ^0 being the map $(a_0, a_1, \cdots) \mapsto (1, 0, 0, \cdots)$)

$$\lambda^k(a_0, a_1, a_2, ...) := (\lambda_0^k(a_0), \sum_{i=0}^{k-1} \lambda_0^i(a_0)\lambda_1^{k-i}(a_1), \sum_{i=0}^{k-1} \lambda_0^i(a_0)\lambda_2^{k-i}(a_2), ...)$$

where $\lambda_n^i : M_n \to M_n$ are group homomorphisms for all $n \geq 1$ and all $i \geq 1$. Then the R-algebra M_* automatically satisfies 1)-3) of the definition of lambda ring. In this situation, to check that M_* is a lambda ring we only need to verify that for any $n \geq 1$, the groups M_n are R-lambda algebras. This really amounts to check axioms 4) and 5) for elements $x \in M_n$ and $y \in M_0$. See also the proof of [HKT17, Theorems 7.1 and 8.18] for more details about why it suffices to check this.

Given a lambda ring, one can always define the so called Adams operations, which are very useful for many purposes.

Definition C.4. Let R be a lambda ring. For each $n \geq 1$ we can define the nth ***Adams operation*** ψ^n by recursion as $\psi^1(x) = x$, $\psi^2(x) = x^2 - 2\lambda^2(x)$, $\psi^k(x) = \lambda^1(x)\psi^{k-1}(x) - \lambda^2(x)\psi^{k-2}(x) + ... + (-1)^k\lambda^{k-1}(x)\psi^1(x) + (-1)^{k+1}k\lambda^k(x)$ (these are called Newton formulas, see [Yau10, 3.10]).

Remark C.5. Consider a ring R with a family of lambda operations satisfying only 1)-2) of Definition C.1. Then Adams operations can be defined via the Newton formulas as before and they coincide with the ones defined in [SGA71, V 7.1] or in [Yau10, 3.1]. This gives that ψ^n are group homomorphism under these assumptions on the given lambda operations. If R is a lambda-ring, then each Adams operation ψ^n is a ring homomorphism and for each $m, n \geq 1$, we have $\psi^m\psi^n = \psi^{nm} = \psi^n\psi^m$ ([Yau10, 3.6, 3.7]). If R is $\mathbb{Z}$-torsion free, then the Adams operations uniquely determine the lambda ring structure over R used to define them ([Yau10, Theorem 3.15]).

Definition C.6. ([Yau10, Definition 3.44] and probably having origin in [Knu73, page 49]). A commutative ring R is called a ***ψ-ring*** if it is equipped with ring endomorphisms $\psi^k : R \to R$ for $k \geq 1$ such that $\psi^1 = id_R$ and for each $m, n \geq 1$ $\psi^m\psi^n = \psi^n\psi^m = \psi^{mn}$. If R is noncommutative, we say that it is a ***noncommutative ψ-ring*** if as in the commutative case, it is equipped with ring endomorphisms $\psi^k : R \to R$ for $k \geq 1$ such that $\psi^1 = id_R$ and for each $m, n \geq 1$, $\psi^m\psi^n = \psi^n\psi^m = \psi^{mn}$.

Remark C.7. Usually, a lambda ring is assumed to be commutative and unital. However these two condition might be relaxed. Indeed, in the works [Kra80] and [Sou85] one find the notion of lambda ring without the unit being defined as a ring satisfying the axiom of Definition C.1 not involving the unit, see [Sou85, page 512] for example. This is due to the fact that, strictly speaking, in a pointed category the notion of ring makes perfect sense if we do not ask the multiplicative unit to exist because the additive and the multiplicative units cannot be characterized at the same time via ***pointed*** maps from the zero object to the ring we are considering, and those authors were working mainly in pointed (homotopy) categories. Also, as far as we know, there isn't a well developed theory, or even a notion, of lambda rings in the context of noncommutative rings. The problem, roughly speaking, is that the axioms of a lambda rings involves symmetric polynomials that do not easily fit in the context of noncommutative rings. To have a feeling of the issues, the reader can try to make sense of axiom 2) in this context for example. However, the definition of ψ-ring easily extends to the noncommutative case. Indeed the only definition of "noncommutative lambda ring" we have been able to find in literature is the one contained in [Pat95, Definition I.1] that agrees with our definition of noncommutative ψ-ring. Notice that in [Pat95] the only noncommutative rings considered are noncommutative R-algebras for some commutative ring R containing the rationals.

The previous definitions allow us to make sense of the notion of lambda ring in any category with finite products and a terminal object, sometimes the terminal object is referred as the empty product. This can be done using the machinery of [Rio10, Section 2] in the context of lambda-rings. We fix in this section such a category $\mathcal{C}$. Since we do not know any explicit reference for the notion of lambda ring object is such category besides the one that can be given using the reasoning of [Rio10, Section 2], we think it is worthwhile to spell out its structure here. We will denote the terminal object of $\mathcal{C}$ by $*$. Suppose we are given a commutative unital ring object K in $\mathcal{C}$. We define this notion using the machinery of [Rio10, Section 2], see also [Bor94, Section 3.2 page 125], i.e. as a datum $(K, +, -, \cdot, 0, 1)$ where K is an object of $\mathcal{C}$, $+ : K \times K \to K$ and $\cdot : K \times K \to K$ represent the additive and the multiplicative laws of K, $- : K \to K$ denotes the inverse for the group structure and the two maps $0, 1 : * \to K$ represent the additive and the multiplicative neutral elements. These maps satisfy the usual axioms required from the definition of commutative ring object, i.e. $(K, +, -, 0)$ is an abelian group object in $\mathcal{C}$, $(K, \cdot, 1)$ is a commutative monoid object in $\mathcal{C}$ and we require the obvious diagram expressing the right and left distributivity of the multiplication with respect of the addition to commute. We will write a polynomial of degree m in n variables with integer coefficients as

$$P = \sum_{|J| \leq m} a_J x^J, \quad J = (j_1, \dots, j_n), \quad |J| = \sum_{i=1}^{n} j_i \leq m, \quad x^J = x_1^{j_1} \cdots x_n^{j_n} \tag{5}$$

here the x_i are the variables and $a_J \in \mathbb{Z}$ for every J. Now, given an integer $q \in \mathbb{Z}$ we define the multiplication by q as a map $\cdot q : K \to K$ to be the zero map if $q = 0$, and as the following composition if $q > 0$

$$K \xrightarrow{\text{diagonal}} K^q := K \underbrace{\times \cdots \times}_{q-1 \text{ times}} K \xrightarrow{+} K$$

which is well defined because of the associativity of the group law. If $q < 0$ we define the map in the same way but we postcompose with the map $- : K \to K$. We can do something analogue with the operation "raising to the power of j" for any $j \in \mathbb{N}$. If $j = 0$ we define this map as $1 : K \to K$. Otherwise we define the map $\cdot^j : K \to K$ as the following composition

$$K \xrightarrow{\text{diagonal}} K^j \xrightarrow{\cdot} K$$

also here this map is well defined because of the associativity of the multiplicative law. With the same process, we can define for every multivariable x^J of length n and degree m as above a map

$$x^J : K^n \to K$$

by considering the composition

$$K^n \xrightarrow{\cdot^{j_1} \times \cdots \times \cdot^{j_n}} K^n \xrightarrow{\cdot} K$$

which is well defined because of the axioms of commutative ring. Post composing the previous map with $a\cdot$ for any integer a gives us maps

$$ax^J : K^n \to K$$

Now, suppose we have a polynomial $P = \sum_{|J| \leq m} a_J x^J$ in n variables and of degree m as in (5). Denote the number of summands in P by q_P. We can define a map $P : K^n \to K$ as follows

$$K^n \xrightarrow{\times_{|J| \leq m} a_J x^J} K^{q_P} \xrightarrow{+} K$$

which is well defined because of the ring axioms (associativity, distributivity etc.). The last step to make sense of the lambda ring axioms is then to consider a family of maps $\lambda^r : K \to K$ with $r \geq 0$. We can write expressions using these operations as variables. For example suppose we want to formalize the axiom $\lambda^r(x+y) = \sum_{i+j=r} \lambda^i(x)\lambda^j(y)$ as the equality between two maps $K \times K \to K$. We then interpret the left hand side as the composition

$$K \times K \xrightarrow{+} K \xrightarrow{\lambda^r} K$$

For the right hand side, we see it as the composition

$$K \times K \xrightarrow{(\lambda^0 \times \cdots \times \lambda^r) \times (\lambda^0 \times \cdots \times \lambda^r)} K^{r+1} \times K^{r+1} \xrightarrow{P} K$$

where P is the polynomial of degree 2 in $2(r+1)$ variables involved in the right hand side. If a map $K \times K \to K$ is built in this way, we will denote it as P^λ. Asking if the axiom 2) of lambda ring holds then amount to ask if those two maps are equal. The same can be done for the remaining axioms of lambda ring: they all involve polynomials with coefficients in $\mathbb{Z}$. We can then give the following:

Definition C.8. A lambda ring object, or simply lambda ring, in a category $(\mathcal{C}, \times)$ with finite products is the datum of a commutative (unital) ring object $(K, +, -, \cdot, 0, 1)$ in it together with a family of morphisms $\{\lambda^n : K \to K\}_{n \in \mathbb{N}}$ in $\mathcal{C}$ such that the axioms 1)-5) of definition C.1 hold, provided we make sense of the terms involved as we explained above. Remark that this definition coincides with the one we would get by using the machinery of [Rio10, Section 2]. A ψ-ring object in $(\mathcal{C}, \times)$ is the datum of a ring object (commutative or not) together with a family of ring homomorphisms ψ^k, $k \geq 1$, satisfying the formal properties of Definition C.6.

Notice that all the arrows induced by the polynomials involved in the axioms 2), 4) and 5) of the definition of a lambda ring are naturally pointed, i.e. they are pointed maps $(K^p, 0) \to (K, 0)$ for some suitable p.

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