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ON THE ANALYTIC SINGULAR SUPPORT FOR THE SOLUTIONS OF A CLASS OF DEGENERATE ELLIPTIC OPERATORS

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Abstract. We study a class of degenerate elliptic operators (which is a slight extension of the sums of squares of real analytic vector fields satisfying Hörmander Condition). We show that, in dimension 2 and 3, for every operator L in such a class and for every distribution u such that Lu is real analytic, the analytic singular support of u, $\operatorname{sing\,supp} u$, is a "negligible" set. In particular, we provide (optimal) upper estimates for the Hausdorff dimension of $\operatorname{sing\,supp} u$. Finally, we show that in dimension $n \geq 4$, there exists an operator in such a class and a distribution u such that $\operatorname{sing\,supp} u$ is of dimension n.

1. Introduction and statement of the results

We study the (local) analytic regularity for a class of degenerate elliptic operators with real analytic coefficients. In order to be definite, throughout this paper we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain ($n \geq 2$), let $N \geq 2$ be an integer and let

$$X_j(x, D) = \sum_{k=1}^{n} a_{jk}(x) D_k, \qquad (j = 1, \dots, N)$$

where $a_{jk} \in C^\omega(\Omega; \mathbb{R})$, $k = 1, \dots, n$, $j = 1, \dots, N$ and $D_k = \frac{\partial_{x_k}}{\sqrt{-1}}$.

We assume

(H) for every $x \in \Omega$ the dimension of the Lie algebra generated by $X_1, \dots, X_N$ and their (possibly iterated) commutators is n.

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We consider the operator

$$L(x,D)=\sum_{i,j=1}^{N} X_i(x,D)b_{ij}(x)X_j(x,D)+ \sum_{j=1}^{N} c_j(x)X_j(x,D)+d(x), \tag{1.1}$$

with $b_{ij}, c_j, d \in C^{\omega}(\Omega;\mathbb{C})$, $i,j=1,\dots,N$, and

$$[b_{ij}]_{i,j=1,\dots,N}+[\bar{b}_{ji}]_{i,j=1,\dots,N}>c,$$

where c is a positive constant. We consider the following problem:

(P) Let $u \in \mathcal{D}'(\Omega)$ such that Lu is real analytic on a subdomain $V \subset \Omega$. Is u real analytic in V possibly except on a negligible set?

We recall that, even taking in (1.1) $b_{ij}=\delta_{ij}$ and $b_j=d=0$, the problem of the analytic regularity of the solutions of $Lu=f$ with f real analytic is open. It is well-known that if L is elliptic on Ω, then $\operatorname{sing\,supp} u=\emptyset$ (this is a classical result see e.g. [4], page 207). Here $\operatorname{sing\,supp} u$ is the analytic singular support, i.e. $x_0 \notin \operatorname{sing\,supp} u$ if and only if u is real analytic near x_0. We point out that, as remarked in [9] page 149, by assuming a constant rank condition, a second order degenerate elliptic operator can be written as an operator sum of squares of vector fields. Let us also recall that, for sums of squares, if the coefficients of L are real analytic, Hörmander Condition (H) is a necessary and sufficient condition for the C^∞ hypoellipticity of L (see [8], Théorème 2.2). On the other hand, the real analyticity of Lu does not imply that u be real analytic (see [3]). Then, in general (without additional assumptions on the operator L), one cannot expect that u be a real analytic function whenever Lu is.

We refer the interested reader to [6] and [2] for an accurate description of the problem of the analytic hypoellipticity for operators sums of squares and an updated description of the available regularity results. In this paper, we take a different attitude: instead of looking for conditions ensuring the absence of the analytic singularities, we show that, in low dimension, the set of the analytic singularities is negligible (of Hausdorff dimension 0 for $n=2$ and 2 for $n=3$). More precisely, we prove the following

Theorem 1.1. *Let L be an operator of the form* (1.1) *and let u be a distribution*[1] *defined on Ω such that Lu is real analytic on a domain $V \subset \Omega$. Then,*

[1] Really, by a result of Hörmander [9], $u \in C^\infty(V)$.

(i) If $n = 2$, u is real analytic on V possibly except on a discrete set.
(ii) If $n = 3$, $\operatorname{sing\,supp} u \cap V$ is contained in an analytic set of dimension 2.

It is well-known that the problem of regularity is a microlocal problem (i.e. it is appropriately formulated in the phase space taking into account both the singular points and the related frequencies). We point out that even if our results are stated in a local form they are in essence microlocal. Indeed, they are based on the following (classical) facts:

(1) the analytic singular support of u is the projection (on the base) of the analytic wave front set of u, $WF(u)$ (see [10] Definition 3.1 and Theorem 3.2);
(2) $WF(u) \subseteq WF(Lu) \cup \mathrm{Char}(L)$ (see [10] Theorem 5.4),

where

$$\mathrm{Char}(L) = \{(x, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\}) \mid X_j(x, \xi) = 0,\ j = 1, \dots, N\}.$$

More precisely, we can reduce our vector fields to local standard forms and, since we are working in low dimension, we can remove from the characteristic set suitable regions (due to the fact that these regions do not intersect the analytic wave front set of the distribution u). In this way, we end up with a subset of the characteristic set whose projection on the base–containing the analytic singular support of u– is suitably small.

We recall that the analytic singular support of a solution of $Lu = f$, with f real analytic, may be the empty set. For instance, for $n = 2$, it is well-known that the operator $L = D_1^2 + x_1^{2p} D_2^2$, where p is a positive integer, is analytic hypoelliptic. On the other hand, for $L = D_1^2 + (x_1^2 + x_2^2) D_2^2$, there exists a solution of the equation $Lu = f$, with f real analytic, such that u is not real analytic at the origin of $\mathbb{R}^2$ (see [12]), i.e. $\operatorname{sing\,supp} u = \{(0, 0)\}$ (because of L is elliptic except at $(0, 0)$). Let us also recall that, in the case $n \geq 3$, the projection on the base of $\mathrm{Char}(L)$, $\pi(\mathrm{Char}(L))$, may be the whole $\mathbb{R}^n$. For instance, consider $L = D_1^2 + (D_2 + x_1 D_3)^2$. Then, $\pi(\mathrm{Char}(L)) = \mathbb{R}^3$ but every solution of $Lu = f$ is real analytic where f is real analytic (this can be seen as a very special case of a more general result see [15, 14]). It is also well-known, that there exists a solution v (see (3.19)) of $(D_1^2 + D_2^2 + x_1^2 D_3^2) v = 0$ such that $\operatorname{sing\,supp} v \neq \emptyset$ (see [3]). One can easily show that, for such a function, $\operatorname{sing\,supp} v$ is an analytic set of dimension 1. On the other hand, in Section 3 we prove that, without additional assumptions, Theorem 1.1(ii) is optimal. (The optimality of Theorem 1.1(i) is a consequence of [12].) More precisely, we have the following

Proposition 1.1. *There exists a solution of equation*

$$(D_1^2 + D_2^2 + x_1^2 D_3^2)u = 0 \quad \text{in} \quad \mathbb{R}\times]-\infty, 2[\times\mathbb{R},$$

such that $\operatorname{sing\,supp} u = \{(x_1, x_2, x_3) \in \mathbb{R}\times]-\infty, 2[\times\mathbb{R} \mid x_1 = 0\}$.

For $n \geq 4$, the analytic singular support may be a large set. Indeed, we have the

Proposition 1.2. *Let $n \geq 4$, then there exists a solution of equation*

$$\left(D_1^2 + (D_2 + x_1 D_3)^2 + \sum_{j=4}^{n} D_j^2\right) u = 0$$

in $]-\infty, -2[\times\mathbb{R}^2\times]-\infty, 2[\times\mathbb{R}^{n-4}$, *such that*

$$\operatorname{sing\,supp} u =]-\infty, -2[\times\mathbb{R}^2\times]-\infty, 2[\times\mathbb{R}^{n-4}.$$

2. Proof of Theorem 1.1

Let us begin the proof of Theorem 1.1 by recalling a very special case of a known result (see e.g. [1] page 61).

Theorem 2.2. *(i) Let $\Omega \subset \mathbb{R}^2$ be an open set and let*

$$X_1 = D_{x_1}, \qquad X_j = a_j(x_1, x_2) D_{x_2}, \qquad j = 2, \dots, N, \tag{2.2}$$

be real analytic vector fields satisfying Hörmander Condition. Consider an operator L as in (1.1)*, suppose that* $\operatorname{Char}(L)$ *is a (real analytic) symplectic manifold and let* $x_0 \in \pi(\operatorname{Char}(L)) = \{x \in \Omega \mid \sum_{s=2}^{N} a_s^2(x) = 0\}$. *If there exist $\ell \in \{2, \dots, N\}$, m a positive integer and U, a neighborhood of x_0 in $\mathbb{R}^2$, such that the functions in* (2.2) *satisfy*

1. $\partial_1^i a_j(x) = 0$*, for every* $x \in U \cap \pi(\operatorname{Char}(L))$*, for every* $j \in \{2, \dots, N\}$ *and* $i \in \{1, \dots, m-1\}$*;*
2. $\partial_1^m a_\ell(x_0) \neq 0$.

Then $x_0 \notin \operatorname{sing\,supp} u$ *if* $x_0 \notin \operatorname{sing\,supp} Lu$.

(ii) Let $\Omega \subset \mathbb{R}^3$ an open set and let $X_1 = D_{x_1}$, $X_2 = D_2 + a(x_1, x_2, x_3) D_{x_3}$ be real analytic vector fields satisfying Hörmander Condition. Consider an operator L as in (1.1) *and suppose that* $\operatorname{Char}(L)$ *is a (real analytic) symplectic manifold. Then*

$$(x_0, \xi_0) \notin WF(u) \quad \text{if} \quad (x_0, \xi_0) \notin WF(Lu).$$

2.1. **Proof of Theorem 1.1(i)** (i.e. $n = 2$). By Assumption (H), for every point $x \in \Omega$ there exist a least one vector field, say X_1, and an open subset $W \subset V$, such that X_1 is not singular on W. Then, possibly reducing W, we may rectify the vector field X_1, using a real analytic change of coordinates, and we may assume that

$$X_1 = D_1$$

Because of the compactness of $\overline{\Omega}$, in order to prove the result, it suffices to show that a solution u has at most finitely many points of non analyticity in such a set W. By means of a linear substitution, with real analytic coefficients, acting on the vector fields we can assume that the remaining vector fields are of the form

$$a(x_1, x_2)D_1 \quad \text{or} \quad b(x_1, x_2)D_2$$

(for suitable real analytic functions a and b). We observe that Hörmander Condition is retained by non-singular linear substitutions with smooth coefficients. Let us also point out that the operator L w.r.t. the "new" vector fields takes the form (1.1) for suitable coefficients. Then the characteristic set of L is given by

$$\begin{aligned}\mathrm{Char}(L) = \{(x_1, x_2, 0, \xi_2) \mid (x_1, x_2) \in W, \quad \xi_2 \neq 0,& \\ b_j(x_1, x_2) = 0 \qquad \forall j \in J\},&\end{aligned}$$

where $J \subseteq \{2, \dots, N\}$ and $b_j(x_1, x_2)$ is a real analytic function on W, for every $j \in J$. Set

$$B(x) = \sum_{j \in J} [b_j(x)]^2.$$

By Assumption (H), for every $\bar{x} \in W$ there exists $k \in \mathbb{N}$ such that

$$\begin{cases} \partial_{x_1}^j B(\bar{x}) = 0, \quad \forall j = 1, \dots, k-1, \\ \partial_{x_1}^k B(\bar{x}) \neq 0. \end{cases} \tag{2.3}$$

Let us take $\bar{x} \in W \cap \pi(\mathrm{Char}(L))$. Without loss of generality we may assume that $\bar{x} = 0$ and that

$$W =]-\delta, \delta[\times]-\delta, \delta[.$$

We observe that

$$W \cap \pi(\mathrm{Char}(L)) = \{x \in W \mid B(x) = 0\}.$$

Using (2.3) and the Weierstrass Preparation Theorem, we find, possibly reducing W,

$$B(x) = B(x_1, x_2) = e(x_1, x_2) \left(x_1^k + \sum_{j=1}^{k} B_j(x_2) x_1^{k-j} \right), \tag{2.4}$$

for $(x_1, x_2) \in W$ where e and B_j, $j = 1, \dots, k$, are real analytic functions with $B_j(0) = 0$, $j = 1, \dots, k$, and $e(x_1, x_2) \neq 0$ for every $(x_1, x_2) \in W$. Set

$$f(x_1, x_2) = x_1^k + \sum_{j=1}^{k} B_j(x_2) x_1^{k-j}. \tag{2.5}$$

Then, we have that

$$W \cap \pi(\mathrm{Char}(L)) = \{x \in W \mid f(x) = 0\}. \tag{2.6}$$

By the Fundamental Theorem of Algebra, we have that there exist $z_1(x_2), \dots, z_k(x_2) \in \mathbb{C}$ such that $f(z_j(x_2), x_2) = 0$, for $j = 1, \dots, k$ and for every $x_2 \in]-\delta, \delta[$. Then, the discriminant is given by

$$D(x_2) = \Pi_{i<j}(z_i(x_2) - z_j(x_2))^2 \qquad (x_2 \in]-\delta, \delta[).$$

We observe that $D(\cdot)$ is real analytic on $]-\delta, \delta[$. (Indeed, it is a symmetric polynomial w.r.t. the roots z_ℓ. Then it can be written as a polynomial of k variables evaluated at the elementary symmetric functions $s_i^{(k)}(x_2) = \sum_{1 \leq h_1 < \dots < h_i \leq k} z_{h_1}(x_2) \dots z_{h_i}(x_2)$, $i = 1, \dots, k$. Then the analytic regularity of $D(\cdot)$ is a direct consequence of the identities $s_j^{(k)}(x_2) = (-1)^j B_j(x_2)$, $j = 1, \dots, k$.)

Let us suppose that $D(x_2) = 0$ for every $x_2 \in]-\delta, \delta[$ (otherwise we may directly assume that f is of the same form as $\tilde{f}$ below). Then by the results in [11] (Théorème page 18) there exists $\tilde{f}$ of the form (2.5) but of degree (w.r.t. x_1) strictly less than f such that

$$\mathcal{A} := \{(x_1, x_2) \in]-\delta, \delta[^2 \mid f(x_1, x_2) = 0\} = \\ \{(x_1, x_2) \in]-\delta, \delta[^2 \mid \tilde{f}(x_1, x_2) = 0\},$$

and the discriminant of $\tilde{f}$, $\tilde{D}(\cdot)$, is not identically zero on $]-\delta, \delta[$. We have that there exist $\tilde{k}$ functions, $\tilde{z}_j(x_2)$, such that

$$\tilde{f}(x_1, x_2) = \Pi_{j=1}^{\tilde{k}}(x_1 - \tilde{z}_j(x_2)).$$

Then, the set $\mathcal{A}$ can be stratified as follows: since $\tilde{D}(\cdot)$ is real analytic, it vanishes (at most) at finitely many points $\tilde{x}_2^i$, $i = 1, \dots, \ell$. Let

$$w_{ij} = (\tilde{z}_j(\tilde{x}_2^i), \tilde{x}_2^i) \in \mathcal{A},\ i = 1, \dots, \ell \text{ and } j = 1, \dots, \tilde{k}, \tag{2.7}$$

(these are the strata of dimension 0 where we have no control on the singularities of u). Let

$$(y_1, y_2) \in \mathcal{A} \setminus \{w_{ij} \mid, i = 1, \dots, \ell,\ j = 1, \dots, \tilde{k}\}.$$

Then there exist $\varepsilon > 0$ and $j \in \{1, \dots, \tilde{k}\}$ such that $y_1 = \tilde{z}_j(y_2)$, $\tilde{z}_j(\cdot)$ is a real analytic function in $]y_2 - \varepsilon, y_2 + \varepsilon[$, and

(2.8) $Q_\varepsilon(y) \cap \pi(\mathrm{Char}(L)) = Q_\varepsilon(y) \cap \{(\tilde{z}_j(x_2), x_2) \mid x_2 \in]y_2 - \varepsilon, y_2 + \varepsilon[\}.$

where

$$Q_\varepsilon(y) :=]y_1 - \varepsilon, y_1 + \varepsilon[\times]y_2 - \varepsilon, y_2 + \varepsilon[.$$

The proof of Theorem 1.1(i) is completed if we show that

$$Q_\varepsilon(y) \cap \{(\tilde{z}_j(x_2), x_2) \mid x_2 \in]y_2 - \varepsilon, y_2 + \varepsilon[\} \cap \operatorname{sing\,supp} u = \emptyset. \tag{2.9}$$

We observe that $\mathrm{Char}(L) \cap \pi^{-1}(Q_\varepsilon(y))$ is a symplectic manifold. In order to apply Theorem 2.2(i) we need to check Assumptions (1) and (2). By the definition of the set $\mathcal{A}$ and (2.8), we deduce that

$$f(x_1, x_2) = e_\sharp(x_1, x_2)(x_1 - \tilde{z}_j(x_2))^{2i}, \qquad (x_1, x_2) \in Q_\varepsilon(y), \tag{2.10}$$

for a suitable $i \in \mathbb{N}$ and with $e_\sharp(\cdot)$ strictly positive in $Q_\varepsilon(y)$. ((2.10) can be obtained by dividing f by $x_1 - \tilde{z}_j(x_2)$ as many times as possible.) Then, we find that

$$b_\ell(x_1, x_2) = e_\ell(x_1, x_2)(x_1 - \tilde{z}_j(x_2))^i, \qquad (\ell \in J)$$

where e_ℓ are real analytic functions. Hence, there exists (at least) an index $\ell_0 \in J$ such that e_{ℓ_0} is always different from zero on $Q_\varepsilon(y)$. This implies that Conditions (1) and (2) of Theorem 2.2(i) are satisfied with $m = i$. Then, by Theorem 2.2(i), we deduce that (2.9) holds and we conclude that $\operatorname{sing\,supp} u \cap V \subseteq \{w_{ij} \mid, i = 1, \dots, \ell,\ j = 1, \dots, \tilde{k}\}$ (with w_{ij} as in (2.7)).

2.2. **Proof of Theorem 1.1(ii)** (i.e. $n = 3$). As in the case of $n = 2$, we may assume that

$$\begin{cases} X_1 = D_1, \\ X_j = \sum_{s=2}^3 a_{js}(x) D_s, \quad j = 2, \dots, N, \end{cases} \tag{2.11}$$

for suitable real analytic functions a_{js}, $j = 2, \dots, N$ and $s = 2, 3$. There are two cases: either there exists a point $x_0 \in V$ such that at x_0 three of the vector fields $X_1, \dots, X_N$ are linearly independent or we are in the complementary case.

In the first case, possibly renaming the vector fields, we may assume that X_1, X_2 and X_3 are linearly independent at x_0. Then, we consider the real analytic function (defined on V)

$$f(x) = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{22}(x) & a_{23}(x) \\ 0 & a_{32}(x) & a_{33}(x) \end{bmatrix}.$$

We have the decomposition

$$V = \{x \in V \mid f(x) \neq 0\} \cup \{x \in V \mid f(x) = 0\},$$

where $\{x \in V \mid f(x) = 0\}$ is an analytic set of dimension 2.

Hence, since L is elliptic (then analytic hypoelliptic) on the set $\{x \in V \mid f(x) \neq 0\}$, we conclude that

$$\operatorname{sing\,supp} u \cap V \subseteq \{x \in V \mid f(x) = 0\},$$

and the proof is completed.

Then, we may assume that, in V, there are at most two linearly independent vector fields. Set

$$\mathcal{A} = \left\{ x \in V \mid \sum_{j=2}^{N} \sum_{s=2}^{3} (a_{js}(x))^2 = 0 \right\}.$$

We observe that, by Hörmander Condition, $V \setminus \mathcal{A} \neq \emptyset$. For any $x_0 \in V \setminus \mathcal{A}$ there exists a component, say $a_{22}(\cdot)$, such that $a_{22}(x_0) \neq 0$. Let us consider the open set

$$V_1 = \{x \in V \mid a_{22}(x) \neq 0\}.$$

Modulo a (non-singular) substitution with real analytic coefficients, we may assume that

$$\begin{cases} X_1 = D_1, \\ X_2 = D_2 + b_{23}(x) D_3 \\ X_j = b_{j3}(x) D_3, \quad j = 3, \dots, N, \end{cases} \tag{2.12}$$

for suitable real analytic functions b_{j3}, $j = 2, \dots, N$.

We remark that, by this substitution, the operator L can be rewritten once more in the form (1.1). Let us also point out that the characteristic set is (clearly) unchanged and that also the new vector fields satisfy Hörmander Condition.

Case 1: $N = 2$.

Then

$$\operatorname{Char}(L) \cap \pi^{-1}(V_1) = \{(x, 0, -b_{23}(x)\xi_3, \xi_3) \mid \text{ with } x \in V_1,\ \xi_3 \neq 0\}$$

and

$$\operatorname{Char}(L) \cap \pi^{-1}(V_1) = A \cup B$$

where

$$A = \{(x, 0, -b_{23}(x)\xi_3, \xi_3) \mid \text{ with } x \in V_1,\ \xi_3 \neq 0, \partial_{x_1} b_{23}(x) \neq 0\}$$

and

$$B = \{(x, 0, -b_{23}(x)\xi_3, \xi_3) \mid \text{ with } x \in V_1,\ \xi_3 \neq 0, \partial_{x_1} b_{23}(x) = 0\}.$$

Then, by Theorem 2.2(ii), $A \cap WF(u) = \emptyset$, hence

$$\pi(A) \cap \operatorname{sing\,supp} u = \emptyset$$

and $\pi(B) \subseteq \{x \in V_1 \mid \partial_{x_1} b_{23}(x) = 0\}$. By Hörmander Condition $\partial_{x_1} b_{23}(x)$ cannot be identically zero and we obtain that

$$\operatorname{sing\,supp} u \cap V_1 \subseteq \pi(B) \subseteq \{x \in V_1 \mid \partial_{x_1} b_{23}(x) = 0\}.$$

Then, we conclude that

$$\operatorname{sing\,supp} u \cap V \subseteq \{x \in V \mid a_{22}(x)\partial_{x_1} b_{23}(x) = 0\}.$$

It remains to consider the complementary case:

Case 2: $N > 2$.

We can assume that there exists $j \in \{3, \dots, N\}$ such that $b_{j3}(x)$ is not identically zero on V_1. Then, we conclude that

$$\operatorname{sing\,supp} u \cap V_1 \subseteq \pi(\mathrm{Char}(L)) \cap V_1 \subseteq \{x \in V_1 \mid b_{j3}(x) = 0\},$$

and

$$\operatorname{sing\,supp} u \cap V \subseteq \{x \in V \mid a_{22}(x) b_{j3}(x) = 0\}.$$

This completes our proof. $\square$

3. Proof of Proposition 1.1

We consider the function

$$u(x_1, x_2, x_3) = \sum_{n=1}^{\infty} e^{(x_2-2)\sqrt{2^n} - \frac{x_1^2}{2} 2^n} \cos(x_3 2^n), \tag{3.13}$$

for $(x_1, x_2, x_3) \in \mathbb{R} \times]-\infty, 2[\times \mathbb{R}$. A direct computation shows that

$$(D_1^2 + D_2^2 + x_1^2 D_3^2) u(x_1, x_2, x_3) = 0 \quad \text{in} \quad \mathbb{R} \times]-\infty, 2[\times \mathbb{R}. \tag{3.14}$$

We claim that $\operatorname{sing\,supp} u = \{0\} \times]-\infty, 2[\times \mathbb{R}$. We recall that

$$\operatorname{sing\,supp} u \subseteq \{(0, x_2, x_3) \mid (x_2, x_3) \in]-\infty, 2[\times \mathbb{R}\}$$

(in other words, the analytic singular support of u is contained in the projection on the base of the characteristic manifold). Then, in order to complete the proof we show that for every point of the form $(0, x_2, x_3)$, with $(x_2, x_3) \in]-\infty, 2[\times \mathbb{R}$, we have that $(0, x_2, x_3) \in \operatorname{sing\,supp} u$. For this purpose let us begin with an elementary remark

Lemma 3.1. *The set $E = \{\pi m / 2^n \mid m \in \mathbb{Z},\, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.*

Proof. For every $x \in \mathbb{R}$ we have that

$$\frac{[x 2^n / \pi]}{2^n} = \frac{x}{\pi} - \frac{\{x 2^n / \pi\}}{2^n}, \qquad (n \in \mathbb{N})$$

(here $[a]$ denotes the integer part of a while $\{a\}$ stands for the fractional part of a). Then

$$\left|\pi\frac{[x2^n/\pi]}{2^n}-x\right|\leq\pi\frac{|\{x2^n/\pi\}|}{2^n}\leq\frac{\pi}{2^n},\qquad(n\in\mathbb{N})$$

and the conclusion follows. □

Then it suffices to show that u is not analytic at every point of the form $(0,x_2,x_3)$ with $x_2\in]-\infty,2[$ and $x_3\in E$. Let $(0,x_2,x_3)$ with $x_2\in]-\infty,2[$ and $x_3\in E$ we claim that

$$\limsup_{h\to\infty}\left(\frac{|D_3^h u(0,x_2,x_3)|}{h!}\right)^{\frac{1}{h}}=+\infty. \tag{3.15}$$

We observe that (3.15) implies that

$$\{0\}\times]-\infty,2[\times E\subseteq \operatorname{sing\,supp} u\quad(\subseteq\{0\}\times]-\infty,2[\times\mathbb{R}),$$

i.e. $\operatorname{sing\,supp} u=\{0\}\times]-\infty,2[\times\mathbb{R}$ since it is a closed set (and $\overline{E}=\mathbb{R}$). Then, let $x_3=\pi m/2^\ell$ (with $m\in\mathbb{Z}$ and $\ell\in\mathbb{N}$) and let $k>\max\{2,\ell\}$. We have that

$$D_3^{2^k}u(0,x_2,x_3)=\sum_{n=1}^{\infty}e^{(x_2-2)\sqrt{2^n}}(2^n)^{2^k}\cos(x_3 2^n). \tag{3.16}$$

(Here we used the fact that, for $k\geq 2$, $D^{2^k}\cos x=\cos x$.) Then, we find that

$$\begin{aligned}D_3^{2^k}u(0,x_2,x_3)&=\sum_{n=\ell+1}^{\infty}e^{(x_2-2)\sqrt{2^n}}(2^n)^{2^k}\\&+\sum_{n=1}^{\ell}e^{(x_2-2)\sqrt{2^n}}(2^n)^{2^k}\cos(x_3 2^n)\geq\sum_{n=\ell+1}^{\infty}e^{(x_2-2)\sqrt{2^n}}(2^n)^{2^k}-\ell(2^\ell)^{2^k}.\end{aligned} \tag{3.17}$$

(Here we used the fact that $\cos(2^n x_3)=\cos(2^{n-\ell}\pi m)=1$ for $n>\ell$.) In view of (3.17), we deduce that

$$|D_3^{2^k}u(0,x_2,x_3)|+\ell(2^\ell)^{2^k}\geq e^{(x_2-2)\sqrt{2^{2k}}}(2^{2k})^{2^k}. \tag{3.18}$$

Using (3.18), the fact that $(2^\ell)^{2^k}/(2^k!) = o(1)$, as $k \to \infty$, and $2^k \geq (2^k!)^{\frac{1}{2^k}}$, we conclude that

$$\limsup_{h\to+\infty}\left(\frac{|D_3^h u(0,x_2,x_3)|}{h!}\right)^{\frac{1}{h}} \geq \lim_{k\to\infty}\left(\frac{|D_3^{(2^k)}u(0,x_2,x_3)|}{2^k!}\right)^{\frac{1}{2^k}} =$$

$$\lim_{k\to\infty} e^{x_2-2}\frac{2^{2k}}{(2^k!)^{\frac{1}{2^k}}} \geq \lim_{k\to\infty} e^{x_2-2}2^k = +\infty.$$

Remark 3.1. *We point out that the construction of the function u is based on some theoretical considerations which we think will shed some light on Formula* (3.13)*. It is well-known that a non-analytic solution of* (3.14) *is given by*

$$v(x_1,x_2,x_3) = \int_0^{+\infty} e^{i\rho x_3} e^{-\frac{x_1^2}{2}\rho} e^{(x_2-1)\sqrt{\rho}}\, d\rho. \tag{3.19}$$

Furthermore, one can show that $\operatorname{sing\,supp} v = \{0\}\times]-\infty,1[\times\{0\}$*. In order to construct the function u the idea consists in taking a superposition of the integrand in* (3.19) *of the form*

$$w(x_1,x_2,x_3) = \int_0^{+\infty} e^{i\rho x_3} e^{-\frac{x_1^2}{2}\rho} e^{(x_2-1)\sqrt{\rho}}\, \hat{g}(\rho)\, d\rho,$$

for a suitable g which is not analytic at any point of $\mathbb{R}$. A classical example of such a function is

$$g(x) = \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-\sqrt{2^n}}\cos(2^n x).$$

We claim that if $(0,0,x_3) \in \operatorname{sing\,supp} w$ for every $x_3 \in \mathbb{R}$, then $(0,x_2,x_3) \in \operatorname{sing\,supp} w$, for every $(x_2,x_3) \in]-\infty,1[\times\mathbb{R}$. Indeed, if $(0,0,\bar{x}_3) \in \operatorname{sing\,supp} w$, for a suitable $\bar{x}_3 \in \mathbb{R}$, then there exists $\bar{\xi}_3 \neq 0$ such that $(0,0,\bar{x}_3,0,0,\bar{\xi}_3) \in WF(w)$ and

$$F = \{(0,x_2,\bar{x}_3,0,0,\bar{\xi}_3) \mid x_2 \in]-\infty,1[\}$$

is the Hamiltonian leaf through the point $(0,0,\bar{x}_3,0,0,\bar{\xi}_3)$. Then, it is well-known that if $F\cap WF(w) \neq \emptyset$ then $F \subset WF(w)$ (see [13] *Theorem 4.2). Hence, our claim follows from the fact that $\pi(F) \subseteq \operatorname{sing\,supp} w$. A direct computation shows that*

$$w(0,0,x_3) = \sum_{n=1}^{\infty} e^{-2\sqrt{2^n}} e^{ix_3 2^n}, \qquad x_3 \in \mathbb{R}$$

(the real part of $w(0,0,x_3)$ modulo an irrelevant factor 2 in the first exponential behaves like the function g) and we conclude that w is not

real analytic at $(0,0,x_3)$ *for every* $x_3 \in \mathbb{R}$. *The function* u *given in* (3.13) *is the real part of* w.

4. Proof of Proposition 1.2

We observe that in order to prove Proposition 1.2, it suffices to consider the case of $n=4$. Indeed, if we find a function $u = u(x_1,x_2,x_3,x_4)$ as in the statement of the proposition, then, for $n>4$, by taking $v(x_1,\dots,x_n) = u(x_1,x_2,x_3,x_4)$ the conclusion follows. Then let us consider the equation

$$\left(D_1^2 + (D_2 + x_1 D_3)^2 + D_4^2\right) u = 0 \text{ in }]-\infty,-2[\times\mathbb{R}^2\times]-\infty,2[, \tag{4.20}$$

and the function

$$u(x_1,x_2,x_3,x_4) = \sum_{n,m,s=1}^{\infty} e^{-2s+(x_4-2)\sqrt{2^n}-\sqrt{2^m}-\frac{(x_1+s2^{m-n})^2}{2}2^n} \cos(x_2 s 2^m + x_3 2^n), \tag{4.21}$$

where $(x_1,x_2,x_3,x_4) \in]-\infty,-2[\times\mathbb{R}^2\times]-\infty,2[$. A direct computation shows that u is a solution of (4.20). As in the proof of Proposition 1.1 it suffices to show that the function u is not real analytic on a dense set in $]-\infty,-2[\times\mathbb{R}^2\times]-\infty,2[$. For this purpose let us take

$$\bar{x} = \left(-\frac{\bar{s}_1}{2^{\bar{s}_2}}, \pi\frac{\bar{m}_1}{2^{\bar{m}_2}}, \pi\frac{\bar{n}_1}{2^{\bar{n}_2}}, \bar{x}_4\right) \tag{4.22}$$

with $\bar{s}_1,\bar{s}_2,\bar{m}_2,\bar{n}_2 \in \mathbb{N}$, $\bar{m}_1,\bar{n}_1 \in \mathbb{Z}$, $\bar{x}_4 < 2$ and $\bar{s}_1/2^{\bar{s}_2} > 2$. Arguing as in the proof of Proposition 1.1, one can show that the set of all the points as $\bar{x}$ is dense in $]-\infty,-2[\times\mathbb{R}^2\times]-\infty,2[$. We want to show that

$$\limsup_{h\to\infty} \left(\frac{|D_3^h u(\bar{x})|}{h!}\right)^{\frac{1}{h}} = +\infty. \tag{4.23}$$

For

$$k > \max\{2, \bar{n}_2, \bar{m}_2, \bar{s}_2\}, \tag{4.24}$$

we have that

$$D_3^{2^k} u(\bar{x}) = \sum_{n,m,s=1}^{\infty} e^{-2s+(\bar{x}_4-2)\sqrt{2^n}-\sqrt{2^m}-\frac{(\bar{x}_1+s2^{m-n})^2}{2}2^n} \cdot (2^n)^{2^k} \cos(\bar{x}_2 s 2^m + \bar{x}_3 2^n). \tag{4.25}$$

We observe that, in order to prove (4.23), it suffices to show that

$$\lim_{k\to\infty} \left(\frac{|D_3^{2^k} u(\bar{x})|}{2^k!}\right)^{\frac{1}{2^k}} = +\infty. \tag{4.26}$$

We split the sum in (4.25) into two sums:

$$\text{the sum for } n, m, s \geq 1 \text{ such that } 2^n \geq s2^m, \tag{4.27}$$

$$\text{the sum for } n, m, s \geq 1 \text{ such that } 2^n < s2^m. \tag{4.28}$$

We have that

$$I + \left(\frac{|D_3^{2^k} u(\bar{x})|}{2^k!} \right)^{\frac{1}{2^k}} \geq II$$

where

$$I = \left(\frac{\sum'_{n,m,s} e^{\psi_{nms}} (2^n)^{2^k} |\cos(\bar{x}_2 s 2^m + \bar{x}_3 2^n)|}{2^k!} \right)^{\frac{1}{2^k}}$$

and

$$II = \left(\frac{\left| \sum''_{n,m,s} e^{\psi_{nms}} (2^n)^{2^k} \cos(\bar{x}_2 s 2^m + \bar{x}_3 2^n) \right|}{2^k!} \right)^{\frac{1}{2^k}},$$

with

$$\psi_{nms} = -2s + (\bar{x}_4 - 2)\sqrt{2^n} - \sqrt{2^m} - \frac{(\bar{x}_1 + s2^{m-n})^2}{2} 2^n.$$

(Here we denote by $\sum'_{n,m,s}$ the sum for $n, m, s \geq 1$ such that (4.27) holds, while $\sum''_{n,m,s}$ stands for the analogous sum under Condition (4.28).) We claim that I yields a bounded contribution to the limit. Indeed, if we take n, m and s as in (4.27), we have that

$$|\bar{x}_1 + s2^{m-n}| \geq |\bar{x}_1| - 1 \geq 1 \tag{4.29}$$

(in the last inequality we used the fact that $\bar{x}_1 < -2$). Hence, by (4.29), we find

$$I \leq \left(\frac{\sum'_{n,m,s} e^{-2s + (\bar{x}_4 - 2)\sqrt{2^n} - \sqrt{2^m} - \frac{(\bar{x}_1 + s2^{m-n})^2}{2} 2^n} (2^n)^{2^k}}{2^k!} \right)^{\frac{1}{2^k}}$$

$$\leq \left(\frac{\sum'_{n,m,s} e^{-2s + (\bar{x}_4 - 2)\sqrt{2^n} - \sqrt{2^m} - 2^{n-1}} (2^n)^{2^k}}{2^k!} \right)^{\frac{1}{2^k}}.$$

By using the elementary inequality

$$e^{-\frac{t}{2}} t^a \leq e^{-a} (2a)^a, \qquad (a > 1,\ t \geq 0),$$

we have

$$e^{-2^{n-1}} (2^n)^{2^k} \leq e^{-2^k} (2^{k+1})^{2^k}.$$

Then

$$I \leq \left(\frac{\sum'_{n,m,s} e^{-2s+(\bar{x}_4-2)\sqrt{2^n}-\sqrt{2^m}} e^{-2^k} (2^{k+1})^{2^k}}{2^k!} \right)^{\frac{1}{2^k}}$$

$$\leq \left(\sum_{n,m,s}{}' e^{-2s+(\bar{x}_4-2)\sqrt{2^n}-\sqrt{2^m}} \right)^{\frac{1}{2^k}} \left(\frac{e^{-2^k} (2^{k+1})^{2^k}}{2^k!} \right)^{\frac{1}{2^k}}.$$

Using the Stirling formula, we get

$$\left(\frac{e^{-2^k} (2^{k+1})^{2^k}}{2^k!} \right)^{\frac{1}{2^k}} \sim \left(\frac{e^{-2^k} (2^{k+1})^{2^k}}{(2^k)^{2^k} e^{-2^k} \sqrt{\pi 2^{k+1}}} \right)^{\frac{1}{2^k}} = \frac{2}{(\pi 2^{k+1})^{\frac{1}{2^{k+1}}}} \to 2$$

as $k \to \infty$. Hence, we obtain that I is bounded.

Let us consider the term II: we split the sum $\sum''$ into three parts $\Sigma''_{(i)}$, $i = 1, 2, 3$, as described below. To each of these sums corresponds $II_{(i)}$ (which is defined as II with $\sum''$ replaced by $\sum''_{(i)}$). Then, we have

$$II + II_{(1)} + II_{(2)} \geq II_{(3)}. \tag{4.30}$$

Estimate of $II_{(1)}$: in $\sum''_{(1)}$ we take the sum $\sum''$ with the additional constraint $n \leq \bar{n}_2$ (we recall that $\bar{n}_2$ is defined in (4.22), in particular it depends only on the point $\bar{x}$). We find that

$$II_{(1)} \leq \left(\sum_{n,m,s}{}'' e^{-2s+(\bar{x}_4-2)\sqrt{2^n}-\sqrt{2^m}} \right)^{\frac{1}{2^k}} \left(\frac{(2^{\bar{n}_2})^{2^k}}{2^k!} \right)^{\frac{1}{2^k}},$$

with the RHS uniformly bounded w.r.t. k.

Estimate of $II_{(2)}$: we take the sum for $n, m, s \geq 1$, with $2^n < s2^m$ and the additional constraints $n > \bar{n}_2$ and $m \leq \bar{m}_2$. Then, by (4.28), we find

$$II_{(2)} \leq \left(\sum\nolimits''_{(2)} e^{-2s+(\bar{x}_4-2)\sqrt{2^n}-\sqrt{2^m}} \frac{(2^n)^{2^k}}{2^k!} \right)^{\frac{1}{2^k}}$$

$$\leq \left(\sum\nolimits''_{(2)} e^{-s+(\bar{x}_4-2)\sqrt{2^n}-\sqrt{2^m}} \cdot \frac{e^{-s} s^{2^k} (2^m)^{2^k}}{2^k!} \right)^{\frac{1}{2^k}}$$

Using, once more, the elementary inequality $e^{-s}s^{2^k} \le e^{-2^k}(2^k)^{2^k}$ and the fact that $m \le \bar{m}_2$, we have

$$II_{(2)} \le \left(\sum\nolimits''_{(2)} e^{-s+(\bar{x}_4-2)\sqrt{2^n}-\sqrt{2^m}} \cdot \frac{e^{-2^k}(2^k)^{2^k}}{2^k!} \right)^{\frac{1}{2^k}} 2^{\bar{m}_2}$$

which is once more a uniformly bounded term w.r.t. k. It remains to consider

Estimate of $II_{(3)}$: we take the sum for $n, m, s \ge 1$, with $2^n < s2^m$ and the additional constraints $n > \bar{n}_2$, $m > \bar{m}_2$. Then, we have

$$II_{(3)} = \left(\frac{\left| \sum''_{(3)} e^{\psi_{nms}} (2^n)^{2^k} \cos(\bar{x}_2 s 2^m + \bar{x}_3 2^n) \right|}{2^k!} \right)^{\frac{1}{2^k}}$$

$$= \left(\frac{\sum''_{(3)} e^{-2s+(\bar{x}_4-2)\sqrt{2^n}-\sqrt{2^m}-\frac{(\bar{x}_1+s2^{m-n})^2}{2}2^n} (2^n)^{2^k}}{2^k!} \right)^{\frac{1}{2^k}}$$

(since $\cos(\bar{x}_2 s 2^m + \bar{x}_3 2^n) = 1$ for $n > \bar{n}_2$ and $m > \bar{m}_2$). We recall that $\bar{s}_1/2^{\bar{s}_2} > 2$ (see (4.22)), then we may choose $s = \bar{s}_1$ and

$$n - m = \bar{s}_2 \quad (> 0). \tag{4.31}$$

Indeed, with this choice, we have, in view of (4.22), that $\bar{x}_1 + s2^{m-n} = 0$, whence $2^n < s2^m$. Then, we find

$$II_{(3)} \ge \left(\frac{e^{-2\bar{s}_1+(\bar{x}_4-2)\sqrt{2^n}-\sqrt{2^m}} (2^n)^{2^k}}{2^k!} \right)^{\frac{1}{2^k}}, \tag{4.32}$$

for every $n > \bar{n}_2$, $m > \bar{m}_2$ with $n - m = \bar{s}_2$.

In particular, we may take

$$n = 2k \quad \text{and} \quad m = 2k - \bar{s}_2 \quad \text{in (4.32)},$$

(we observe that, by (4.24), $n = 2k > \bar{n}_2$ and $m = 2k - \bar{s}_2 > \bar{m}_2$). Furthermore, by (4.31), $\sqrt{2^n} > \sqrt{2^m}$ and we find that

$$II_{(3)} \ge \left(\frac{e^{-2\bar{s}_1+(\bar{x}_4-2)\sqrt{2^n}-\sqrt{2^m}} (2^n)^{2^k}}{2^k!} \right)^{\frac{1}{2^k}}$$

$$\ge \left(e^{-2\bar{s}_1} \left(\frac{(2^k)^{2^k}}{2^k!} \right) (2^k)^{2^k} e^{(\bar{x}_4-3)\sqrt{2^{2k}}} \right)^{\frac{1}{2^k}}.$$

By the estimate

$$\left(\frac{(2^k)^{2^k}}{2^k!}\right) \geq 1,$$

we deduce that

$$II_{(3)} \geq e^{-\frac{\bar{s}_1}{2^{k-1}}+\bar{x}_4-3}\, 2^k.$$

Then, we conclude

$$\lim_{k\to+\infty} II_{(3)} \geq \lim_{k\to+\infty} e^{-\frac{\bar{s}_1}{2^{k-1}}+\bar{x}_4-3}\, 2^k = +\infty,$$

i.e.

$$\lim_{k\to\infty}\left(\frac{|D_3^{2^k}u(\bar{x})|}{2^k!}\right)^{\frac{1}{2^k}} = +\infty.$$

This completes our proof of Proposition 1.2.

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