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Consumption in Incomplete Markets*

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Abstract We develop a method to find approximate solutions, and their accuracy, to consumption-investment problems with isoelastic preferences and infinite horizon, in incomplete markets where state variables follow a multivariate diffusion. We construct upper and lower *contractions*, fictitious complete markets in which state variables are fully hedgeable, but their dynamics is distorted. Such contractions yield pointwise upper and lower bounds for both the value function and the optimal consumption of the original incomplete market, and their optimal policies are explicit in typical models. Approximate consumption-investment policies coincide with the optimal one if the market is complete or utility logarithmic.

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1 Introduction

In an incomplete market consumers face risks that they cannot hedge entirely through dynamic trading. Such residual risks jointly affect consumption and investment decisions, as they hinder the ability to smooth consumption over time and to hedge against future changes in investment opportunities. Unlike a complete market, where the first-order condition identifies the optimal consumption plan, hence its unique self-financing investment policy, in an incomplete market consumption and investment choices are intertwined in a simultaneous system, and far less understood. A voluminous literature[1] in optimal consumption and portfolio choice offers alternative characterizations of optimal strategies in diffusion models, but explicit solutions are confined to those that are either dynamically complete or without intertemporal consumption[2].

We start with a general existence result of solutions to the associated HJB equation and a verification theorem for consumption and investment problems with isoelastic utilities in a Markov setting. For incomplete markets with consumption, which are the focus of this paper, verification *per se* has a limited practical scope in view of the scarcity of closed-form solutions. The lack of explicit solutions has led applied researchers to investigate approximate policies[3], thereby raising the twin theoretical questions of whether such policies are close to optimal, and whether they are similar to the unknown optimal

[1] The continuous time literature begins with the work of Merton on constant [32] and stochastic [33] investment opportunities. Much of the following research, among which we mention Karatzas et al. [28], Cox and Huang [9], Karatzas et al. [29], He and Pearson [23], Duffie et al. [11], aims at characterizing optimal policies either with martingale or with control methods. Kim and Omberg [30], Zariphoupoulou [39] find explicit solutions in an incomplete market without consumption and Wachter [38] in a complete market with consumption. Liu [31] extends these results to a wide class of quadratic models, in which optimal policies are given in terms of solutions of Riccati differential equations, again in models that are either complete with consumption, or incomplete but without consumption. Existence and verification theorem for optimal consumption-investment problem in different settings of incomplete markets are discussed in Fleming and Hernández-Hernández [14], Fleming and Pang [15], Castañeda-Leyva and Hernández-Hernández [6], Hata and Sheu [20, 21], with constraints on model parameters and admissible strategies. Rogers [37] offers a recent survey of the portfolio choice literature.

[2] An exception is logarithmic utility, for which the optimal portfolio is myopic, intertemporal hedging is absent, and the consumption-wealth ratio constantly equals the time-preference rate, hence is insensitive to market completeness and asset dynamics, see Goll and Kallsen [17] for a general statement.

[3] Campbell and Viceira [5, 4] study the policies resulting from a log-linear approximation of the budget constraint, and investigate the impact of stochastic investment opportunities on consumption and investment policies, while the accuracy of the approximation is not analyzed in detail. Recently, Pohl et al. [34] show that such log-linear approximations lead to large numerical errors in asset pricing models. Haugh et al. [22] calculate numerical upper bounds for the maximum power utility from terminal wealth by adding artificial assets that complete the market, in the spirit of He and Pearson [23] and Cvitanić and Karatzas [10]. Bick et al. [3] employ a similar approach with intertemporal consumption, assuming deterministic risk premia for unhedgeable risks.

policy. As an approximate policy is explicit by design, the answer to these questions should also be explicit.

The central idea of our approximation is a market *contraction*, a fictitious complete market in which state shocks are fully hedgeable but state dynamics is distorted, and the solution is explicit in typical models. Importantly, a contraction is *not* a completion of the original market (He and Pearson [23], Cvitanić and Karatzas [10]). In fact, in a contraction the state-variable dynamics is not even equivalent to the original market, as the covariance structure is altered to remove unhedgeable shocks. We find lower and upper contractions, which yield respective pointwise bounds on the problem's value function and optimal consumption policy.

The upper contraction is a complete market in which excess returns and the safe rate follow the same dynamics as in the original market, while shocks to state variables are spanned by asset returns but exhibit a preference-dependent distortion in their drift. The corresponding value function yields an upper bound for its counterpart in the original market, and is a super-solution of the original HJB equation when risk aversion is greater than one (a sub-solution for risk aversion below one). In the lower contraction, excess returns and state variables follow the same dynamics as in the upper contraction, but the safe rate is lower, and the resulting decline in investment opportunities yields a lower bound for the value function in the original market.

With these tools we derive in closed form the upper bound of the certainty equivalent loss, which is the fraction of wealth lost by adopting the approximate explicit policy instead of the unknown exact policy. In particular, the loss is zero if the utility is logarithmic or the market complete. While this result offers a rigorous quantitative statement on the approximate optimality of the proposed policy, we also show that the unknown optimal consumption policy lies pointwise between the consumption policies in the market contractions, and thereby obtain a bound for the gap between approximate consumption and its unknown optimum.

We bring these results to life by showing their potential in an incomplete market with a constant interest rate but stochastic, partially unhedgeable risk premia. We show that for typical parameter values reported in empirical studies, our approximate policies lead to a certainty equivalent loss of few percentage points of wealth, even for a fully incomplete market, in which shocks to risk premia are uncorrelated with asset returns. In this market allocation to stocks is substantially higher than in a static market, i.e., intertemporal hedging is positive. Intertemporal hedging also declines with time-preference, a result that is qualitatively consistent with the interpretation of stronger time-preference as a shorter horizon.

The paper is organized as follows: Section 2 introduces the model and the basic assumptions that drive the main results. Section 3 contains the existence of solutions to the HJB equation by constructing sub- and super-solutions (Theorem 3.2) and the main verification theorem (Theorem 3.3), which holds under rather general assumptions on the dynamics of asset prices and state variables, and replaces traditional transversality conditions with an ergodicity criterion that is easily verified when the solution to the HJB equation is available.

Section 4 contains the results on approximate optimality. The main idea is to look for an explicit solution to the HJB equation of market contractions and then derive from such a solution upper and lower bounds to the unknown value function of the original problem. This plan is carried out by Theorems 4.1 and 4.2, which yield respectively an upper bound and a lower bound. The lower bound is obtained from the approximate investment policy, which is optimal in the lower market contraction. Analogously, the upper bound corresponds to the shadow risk premium of unhedgeable risk implied by the upper contraction, which yields an approximate pricing measure. These bounds come together in Proposition 4.4, which translates them into an upper bound for the certainty equivalent loss as a fraction of the initial wealth, also derived in closed form.

A feature of the results in this paper is that they do not require the existence of an optimal policy for the original optimization problem, but offer tools to obtain it. Starting from some suboptimal but reasonable strategies, Lemma 3.1 and Theorem 3.2 yield the existence of a solution to the HJB equation of the original problem. Further, the assumptions of the verification result (Theorem 3.3) can be checked through estimates obtained by the value functions in the upper and lower contractions. Corollary 4.5 shows that the consumption rate must remain between the ones implied by the primal and dual approximations. This result is important because it implies that comparative statics performed on the approximate explicit solutions are also relevant for the unknown solution itself.

Section 5 discusses the applications to the model with stochastic risk premia. We find the approximate solutions in closed form, derive the certainty equivalent loss, and perform comparative statics with respect to preferences and market parameters. Using realistic parameter values, we find that the welfare loss from the use of the approximate strategies rather than the optimal one is equivalent to a loss in the initial capital of less than 3%.

The results in this paper are applied in Guasoni and Wang [19] to a market where interest rates follow the Vasicek model and are partially unhedgeable, but risk premia are constant. In that setting, Theorems 4.1 and 4.2 below yield approximate solutions to the optimal investment and consumption strategies and their error bound and Corollary 4.5 yields the existence of a smooth solution to the HJB equation of the original problem. By contrast, the model in Guasoni and Wang [19] requires a different argument to verify that the solution to the HJB equation is indeed the value function of the original problem.

2 Model

2.1 Market

The financial market includes a safe asset S_t^0 and n risky assets $(S_t^i)_{t\geq 0}^{1\leq i\leq n}$, such that $S_t^i > 0$ a.s. for all $0 \leq i \leq n$ and $t \geq 0$. Investment opportunities depend on k state variables $Y = (Y_t^i)_{t\geq 0}^{1\leq i\leq k}$, such that $Y_t \in E$ a.s. for all $t \geq 0$, where $E \subseteq \mathbb{R}^k$ is an open connected set. The first goal is to define a model in which assets and state variables have the joint dynamics:

$$\begin{aligned}
\frac{dS_t^0}{S_t^0} =& r(Y_t)dt, && (2.1)\\
\frac{dS_t^i}{S_t^i} =& r(Y_t)dt + dR_t^i, && 1 \leq i \leq n,\\
dR_t^i =& \mu_i(Y_t)dt + \sum_{j=1}^{n} \sigma_{ij}(Y_t)dZ_t^j, && 1 \leq i \leq n,\\
dY_t^i =& b_i(Y_t)dt + \sum_{j=1}^{k} a_{ij}(Y_t)dW_t^j, && 1 \leq i \leq k,\\
d\langle R^i, Y^j\rangle_t =& \Upsilon_{ij}(Y_t)dt, && 1 \leq i \leq n, 1 \leq j \leq k. \quad (2.2)
\end{aligned}$$

where $Z = (Z_t^i)_{t\geq 0}^{1\leq i\leq n}$ and $W = (W_t^i)_{t\geq 0}^{1\leq i\leq k}$ are multivariate Brownian Motions with correlation matrix $\rho(Y_t) = (\rho_{ij}(Y_t))_{1\leq i\leq n, 1\leq j\leq k}$. Denote the covariance matrices by $\Sigma = \sigma\sigma' = d\langle R, R\rangle_t/dt$, $A = aa' = d\langle Y, Y\rangle_t/dt$ and $\Upsilon = \sigma\rho a' = d\langle R, Y\rangle_t/dt$ (henceforth, the prime sign denotes matrix transposition). Then, $\Upsilon'\Sigma^{-1}\Upsilon$ characterizes the degree of incompleteness of the market, with $\Upsilon'\Sigma^{-1}\Upsilon = A$ corresponding to a complete market where state variables are perfectly replicable, and $\Upsilon = 0$ to a fully incomplete market where hedging is impossible, as asset returns and state variables shocks are uncorrelated. To ease notation, henceforth the argument Y_t is omitted from all model coefficients, unless ambiguity arises.

With this notation, the joint dynamics (2.1)-(2.2) of R and Y is defined in terms of the solution to a martingale problem on the canonical probability space. Denote by $C^m(E, \mathbb{R}^k)$ ($C^{m,\alpha}(E, \mathbb{R}^k)$) the class of $\mathbb{R}^k$ valued functions on E with locally α-Hölder continuous partial derivatives of m-th order. The superscripts are dropped for $m = 0$ or $k = 1$, so that $C(E, \mathbb{R})$ denotes $C^0(E, \mathbb{R}^1)$. $\mathbb{R}$ is replaced by $\mathbb{R}_+$ to denote the set of non-negative real numbers. The following assumption ensures that the model coefficients are regular and non-degenerate, and that (2.1)-(2.2) identify a unique probability measure:

Assumption 2.1 (Well-Posedness) Assume that:

(i) For some $\alpha \in (0,1)$, $r \in C^{1,\alpha}(E, \mathbb{R}^k)$, $b \in C^{1,\alpha}(E, \mathbb{R}^k)$, $\mu \in C^{1,\alpha}(E, \mathbb{R}^n)$, $A \in C^{2,\alpha}(E, \mathbb{R}^{k\times k})$, $\Sigma \in C^{2,\alpha}(E, \mathbb{R}^{n\times n})$ and $\Upsilon \in C^{2,\alpha}(E, \mathbb{R}^{n\times k})$. A and Σ are (strictly) positive definite for all $y \in E$.

(ii) Let $x = (z, y)$ where $z \in \mathbb{R}^n$, $y \in \mathbb{R}^k$,

$$\tilde{A}(x) = \begin{pmatrix} \Sigma(y) & \Upsilon(y) \\ \Upsilon'(y) & A(y) \end{pmatrix}, \text{ and } \tilde{b}(x) = \begin{pmatrix} \mu(y) \\ b(y) \end{pmatrix}.$$

There is a unique solution P to the martingale problem on $\mathbb{R}^n \times E$ with its Borel σ-algebra for[4]:

$$L = \frac{1}{2}\sum_{i,j=1}^{n+k} \tilde{A}_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n+k} \tilde{b}_i(x)\frac{\partial}{\partial x_i}.$$

Note that when (i) holds, (ii) is equivalent to the existence and uniqueness of the weak solution to the stochastic differential equation (SDE) with the infinitesimal generator L (Karatzas and Shreve [27, 5.4.C]). Note that the measure P depends on the initial value $Y_0 = y$, but such dependence is omitted for brevity unless ambiguity arises. $\mathbb{E}$ denotes the expectation under P.

Let $\mathcal{B} = (\mathcal{B}_t)_{t\geq 0}$, where $\mathcal{B}_t = \sigma\{G_s, 0 \leq s \leq t\}$, be the filtration generated by the coordinate process $G \in C([0,\infty), \mathbb{R}^n \times E)$, and define $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ as the right-continuous envelope of $\mathcal{B}$, i.e. $\mathcal{F}_t = \mathcal{B}_{t+} = \bigcap_{s>t} \mathcal{B}_s$. As P solves the martingale problem for L, the process $D_t^f = f(G_t) - f(G_0) - \int_0^t Lf(G_s)ds$ is a $(\mathcal{B}, P)$-martingale for any $f \in C^2(E, \mathbb{R})$ with compact support. Hence, it is also a $(\mathcal{F}, P)$-martingale (Revuz and Yor [36, Theorem II.2.8]). Then, it follows that there exist Brownian Motions Z and W such that G is an Itô process with diffusion coefficient $\tilde{A}$ and drift $\tilde{b}$ [36, Proposition VII.2.4 and Theorem VII.2.7][5]. Finally, constructing the processes R and Y as the projections of G on the first n and last k coordinates respectively (so that $G = (R, Y)$), it follows that R and Y satisfy (2.1)-(2.2) as desired.

The next assumption excludes arbitrage through the existence of a martingale measure:

Assumption 2.2 (Martingale Measure) There exists a probability $\tilde{P}$, such that $\tilde{P}|_{\mathcal{F}_t}$ and $P|_{\mathcal{F}_t}$ are equivalent for every $t \in [0, \infty)$, and S/S^0 is a $\tilde{P}$ local-martingale.

In this model, for every $\mathcal{F}$-adapted, $\mathbb{R}^k$-valued process η, referred to as risk-premium for unhedgeable risk, the discount factor

$$M_t^\eta = e^{-\int_0^t r_s ds}\mathcal{E}\left(\int_0^\cdot -\left(\mu'\Sigma^{-1} + \eta_s'\Upsilon'\Sigma^{-1}\right)\sigma dZ_s + \int_0^\cdot \eta_s' a dW_s\right)_t$$

makes $M^\eta S$ a local martingale. Denote $\mathcal{R}$ as the set of all such risk-premia. Finally, the domain E satisfies the following condition, which holds in virtually all models.

Assumption 2.3

i) E is star-shaped[6] with respect to some $y_0 \in E$.

[4] Formally, $\tilde{A}$ and $\tilde{b}$ are functions defined on $\mathbb{R}^n \times E$, though they only depend on the last k coordinates in the set E. This is also the case in the definitions of martingale problems in the rest of the paper.

[5] Theorem VII.2.7 in Revuz and Yor [36] requires an extension of the probability space when the coefficients A and Σ vanish. Such extension is not required here, as both coefficients are strictly positive definite.

[6] E is star-shaped with respect to $y_0 \in E$ if for each $y \in E$, the line segment $\{\alpha y + (1 - \alpha)y_0, \alpha \in [0, 1]\} \subset E$. This is always the case if E is convex.

ii) there exists a sequence of open, bounded, connected subset $E_n \subset E$, $n \geq 1$, each star-shaped with respect y_0. ∂E_n is $C^{2,\alpha}$, $\bar{E}_n \subsetneq E_{n+1}$ and $\bigcup_{n\geq 1} E_n = E$.

Assumptions 2.1, 2.2, and 2.3 apply throughout the paper, without further reference.

2.2 Preferences

With initial wealth x, an agent trades according to a portfolio $\pi_t = (\pi_t^i)_{t\geq 0}^{1\leq i\leq n}$, which represents the proportions of wealth X invested in each risky asset, and consumes at a continuous rate $c_t = l_t X_t$, at time t. The corresponding wealth $X^{\pi,l}$ satisfies the dynamics:

$$\frac{dX_t^{\pi,l}}{X_t^{\pi,l}} = rdt + \pi_t' dR_t - l_t dt.$$

The agent's goal is to maximize the expected power utility from consumption on an infinite horizon:

$$\max_{(\pi,l)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right], \tag{2.3}$$

where $\gamma > 0$ and $\gamma \neq 1$ is the agent's relative risk aversion and $\beta > 0$ is the time preference parameter. The choice of π and l are restricted to the admissible set $\mathcal{A}$:

Definition 2.4 The set of admissible investment and consumption policies $\mathcal{A}$ consists of all pairs of $\mathcal{F}$-adapted processes (π, l), where π is integrable with respect to R and the corresponding consumption and wealth process $l, X^{\pi,l}$ satisfy $l_t, X_t^{\pi,l} \geq 0$ a.s. for all $t \geq 0$.

3 Optimality

This section provides the existence of a smooth solution to the HJB equation and a verification theorem that identifies it with the value function. To find the optimal policy, first conjecture that the value function

$$V(x,y) = \sup_{(\pi,l)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt \,\middle|\, X_0 = x, Y_0 = y\right]$$

is of the form $V(x,y) = \frac{x^{1-\gamma}}{1-\gamma}g(y)^\gamma$, which reflects the usual homogeneity with respect to wealth, and is chosen so that g^{-1} coincides with the candidate optimal consumption-wealth ratio. In terms of g, the Hamilton-Jacobi-Bellman (HJB) equation for this optimization problem is

$$
\begin{aligned}
r+\frac{\beta}{\gamma-1} = \frac{\gamma\nabla g' b}{(\gamma-1)g} + \frac{\gamma\,\mathrm{tr}(AD^2g)}{2(\gamma-1)g} + \frac{\gamma\nabla g' A\nabla g}{2g^2} \\
- \sup_{\pi,l}\left(\pi'\mu - \frac{\gamma}{2}\pi'\Sigma\pi + \gamma\pi'\Upsilon\frac{\nabla g}{g} + \frac{g^{-\gamma}l^{1-\gamma}}{1-\gamma} - l\right). \quad (3.1)
\end{aligned}
$$

and the first order conditions are

$$
\hat{\pi}(y) = \frac{1}{\gamma}\Sigma^{-1}(y)\mu(y) + \Sigma^{-1}(y)\Upsilon(y)\frac{\nabla g(y)}{g(y)}, \qquad \hat{l}(y) = g(y)^{-1}. \quad (3.2)
$$

Substituting (3.2) into the HJB equation yields $\mathcal{H}(y,g,\nabla g, D^2 g) = 0$, where

$$
\begin{aligned}
\mathcal{H}(y,g,\nabla g,D^2g) = g^{-1} + \frac{\nabla g'\left(b + \frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\mu\right)}{g} \\
- \frac{(1-\gamma)\nabla g'\left(A - \Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g}{2g^2} \\
+ \frac{\mathrm{tr}(AD^2g)}{2g} - \left(\frac{\beta}{\gamma} + \left(1-\frac{1}{\gamma}\right)\left(\frac{\mu'\Sigma^{-1}\mu}{2\gamma} + r\right)\right). \quad (3.3)
\end{aligned}
$$

3.1 Existence

This subsection establishes the existence of a solution to the HJB equation to a general domain $E \subset \mathbb{R}^k$ exploiting the sub- and super-solutions generated by some admissible trading strategy (π, l) and risk premium η.

First, define g_1 as the lower bound on the value function corresponding to some strategy (π, l):

$$
\frac{x^{1-\gamma}}{1-\gamma}g_1^{\gamma}(y) = \mathbb{E}\left[\int_0^{\infty} e^{-\beta t}\frac{(l_t X_t)^{1-\gamma}}{1-\gamma}dt\right] \quad (3.4)
$$

Similarly, define $g_2(y)$ as the upper bound on the value function corresponding to the risk premium η (cf. Lemma 6.1 in the Appendix):

$$
g_2(y) = \mathbb{E}\left[\int_0^{\infty} e^{-\frac{\beta}{\gamma}t}\left(M_t^{\eta}\right)^{\frac{\gamma-1}{\gamma}}\,dt\right]. \quad (3.5)
$$

The following lemma shows that g_1 and g_2, which depend on the consumption-investment strategy (l, π) and the risk premium η, form a pair of sub- and super-solutions for the HJB equation $\mathcal{H}(y,g,\nabla g,D^2g) = 0$.

Lemma 3.1 *Assume that there exists* $l \in C^{\alpha}(E,\mathbb{R}_+)$, $\pi \in C^{\alpha}(E,\mathbb{R}^d)$ *and* $\eta \in C^{\alpha}(E,\mathbb{R}^k)$, *such that*

(i) $-\frac{\beta}{1-\gamma} + \pi\mu - \frac{\gamma\pi'\Sigma\pi}{2} - l + r \leq 0$,
(ii) $-\frac{\beta}{\gamma-1} - r - \frac{\mu'\Sigma^{-1}\mu + \eta'(A-\Upsilon'\Sigma^{-1}\Upsilon)\eta}{2\gamma} \leq 0$, *and*
(iii) the corresponding g_1 *and* g_2 *in* (3.4)-(3.5) *are continuous.*

If $\gamma \in (0,1)$, *then* $\mathcal{H}(y, g_2, \nabla g_2, D^2 g_2) \leq 0 \leq \mathcal{H}(y, g_1, \nabla g_1, D^2 g_1)$ *and* $g_1 \leq g_2$. *If* $\gamma \in (1,\infty)$, *then* $\mathcal{H}(y, g_1, \nabla g_1, D^2 g_1) \leq 0 \leq \mathcal{H}(y, g_2, \nabla g_2, D^2 g_2)$ *and* $g_2 \leq g_1$.

Checking the assumptions of this lemma in concrete models is typically straightforward. Simple choices are $\pi = 0$ (a safe portfolio), l (consumption proportional to current wealth), and η (premia for unhedgeable risks) large enough, which should depend on the model parameters as functions of Y. The sub- and super-solutions from the previous lemma in turn yield the existence of a solution to the HJB equation.

Theorem 3.2 *If there exists an ordered pair* $\bar{g} \geq \underline{g}$ *of sub- and super-solutions such that* $\mathcal{H}(y, \bar{g}, \nabla\bar{g}, D^2\bar{g}) \leq 0 \leq \mathcal{H}(y, \underline{g}, \nabla\underline{g}, D^2\underline{g})$, *then there exists* $g \in C^2(E, \mathbb{R}_+)$ *that solves the original HJB equation* (3.3) *and* $\underline{g} \leq g \leq \bar{g}$.

Lemma 3.1 implies that an ordered pair of sub- and super-solutions of the HJB equation can be constructed by choosing appropriate investment-consumption strategy (π, l) and risk premium η, which satisfy the mild conditions (i)-(iii). Then, Theorem 3.2 shows that a solution can be constructed between the pair. Note that Lemma 3.1 can also be applied to the market contractions in Section 4 to establish the existence of solutions to the associated HJB equations, which are sub- and super-solutions of the original HJB equation (3.3). Then the existence of a solution to (3.3) follows by Theorem 3.2, even without checking the assumptions of Lemma 3.1 in the original market.

In other words, the existence of a solution to the HJB equation of the incomplete market can be checked either by exhibiting a pair of a reasonable consumption-investment strategy and a risk premium (Lemma 3.1), or by exhibiting the explicit solutions to the market contractions in the next section.

3.2 Verification

Having settled the existence of a twice-differentiable solution g to the HJB equation, the next theorem establishes that this is indeed the reduced value function of the original market. First we state the theorem, then we discuss how its assumptions can be checked using the market *contractions* introduced in this paper.

Theorem 3.3 *Let* $g \in C^2(E, \mathbb{R}_+)$ *be the solution to* (3.3), *and assume that* *(i) there is a unique solution $\hat{P}$ to the martingale problem on $\mathbb{R}^n \times E$ for:*

$$\hat{L} = \frac{1}{2}\sum_{i,j=1}^{n+k} \tilde{A}_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n+k} \hat{b}_i(x)\frac{\partial}{\partial x_i},$$
$$\tilde{A} = \begin{pmatrix} \Sigma & \Upsilon \\ \Upsilon' & A \end{pmatrix} \quad \hat{b} = \begin{pmatrix} \frac{\mu}{\gamma} + \Upsilon\frac{\nabla g}{g} \\ b + \frac{(1-\gamma)\Upsilon'\Sigma^{-1}\mu}{\gamma} + \left(\gamma A + (1-\gamma)\Upsilon'\Sigma^{-1}\Upsilon\right)\frac{\nabla g}{g} \end{pmatrix}.$$

(ii) $\int_0^\infty g(Y_t)^{-1}dt = \infty$ $\hat{P}$-a.s.

Then, the controls $\hat{\pi} : \mathbb{R}^k \mapsto \mathbb{R}^n, \hat{l} : \mathbb{R}^k \mapsto \mathbb{R}$ defined by (Henceforth, we omit the argument of the functions π, l, η unless ambiguity arises.)

$$\hat{\pi}(y) = \frac{\Sigma^{-1}(y)\mu(y)}{\gamma} + \Sigma^{-1}(y)\Upsilon(y)\frac{\nabla g(y)}{g(y)}, \qquad \hat{l}(y) = g(y)^{-1} \tag{3.6}$$

are optimal for (2.3) *and its value function is* $\frac{x^{1-\gamma}}{1-\gamma}g(y)^\gamma$.

The value of this theorem hinges on the ability to check its assumptions when the solution g to the HJB equation is not known explicitly. This task is accomplished by Proposition 4.6 and Corollary 4.5 below using the explicit solutions to the market *contractions* introduced in this paper. In particular, Proposition 4.6 yields the solution to the martingale problem under the generator $\hat{L}$, the myopic probability measure (cf. Guasoni and Robertson [18]). Likewise, Corollary 4.5 offers an explicit lower bound for g^{-1}, from which the condition in (ii) above can be checked.

In theory, the verification Theorem 3.2 reduces the study of the consumption-investment problem to the solution of the HJB equation (3.3), and it seems natural to proceed with numerical methods from this point. The difficulty with this approach is that equation (3.3) may have several solutions (even in the case of a complete market), and the one that corresponds to the value function is not identified by boundary conditions that fit numerical methods.

Instead, the correct solution is identified by the conditions (i) and (ii) in Theorem 3.2 above. The (non) integrability condition (ii) alone is satisfied by any solution bounded away from zero, and is not enough to isolate the value function, which is the only one to also satisfy condition (i). Unfortunately, neither of these conditions neatly fits available numerical methods. For example, in the basic case of a scalar state variable ($k = 1$) the HJB equation reduces to a nonlinear ODE, which could be tackled with a numerical ODE solver if initial conditions for the function g and its derivative were available at some point, but (i) and (ii) above do not imply any particular boundary values.

Alternative numerical methods would require additional theoretical results. One possibility would be to consider a large but finite horizon T instead, thereby increasing the dimension of the problem from k to $k + 1$, and then use a numerical scheme for nonlinear PDEs, relying on a convergence result akin to Barles and Souganidis [1]. With a finite horizon, the reduced value function $g_T(t, y)$ has the clear terminal condition $g_T(T, y) = 1$ for all $y \in E$, but there is no guarantee that, at T increases, the solution of such an equation converges to the solution of the infinite-horizon problem, which may not even be well-posed (see Jin [26] and Dybvig et al. [12] for positive and negative results, respectively).

A further possibility would be to employ a value recursion method, starting with some initial guess for the value function, computing numerically the corresponding optimal c, π to solving the resulting linear equation (3.1) for a new approximate value function, iterating the procedure until it converges to the required accuracy. Yet, even such a method would require the specification of boundary conditions for each iteration, and convergence results for value iteration are scarce (see Rogers [37]).

The next section provides a method to seek approximate closed-form solutions to the optimal consumption-investment problem through the introduction of upper and lower *market contractions*, fictitious complete markets which typically admit closed-form solutions, and which yield upper and lower bounds for the reduced value function of the original problem, thereby facilitating comparative statics. The resulting consumption-investment strategies are often nearly optimal (cf. the application in section 5) and, if higher numerical accuracy is required, they may also be used as starting points for iterative schemes.

In addition, such contractions provide pointwise upper and lower bounds for the optimal policies (cf. Corollary 4.5 below), which can serve as a diagnostic tool to evaluate the accuracy of numerical methods by checking whether numerical solution lie within the prescribed bound.

A further insight of the market contractions is to provide a concrete worst-case reference model, which allows to estimate the extent to which incompleteness reduces agents' utility: the lower contraction in subsection 4.2 below shows that an incomplete market cannot be worse than a similar complete model, in which state variables are exactly replicable, but the interest rate is reduced. This feature is in contrast to usual market completions, which only provide optimistic comparisons.

4 Approximate Solutions

This section proposes an approximation method by solving the optimal investment and consumption problems in *market contractions*, fictitious complete markets in which state-variables are perfectly hedgeable. Unlike the completions of Karatzas et al. [29] and He and Pearson [23], contractions do not arise from the addition of fictitious assets to the original market under the same probability measure. By contrast, the probability measure under which a contraction is defined is not even equivalent to the probability measure of the original market because asset-state correlations are different.

As in complete markets the value function is often explicit (see e.g. Liu [31]), approximate consumption-investment policies and risk premia follow, along with their approximation error, which is also explicit.

4.1 Upper Bound

For the original market in equations (2.1)-(2.2), consider the following fictitious complete market:

$$
\begin{aligned}
\frac{dS_t^0}{S_t^0} &= r dt && (4.1)\\
\frac{dS_t^i}{S_t^i} &= r dt + dR_t^i, && 1 \le i \le n,\\
dR_t^i &= \mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dZ_t^j, && 1 \le i \le n,\\
dY_t^i &= \left(b + \left(1 - \frac{1}{\gamma}\right)\left(\tilde{\Upsilon} - \Upsilon\right)' \Sigma^{-1}\mu\right)_i dt + \sum_{j=1}^{k} a_{ij} dW_t^j, && 1 \le i \le k\quad (4.2)\\
d\langle R, Y\rangle_t &= \tilde{\Upsilon} dt, && (4.3)
\end{aligned}
$$

where $\tilde{\Upsilon} = \Upsilon((\Upsilon'\Sigma^{-1}\Upsilon)\#A)^{-1}A$, and $B\#C = C^{1/2}(C^{-1/2}BC^{-1/2})^{1/2}C^{1/2}$ is the geometric average of two positive-definite matrices B, C.[7] Thus, in this new market, the asset-state covariance matrix is $\tilde{\Upsilon}$ rather than Υ. Because $\tilde{\Upsilon}'\Sigma^{-1}\tilde{\Upsilon} = A$, this market is complete. If the original market is complete ($\Upsilon'\Sigma^{-1}\Upsilon = A$), then $(\Upsilon'\Sigma^{-1}\Upsilon)\#A = A$ and hence $\tilde{\Upsilon} = \Upsilon$.

The intuition of such a construction starts from the condition for market completeness, which is $\tilde{\Upsilon}'\Sigma^{-1}\tilde{\Upsilon} = A$. Thus, to construct a complete market that shares the same $n \times n$ covariance matrix of returns Σ and the same $k \times k$ covariance matrix A of state variables – and which is complete – it suffices to find an $n \times k$ covariance matrix $\tilde{\Upsilon}$ that satisfies the above condition. Furthermore, the objective is to find such a covariance matrix so that it is close to the original covariance matrix Υ. Thus, set $\tilde{\Upsilon} = \Upsilon X$ for some $k \times k$ matrix X and rewrite the market completeness condition as

$$X'\Upsilon\Sigma^{-1}\Upsilon X = A.$$

Multiplying both from the left and the right by A^{-1}, and setting $Y = XA^{-1}$, it follows that

$$Y'\Upsilon'\Sigma^{-1}\Upsilon Y = A^{-1},$$

[7] This definition of geometric mean for matrices, credited to Pusz and Woronowicz [35], implies that $B\#C = C\#B$ is the unique positive-definite solution X to the matrix equations $XB^{-1}X = C$ and $XC^{-1}X = B$. Extending the definition by continuity, $B\#C$ is defined also if B or C are positive-semidefinite (Bhatia [2, Chapter 4]). See also Horn and Johnson [25].

and the unique positive-definite solution of this equation is precisely

$$Y = (\Upsilon'\Sigma^{-1}\Upsilon)^{-1}\#A^{-1} = ((\Upsilon'\Sigma^{-1}\Upsilon)\#A)^{-1},$$

which in turn yields

$$\tilde{\Upsilon} = \Upsilon X = \Upsilon Y A = \Upsilon((\Upsilon'\Sigma^{-1}\Upsilon)\#A)^{-1}A,$$

which is the expression for $\tilde{\Upsilon}$ introduced above. Without adjustments, such new covariance matrix would bias the investor's welfare through the returns from intertemporal hedging demand, as attested by the term $\frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\mu$ in (3.3), where Υ would be replaced by $\tilde{\Upsilon}$.

Thus, the dynamics in (4.2) includes an additional drift that offsets the impact of such bias on welfare, while preserving completeness. Note that such correction vanishes if the original market is complete ($\tilde{\Upsilon} = \Upsilon$) or utility logarithmic ($\gamma = 1$). The crucial point of Theorem 4.1 below is that the market contraction, while remaining close to the original model, is slightly more favorable, and hence provides an upper bound for the original value function.

Intuitively, such an improvement stems from completeness: In the original market, the k portfolios $\Sigma^{-1}\Upsilon$ imperfectly hedge the state variables, while in the contraction the k portfolios obtained from the rows of $\Sigma^{-1}\tilde{\Upsilon}$ hedge perfectly. Improved hedging performance is irrelevant for logarithmic investors ($\gamma = 1$), while it is attractive to investors with very high (respectively, very low) risk aversion, who seek to mitigate (respectively, intensify) portfolio exposure to investment opportunities, without generating the unhedgeable shocks that market incompleteness entails.

The assumptions of Theorems 4.1 and 4.2 below are straightforward to check in applications. Indeed, while the HJB equation of the consumption-investment model in the original market is intractable, the corresponding equations for market contractions often admit explicit solutions, which means that conditions (i) and (ii) below ((ii) and (iii) in Theorem 4.2) are verified by direct calculations. Section 5 below demonstrates the analysis in a concrete model.

Theorem 4.1 *Let the assumptions in Lemma 3.1 hold with Υ replaced by $\tilde{\Upsilon}$, and let $g^d \in C^2(E, \mathbb{R}_+)$ be the solution to the HJB equation for the market in* (4.1)-(4.3)*:*

$$\begin{aligned} 0 = \mathcal{H}^d(y, g, \nabla g, D^2 g) = g^{-1} + \frac{\nabla g'}{g}\left(b - \left(1 - \frac{1}{\gamma}\right)\Upsilon'\Sigma^{-1}\mu\right) \\ + \frac{\operatorname{tr}\left(A D^2 g\right)}{2g} - \left(\frac{\beta}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\left(r + \frac{\mu'\Sigma^{-1}\mu}{2\gamma}\right)\right), \quad (4.4) \end{aligned}$$

which exists by Theorem 3.2. Assume that:

(i) There exists a unique solution $\bar{P}^d$ to the martingale problem on $\mathbb{R}^n \times E$

$$\bar{L}^d = \frac{1}{2}\sum_{i,j=1}^{2} \tilde{A}_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{2} \bar{b}_i^d(x)\frac{\partial}{\partial x_i},$$

where

$$\tilde{A} = \begin{pmatrix} \Sigma & \Upsilon \\ \Upsilon' & A \end{pmatrix}, \qquad \bar{b}^d = \begin{pmatrix} \frac{\mu}{\gamma} + \frac{\Upsilon \nabla g^d}{g^d} \\ b + \frac{(1-\gamma)\Upsilon' \Sigma^{-1}\mu}{\gamma} + \frac{\left(\gamma A + (1-\gamma)\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g^d}{g^d} \end{pmatrix}.$$

(ii) $\int_0^\infty g^d(Y_t)^{-1}dt = \infty$ $\bar{P}^d$-a.s.

Then, g^d is sub-solution of the original HJB *equation* (3.3) *if* $0 < \gamma < 1$ *and super-solution if $\gamma > 1$, and the upper bound in the original market with $\eta = \frac{\gamma \nabla g^d}{g^d}$ satisfies*

$$\frac{x^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\frac{\beta}{\gamma}t}\left(M_t^\eta\right)^{\frac{\gamma-1}{\gamma}}dt\right]^\gamma \le \frac{x^{1-\gamma}}{1-\gamma} g^d(y)^\gamma. \tag{4.5}$$

The main message of this theorem is that an explicit solution g^d to the HJB equation of the market contraction (4.1)-(4.3) yields a stochastic discount factor M^η that admits an explicit upper bound in terms of g^d itself. Note that if $\Upsilon'\Sigma^{-1}\Upsilon = A$ or $\gamma = 1$, then g^d also solves the original HJB equation $\mathcal{H}(y, g, \nabla g, D^2 g) = 0$. In this case, Theorem 3.3 implies that the bound in (4.5) coincides with the value function, hence the estimate is sharp.

4.2 Lower Bound

The next step is to find approximate policies $(\pi, l) \in \mathcal{A}$ for the utility maximization problem, so that the corresponding lower bound is close to the right-hand side in (4.5). We achieve this by altering the safe rate in (4.1)-(4.3). The idea is that a (state-dependent) decrease ϕ in the safe rate is enough to turn the upper bound in the above contraction into a lower bound. The safe rate needs to drop at least by the amount specified in condition (i), while its functional form is flexible. This means that ϕ can be chosen in each model as to achieve tractability.

For a given function $\phi \in C(E, \mathbb{R}_+)$, consider the fictitious market

$$\begin{aligned}
\frac{dS_t^0}{S_t^0} &= (r-\phi)\,dt && (4.6)\\
\frac{dS_t^i}{S_t^i} &= (r-\phi)\,dt + dR_t^i, && 1 \le i \le n,\\
dR_t^i &= \mu_i dt + \sum_{j=1}^n \sigma_{ij} dZ_t^j, && 1 \le i \le n,\\
dY_t &= \left(b + \left(1-\frac{1}{\gamma}\right)\left(\tilde{\Upsilon} - \Upsilon\right)' \Sigma^{-1}\mu\right) dt + a dW_t, &&\\
d\langle R, Y\rangle_t &= \tilde{\Upsilon} dt, && (4.7)
\end{aligned}$$

where $\tilde{\Upsilon}$ is defined after (4.3). Here excess returns of risky assets and state variable are the same as in the upper contraction (4.1)-(4.3), but the safe rate is reduced by ϕ. The solution to the HJB equation in this complete market suggest candidate investment and consumption policies, from which Theorem 4.2 derives a lower bound for the value function in the original market.

Theorem 4.2 *Let the assumptions in Lemma 3.1 hold with Υ and r replaced by Υ' and $r-\phi$, respectively, and $g^p \in C^2(E, \mathbb{R}_+)$, which exists by Theorem 3.2, be the solution to $\mathcal{H}^d(y, g\nabla g, D^2 g) + (1-1/\gamma)\phi = 0$, which is the HJB equation for the market in (4.6)-(4.7), and the operator $\mathcal{H}^d$ is defined in (4.4). Assume that*

(i) $\phi \ge \gamma \frac{(\nabla g^p)'\left(A - \Upsilon' \Sigma^{-1}\Upsilon\right)\nabla g^p}{2(g^p)^2}$.

(ii) There exists a unique solution $\bar{P}^p$ to the martingale problem on $\mathbb{R}^n \times E$

$$\bar{L}^p = \frac{1}{2}\sum_{i,j=1}^2 \tilde{A}_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^2 \bar{b}_i^p(x)\frac{\partial}{\partial x_i},$$

where

$$\tilde{A} = \begin{pmatrix} \Sigma & \Upsilon \\ \Upsilon' & A \end{pmatrix}, \qquad \bar{b}^p = \begin{pmatrix} \frac{\mu}{\gamma} + \frac{\Upsilon \nabla g^p}{g^p} \\ b + \frac{(1-\gamma)\Upsilon'\Sigma^{-1}\mu}{\gamma} + \frac{\left(\gamma A + (1-\gamma)\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g^p}{g^p} \end{pmatrix}.$$

(iii) $\int_0^\infty g^p(Y_t)^{-1} dt = \infty$ *$\bar{P}^p$-a.s.*

Then, with $l = (g^p)^{-1}$ and $\pi = \frac{\Sigma^{-1}\mu}{\gamma} + \frac{\Sigma^{-1}\Upsilon\nabla g^p}{g^p}$, the expected utility in the original market satisfies:

$$\frac{x^{1-\gamma}}{1-\gamma} g^p(y)^\gamma \le \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{\left(l X_t^{\pi,l}\right)^{1-\gamma}}{1-\gamma} dt\right]. \tag{4.8}$$

If (i) holds with equality, so does (4.8). Furthermore, if $0 < \gamma < 1$, g^p is a super-solution of the original HJB *equation* (3.3)*, and $g^p \le g^d$. If $\gamma > 1$, then g^p is a sub-solution and $g^p \ge g^d$.*

Unlike Theorem 4.1, the fictitious market in Theorem 4.2 depends on the choice of the safe rate drop ϕ, and the question is how to choose such ϕ to ensure that an explicit solution is available. As shown in Section 5, an effective method is to select ϕ of the same functional form (i.e., linear, quadratic, etc.) as the inhomogeneous term $\frac{\beta}{\gamma} + (1 - \frac{1}{\gamma})(r + \frac{\mu' \Sigma^{-1} \mu}{2\gamma})$, which is the optimal consumption-wealth ratio in a hypothetical market in which drifts and covariances remain at their current values indefinitely. Then, the parameters can be chosen as to satisfy condition (i) above.

4.3 Performance and Policy Bounds

The combination of the upper and lower bounds obtained above yields precise comparisons between the approximate policy and the unknown optimizer and their respective performances. Proposition 4.4 below states an upper bound for the performance gap between the approximate policy and the unknown optimum. In the same spirit, Corollary 4.5 shows that the optimal consumption lies between the consumption policies obtained in the two market contractions, pointwise.

Definition 4.3 (Certainty Equivalent Loss) For an agent with risk aversion γ, time preference parameter β and initial wealth $x > 0$, let $V(x, y)$ be the value function, i.e. the maximum utility from consumption. Then, the (relative) Certainty Equivalent Loss $\mathrm{CEL}(\pi, l)$ of any $(\pi, l) \in \mathcal{A}$ is defined by the equation:

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t} \frac{\left(l_t X_t^{\pi,l}\right)^{1-\gamma}}{1-\gamma} dt\right] = V\left(x(1 - \mathrm{CEL}(\pi, l)), y\right).$$

From this definition, the utility from consumption by adopting a suboptimal policy (π, l) is equivalent to losing a fraction $\mathrm{CEL}(\pi, l)$ of the initial wealth while using the optimal policy. $\mathrm{CEL}(\pi, l)$ takes values in $[0, 1]$. If $\mathrm{CEL}(\pi, l) = 0$, the utility is maximal and (π, l) is optimal, while $\mathrm{CEL}(\pi, l) = 1$ indicates the extreme suboptimality of a total loss. The next proposition combines the bounds above to obtain an estimate of the CEL.

Proposition 4.4 *If Theorem 4.1 and 4.2 hold, the* CEL *of the strategy* (π, l), *where* $\pi = \frac{\Sigma^{-1}\mu}{\gamma} + \frac{\Sigma^{-1}\Upsilon\nabla g^p}{g^p}$ *and* $l = (g^p)^{-1}$, *satisfies:*

$$0 \le \mathrm{CEL}(\pi, l) \le 1 - \left(\frac{g^p(y)}{g^d(y)}\right)^{\frac{\gamma}{1-\gamma}}.$$

If the market is complete or utility logarithmic, then $g^d = g^p$, both of which solve the original HJB equation (In this case, one can set $\phi = 0$ in Theorem 4.2.) $\mathcal{H}(y, g, \nabla g, D^2 g) = 0$, and hence $\mathrm{CEL}(\hat{\pi}, \hat{l}) = 0$. Thus, in the case of complete markets or logarithmic utility, the approximation recovers the optimizer, and the above proposition suggests that when $\Upsilon'\Sigma^{-1}\Upsilon$ is close to A or γ is close to 1, the error should be small.

Note also that the bound above can also be used on a finite horizon setting, in which case it yields a higher certainty equivalence loss CEL_T. A similar argument as in the proof of the Proposition 4.4 then yields

$$\mathrm{CEL}_T(\pi, l) \leq 1 - \left(\frac{g^p(y)}{g^d(y)}\right)^{\frac{\gamma}{1-\gamma}} \frac{\left(1 - \mathbb{E}_{\bar{P}^p}\left[e^{-\int_0^T g^p(Y_s)^{-1}ds} dt\right]\right)^{\frac{1}{1-\gamma}}}{\left(1 - \mathbb{E}_{\bar{P}^d}\left[e^{-\int_0^T g^d(Y_s)^{-1}ds}\right]\right)^{\frac{\gamma}{1-\gamma}}},$$

which reflects the overall loss that stems from using the strategy which is optimal on the market contraction with an infinite horizon instead of the optimal strategy (i.e., in the true model with the finite horizon). As the horizon increases, this bound recovers the one in the Proposition 4.4.

4.4 Establishing Existence

Note that Theorems 4.1 and 4.2 do not require the verification Theorem 3.3 for the original market to hold. More importantly, the reduced value functions g^p and g^d identified in these theorems help establish that the conditions of Theorem 3.3 hold. Because g^d and g^p form a pair of ordered sub- and super-solution to the original HJB equation, Theorem 3.2 implies the following result.

Corollary 4.5 *Let $g^d(y)$ and $g^p(y)$ be defined as in Theorem 4.1 and 4.2, respectively. There exists $g \in C^2(E, \mathbb{R}_+)$ that solves the original HJB equation* (3.3)*:*

$$\begin{cases} g^p(y)^{-1} \leq g(y)^{-1} \leq g^d(y)^{-1}, \ \gamma > 1, \\ g^d(y)^{-1} \leq g(y)^{-1} \leq g^p(y)^{-1}, \ 0 < \gamma < 1. \end{cases}$$

The importance of this result is twofold: First, it offers a lower bound for the unknown g^{-1} in terms of the explicit $(g^d)^{-1}$ or $(g^p)^{-1}$, which can be used to check condition (ii) of Theorem 3.3. Second, it shows that the optimal consumption policies in the market contractions are not only close to the unknown optimal consumption, i.e., they are pointwise upper and lower bounds.

Finally, these bounds also help check condition (i) in Theorem 3.3 (the solution to the martingale problem) because g^p and g^d control the behavior of g near the boundary of the domain E.

Proposition 4.6 *For $0 < \gamma < 1$, let $\underline{g} = g^p$ and $\bar{g} = g^d$. For $\gamma > 1$, let $\underline{g} = g^d$ and $\bar{g} = g^p$. If*

(i) $\sup\limits_{y \in E}(\underline{g}^{-1} - \bar{g}^{-1}) < \infty$,

(ii) $\lim_{n\uparrow\infty} \inf_{y \in E \setminus E_n}(\ln \bar{g} - \ln \underline{g}) = \infty$,

then the martingale problem for $\hat{L}$ in Theorem 3.3 has a unique solution.

The next section brings the theoretical results to life by examining a concrete model in detail.

5 Application

The approximate HJB equations in Theorem 4.1 and 4.2 have closed-form solutions in a range of portfolio choice models considered in the literature, such as Kim and Omberg [30], Wachter [38], and the stochastic interest rate model in Guasoni and Wang [19]. This section discusses in detail an application to a model with stochastic risk premia and volatilities, investigating their effects on investment and consumption policies and sensitivities to parameters. The model lies outside of the scope of Liu [31]'s results, as it combines intertemporal consumption with incomplete markets.

Consider a market in which the safe rate $r > 0$ is constant, while both the expected return and volatility of the risky asset depend on one state variable Y, which follows a Feller [13] diffusion:

$$\begin{aligned} \frac{dS_t}{S_t} =& rdt + dR_t, & (5.1)\\ dR_t =& \mu Y_t dt + \sigma\sqrt{Y_t} dZ_t, \\ dY_t =& b(\theta - Y_t)dt + a\sqrt{Y_t} dW_t, \\ d\langle Z, W\rangle_t =& \rho dt, & (5.2) \end{aligned}$$

where $\mu, \sigma, \rho, \theta, b, a$ are constants, all strictly positive with the exception of $\rho \in [0, 1]$. (Note the slight but harmless abuse of notation, as in the previous sections μ, σ, r, ρ, b, and a were functions of Y, while in this section they are constants.) Z and W are 1-dimensional Brownian Motions. Denote $\Upsilon = \sigma\rho a$, $\Sigma = \sigma^2$ and $A = a^2$. Assume $b\theta \geq \frac{A}{2}$, so that starting from $Y_0 = y > 0$, $Y_t > 0$ a.s. for all $t \geq 0$ (cf. Cox et al. [8]). $\rho^2 < 1$, so the market is incomplete. The agent has risk aversion $\gamma > 1$.

The next lemma computes closed-form solutions up to an integral, to the HJB equations in Theorem 4.1 and 4.2.

Lemma 5.1 *For constants $k \in \mathbb{R}$, $K > 0$, $A > 0$, $\lambda > 0$ and $c > 0$, the* ODE

$$1 + (-ky + \lambda)\, g_y + \frac{1}{2} A y g_{yy} - (cy + K)g = 0 \qquad (5.3)$$

has the solution $g(y) = \int_0^\infty e^{C(t)-B(t)y} dt$, where

$$C(t) = -\frac{2\lambda}{A}\left(\ln\left((k+\alpha)e^{\alpha t} - k + \alpha\right) - \ln 2\alpha - \frac{1}{2}(k+\alpha)t\right) - Kt, \quad (5.4)$$

$$B(t) = 2c\frac{e^{\alpha t} - 1}{e^{\alpha t}(k+\alpha) - k + \alpha}, \quad \alpha = \sqrt{k^2 + 2cA}. \qquad (5.5)$$

The next proposition shows that the assumptions in Theorem 4.1 and 4.2 hold.

Proposition 5.2 *The model* (5.1)-(5.2) *satisfies the assumptions in Theorem 4.1 with $g^d(y) = \int_0^\infty e^{C(t)-B(t)y}dt$, where*

$$\begin{aligned}
B(t) &= 2c\frac{e^{\alpha t}-1}{e^{\alpha t}(k+\alpha)-k+\alpha}, \\
C(t) &= -\frac{2b\theta}{A}\left(\ln\left((k+\alpha)e^{\alpha t}-k+\alpha\right)-\ln 2\alpha-\frac{1}{2}(k+\alpha)t\right)-Kt, \\
c &= \frac{(\gamma-1)\mu^2}{2\gamma^2\Sigma},\ \alpha=\sqrt{k^2+2cA}, \\
k &= b-\frac{(1-\gamma)\rho a\mu}{\gamma\sigma}, \quad K=\frac{\beta+(\gamma-1)r}{\gamma}.
\end{aligned}$$

The model also satisfies the assumptions in Theorem 4.2 with $\phi(y) = \gamma A(1-\rho^2)Qy$ and $g^p(y) = \int_0^\infty e^{\bar{C}(t)-\bar{B}(t)y}dt$, where

$$\begin{aligned}
\bar{B}(t) &= 2\bar{c}\frac{e^{\bar{\alpha} t}-1}{e^{\bar{\alpha} t}(k+\bar{\alpha})-k+\bar{\alpha}}, \\
\bar{C}(t) &= -\frac{2b\theta}{A}\left(\ln\left((k+\bar{\alpha})e^{\bar{\alpha} t}-k+\bar{\alpha}\right)-\ln 2\bar{\alpha}-\frac{1}{2}(k+\bar{\alpha})t\right)-Kt, \\
\bar{c} &= \frac{(\gamma-1)\mu^2}{2\gamma^2\Sigma}-(\gamma-1)A\left(1-\rho^2\right)Q>0,\ \bar{\alpha}=\sqrt{k^2+2\bar{c}A},
\end{aligned}$$

and the constant Q solves the equation $\frac{2\bar{c}^2}{(k+\bar{\alpha})^2} = Q$. (There are multiple solutions of Q, but only the one leading to the largest lower bound is used.)

These explicit formulas allow to establish the following analytical approximate properties of the optimal consumption policy.

Corollary 5.3 *The approximate consumption-wealth ratio $l(Y_t) = g^p(Y_t)^{-1}$ is increasing in Y_t, r, and β.*

Proof. $g^p(y)$ is decreasing in y, because y's coefficient $-\bar{B}(t)$ in the exponential function is non-positive. Since r and β only appear in $\bar{C}(t)$, and both have negative coefficients, $g^p(y)$ is decreasing in r and β. Thus $l(Y_t) = g^p(Y_t)^{-1}$ is increasing in Y_t, r, and β.

In this model of stochastic risk premia, a higher Y corresponds to better investment opportunities (higher Sharpe ratio), and has two opposite effects on the agent's consumption. By the income effect, the agent expects to earn more from investments, and hence is willing to consume more. By the substitution effect, the agent is willing to consume less to take advantage of the better investment opportunities. Corollary 5.3 shows that the income effect dominates in this model. For the same reason, a higher constant interest rate, which indicates better investment opportunities, leads to an increase in consumption. Finally, as β increases, the agent becomes more impatient, hence prefers consuming early. Corollary 5.3 confirms this intuition, because an increase in the consumption-wealth ratio means an increase of consumption in

Parameter	μ	σ	b	θ	a	r	ρ
Value	1.66	1.00	0.088	0.035	0.031	0.013	-0.84

Table 5.1 Parameters for the model of stochastic risk premia, with the real dividend yield of S&P composite as the state variable, estimated from Shiller's data, by matching the quadratic variation and the first two moments of the state variable's stationary distribution.

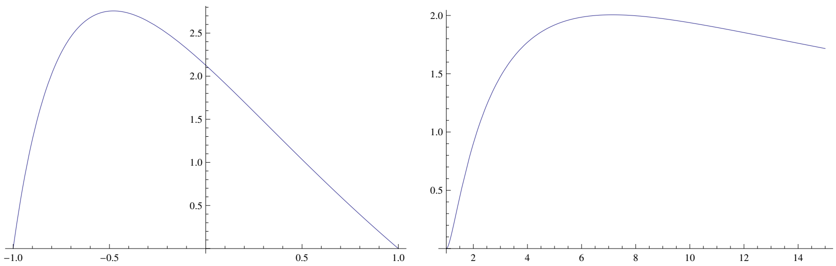

Fig. 5.1 Upper bound of the CEL (vertical axis, in percent) of approximate policies in the model of stochastic risk premia, against the correlation coefficient $-1 \leq \rho \leq 1$ (left panel, $\gamma = 4$), and relative risk aversion $1 \leq \gamma \leq 15$ (right panel). Other market parameters are in Table 5.1, while $\beta = 0.02$, $Y = \theta$.

dollar amount in the short run, which reduces the agent's wealth and future consumption.

Let R be the real return of the S&P composite, r the real interest rate, and Y the aggregate real dividend yield of the S&P composite index[8]. Parameters for the model (5.1)-(5.2), as summarized in Table 5.1, are estimated using Shiller's data[9] by matching the quadratic variation and moments of the state variable's stationary distribution[10].

Figure 5.1 shows the upper bound of the CEL of approximate policies $\pi = \frac{\Sigma^{-1}\mu}{\gamma} + \Sigma^{-1}\Upsilon\frac{g^p_y}{g^p}$, and $l = (g^p)^{-1}$ in Proposition 4.4. In the left panel, with $-1 \leq \rho \leq 1$, $\gamma = 4$, $\beta = 0.02$, Y at the mean of its stationary distribution and other market parameters in Table 5.1, the graph confirms that in a complete market, i.e. if $\rho = 1$ or -1, the approximate policies have a CEL $= 0$, thus are optimal. The largest CEL appears around $\rho = -0.5$, with a loss of less than 2.8% of the initial wealth. Results with $\gamma = 2$, 6, 8 and 10 (not reported in detail) show the same pattern, and the largest CEL is always less than 2.9%, which indicate that the performance of the approximate policies is very close to optimal.

In the right panel, with $1 \leq \gamma \leq 15$, $\beta = 0.02$, Y at the mean of its stationary distribution and other market parameters in Table 5.1, the graph confirms that with logarithmic utility, i.e. if $\gamma = 1$, the approximate policies

[8] That is, the ratio between the sum of all dividends distributed in a calendar year by the companies included in the index, and the sum of their market capitalizations at the end of the same year. Note that the variable Y here is used as a state variable, as the real return reflects both price changes and dividend distributions.

[9] http://www.econ.yale.edu/ shiller/data/chapt26.xlsx.

[10] In the model (5.1)-(5.2), Y is a square root process. Thus, the stationary distribution of Y is a Gamma distribution, with shape parameter $2b\theta/a^2$ and scale parameter $a^2/2b$.

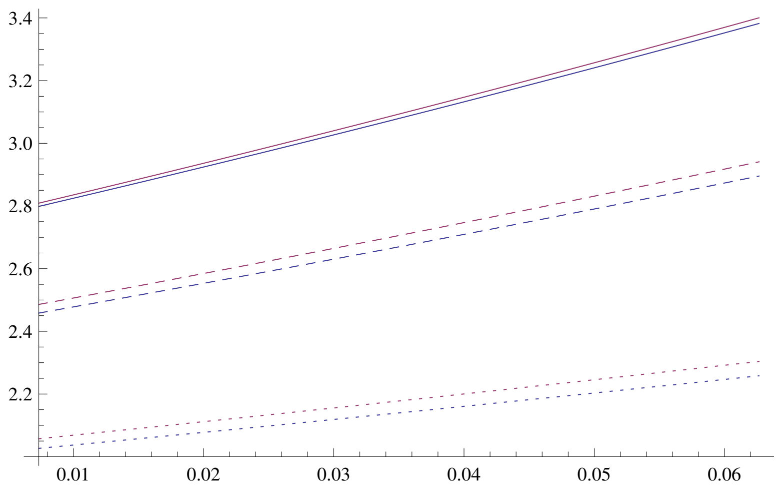

Fig. 5.2 Lower and upper bounds of the optimal consumption-wealth ratio (vertical axis, in percent) in the model of stochastic risk premia against the state variable Y (horizontal axis) within two standard deviations from the mean of its stationary distribution, with market parameters in Table 5.1, $\beta = 0.02$ and $\gamma = 2$ (solid line), 4 (dashed line), and 8 (dotted line), respectively.

are optimal. $\gamma > 1$ has two opposite effects on the CEL: i) As γ increases, the ODE which g^p solves in Theorem 4.2 deviates further from the original HJB equation. Thus, the the CEL of the approximate policies based on g^p is larger; ii) As the agent becomes more risk averse, less wealth is invested in the risky asset. Thus, the non-traded risk in the stochastic investment opportunities has less impact on welfare when using approximate policies, and the CEL decreases. The combined effect leads the CEL first to increase, until reaching the maximum around $\gamma = 6$, with a loss of less than 2% of the initial wealth, and then decrease with γ. This figure also confirms the result in the left panel that the performance of the approximate policies is very close to optimal.

Figure 5.2 shows the lower and upper bounds of optimal consumption-wealth ratio (when it exists) for Y within two standard deviations from the mean of its stationary distribution, with parameters in Table 5.1, and $\gamma = 2$, 4, and 8. The upper and lower bounds for each value of γ are very close, with the largest error appearing at $\gamma = 8$, of less than 2 basis points.

6 Conclusion

This paper tackles the intractability of consumption-investment problems in incomplete markets. We introduce market *contractions*, complete markets in which returns follow the original dynamics, while state variables are perfectly hedgeable, leading to slightly more favorable (upper contraction) and unfavorable (lower) investment opportunities. The dynamics in the lower contraction differs from the upper one only by a shift in the safe rate.

Explicit optimal policies in the market contractions yield (i) an approximate consumption and investment policy, from the optimal policy in the lower contraction, (ii) an approximate pricing measure, from the risk premium implied by the upper contraction, and (iii) an estimate on the precision of the approximation, from both solutions. Furthermore, (iv) the unknown consumption in the original market is bounded pointwise within the consumptions in the upper and lower contractions, and (v) the approximate policies yield a criterion to obtain a verification theorem in the original market.

Appendix: Proofs

We first recall a well-known duality property of wealth processes and stochastic discount factors. For any $\eta \in \mathcal{R}$ and $(\pi, l) \in \mathcal{A}$,

$$\frac{d\left(M_t^\eta X_t^{\pi,l}\right)}{M_t^\eta X_t^{\pi,l}} = \left(\pi_t' - \mu'\Sigma^{-1} - \eta_t'\Upsilon'\Sigma^{-1}\right)\sigma dZ_t + \eta_t' a dW_t - l_t dt.$$

Thus, $M_t^\eta X_t^{\pi,l} + \int_0^t M_s^\eta c_s ds$ is a non-negative local martingale, and therefore a super-martingale. Thus, $\mathbb{E}[\int_0^t M_s^\eta c_s ds] \le x$, and in the limit as $t \uparrow \infty$, $\mathbb{E}[\int_0^\infty M_s^\eta c_s ds] \le x$. As this inequality holds true for all $\eta \in \mathcal{R}$,

$$\sup_{\eta \in \mathcal{R}} \mathbb{E}\left[\int_0^\infty M_t^\eta c_t dt\right] \le x. \tag{6.1}$$

The next Lemma establishes an upper bound, uniform for any policy $(\pi, l) \in \mathcal{A}$, of the expected utility from consumption up to a horizon T (cf. [18, Lemma 5] for expected utility from terminal wealth). We refer to the left-hand side of (6.2) as the primal bound, and to the right-hand side as the dual bound. The limits of the primal and dual bounds as $T \uparrow \infty$ give the lower and upper bounds for the value function. If there exist $(\hat{\pi}, \hat{l}) \in \mathcal{A}$ and $\hat{\eta} \in \mathcal{R}$ such that (with $\hat{c} = \hat{l} X^{\hat{\pi},\hat{l}}$)

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{\hat{c}_t^{1-\gamma}}{1-\gamma}dt\right] = \frac{x^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\frac{\beta}{\gamma}t}\left(M_t^{\hat{\eta}}\right)^{\frac{\gamma-1}{\gamma}} dt\right]^\gamma,$$

then $\hat{\pi}$, $\hat{l}$ and $\hat{\eta}$ are the optimal portfolio, consumption and market price of non-traded risk, respectively.

Lemma 6.1 *For any $(\pi, l) \in \mathcal{A}$, $\eta \in \mathcal{R}$ and $T > 0$,*

$$\mathbb{E}\left[\int_0^T e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] \le \frac{x^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\int_0^T e^{-\frac{\beta}{\gamma}t}\left(M_t^{\eta}\right)^{\frac{\gamma-1}{\gamma}} dt\right]^\gamma. \tag{6.2}$$

Proof. Recall that for any differentiable, strictly increasing, and strictly concave f on $(0,\infty)$, and $z>0$, $\sup_{x>0}(f(x)-xz) = f((f')^{-1}(z)) - (f')^{-1}(z)z$. Let $f(x) = e^{-\beta t}\frac{x^{1-\gamma}}{1-\gamma}$, then $(f')^{-1}(z) = e^{-\frac{\beta t}{\gamma}} z^{-\frac{1}{\gamma}}$. Replacing x by c_t and z by yM_t^η, it follows that, setting $q = \frac{\gamma-1}{\gamma}$,

$$e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma} \le e^{-\frac{\beta}{\gamma}t}\frac{(yM_t^\eta)^q}{1-\gamma} - e^{-\frac{\beta}{\gamma}t}(yM_t^\eta)^q + yM_t^\eta c_t, \qquad \text{for all } y>0,$$

whence, integrating, and recalling (6.1),

$$\mathbb{E}\left[\int_0^T e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] \le y^q\frac{\gamma}{1-\gamma}\mathbb{E}\left[\int_0^T e^{-\frac{\beta}{\gamma}t}\left(M_t^\eta\right)^q dt\right] + y\mathbb{E}\left[\int_0^T M_t^\eta c_t dt\right].$$

The right-hand side reaches its minimum at $\hat{y} = x^{-\gamma}/\mathbb{E}[\int_0^T e^{-\frac{\beta}{\gamma}t}(M_t^\eta)^q dt]^{-\gamma}$, and the claim follows by substituting this value.

Proof of Lemma 3.1 Consider the differential equation

$$\left(-\frac{\beta}{1-\gamma} + \pi\mu - \frac{\gamma\pi'\Sigma\pi}{2} - l + r\right) f + \frac{\nabla f' b}{1-\gamma} + \pi'\Upsilon\nabla f + \frac{1}{2(1-\gamma)}\operatorname{tr}(AD^2 f) = -\frac{l^{1-\gamma}}{1-\gamma}. \quad (6.3)$$

From Theorem 6.6.13 in Gilbarg and Trudinger [16], assumption (i) and (iii) imply that equation (6.3) with the boundary condition $f_n(y) = u_1(y) = g_1(y)^\gamma$ on ∂E_n, has a solution $f_n \in C^{2,\alpha}(E_n)$. By Itô's lemma,

$$\begin{aligned} d\left(e^{-\beta t}\frac{X_t^{1-\gamma}}{1-\gamma}f_n(Y_t)\right) &= e^{-\beta t}X_t^{1-\gamma}\left(\frac{\nabla f_n' b}{1-\gamma} + \frac{1}{2(1-\gamma)}\operatorname{tr}(AD^2 f_n)\right. \\ &\quad \left. + \left(-\frac{\beta}{1-\gamma} + \pi(Y_t)\mu - \frac{\gamma\pi(Y_t)'\Sigma\pi(Y_t)}{2} - l(Y_t) + r\right) f_n\right) dt \\ &\quad + e^{-\beta t}X_t^{1-\gamma}\pi'\Upsilon\nabla f_n + e^{-\beta t}X_t^{1-\gamma}\frac{\nabla f_n' a}{1-\gamma}dW_t + e^{-\beta t}X_t^{1-\gamma}\pi'\sigma dZ_t \end{aligned}$$

For any initial value $x\in\mathbb{R}^n, y\in E$, there exists n, such that $\frac{1}{n} < x < n$ and $y \in E_n$. Let $\tau_n = \inf\{t\ge 0 : (X_t, Y_t) \notin (\frac{1}{n}, n)\times E_n\}$. Because f_n satisfies (6.3) in the bounded domain E_n,

$$\frac{x^{1-\gamma}}{1-\gamma}f_n(y) = \mathbb{E}\left[e^{-\beta\tau_n}\frac{X_{\tau_n}^{1-\gamma}}{1-\gamma}u_1(Y_{\tau_n}) + \int_0^{\tau_n}\frac{e^{-\beta t}X_t^{1-\gamma}l(Y_t)^{1-\gamma}}{1-\gamma}dt\right]$$

The existence of unique solution to the martingale problem from Assumption 2.1(ii) implies that (X,Y) never explode. Now, the same argument as in the proof of Theorem 1 in Heath and Schweizer [24], together with the local Hölder continuity of the model parameters, imply that (X,Y) is a strong Markov process. Thus, for some finite $T>0$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{(l_tX_t)^{1-\gamma}}{1-\gamma}dt\middle|\mathcal{F}_{\tau_n\wedge T}\right]\\
&\quad=\mathbb{E}\left[\int_{\tau_n\wedge T}^\infty e^{-\beta t}\frac{(l_tX_t)^{1-\gamma}}{1-\gamma}dt\middle|\mathcal{F}_{\tau_n\wedge T}\right]+\int_0^{\tau_n\wedge T}e^{-\beta t}\frac{(l_tX_t)^{1-\gamma}}{1-\gamma}dt\\
&\qquad=e^{-\beta\tau_n\wedge T}\frac{X_{\tau_n\wedge T}^{1-\gamma}}{1-\gamma}u_1(Y_{\tau_n\wedge T})+\int_0^{\tau_n\wedge T}e^{-\beta t}\frac{(l_tX_t)^{1-\gamma}}{1-\gamma}dt.
\end{aligned}$$

The tower property of conditional expectation implies

$$\begin{aligned}
\frac{x^{1-\gamma}}{1-\gamma}u_1(y)&=\mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{(l_tX_t)^{1-\gamma}}{1-\gamma}dt\right]\\
&=\mathbb{E}\left[e^{-\beta\tau_n\wedge T}\frac{X_{\tau_n\wedge T}^{1-\gamma}}{1-\gamma}u_1(Y_{\tau_n\wedge T})+\int_0^{\tau_n\wedge T}e^{-\beta t}\frac{(l_tX_t)^{1-\gamma}}{1-\gamma}dt\right].
\end{aligned}$$

Letting $T\to\infty$, by the dominated convergence theorem (for the first term in the the expectation) and the monotone convergence theorem (for the second term), the right-hand side converges to

$$\mathbb{E}\left[e^{-\beta\tau_n}\frac{X_{\tau_n}^{1-\gamma}}{1-\gamma}u_1(Y_{\tau_n})+\int_0^{\tau_n}\frac{e^{-\beta t}X_t^{1-\gamma}l(Y_t)^{1-\gamma}}{1-\gamma}dt\right]=\frac{x^{1-\gamma}}{1-\gamma}f_n(y).$$

As this equality holds for every n, it follows that $u_1(y)\in C^2(E)$ and solves (6.3) in E, whence $g_1=u_1^{\frac{1}{\gamma}}$ solves:

$$\begin{aligned}
&r+\pi\mu-\frac{\gamma\pi'\Sigma\pi}{2}+\frac{\beta}{\gamma-1}+\frac{\gamma\nabla g_1'b}{(1-\gamma)g_1}\\
&\qquad+\gamma\pi'\Upsilon\frac{\nabla g_1}{g_1}+\frac{\gamma\operatorname{tr}(AD^2g_1)}{2(1-\gamma)g_1}-\frac{\gamma\nabla g_1'A\nabla g_1}{2g_1^2}+\frac{g_1^{-\gamma}l^{1-\gamma}}{1-\gamma}-l=0.
\end{aligned}$$

Thus

$$\begin{aligned}
&r+\frac{\beta}{\gamma-1}+\frac{\gamma\nabla g_1'b}{(1-\gamma)g_1}+\frac{\gamma\operatorname{tr}(AD^2g_1)}{2(1-\gamma)g_1}-\frac{\gamma\nabla g_1'A\nabla g_1}{2g_1^2}+\\
&\qquad\sup_{\pi,l}\left(\pi'\mu-\frac{\gamma}{2}\pi'\Sigma\pi+\gamma\pi'\Upsilon\frac{\nabla g_1}{g_1}+\frac{g_1^{-\gamma}l^{1-\gamma}}{1-\gamma}-l\right)\geq 0.
\end{aligned}$$

Then, $\mathcal{H}(y, g_1, \nabla g_1, D^2 g_1) \geq 0$ for $0 < \gamma < 1$ and $\mathcal{H}(y, g_1, \nabla g_1 g, D^2 g_1) \leq 0$ for $\gamma > 1$. On the other hand, consider the differential equation:

$$\left(-\frac{\beta}{\gamma-1} - r - \frac{1}{2\gamma}\mu'\Sigma^{-1}\mu - \frac{1}{2\gamma}\eta' A\eta + \frac{\eta'\Upsilon'\Sigma^{-1}\Upsilon\eta}{2\gamma}\right) f$$
$$+ \nabla f'\left(\frac{\gamma b}{\gamma-1} - \Upsilon'\Sigma^{-1}(\mu + \Upsilon\eta) + A\eta\right) + \frac{\gamma \operatorname{tr}\left(AD^2 f\right)}{2(\gamma-1)} = \frac{\gamma}{1-\gamma}. \quad (6.4)$$

With similar arguments as above, Assumption (ii) and (iii) imply that there exists a unique solution h_n in E_n with the boundary condition $h_n = g_2$ on ∂E_n. By Itô's lemma,

$$d\left(e^{-\frac{\beta}{\gamma}t}\left(M_t^\eta\right)^{\frac{\gamma-1}{\gamma}} h_n(Y_t)\right) = \frac{\gamma-1}{\gamma} e^{-\frac{\beta}{\gamma}t}\left(M_t^\eta\right)^{\frac{\gamma-1}{\gamma}} \times$$
$$\left(\left(-\frac{\beta}{\gamma-1} - r - \frac{\mu'\Sigma^{-1}\mu}{2\gamma} - \frac{\eta' A\eta}{2\gamma} + \frac{\eta'\Upsilon'\Sigma^{-1}\Upsilon\eta}{2\gamma}\right) h_n + \frac{\gamma\nabla h_n' b}{\gamma-1} + \frac{\gamma \operatorname{tr}\left(AD^2 h_n\right)}{2(\gamma-1)}\right) dt$$
$$+ \frac{\gamma-1}{\gamma} e^{-\frac{\beta}{\gamma}t}\left(M_t^\eta\right)^{\frac{\gamma-1}{\gamma}}\left(-\left(\mu'\Sigma^{-1} + \eta'\Upsilon'\Sigma^{-1}\right)\Upsilon + \eta' A\right)\nabla h_n dt$$
$$+ \frac{\gamma-1}{\gamma} e^{-\frac{\beta}{\gamma}t}\left(M_t^\eta\right)^{\frac{\gamma-1}{\gamma}} h_n\left(-\left(\mu'\Sigma^{-1} + \eta'\Upsilon'\Sigma^{-1}\right)\sigma dZ_t + \eta' a dW_t\right)$$
$$+ e^{-\frac{\beta}{\gamma}t}\left(M_t^\eta\right)^{\frac{\gamma-1}{\gamma}} \nabla h_n' a dW_t$$

For any initial value $y \in E$, there exists n, such that $y \in E_n$. Define thus $\tau_n = \inf\{t \geq 0 : Y_t \notin E_n\}$. As h_n satisfies (6.4) in the bounded domain E_n,

$$h_n(y) = \mathbb{E}\left[e^{-\frac{\beta}{\gamma}\tau_n}\left(M_{\tau_n}^\eta\right)^{\frac{\gamma-1}{\gamma}} g_2(Y_{\tau_n}) + \int_0^{\tau_n} e^{-\frac{\beta}{\gamma}t}\left(M_t^\eta\right)^{\frac{\gamma-1}{\gamma}} dt\right]$$

The fact that Y never explodes,together with the local Hölder continuity of the model parameters, imply that Y is a strong Markov process. Thus, similar to the argument above for u_1, $h_n = g_2$ in E_n. Because this holds for every n, g_2 solves (6.4) in E, or equivalently,

$$0 = r + \frac{\beta}{\gamma-1} + \frac{\gamma\nabla g_2' b}{(1-\gamma)g_2} + \frac{\gamma \operatorname{tr}\left(AD^2 g_2\right)}{2(1-\gamma)g_2} + \frac{\mu'\Sigma^{-1}\mu}{2\gamma} + \frac{\nabla g_2'\Upsilon'\Sigma^{-1}\mu}{g_2}$$
$$+ \frac{\gamma g_2^{-1}}{1-\gamma} + \frac{\eta' A\eta}{2\gamma} - \frac{\eta'\Upsilon'\Sigma^{-1}\Upsilon\eta}{2\gamma} + \frac{\nabla g_2'}{g_2}\left(\Upsilon'\Sigma^{-1}\Upsilon - A\right)\eta.$$

Note also that

$$\inf_\eta\left(\frac{\eta' A\eta}{2\gamma} - \frac{\eta'\Upsilon'\Sigma^{-1}\Upsilon\eta}{2\gamma} + \frac{\nabla g_2'}{g_2}\left(\Upsilon'\Sigma^{-1}\Upsilon - A\right)\eta\right) = -\frac{\gamma\nabla g_2'\left(A - \Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g_2}{2g_2^2},$$

thus

$$r+\frac{\beta}{\gamma-1}+\frac{\gamma\nabla g_2'b}{(1-\gamma)g_2}+\frac{\gamma\operatorname{tr}\left(AD^2g_2\right)}{2(1-\gamma)g_2}-\frac{\gamma\nabla g_2'A\nabla g_2}{2g_2^2}$$
$$+\frac{\mu'\Sigma^{-1}\mu}{2\gamma}+\frac{\nabla g_2'\Upsilon'\Sigma^{-1}\mu}{g_2}+\frac{\gamma g_2^{-1}}{1-\gamma}+\frac{\gamma\nabla g_2'\Upsilon'\Sigma^{-1}\Upsilon\nabla g_2}{2g_2^2}\leq 0.$$

Since $\sup_{\ell}\left(\frac{g_2^{-\gamma}\ell^{1-\gamma}}{1-\gamma}-\ell\right)=\frac{\gamma g_2^{-1}}{1-\gamma}$, and

$$\sup_{\pi}\left(\pi'\mu-\frac{\gamma}{2}\pi'\Sigma\pi+\gamma\pi'\Upsilon\frac{\nabla g_2}{g_2}\right)=\frac{\mu'\Sigma^{-1}\mu}{2\gamma}+\frac{\nabla g_2'\Upsilon'\Sigma^{-1}\mu}{g_2}+\frac{\gamma\nabla g_2'\Upsilon'\Sigma^{-1}\Upsilon\nabla g_2}{2g_2^2},$$

it follows that

$$r+\frac{\beta}{\gamma-1}+\frac{\gamma\nabla g_2'b}{(1-\gamma)g_2}+\frac{\gamma\operatorname{tr}(AD^2g_2)}{2(1-\gamma)g_2}-\frac{\gamma\nabla g_2'A\nabla g_2}{2g_2^2}+$$
$$\sup_{\pi,l}\left(\pi'\mu-\frac{\gamma}{2}\pi'\Sigma\pi+\gamma\pi'\Upsilon\frac{\nabla g_2}{g_2}+\frac{g_2^{-\gamma}l^{1-\gamma}}{1-\gamma}-l\right)\leq 0.$$

Then, $\mathcal{H}(y,g_2,\nabla g_2,D^2g_2)\leq 0$ if $0<\gamma<1$ and $\mathcal{H}(y,g_2,\nabla g_2,D^2g_2)\geq 0$ if $\gamma>1$. Finally, Lemma 6.1 implies that $\frac{x^{1-\gamma}}{1-\gamma}g_1^{\gamma}\leq\frac{x^{1-\gamma}}{1-\gamma}g_2^{\gamma}$. Thus if $0<\gamma<1$, $g_1\leq g_2$ and if $\gamma>1$, $g_1\geq g_2$. □

Lemma 3.1 is conceptually close to the result in Heath and Schweizer [24], where the equivalence is shown between a Feynman-Kač functional and the solution to a partial differential equation with a terminal condition. The difference is that (i) in the present setting the horizon is infinite, therefore the associated HJB equation does not have such a terminal condition; and (ii) the equivalence here is established for both the primal and dual bounds of the value function. In addition, the comparison between the primal and dual bounds is used for the existence result in Theorem 3.2, which relies on a method of sub- and super-solution akin to to Hata and Sheu [20], Gilbarg and Trudinger [16], whereby solutions are established first locally and then globally.

Proof of Theorem 3.2. With $u=\gamma\ln g$, rewrite the HJB equation as $\mathcal{G}(y,u,\nabla u,D^2u)=0$, where

$$\mathcal{G}(y,u,\nabla u,D^2u)=\gamma e^{-\frac{u}{\gamma}}+\nabla u'\left(b+\frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\mu\right)+\frac{1}{2}\operatorname{tr}\left(AD^2u\right)$$
$$+\frac{1}{2}\nabla u'\left(A+\frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla u-\beta+\frac{(1-\gamma)\mu'\Sigma^{-1}\mu}{2\gamma}+(1-\gamma)r, \quad (6.5)$$

and $\underline{u}=\gamma\ln\underline{g}$ and $\bar{u}=\gamma\ln\bar{g}$ are super-solutions and sub-solution, respectively. It suffices to show that a classical solution to $\mathcal{G}(y,u,\nabla u,D^2u)=0$ exists.

For each $n\in\mathbb{N}$, since A is positive definite and continuous, the eigenvalues of A are bounded (away from 0) in E_n. Thus there exists $\underline{\lambda}_n<\bar{\lambda}_n$, such that for any $x \in \mathbb{R}^k$ and $y \in E_n$, $\underline{\lambda}_n \sum_{i=1}^{k} x_i^2 \leq \sum_{i,j=1}^{k} A_{ij}(y) x_i x_j \leq \bar{\lambda}_n \sum_{i=1}^{k} x_i^2$. Then Lemma 6.2 below implies that there exists a solution u_n in $\bar{E}_n$ to the boundary value problem

$$\begin{aligned} &\mathcal{G}(y, u, \nabla u, D^2 u) = 0, y \in E_n \\ &u|_{\partial E_n} = \underline{u}|_{\partial E_n}. \end{aligned}$$

Since $\underline{u} \leq \bar{u}$, from the comparison principle (cf. Theorem 10.1 in Gilbarg and Trudinger [16]), $\underline{u} \leq u_n \leq \bar{u}$ in E_n. The same holds for every $m \geq n$, and thus $\{u_m\}_{m \geq n}$ are uniformly bounded in E_n. Because $\bar{E}_n \subsetneq E_{n+1}$, Theorem 13.6 in Gilbarg and Trudinger [16]) implies that for $m \geq n+1$, there exists $\alpha' \in (0, 1]$, such that $[\nabla u_m]_{\alpha', E_n}$ is bounded above by a constant C, where $[f]_{\alpha, \Omega} = \sup_{x,y \in \Omega, x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$, C and α' only depend on $\max_{E_{n+1}} |u_m|$, $\underline{\lambda}_{n+1}$, $\bar{\lambda}_{n+1}$, and are independent of m. Without loss of generality, assume $\alpha = \min(\alpha, \alpha')$ (otherwise reset α to the minimum).

Then consider u_m's as solutions to the following linear problem:

$$\mathcal{J}(y, u, \nabla u, D^2 u) = f(y),$$

where

$$\begin{aligned} &\mathcal{J}(y, u, Du, D^2 u) = \nabla u' \left(b + \frac{(1-\gamma)}{\gamma} \Upsilon' \Sigma^{-1} \mu \right) + \operatorname{tr}\left(A D^2 u\right), \\ &f(y) = -\gamma e^{-\frac{u}{\gamma}} - \nabla u' \left(A + \frac{(1-\gamma)}{\gamma} \Upsilon' \Sigma^{-1} \Upsilon \right) \nabla u + \beta - \frac{(1-\gamma)\mu' \Sigma^{-1} \mu}{2\gamma} - (1-\gamma) r. \end{aligned}$$

Since ∇u_m is α-Hölder continuous in E_n, so is f, for every $m \geq n+1$. Then, the Schauder interior estimates (see Corollary 6.3 in Gilbarg and Trudinger [16]) imply that for $m \geq n+1$, with $d = \operatorname{dist}(E_n, \partial E_{n+1})$,

$$\begin{aligned} d \max_{E_n} |\nabla u_m| + d^2 \max_{E_n} |D^2 u_m| + d^{2+\alpha} [D^2 u_m]_{\alpha, E_n} \\ \leq D \left(\max_{E_{n+1}} |u_m| + \max_{E_{n+1}} |f| + [f]_{\alpha, E_{n+1}} \right), \end{aligned}$$

where the constant D is independent of m. Thus, in any compact set E_n, for $m \geq n+1$, u_m's are uniformly bounded, ∇u_m and $D^2 u_m$ are equicontinuous. From Arzelà-Ascoli theorem, u_m's (up to a subsequence) converges locally uniformly to a function u, and on each E_n, $\nabla u_m \to \nabla u$ and $D^2 u_m \to D^2 u$ uniformly, as $m \uparrow \infty$. Thus u is a classical solution to (6.5), and $\underline{u} \leq u \leq \bar{u}$. □

Lemma 6.2 *There exists a solution to the boundary value problem*

$$\begin{aligned} &\mathcal{G}(y, u, \nabla u, D^2 u) = 0, y \in E_n \\ &u|_{\partial E_n} = \underline{u}|_{\partial E_n}. \end{aligned}$$

Proof. This proof follows an idea similar to Hata and Sheu [20] and we discuss the case of $0<\gamma<1$. The case of $\gamma>1$ follows similarly. By Theorem 3.4 in Hata and Sheu [20], it suffices to prove solutions to the following two boundary value problems are bounded, uniformly in $\tau\in[0,1]$:

$$\begin{aligned}&\mathcal{G}^\tau(y,u,\nabla u,D^2u)=0, y\in E_n\\&u|_{\partial E_n}=\tau\underline{u}|_{\partial E_n},\end{aligned}\tag{6.6}$$

where $\mathcal{G}^\tau$ is defined by replacing γ with $1-\tau(1-\gamma)$ in $\mathcal{G}$, and

$$\begin{aligned}&\bar{\mathcal{G}}^\tau(y,u,\nabla u,D^2u)=0, y\in E_n\\&u|_{\partial E_n}=0,\end{aligned}\tag{6.7}$$

where $\bar{\mathcal{G}}^\tau=\tau e^{-u}+\tau\nabla u'b+\frac{1}{2}\mathrm{tr}(AD^2u)+\frac{\tau}{2}\nabla u'A\nabla u-\tau\beta$. For (6.6), first note that the constant function $\underline{f}_n=-\sup_{y\in\partial E_n}|\underline{u}|-\ln\max(\frac{C_n}{\gamma},1)$, where

$$C_n=\sup_{y\in\bar{E}_n,\tau\in[0,1]}\left(\beta-\frac{\tau(1-\gamma)\mu'\Sigma^{-1}\mu}{2(1-\tau(1-\gamma))}-\tau(1-\gamma)r\right),$$

is a super-solution, and $\tau\underline{u}\geq\underline{f}_n$ for $y\in\partial E_n$. From the comparison principle, for any solution $u_{n,\tau}$ to (6.6), $u_{n,\tau}\geq\underline{f}_n$.

For an upper bound, check the linear equation, which by Theorem 8.34 in Gilbarg and Trudinger [16], has a solution in $C^{1,\alpha}$,

$$\begin{aligned}\nabla f(y)'b+\frac{1}{2}\mathrm{tr}\left(AD^2f(y)\right)-\beta f(y)=&0 \text{ for } y\in E_n,\\ f(y)=&1 \text{ for } y\in\partial E_n.\end{aligned}$$

By the Feyman-Kac formula, the solution is $\mathbb{E}_y[e^{-\beta\theta_n}]$, where θ_n is the hitting time of ∂E_n by Y_t and $\mathbb{E}_y$ indicates the expectation with $Y_0=y$. Then the solution to the following equation

$$\begin{aligned}\nabla f(y)'b+\frac{1}{2}\mathrm{tr}\left(AD^2f(y)\right)-\beta f(y)+1=&0 \text{ for } y\in E_n,\\ f(y)=&1 \text{ for } y\in\partial E_n\end{aligned}$$

is $\bar{f}_n=\frac{1}{\beta}+(1-\frac{1}{\beta})\mathbb{E}_y[e^{-\beta\theta_n}]$, and Theorem 3.8 in Hata and Sheu [20] yields $e^{u_{n,\tau}}\leq\tau e^{\underline{u}}+(1-\tau)\bar{f}_n$, which is bounded from above.

For (6.7), let u_n^0 be a solution. Note that $u_1=-\ln\beta$ is a solution with boundary condition $-\ln\beta$ on ∂E_n. When $0<\beta\leq1$, $-\ln\beta\geq0$ and by comparison principle, $u_n^0\leq u_1$. Similarly, $u_2=0$ is a super-solution, while $u_2\leq u_n^0$. When $\beta>1$, $-\ln\beta<0$, and $u_1\leq u_n^0\leq u_2$.

Proof of Theorem 3.3. First we prove the following equalities

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{-\beta t}\frac{\hat{c}_t^{1-\gamma}}{1-\gamma}dt\right] =&\frac{x^{1-\gamma}}{1-\gamma}g(y)^\gamma\left(1-\mathbb{E}_{\hat{P}}\left[e^{-\int_0^T g(Y_s)^{-1}ds}\right]\right),\\ \frac{x^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\int_0^T e^{-\frac{\beta}{\gamma}t}\left(M_t^{\hat{\eta}}\right)^{\frac{\gamma-1}{\gamma}}dt\right]^\gamma =&\frac{x^{1-\gamma}}{1-\gamma}g(y)^\gamma\left(1-\mathbb{E}_{\hat{P}}\left[e^{-\int_0^T g(Y_s)^{-1}ds}\right]\right)^\gamma.\end{aligned}$$

Since $X_t^{\pi,l} = xe^{\int_0^t(r+\pi_s'\mu-\frac{\pi_s'\Sigma\pi_s}{2}-l_s)ds+\pi_s'\sigma dZ_s}$,

$$e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma} = \frac{x^{1-\gamma}}{1-\gamma}l_t^{1-\gamma}e^{(1-\gamma)\int_0^t\left(\left(r+\frac{\beta}{\gamma-1}+\pi_s'\mu-\frac{\pi_s'\Sigma\pi_s}{2}-l_s\right)ds+\pi_s'\sigma dZ_s\right)}. \tag{6.8}$$

Then, substituting $\pi = \frac{\Sigma^{-1}\mu}{\gamma}+\Sigma^{-1}\Upsilon\frac{\nabla g}{g}$ and $l = g^{-1}$, the integral in the last exponential function above:

$$\begin{aligned}&(1-\gamma)\int_0^t\left(\left(r+\frac{\beta}{\gamma-1}+\pi_s'\mu-\frac{\pi_s'\Sigma\pi_s}{2}-l_s\right)ds+\pi_s'\sigma dZ_s\right)\\ =&-\int_0^t g^{-1}ds-\gamma\int_0^t\left(\frac{\nabla g'b}{g}+\frac{\operatorname{tr}(AD^2g)}{2g}-\frac{\nabla g'A\nabla g}{2g^2}\right)ds-\gamma\int_0^t\frac{\nabla g'a}{g}dW_s\\ &+\gamma\int_0^t\mathcal{H}(Y_s,g,\nabla g,D^2g)ds+\ln D_t,\end{aligned}$$

where

$$\begin{aligned}D_t =&\mathcal{E}\left(\int_0^\cdot\left(\frac{1-\gamma}{\gamma}\Sigma^{-1}\mu+\frac{(1-\gamma)\Sigma^{-1}\Upsilon\nabla g}{g}\right)'\sigma\bar{\rho}dB_s\right)_t\\ &\mathcal{E}\left(\int_0^\cdot\left(\frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\mu+\left(\gamma A+(1-\gamma)\Upsilon'\Sigma^{-1}\Upsilon\right)\frac{\nabla g}{g}\right)'(a')^{-1}dW_s\right)_t.\end{aligned}$$

From Lemma 6.3 below, D is an $(\mathcal{F}_t,P)$-martingale and $\hat{P}|_{\mathcal{F}_t} = D_tP|_{\mathcal{F}_t}$.

Since g solves the HJB equation, $\mathcal{H}(y,g,\nabla g,D^2g) = 0$. Also, note that, by Itô's formula,

$$\int_0^t\left(\frac{\nabla g'b}{g}+\frac{\operatorname{tr}(AD^2g)}{2g}-\frac{\nabla g'A\nabla g}{2g^2}\right)ds+\int_0^t\frac{\nabla g'a}{g}dW_s = \ln g(Y_t)-\ln g(y).$$

Hence (6.8) equals to $\frac{x^{1-\gamma}}{1-\gamma}g(y)^\gamma g(Y_t)^{-1}e^{-\int_0^t g^{-1}(Y_s)ds}D_t$. Then, with the candidate portfolio and consumption in (3.6),

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] =&\frac{x^{1-\gamma}}{1-\gamma}g(y)^\gamma\mathbb{E}\left[\int_0^T g(Y_t)^{-1}e^{-\int_0^t g(Y_s)^{-1}ds}D_tdt\right]\\ =&\frac{x^{1-\gamma}}{1-\gamma}g(y)^\gamma\left(1-\mathbb{E}_{\hat{P}}\left[e^{-\int_0^T g(Y_s)^{-1}ds}\right]\right),\end{aligned}$$

where $\mathbb{E}_{\hat{P}}$ indicates the expectation under $\hat{P}$, and the first equality is proven.

On the other hand, plugging the candidate $\eta = \frac{\gamma \nabla g}{g}$ and following similar calculations, with $q = \frac{\gamma-1}{\gamma}$,

$$e^{-\frac{\beta}{\gamma}t}\left(M_t^\eta\right)^q = \frac{g(y)}{g(Y_t)} e^{-\int_0^t g(Y_s)^{-1} ds} D_t.$$

Thus,

$$\begin{aligned}\frac{x^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\int_0^T e^{-\frac{\beta}{\gamma}t}\left(M_t^\eta\right)^{\frac{\gamma-1}{\gamma}} dt\right]^\gamma &= \frac{x^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\int_0^T \frac{g(y)}{g(Y_t)} e^{-\int_0^t g(Y_s)^{-1} ds} D_t dt\right]^\gamma \\ &= \frac{x^{1-\gamma}}{1-\gamma} g(y)^\gamma \left(1 - \mathbb{E}_{\hat{P}}\left[e^{-\int_0^T g(Y_s)^{-1} ds}\right]\right)^\gamma.\end{aligned}$$

which concludes the proof of the second equality. Now, by the monotone convergence theorem,

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma} dt\right] = \lim_{T\to\infty}\mathbb{E}\left[\int_0^T e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma} dt\right]$$

for any $(\pi, l) \in \mathcal{A}$. Thus, with $(\hat{\pi}, \hat{l}, \hat{\eta})$ in (3.6),

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{\hat{c}_t^{1-\gamma}}{1-\gamma} dt\right] = \frac{x^{1-\gamma}}{1-\gamma} g(y)^\gamma \lim_{T\to\infty}\left(1 - \mathbb{E}_{\hat{P}}\left[e^{-\int_0^T g(Y_s)^{-1} ds}\right]\right) \leq$$

$$\frac{x^{1-\gamma}}{1-\gamma} g(y)^\gamma \lim_{T\to\infty}\left(1 - \mathbb{E}_{\hat{P}}\left[e^{-\int_0^T g(Y_s)^{-1} ds}\right]\right)^\gamma = \frac{x^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\frac{\beta t}{\gamma}}\left(M_t^{\hat{\eta}}\right)^q dt\right]^\gamma,$$

and the equality holds if and only if $\lim_{T\to\infty}(1 - \mathbb{E}_{\hat{P}}[e^{-\int_0^T g(Y_s)^{-1} ds}]) = 1$.

Since $1 - e^{-\int_0^T g(Y_s)^{-1} ds}$ is non-negative and increasing in T, by the monotone convergence theorem,

$$\lim_{T\to\infty}\left(1 - \mathbb{E}_{\hat{P}}\left[e^{-\int_0^T g(Y_s)^{-1} ds}\right]\right) = 1 - \mathbb{E}_{\hat{P}}\left[e^{-\int_0^\infty g(Y_s)^{-1} ds}\right].$$

Then, the inequality (6.9) becomes an equality, i.e., $(\hat{\pi}, \hat{l})$ in 3.6 is optimal, when $\mathbb{E}_{\hat{P}}[e^{-\int_0^\infty g(Y_s)^{-1} ds}] = 0$, which is equivalent to $\int_0^\infty g(Y_s)^{-1} ds = \infty$ $\hat{P}$-a.s., and in this case, both sides of (6.9) are equal to $\frac{x^{1-\gamma}}{1-\gamma} g(y)^\gamma$. □

Lemma 6.3 *Assume for some $f \in C^1(E, \mathbb{R})$, there exists a unique solution $\hat{P}$ to the martingale problem on $\mathbb{R}^n \times E \ni x = (z, y)$*

$$\hat{L} = \frac{1}{2}\sum_{i,j=1}^{n+k}\tilde{A}_{i,j}(y)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n+k}\hat{b}_i(y)\frac{\partial}{\partial x_i}, \tag{6.9}$$

$$\tilde{A}(y) = \begin{pmatrix} \Sigma(y) & \Upsilon(y) \\ \Upsilon'(y) & A(y) \end{pmatrix}, \quad \hat{b} = \begin{pmatrix} \frac{\mu}{\gamma} + \Upsilon\frac{\nabla f}{f} \\ b + \frac{(1-\gamma)\Upsilon'\Sigma^{-1}\mu}{\gamma} + \left(\gamma A + (1-\gamma)\Upsilon'\Sigma^{-1}\Upsilon\right)\frac{\nabla f}{f} \end{pmatrix}.$$

Then,

$$D_t = \mathcal{E}\left(\int_0^\cdot \left(\frac{1-\gamma}{\gamma}\Sigma^{-1}\mu + \frac{(1-\gamma)\Sigma^{-1}\Upsilon\nabla f}{f}\right)'\sigma\bar{\rho}dB_s\right)_t$$
$$\mathcal{E}\left(\int_0^\cdot \left(\frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\mu + \left(\gamma A + (1-\gamma)\Upsilon'\Sigma^{-1}\Upsilon\right)\frac{\nabla f}{f}\right)'(a')^{-1}dW_s\right)_t$$

is an $(\mathcal{F}_t, P)$-martingale. Furthermore, for any $t < \infty$, $\hat{P}|_{\mathcal{F}_t} = D_t P|_{\mathcal{F}_t}$.

Proof. Since Assumption 1 holds and $\frac{\nabla f}{f}$ is locally bounded, Theorem 2.4 and Remark 2.5 in Cheridito et al. [7] imply that there exists a $(\mathcal{B}_t, P)$-martingale $\hat{D}_t$, such that $\hat{P}|_{\mathcal{B}_\tau} = \hat{D}_\tau P|_{\mathcal{B}_\tau}$ for any finite stopping time (with respect to $\mathcal{B}_t$) τ. Note that from Theorem II.2.8 in Revuz and Yor [36], $\hat{D}_t$ is also an $(\mathcal{F}_t, P)$-martingale. Furthermore, from Proposition VII.2.4 and Theorem VII.2.7 in Revuz and Yor [36], there exist Brownian Motions Z and W adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, such that (2.1)-(2.2) hold (cf. footnote 5).

By definition, $\{\mathcal{F}_t\}_{t\geq 0}$ is the right-continuous envelope of the filtration generated by (R, Y). On the other hand,

$$dR_t = \mu(Y_t)dt + \sigma(Y_t)dZ_t,$$
$$dY_t = b(Y_t)dt + a(Y_t)dW_t.$$

Thus $\{\mathcal{F}_t\}_{t\geq 0}$ coincides with the right-continuous envelope of the filtration generated by (Z, W). Define also the process B_t by

$$dZ_t = \rho(Y_t)dW_t + \bar{\rho}(Y_t)dB_t,$$

where $\rho(y) = \sigma^{-1}(y)\Upsilon(y){a'}^{-1}(y)$, and note that it is a Brownian motion independent of W. Thus, by the martingale representation theorem,

$$\hat{D}_t = \mathcal{E}\left(\int_0^\cdot (d'_{1t}dB_s + d'_{2t}dW_s)\right)_t.$$

for some adapted processes d_1 and d_2. Then, by Girsanov's Theorem,

$$\hat{B}_t = B_t - \int_0^t d_{1t}dt, \qquad \text{and} \qquad \hat{W}_t = W_t - \int_0^s d_{2t}dt$$

are Brownian Motions under $\hat{P}$, and the dynamics of (R, Y) under $\hat{P}$ are

$$dR_t = (\mu + \sigma\bar{\rho}d_{1t} + \sigma\rho d_{2t})\,dt + \sigma d\hat{Z}_t,$$
$$dY_t = (b + ad_{2t})\,dt + ad\hat{W}_t.$$

On the other hand, the infinitesimal generator for (R, Y) under $\hat{P}$ is $\hat{L}$. Thus,

$$\begin{pmatrix} \mu + \sigma\bar{\rho}d_{1t} + \sigma\rho d_{2t} \\ b + ad_{2t} \end{pmatrix} = \hat{b},$$

which implies that

$$\begin{aligned}
d_{1t} =& \bar{\rho}\sigma\left(\frac{1-\gamma}{\gamma}\Sigma^{-1}\mu + \frac{(1-\gamma)\Sigma^{-1}\Upsilon\nabla f}{f}\right),\\
d_{2t} =& a^{-1}\left(\frac{(1-\gamma)\Upsilon'\Sigma^{-1}\mu}{\gamma} + \left(\gamma A + (1-\gamma)\Upsilon'\Sigma^{-1}\Upsilon\right)\frac{\nabla f}{f}\right).
\end{aligned}$$

Thus $D_t = \hat{D}_t$, and D_t is an $(\mathcal{F}_t, P)$-martingale.

Finally, for $t < T < \infty$ and every $A \in \mathcal{F}_t \subset \mathcal{B}_T$, since $\hat{P}|_{\mathcal{B}_T} = D_T P|_{\mathcal{B}_T}$, whence $\hat{P}|_{\mathcal{F}_t} = D_t P|_{\mathcal{F}_t}$.

Proof of Theorem 4.1. Consider the stochastic discount factor M^η with $\eta = \frac{\gamma\nabla g^d}{g^d}$. Then, following similar calculations as for the dual bound in Theorem 3.3 (with $q = \frac{\gamma-1}{\gamma}$),

$$\begin{aligned}
-\frac{\beta t}{\gamma}\ln\left(M_t^\eta\right)^q =& \int_0^t -d\ln g^d(Y_s) + \ln\bar{D}_t + \int_0^t g^d(Y_s)^{-1}ds\\
&+ \int_0^t\left(\mathcal{H}^d(Y_s, g^d, \nabla g^d, D^2 g^d) + \frac{(\gamma-1)\left(\nabla g^d\right)'\left(A - \Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g^d}{2\left(g^d\right)^2}\right)ds
\end{aligned}$$

where

$$\begin{aligned}
\bar{D}_t =& \mathcal{E}\left(\int_0^\cdot\left(\frac{(1-\gamma)\Sigma^{-1}\mu}{\gamma} + \frac{(1-\gamma)\Sigma^{-1}\Upsilon\nabla g^d}{g^d}\right)'\sigma\bar{\rho}dB_s\right)_t\\
&\mathcal{E}\left(\int_0^\cdot\left(\frac{(1-\gamma)\Upsilon'\Sigma^{-1}\mu}{\gamma} + \frac{\left(\gamma A + (1-\gamma)\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g^d}{g^d}\right)'(a')^{-1}dW_s\right)_t.
\end{aligned} \tag{6.10}$$

$\mathcal{H}^d(Y_s, g^d, \nabla g^d, D^2 g^d) = 0$ implies that for any $T < \infty$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-\frac{\beta t}{\gamma}}\left(M_t^\eta\right)^q dt\right]^\gamma =\\
&g^d(y)^\gamma\mathbb{E}\left[\int_0^T e^{\int_0^t -g^d(Y_s)^{-1} + \frac{(\gamma-1)\left(\nabla g^d\right)'\left(A-\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g^d}{2\left(g^d\right)^2}ds} g^d(Y_t)^{-1}\bar{D}_t dt\right]^\gamma.
\end{aligned} \tag{6.11}$$

Since the martingale problem for $\bar{L}^d$ have a unique solution, $\bar{D}$ is a $(\mathcal{F}_t, P)$-martingale by Lemma 6.3, and $\bar{P}^d|_{\mathcal{F}_T} = P|_{\mathcal{F}_T}\bar{D}_T$. Thus, (6.11) equals to

$$g^d(y)^\gamma\mathbb{E}_{\bar{P}^d}\left[\int_0^T e^{\int_0^t -g^d(Y_s)^{-1} + \frac{(\gamma-1)\left(\nabla g^d\right)'\left(A-\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g^d}{2\left(g^d\right)^2}ds} g^d(Y_t)^{-1}dt\right]^\gamma. \tag{6.12}$$

Since $A-\Upsilon'\Sigma^{-1}\Upsilon$ is non-negative definite, when $\gamma>1$(resp. <1), (6.12) is greater (resp. less) than or equal to

$$g^d(y)^\gamma \mathbb{E}_{\bar{P}^d}\left[\int_0^T g^d(Y_s)^{-1}e^{\int_0^t -g^d(Y_s)^{-1}ds}dt\right]^\gamma = g^d(y)^\gamma\left(1-\mathbb{E}_{\bar{P}^d}\left[e^{\int_0^T -g^d(Y_s)^{-1}ds}\right]\right)^\gamma. \quad (6.13)$$

Therefore, since $\int_0^\infty g^d(Y_t)^{-1}dt=\infty$ $\bar{P}^d$-a.s.,

$$\frac{x^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\frac{\beta}{\gamma}}\left(M_t^\eta\right)^q dt\right]^\gamma = \lim_{T\to\infty}\frac{x^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\int_0^T e^{-\frac{\beta}{\gamma}}\left(M_t^\eta\right)^q dt\right]^\gamma$$
$$\le \lim_{T\to\infty}\frac{x^{1-\gamma}}{1-\gamma}g^d(y)^\gamma\left(1-\mathbb{E}_{\bar{P}^d}\left[e^{\int_0^T -g^d(Y_s)^{-1}ds}\right]\right)^\gamma = \frac{x^{1-\gamma}}{1-\gamma}g^d(y)^\gamma.$$

Finally, since g^d solves

$$\mathcal{H}(y,g(y),\nabla g,D^2g)-\frac{(\gamma-1)\nabla g'\left(A-\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g}{2g^2}=0,$$

and $A-\Upsilon'\Sigma^{-1}\Upsilon$ is non-negative definite, when $0<\gamma<1$ (resp. $\gamma>1$), $\mathcal{H}(y,g^d(y),\nabla g^d,D^2g^d)\le(\ge)0$ and g^d is a sub (super)-solution.□

Proof of Theorem 4.2. With $l=(g^p)^{-1}$ and $\pi=\frac{\Sigma^{-1}\mu}{\gamma}+\frac{\Sigma^{-1}\Upsilon\nabla g^p}{g^p}$, following similar calculations as for the primal bound in Theorem 3.3, for every $T<\infty$,

$$\mathbb{E}\left[\int_0^T e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] = \frac{x^{1-\gamma}}{1-\gamma}g^p(y)^\gamma$$
$$\mathbb{E}\left[\int_0^T e^{\int_0^t(1-\gamma)\left(\phi(Y_s)-\frac{\gamma(\nabla g^p)'\left(A-\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g^p}{2(g^p)^2}\right)ds} g^p(Y_t)^{-1}e^{-\int_0^t g^p(Y_s)^{-1}ds}\bar{D}_t dt\right],$$

where $\bar{D}$ is defined in (6.10), with g^d replaced by g^p.

Since the martingale problem for $\bar{L}^p$ both have a unique solution, from Lemma 6.3, $\bar{D}$ is a $(\mathcal{F}_t,P)$-martingale, and $\bar{P}^p|_{\mathcal{F}_T}=P|_{\mathcal{F}_T}\bar{D}_T$. Thus,

$$\mathbb{E}\left[\int_0^T e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] = \frac{x^{1-\gamma}}{1-\gamma}g^p(y)^\gamma\times$$
$$\mathbb{E}_{\bar{P}^p}\left[\int_0^T e^{\int_0^t(1-\gamma)\left(\phi(Y_s)-\frac{\gamma(\nabla g^p)'\left(A-\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g^p}{2(g^p)^2}\right)ds} g^p(Y_t)^{-1}e^{\int_0^t -g^p(Y_s)^{-1}ds}dt\right]. \quad (6.14)$$

Since $\phi \geq \frac{\gamma(\nabla g^p)'(A-\Upsilon'\Sigma^{-1}\Upsilon)\nabla g^p}{2(g^p)^2}$, the above is greater than or equal to

$$\frac{x^{1-\gamma}}{1-\gamma} g^p(y)^\gamma \mathbb{E}_{\bar{P}^p}\left[\int_0^T g^p(Y_t)^{-1} e^{\int_0^t -g^p(Y_s)^{-1}ds} dt\right] = \frac{x^{1-\gamma} g^p(y)^\gamma}{1-\gamma}\left(1 - \mathbb{E}_{\bar{P}^p}\left[e^{\int_0^T -g^p(Y_s)^{-1}ds} dt\right]\right). \quad (6.15)$$

Since $\int_0^\infty g^p(Y_t)^{-1}dt = \infty$ $\bar{P}^p$-a.s.,

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] &= \lim_{T\to\infty}\mathbb{E}\left[\int_0^T e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] \\ &\geq \frac{x^{1-\gamma}}{1-\gamma} g^p(y)^\gamma \lim_{T\to\infty}\left(1 - \mathbb{E}_{\bar{P}}\left[e^{\int_0^T -g^p(Y_s)^{-1}ds}\right]\right) = \frac{x^{1-\gamma} g^p(y)^\gamma}{1-\gamma}.\end{aligned}$$

If $\frac{\gamma(\nabla g^p)'\left(A-\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g^p}{2(g^p)^2} = \phi$, from (6.14), the above inequality becomes equality. Note that g^p solves

$$\mathcal{H}(y, g, \nabla g, D^2 g) - \frac{(1-\gamma)}{\gamma}\left(\phi - \frac{\gamma\nabla g'\left(A - \Upsilon'\Sigma^{-1}\Upsilon\right)\nabla g}{2g^2}\right) = 0.$$

Thus if $0 < \gamma < 1$ (resp. $\gamma > 1$), $\mathcal{H}(y, g^p(y), \nabla g^p, D^2 g^p) \geq (\leq) 0$ and g^p is a super (sub)-solution. Finally, $\frac{x^{1-\gamma}}{1-\gamma} g^p(y)^\gamma \leq \frac{x^{1-\gamma}}{1-\gamma} g^d(y)^\gamma$ by Lemma 6.1 and Theorem 4.1. Thus when $0 < \gamma < 1$ (resp. $\gamma > 1$), $g^p \leq g^d$ ($g^p \geq g^d$). □

Proof of Proposition 4.4

Proof. By Lemma 6.1 and Theorem 4.1

$$V(x, y) \leq \inf_{\eta\in\mathcal{R}} \frac{x^{1-\gamma}}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\frac{\beta}{\gamma}}\left(M_t^\eta\right)^q dt\right]^\gamma \leq \frac{x^{1-\gamma}}{1-\gamma} g^d(y)^\gamma.$$

On the other hand, since $V(x, y)$ is homogeneous in x, by definition of CEL and Theorem 4.2,

$$\begin{aligned}(1 - \mathrm{CEL}(\pi, l))^{1-\gamma} V(x, y) &= V\left(x(1 - \mathrm{CEL}(\pi, l)), y\right) \\ &= \int_0^\infty e^{-\beta t}\frac{\left(l_t X_t^{\pi,l}\right)^{1-\gamma}}{1-\gamma}dt \geq \frac{x^{1-\gamma}}{1-\gamma} g^p(y)^\gamma.\end{aligned}$$

Thus, if $\gamma > 1$, $1 \leq (1 - \mathrm{CEL}(\pi, l))^{1-\gamma} \leq (\frac{g^p(y)}{g^d(y)})^\gamma$, and if $0 < \gamma < 1$, the inequalities are reversed. Therefore, $0 \leq \mathrm{CEL}(\pi, l) \leq 1 - (\frac{g^p(y)}{g^d(y)})^{\frac{\gamma}{1-\gamma}}$.

Proof of Proposition 4.6. First consider the well-posedness of the martingale problem for L^Y, where L^Y is the operator associated to Y (with $u = \gamma \ln g$):

$$\begin{aligned} L^Y =& \frac{1}{2}\sum_{i,j=1}^{k} A_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} \\ &+ \sum_{i=1}^{k}\left(b + \frac{(1-\gamma)\Upsilon'\Sigma^{-1}\mu}{\gamma} + \left(A + \frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla u\right)_i \frac{\partial}{\partial x_i}. \end{aligned}$$

Let $\psi = \gamma \ln \bar{g}$, and $\tilde{u} = \psi - u$, then with $\mathcal{G}$ defined in (6.5),

$$\begin{aligned} & L^Y\tilde{u} = L^Y\psi - L^Y u = L^Y\psi - \mathcal{G}(y,u,\nabla u, D^2 u) \\ & -\frac{1}{2}\nabla u'\left(A + \frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla u + \gamma e^{-\frac{u}{\gamma}} - \beta + \frac{(1-\gamma)\mu'\Sigma^{-1}\mu}{2\gamma} + (1-\gamma)r \\ & = \mathcal{G}(y,\psi,\nabla\psi,D^2\psi) + \nabla u'\left(A + \frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla\psi - \\ & \frac{1}{2}\nabla\psi'\left(A + \frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla\psi \\ & -\frac{1}{2}\nabla u'\left(A + \frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla u + \gamma e^{-\frac{u}{\gamma}} - \gamma e^{-\frac{\psi}{\gamma}} = \mathcal{G}(y,\psi,\nabla\psi,D^2\psi) \\ & -\frac{1}{2}(\nabla u - \nabla\psi)'\left(A + \frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\Upsilon\right)(\nabla u - \nabla\psi)' + \gamma e^{-\frac{u}{\gamma}} - \gamma e^{-\frac{\psi}{\gamma}} \\ & \le \mathcal{G}(y,\psi,\nabla\psi,D^2\psi) + \gamma e^{-\frac{u}{\gamma}} - \gamma e^{-\frac{\phi}{\gamma}} = \gamma\left(\mathcal{H}(y,\bar{g},\nabla\bar{g},D^2\bar{g}) + g^{-1} - \bar{g}^{-1}\right) \end{aligned}$$

where the last inequality holds because $A + \frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\Upsilon$ non-negative definite. Because $\bar{g}$ is a sub-solution to $\mathcal{H}(y,g,\nabla g, D^2 g) = 0$ and

$$\sup_{y\in E}(g^{-1} - \bar{g}^{-1}) < \sup_{y\in E}(\underline{g}^{-1} - \bar{g}^{-1}) < \infty,$$

as assumed in (i), $L^Y\tilde{u} < C$ in E for some constant C. Furthermore, Theorem 4.1 implies that $\tilde{u} = \psi - u = \gamma(\ln\bar{g} - \ln g) \ge 0$. Thus for a positive constant $\lambda \ge C$, $L^Y(\tilde{u}+1) = L^Y\tilde{u} \le C \le \lambda(\tilde{u}+1)$, and

$$\lim_{n\uparrow\infty}\inf_{y\in E\setminus E_n}\tilde{u} \ge \lim_{n\uparrow\infty}\inf_{y\in E\setminus E_n}(\ln\bar{g} - \ln\underline{g}) = \infty.$$

Then Theorem 10.2.1 in Stroock and Varadhan implies that the martingale problem for L^Y is well-posed, and there exists a Brownian Motion W such that [36, Proposition VII.2.4 and Theorem VII.2.7]

$$dY_t = \left(b + \frac{(1-\gamma)\Upsilon'\Sigma^{-1}\mu}{\gamma} + \left(A + \frac{(1-\gamma)}{\gamma}\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla u\right)dt + a dW_t.$$

For the martingale associated to $\hat{L}$, expand the probability space, as to support a Brownian Motion B independent to W. Write $Z_t = \rho W + \bar{\rho}B_t$, where

$\rho\rho' + \bar{\rho}\bar{\rho}' = I_n$, and define R as $dR_t = (\frac{\mu}{\gamma} + \Upsilon\frac{\nabla g}{g})dt + \sigma dZ_t$. Then (R, Y) is the unique weak solution to the stochastic differential equation corresponding to $\hat{L}$ and the martingale problem has a unique solution. □

Proof of Lemma 5.1. Let $g(y) = \int_0^\infty h(y,t)dt$, and suppose that

$$g_y = \int_0^\infty h_y dt \quad \text{and} \quad g_{yy} = \int_0^\infty h_{yy} dt. \tag{6.16}$$

In order to prove that g solves (5.3), it suffices to prove that $h(t, y)$ solves:

$$1 + (-ky + \lambda)\int_0^\infty h_y dt + \frac{Ay\int_0^\infty h_{yy}dt}{2} - (cy + K)\int_0^\infty hdt = 0.$$

The above equation holds true if $h(y,t)$ satisfies:

$$-h_t + (-ky + \lambda)\, h_y + \frac{Ay}{2}h_{yy} - (cy + K)\, h = 0. \tag{6.17}$$

with the boundary condition $h(y,\infty) - h(y, 0) = -1$, for every $y \in E$. Because the above conditions imply that

$$\begin{aligned} 1 + (-ky + \lambda)\int_0^\infty h_y dt + \frac{Ay\int_0^\infty h_{yy}dt}{2} - (cy + K)\int_0^\infty hdt \\ = 1 + \int_0^\infty h_t dt = 1 + h(y,\infty) - h(y, 0) = 0. \end{aligned}$$

Taking derivatives of $h(t, y) = e^{C(t)-B(t)y}$, with $C(t)$ and $B(t)$ defined in (5.4)-(5.5), shows that $h(t, y)$ satisfies (6.17). $C(0) = B(0) = 0$, $C(\infty) = -\infty$ and $B(\infty) = \frac{2c}{k+a}$, which implies $h(y, 0) = 1$ and $h(y,\infty) = 0$, and the boundary condition of h holds, for any $y > 0$.

Finally, since $B'(t) = \frac{4c\alpha^2 e^{\alpha t}}{(e^{\alpha t}(k+\alpha)-k+\alpha)^2} > 0$,

$$0 = B(0) \leq B(t) \leq B(\infty) = \frac{2c}{k + \alpha} \text{for all } t \geq 0,$$

(6.16) holds by Lemma 6.4 below. □

Lemma 6.4 *For the functions $g(y) = \int_0^\infty h(y,t)dt$ and $h(y, t) = e^{C(t)-B(t)y}$, if $B(t) \geq 0$ and is bounded, then $g_y = \int_0^\infty h_y dt$ and $g_{yy} = \int_0^\infty h_{yy} dt$.*

Proof. By definition,

$$g_y = \lim_{\epsilon\to 0}\int_0^\infty \frac{h(t, y + \epsilon) - h(t, y)}{\epsilon}dt = \lim_{\epsilon\to 0}\int_0^\infty \frac{e^{C(t)-B(t)y-B(t)\epsilon} - e^{C(t)-B(t)y}}{\epsilon}dt$$

$$= \lim_{\epsilon\to 0}\int_0^\infty \frac{e^{C(t)-B(t)y}\left(e^{-B(t)\epsilon} - 1\right)}{\epsilon}dt.$$

By the convexity of the exponential function, when $\epsilon > 0$, $e^{-B(t)\epsilon} \geq 1 - B(t)\epsilon$,

$$\left| \frac{e^{-B(t)\epsilon} - 1}{\epsilon} \right| = \frac{1 - e^{-B(t)\epsilon}}{\epsilon} \leq \frac{1 - (1 - B(t)\epsilon)}{\epsilon} = B(t);$$

and, when $\epsilon < 0$, $1 \geq e^{-B(t)\epsilon} + B(t)\epsilon e^{-B(t)\epsilon}$,

$$\left| \frac{e^{-B(t)\epsilon} - 1}{\epsilon} \right| = \frac{1 - e^{-B(t)\epsilon}}{\epsilon} \leq \frac{e^{-B(t)\epsilon} B(t)\epsilon}{\epsilon} = B(t) e^{-B(t)\epsilon}.$$

Since $B(t)$ is bounded, $|\frac{e^{-B(t)\epsilon}-1}{\epsilon}|$ is bounded for ϵ around 0.

Thus, from the dominated convergence theorem,

$$g_y = \lim_{\epsilon \to 0} \int_0^\infty \frac{h(t, y + \epsilon) - h(t, y)}{\epsilon} dt = \int_0^\infty \lim_{\epsilon \to 0} \frac{h(t, y + \epsilon) - h(t, y)}{\epsilon} dt = \int_0^\infty h_y dt.$$

$g_{yy} = \int_0^\infty h_{yy} dt$ follows from a similar argument.

Proof of Proposition 5.2. Assumptions 2.1(i), 2.2 and 2.3 are straight forward, by inspection of the model (5.1)-(5.2), and we omit the proof.

To check Assumption 2.1 (ii), similarly to the argument in Proposition 4.6, it suffices to check the well-posedness of the martingale problem for the operator $L^Y = b(\theta - y)\frac{\partial}{\partial y} + \frac{a^2 y}{2}\frac{\partial^2}{\partial y^2}$, the generator of the state variable Y. By Corollary 5.4.9 in Karatzas and Shreve [27], it is equivalent to the uniqueness of the weak solution to $dY_t = b(\theta - Y_t)dt + a\sqrt{Y_t}dW_t$. Since $b(\theta - y)$ and $a\sqrt{y}$ are Lipschitz continuous on (ϵ, ∞), for any $\epsilon > 0$, there exist a unique weak solution of Y on (ϵ, ∞). Then, since Y is a CIR-process satisfying the parameter restriction $b\theta \geq \frac{A}{2}$ under P, it never reaches 0, there exists a unique solution on $(0, \infty)$.

For additional assumptions in Theorem 4.1, in the model (5.1)-(5.2), the ODE $\mathcal{H}^d(y, g^d, \nabla g^d, D^2 g^d) = 0$ is:

$$\left(g^d\right)^{-1} + \left(-by + b\theta + \frac{(1-\gamma)\rho a \mu y}{\gamma\sigma}\right)\frac{g^d_y}{g^d} + \frac{A y g^d_{yy}}{2g^d} + \frac{(1-\gamma)\mu^2 y}{2\gamma^2 \Sigma} - \frac{\beta}{\gamma} + \frac{(1-\gamma)r}{\gamma} = 0.$$

Since $\gamma > 1$, $\frac{\beta}{\gamma} + \frac{(\gamma-1)r}{\gamma} > 0$ and $\frac{(\gamma-1)\mu^2}{2\gamma^2\Sigma} > 0$, from Lemma 5.1, $g^d(y)$ defined in Proposition 5.2 is the solution of the above ODE.

For the martingale problem for $\bar{L}^d$ in Assumption (i), the corresponding SDE for Y is

$$dY_t = \left(b\theta - \phi^d Y_t\right) dt + a\sqrt{Y_t} d\bar{W}_t, \tag{6.18}$$

where $\phi^d = b - \frac{(1-\gamma)\rho a \mu}{\gamma\sigma} - (\gamma + (1-\gamma)\rho^2) A \frac{g^d_y}{g^d} > 0$, because $0 \leq B(t) \leq \frac{2c}{k+\alpha}$, $0 \geq \frac{g^d_y}{g^d} = \frac{\int_0^\infty -B(t)h(y,t)dt}{\int_0^\infty h(y,t)dt} \geq -\frac{2c}{k+\alpha}$. Thus, $b\theta - \phi y$ and $a\sqrt{y}$ are Lipschitz continuous, and (6.18) has a unique weak solution on (ϵ, ∞) for any $\epsilon > 0$. Then, similar to the argument for L, Lemma 6.5 below shows that Y_t in (6.18) never hits 0 or ∞, and the solution to the martingale problem for $\bar{L}^d$ has a unique solution.

For Assumption (ii), let $G(t) = \ln((k+\alpha)e^{\alpha t} - k + \alpha) - \ln 2\alpha - \frac{1}{2}(k+\alpha)t$. Since $G(0) = 0$, and

$$G'(t) = \frac{(k+\alpha)\alpha e^{at}}{(k+\alpha)e^{\alpha t} - k + \alpha} - \frac{1}{2}(k+\alpha) = (\alpha - k)\left(\frac{1}{2} - \frac{\alpha}{(k+\alpha)e^{\alpha t} - k + \alpha}\right) \geq 0,$$

$$C(t) = -\frac{2b\theta G(t)}{A} - Kt \leq -Kt = -\frac{\beta + (\gamma - 1)r}{\gamma} t.$$

Therefore, as $B(t) \geq 0$ and $y > 0$,

$$g^d(y) < \int_0^\infty e^{C(t)} dt < \int_0^\infty e^{-\frac{\beta+(\gamma-1)r}{\gamma} t} dt = \left(\frac{\beta + (\gamma - 1)r}{\gamma}\right)^{-1}.$$

Thus, $(g^d)^{-1}$ is bounded from below and $\int_0^\infty g^d(Y_t)^{-1} dt = \infty$ $\bar{P}^d$-a.s.

For the additional assumptions in Theorem 4.2, first, from Lemma 6.6 below, there exists a constant Q, such that $\bar{c} = \frac{(\gamma-1)\mu^2}{2\gamma^2 \Sigma} - (\gamma - 1)(1 - \rho^2)AQ > 0$ and $\frac{2\bar{c}^2}{(k+\bar{\alpha})^2} = Q$. For $\phi(y) = \gamma(1-\rho^2)AQy$, the ODE $\mathcal{H}^d(y, g^p, \nabla g^p, D^2 g^p) - \frac{(1-\gamma)\phi}{\gamma} = 0$ becomes:

$$g^p(y)^{-1} + \left(-by + b\theta + \frac{(1-\gamma)\rho a \mu y}{\gamma \sigma}\right) \frac{g_y^p}{g^p} + \frac{A y g_{yy}^p}{2 g^p} - \bar{c} y - \frac{\beta}{\gamma} + \frac{(1-\gamma)r}{\gamma} = 0.$$

Then, since $A > 0$, $\frac{\beta+(\gamma-1)r}{\gamma} > 0$ and $\bar{c} > 0$, from Lemma 5.1, $g^p(y)$ defined in Proposition 5.2 is the solution of the above ODE.

Similar to $B(t)$, $0 \leq \bar{B}(t) \leq \frac{2\bar{c}}{k+\bar{\alpha}}$, $0 \geq \frac{g_y^p}{g^p} \geq -\frac{2\bar{c}}{k+\bar{\alpha}}$. Then similar to the argument for $\bar{L}^d$in Assumption (i) in Theorem 4.1, Lemma 6.5 below implies that the diffusion Y, which follows $dY_t = (b\theta - \phi^p Y_t)dt + a\sqrt{Y_t} d\bar{W}_t$, with $\phi^p = b - \frac{(1-\gamma)\rho a \mu}{\gamma \sigma} - (\gamma + (1-\gamma)\rho^2) A \frac{g_y^p}{g^p}$, never reaches 0 or ∞. Thus Y has a unique weak solution and Assumption (ii) holds.

Note that

$$\phi(y) - \gamma \frac{(\nabla g^p)' \left(A - \Upsilon' \Sigma^{-1} \Upsilon\right) \nabla g^p}{2 (g^p)^2} y = \gamma \left(1 - \rho^2\right) A \left(Q - \frac{1}{2}\left(\frac{g_y^p}{g^p}\right)^2\right) y \geq 0,$$

and Assumption (i) holds. Finally, similar to $C(t)$, $\bar{C}(t) \leq -\frac{\beta+(\gamma-1)r}{\gamma} t$, which implies $\int_0^\infty g^p(Y_t)^{-1} dt = \infty$ $\bar{P}^p$-a.s., and Assumption (iii) holds. □

Lemma 6.5 *If for two constants b_1 and $b_2 > 0$, $b_1 \leq b_t \leq b_2$ for all $t \geq 0$, $\theta \geq \frac{a^2}{2}$ and the stochastic process Y satisfies $dY_t = (\theta - b_t Y_t)dt + a\sqrt{Y_t} dW_t$ with $Y_0 > 0$, then Y never explodes to 0 or ∞.*

Proof. For non-explosion to 0, by the comparison principle, Y is bounded below by Y_2, which follows $dY_{2t} = (\theta - b_2 Y_{2t})dt + a\sqrt{Y_{2t}} dW_t$. Since $\theta \geq \frac{a^2}{2}$, Y_2 never reaches 0, and neither does Y. For non-explosion to ∞, consider n Ornstein-Uhlenbeck processes X_t^i, $i \geq 1$,

$$dX_t^i = -\frac{b_1}{2} X_t^i dt + \frac{a}{2} dW_t^i,$$

where $(W^i)_{i=1}^n$ are n independent Brownian Motions. Let $\tilde{Y}_t = \sum_{i=1}^n (X_t^i)^2$, then

$$d\tilde{Y}_t = \left(\frac{nA}{4} - b_1\tilde{Y}_t\right) dt + a\sqrt{\tilde{Y}_t}\sum_{i=1}^n \frac{X_t^i}{\sqrt{\tilde{Y}_t}} dW_t^i.$$

Note that $\int_0^t \sum_{i=1}^n X_s^i/\sqrt{\tilde{Y}_s} dW_s^i$ is a continuous local martingale starting from 0, and since $\sum_{i=1}^n (X_t^i)^2/\tilde{Y}_t = 1$, its quadratic variation is t. Thus, by Lévy's Theorem, it is a Brownian Motion. Then, let n be large enough such that $\frac{nA}{4} \geq \theta$, by the comparison principle, $\tilde{Y}_t$ with dynamics

$$d\tilde{Y}_t = \left(\frac{nA}{4} - b_1\tilde{Y}_t\right) dt + a\sqrt{\tilde{Y}_t} dW_t$$

dominates Y_{1t} satisfying

$$dY_{1t} = (\theta - b_1 Y_{1t})\, dt + a\sqrt{Y_{1t}} dW_t,$$

which in turn dominates Y_t. Since $\tilde{Y}_t$ is the sum of n independent squared Ornstein-Uhlenbeck processes, which are Gaussian, $\tilde{Y}_t$ never explodes to ∞, and neither does Y_t.

Lemma 6.6 *For the model* (5.1)-(5.2)*, there exists a constant $\hat{Q} > 0$, such that $\bar{c} > 0$, and $\frac{2\bar{c}^2}{(k+\bar{\alpha})^2} = \hat{Q}$, where $\bar{c}$, k and $\bar{\alpha}$ are defined in Proposition 5.2.*

Proof. Consider $U(Q) = \frac{2\bar{c}^2}{(k+\bar{\alpha})^2} - Q$. When $Q = 0$, $\bar{c} = \frac{(\gamma-1)\mu^2}{2\gamma^2\Sigma} > 0$, hence $U = \frac{2\bar{c}^2}{(k+\bar{\alpha})^2} > 0$; when $Q = \frac{\mu^2}{2(1-\rho^2)\gamma^2\Sigma A} > 0$, $\bar{c} = 0$ and $U = -Q < 0$. Since U is continuous in Q, there exists a constant $\hat{Q}$ between 0 and $\frac{\mu^2}{2(1-\rho^2)\gamma^2\Sigma A}$, such that $U(\hat{Q}) = 0$ and since $\bar{d}$ is monotone in Q, $\bar{c}(\hat{Q}) > 0$.

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