Birational geometry of irreducible holomorphic symplectic tenfolds of O'Grady type

BIRATIONAL GEOMETRY OF IRREDUCIBLE HOLOMORPHIC SYMPLECTIC TENFOLDS OF O'GRADY TYPE

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ABSTRACT. In this paper, we analyse the birational geometry of O'Grady ten dimensional manifolds, giving a characterization of Kähler classes and lagrangian fibrations. Moreover, we study symplectic compactifications of intermediate jacobian fibrations of smooth cubic fourfolds.

To Olivier Debarre, whose work on IHS manifolds (and more) has been an inspiration for us.

CONTENTS

INTRODUCTION

An irreducible holomorphic symplectic manifold is a simply connected compact Kähler manifold with a unique up to scalar holomorphic symplectic form. Up to deformation, only few examples are known in each dimension, and the question whether they are all or not is widely open. In dimension 2 though there are only K3 surfaces. In this paper we concentrate on one specific deformation type, the so-called OG10-type, and investigate aspects of its birational geometry. These manifolds are deformation equivalent to the symplectic resolution of singularities of a singular moduli space of sheaves on a K3 surface constructed by O'Grady ([OGr99]).

The geometry of an irreducible holomorphic symplectic manifold X is encoded in the second integral cohomology group $\mathrm{H}^2(X, \mathbb{Z})$. Recall that such a group is torsion

free and has a symmetric non-degenerate bilinear form on it, called the Beauville–Bogomolov–Fujiki form and denoted by $(\bullet, \bullet)$ or, for the associated quadratic form, by $\bullet^2$. The pair $(\mathrm{H}^2(X, \mathbb{Z}), (\bullet, \bullet))$ is a lattice.

We start our investigation by studying lagrangian fibrations. These are maps from an irreducible holomorphic symplectic manifold to a normal non trivial variety, such that all smooth fibres are lagrangian subvarieties. If $f \colon X \to B$ is a lagrangian fibration, then the divisor $f^* \mathcal{O}_B(1)$ is isotropic (i.e. $(f^* \mathcal{O}_B(1))^2 = 0$) and nef. Our first result establishes a birational converse of this fact.

Theorem (Theorem 2.2). *Let X be an irreducible holomorphic symplectic manifold of OG10-type and let $\mathcal{O}(D) \in Pic(X)$ be a primitive non-trivial isotropic line bundle. Assume that the class $[D]$ of $\mathcal{O}(D)$ belongs to the boundary of the birational Kähler cone of X.*

Then, there exists a smooth irreducible holomorphic symplectic manifold Y, a bimeromorphic map $\psi \colon Y \dashrightarrow X$ and a lagrangian fibration $p \colon Y \to \mathbb{P}^5$ such that $\mathcal{O}(D) = \psi^* p^* (\mathcal{O}(1))$.

Recall that the birational Kähler cone of X is the union of the Kähler cones of all the smooth birational models of X which are still irreducible holomorphic symplectic[1].

As a straightforward corollary we get a proof of the weak splitting property conjectured by Beauville ([Bea07]).

Theorem (Corollary 2.4). *Let X be a projective irreducible holomorphic symplectic manifold of OG10-type and let D be an isotropic divisor on it. Let $DCH(X) \subset CH_{\mathbb{Q}}(X)$ be the subalgebra generated by divisor classes. Then the restriction of the cycle class map $cl_{|DCH(X)} \colon DCH(X) \to \mathrm{H}^*(X, \mathbb{Q})$ is injective .*

The next result describes, in a lattice-theoretic way, the ample and movable cones. In the following $\mathcal{C}(X) \subset \mathrm{H}^{1,1}(X, \mathbb{R})$ is the connected component of the cone of positive (with respect to the Beauville–Bogomolov–Fujiki form) classes that contains the Kähler cone.

Theorem (Theorem 3.2, Theorem 5.5). *Let X be an irreducible holomorphic symplectic manifold of OG10-type. The birational Kähler cone of X is an open set in the connected component containing a Kähler class of*

$$\mathcal{C}(X) \setminus \bigcup_{D^2 = -2 \ or \ (D^2 = -6 \ and \ \mathrm{div}(D) = 3)} D^\perp.$$

The Kähler cone of X is the connected component containing a Kähler class of

$$\mathcal{C}(X) \setminus \bigcup_{(0 > D^2 \geq -4) \ or \ (\mathrm{div}(D) = 3 \ and \ 0 > D^2 \geq -24)} D^\perp.$$

Here, $D^\perp$ is the orthogonal complement with respect to the Beauville–Bogomolov–Fujiki form. The most important tool to achieve this result is the classification of prime exceptional divisors (see Proposition 3.1) and wall divisors (see Proposition 5.4). Recall that prime exceptional divisors are irreducible and reduced divisors with negative square ([Mar13]): it is known (see [Mar11]) that the movable

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[1]These birational models have a canonical isometry of their second cohomology with that of X, due to the fact that birational maps are well defined in codimension one as the canonical classes are nef.

cone is contained in a prime exceptional chamber, that is in a chamber where the pairing with prime exceptional divisors does not change sign. In particular, the exceptional chamber considered is given by all divisors whose pairing (using the Beauville–Bogomolov–Fujiki form) is non negative with prime exceptional divisors. Wall divisors have a more technical definition (see Definition 1.9), and they provide a finer subdivision of the positive cone: their orthogonal complements cut it in chambers, and the Kähler cone is the chamber containing a Kähler class in this subdivision.

This classification is obtained in two ways. First we construct in an explicit and geometric way examples of wall divisors, studying the birational transformations associated. Then we prove that these examples are the only possibilities, by using recent results from the minimal model program. This last step is the most technical part of the argument. We use a result of Ch. Lehn and Pacienza on the minimal model program for irreducible holomorphic symplectic manifolds ([LP16]) to reduce the problem to a singular moduli space of sheaves on a K3 surface. Here we can apply a result of Meachan and Zhang ([MZ16]) to show that the divisors we found are all.

Finally, as an application of these results, we study the symplectic compactifications of the twisted intermediate jacobian fibrations associated to a smooth cubic fourfold, as constructed by [Sac20]. In particular, we focus on the uniqueness of such compactifications. In the following V is a smooth cubic fourfold and $U_1 \subset \mathbb{P}\,\mathrm{H}^0(V, \mathcal{O}_V(1))^*$ is the open subset containing linear sections with at worst one node. We denote by $\mathcal{J}^t_{U_1}(V)$ the associated twisted intermediate jacobian fibration ([Voi18]). We recall in Section 6.1 the generalities about these varieties.

Theorem (Theorem 6.6). *Let V be a smooth cubic fourfold. Assume that there exists a smooth compactification X of $\mathcal{J}^t_{U_1}(V)$ such that:*

- *X is an irreducible holomorphic symplectic manifold of OG10-type;*
- *there exists a lagrangian fibration structure $X \to \mathbb{P}^5$ extending the natural fibration $\mathcal{J}^t_{U_1}(V) \to U_1$;*
- *the fibres of $X \to \mathbb{P}^5$ are irreducible.*

Then such a compactification is unique.

We remark that a compactification satisfying the first two conditions in the theorem above always exists, thanks to [Sac20, Theorem 1]. So the only non-trivial condition is the third one: we will see in the proof of the theorem that, for example, if V contains a plane or a rational cubic scroll, then there are reducible fibres.

Structure of the paper. In Section 1 we recall the main definitions and main results that we will need later in the paper. Section 2 deals with lagrangian fibrations and here we prove Theorem 2.2 and Corollary 2.4. In Section 3 we classify prime exceptional divisors and prove Theorem 3.2 on the movable cone. Section 4 contains example of wall divisors and of birational morphisms between moduli spaces of sheaves, where these divisors arise. Section 5 completes the classification of wall divisors and proves Theorem 5.5. Section 6 contains applications of the previous results to the case of compactified intermediate jacobian fibration of smooth cubic fourfolds.

1. PRELIMINARIES

Let S be a projective K3 surface and $v \in \mathrm{H}^{\mathrm{even}}(S, \mathbb{Z})$ an effective and positive Mukai vector (see [Yos01, Definition 0.1]). If $H \in \mathrm{Pic}(S)$ is an ample class, we denote by $M_v(S, H)$ the moduli space of Gieseker H-semistable sheaves F on S such that $ch(F)\sqrt{td_S} = v$.

Theorem 1.1 ([LS06, Rap08, PR13, PR14]). *Suppose that H is v-generic (see [PR13, Section 2.1]) and that $v = 2w$ with $w^2 = 2$. Then the following hold.*

(1) [PR14, Thm. 1.1] *The moduli space $M_v(S, H)$ has either locally factorial or 2-factorial singularities and*

(2) [LS06, Thm. 1.1] *There exists a symplectic desingularisation*

$$\pi_v \colon \widetilde{M}_v(S, H) \longrightarrow M_v(S, H).$$

Moreover, if Σ_v denotes the singular locus of $M_v(S, H)$, then $\widetilde{M}_v(S, H)$ is obtained by blowing up Σ_v with its reduced scheme structure and $\widetilde{M}_v(S, H)$ is an irreducible holomorphic symplectic manifold.

(3) [PR13, Thm 1.7] *There is an isometry*

$$\mathrm{H}^2(\widetilde{M}_v(S, H), \mathbb{Z}) \cong U^3 \oplus E_8(-1)^2 \oplus A_2(-1),$$

where U is the hyperbolic plane and E_8 and A_2 are the lattices associated to the corresponding Dynkin diagrams. Moreover,

$$v^\perp \cong \mathrm{H}^2(M_v(S, H), \mathbb{Z}) \longrightarrow \mathrm{H}^2(\widetilde{M}_v(S, H), \mathbb{Z})$$

is injective, the second map being the pullback π_v^*. Finally, if $\alpha \in v^\perp$ has divisibility 2, we have

$$\frac{\alpha \pm \widetilde{\Sigma}}{2} \in \mathrm{H}^2(\widetilde{M}_v(S, H), \mathbb{Z}).$$

Remark 1.2. Roughly speaking a polarisation is v-generic if any strictly semistable sheaf is extension of stable sheaves of Mukai vector w. We only use a non-generic polarisation in one case (see Example 4.2), where the Mukai vector is $v = (2, 0, -2)$. Let us spell out the definition only in this particular case: a polarisation H is generic if for any divisor class D such that $(D, H) = 0$ and $D^2 \geq -4$, then $D = 0$.

Any irreducible holomorphic symplectic manifold deformation equivalent to a smooth moduli space as in the Theorem is called of OG10-type. We recall in the following example the notation for the moduli spaces used by O'Grady in his original construction.

Example 1.3 ([OGr99]). Assume $v = (2, 0, -2)$. In this case an ample class H is generic if the only divisor D such that $D^2 \geq -4$ and $(D, H) = 0$ is the trivial divisor (cf. [OGr99]). Fixed such a generic ample class, we denote by M_S the

moduli space $M_v(S, H)$. The locus B of non-locally free sheaves is a Weil divisor that is not Cartier (cf. [Per10]). If we denote by $\widetilde{B}$ its strict transform and by $\widetilde{\Sigma}$ the exceptional divisor of the desingularisation, then

$$\mathrm{H}^2(\widetilde{M_S}, \mathbb{Z}) \cong \mathrm{H}^2(S, \mathbb{Z}) \oplus \langle \widetilde{B}, \widetilde{\Sigma} \rangle,$$

where $\widetilde{B}^2 = -2$, $\widetilde{\Sigma}^2 = -6$ and $(\widetilde{B}, \widetilde{\Sigma}) = 3$ (cf. [Rap08]). Finally,

$$v^\perp = \mathrm{H}^2(S, \mathbb{Z}) \oplus \mathbb{Z}(1, 0, 1),$$

where the class $(1, 0, 1)$ corresponds to the Cartier divisor $2B$, and $\pi_v^*(2B) = 2\widetilde{B} + \widetilde{\Sigma}$.

Example 1.4 (Moduli spaces of torsion sheaves). Let S be a very general K3 surface of genus 2, that is $\mathrm{Pic}(S) = \mathbb{Z}H$ where H is an ample line bundle of degree 2. Fix the Mukai vector $v = (0, 2H, s)$. Then the moduli space $M_v = M_v(S, H)$ parametrises sheaves of pure dimension 1. More precisely, M_v contains the relative Picard variety $\mathrm{Pic}^{s+4}(\mathcal{C}/|2H|)$, where $\mathcal{C} \to U \subset |2H|$ is the universal family of smooth curves contained in the linear system $|2H|$.

There is a natural morphism $p \colon M_v \to |2H|$ that assigns to a sheaf its Fitting support; we call this map the *Fitting morphism*.

Notice that M_v is smooth when s is odd, and it is singular when s is even. The last case is the one we are interested in this paper, so from now on we suppose that s is even. If $\pi \colon \widetilde{M_v} \to M_v$ is the symplectic desingularisation described in Theorem 1.1, then the composition $p \circ \pi \colon \widetilde{M_v} \to |2H|$ is a lagrangian fibration.

Remark 1.5. Let $\pi_v \colon \widetilde{M_v}(S, H) \longrightarrow M_v(S, H)$ be the symplectic desinuglarisation of Theorem 1.1. We remark that the inclusion $\pi_v^* \colon \mathrm{H}^2(M_v(S, H), \mathbb{Z}) \to \mathrm{H}^2(\widetilde{M_v}(S, H), \mathbb{Z})$ preserves the intersection product in the following sense. Let $C \subset M_v(S, H)$ be a curve not contained in the singular locus, let $\widetilde{C} \subset \widetilde{M_v}(S, H)$ be its strict transform and $D \in \mathrm{Pic}(M_v(S, H))$ be a Cartier divisor. Then by the projection formula

$$\widetilde{C}.\pi_v^* D = C.D.$$

Definition 1.6. (1) A *monodromy operator* on an irreducible holomorphic symplectic manifold is an isometry of $\mathrm{H}^2(X, \mathbb{Z})$ obtained by composing isometries given by parallel transport along smooth families of deformations of X. The group of monodromy operators is denoted $\mathrm{Mon}^2(X)$.

(2) Given two irreducible holomorphic symplectic manifolds X and Y, a *parallel transport operator* between them is an isometry $p \colon \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(Y, \mathbb{Z})$ obtained by composing isometries given by parallel transport along smooth families of deformations from X to Y. The image $p(D)$ of a class $D \in \mathrm{H}^2(X, \mathbb{Z})$ will be called the parallel transport of D (along p).

The monodromy group is always contained in the index two subgroup $\mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z}))$ of orientation preserving isometries. Recall that an isometry is orientation preserving if it preserves the orientation of the positive three-space generated by the Kähler and symplectic forms (cf. [Mar11, Section 4]). The main tool we are going to use throughout the rest of the paper is the following description of the monodromy group.

Theorem 1.7 ([Ono20, Theorem 5.4]). *Let X be an irreducible holomorphic symplectic manifold of OG10-type. Then*

$$\mathrm{Mon}^2(X) = \mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z})).$$

As in the case of K3 surfaces, interesting properties of divisors are preserved by monodromy transformations. We will be interested in two monodromy invariant classes of divisors: wall divisors and stably prime exceptional divisors.

Definition 1.8. Let X be an irreducible holomorphic symplectic manifold and let $D \in \operatorname{Pic}(X)$ be a divisor. The divisor D is prime exceptional if it is effective, integral and $q(D) < 0$. A divisor is stably prime exceptional if it is prime exceptional on a very general deformation of the pair (X, D).

Definition 1.9. Let X be an irreducible holomorphic symplectic manifold and let $D \in \operatorname{Pic}(X)$ be a primitive divisor. The divisor D is a wall divisor if $q(D) < 0$ and, for all $f \in \operatorname{Mon}^2(X) \cap \operatorname{Hdg}^2(X)$, we have

$$f(D)^\perp \cap \mathcal{BK}_X = \emptyset.$$

In particular, a stably prime exceptional divisor is the multiple of a wall divisor and wall divisors which have an effective multiple are stably prime exceptional divisors. This is due to the fact that an effective prime divisor of negative square is uniruled, and that by the MMP uniruled negative divisors can be contracted ([Bou04, Theorem 4.5] and [HT09, Theorem 10]). By taking together the results of [AV15, BHT15, Mar11, Mon15] we have the following:

Theorem 1.10. *Let X be a irreducible holomorphic symplectic manifold and let $D \in \operatorname{Pic}(X)$ be a divisor. Let Y be an irreducible holomorphic symplectic manifold deformation equivalent to X and let $D' \in \operatorname{Pic}(Y)$ be the image of D through a parallel transport operator. Then, D' is a wall divisor (respectively a stably prime exceptional divisor) if and only if D is.*

An intensive study of the birational geometry of (singular) moduli spaces of O'Grady type has been carried out by Meachan and Zhang [MZ16], they worked in the more general context of Bridgeland stability conditions and moduli spaces of stable objects in the derived category of a K3 surface. We will stick with the notion of Gieseker stability and moduli spaces of sheaves, and our (Bridgeland) stability condition will be a (Gieseker) stability condition parametrized by the choice of an ample class instead of choosing an ample class and a positive class, as is usually done by taking the so called large volume limit, see [Br08, Section 1.4]. For every v-generic stability condition H, there is an open set $\mathcal{C}$ of the space of stability conditions where all moduli spaces are isomorphic. This is called a chamber of the space of stability conditions.

Theorem 1.11. [MZ16, Proposition 5.2 and Theorem 5.3] *Let S be a K3 surface and let $v = 2w$ be a Mukai vector with $w^2 = 2$. Let H be a v-generic ample line bundle on S and let $M_v(S, H)$ be the moduli space of semistable sheaves with Mukai vector v and stability condition H. Let $\mathcal{C}$ be the chamber containing H and $\overline{\mathcal{C}}$ its closure. If τ_0 is a (general) stability condition on $D^b(S)$ in $\overline{\mathcal{C}} \setminus \mathcal{C}$, then the following hold:*

(1) *There is a contraction map $\pi : M_v(S, H) \to M_v(S, \tau_0)$ which contracts precisely the S-equivalence classes of τ_0-semistable objects of $M_v(S, H)$, and there is a rank 2 lattice $T \subset H^{2*}(S, \mathbb{Z})$ generated by the class v and the classes of contracted curves.*

(2) *The map π is a divisorial contraction if and only if there exists a class $s \in T$ with $s^2 = -2$ and $(s, v) = 0$.*

(3) *The map π is a small contraction if and only if there exists a class $s \in T$ with $s^2 = -2$ and $0 < (s, v) \leq 4$.*

Let $Stab^\dagger(S)$ denote the distinguished connected component of the space of Bridgeland stability conditions on $D^b(S)$, and let $\tau \in Stab^\dagger(S)$ be general.

Theorem 1.12. [MZ16, Theorem 7.6] *There is a globally defined and continuous map $\ell \colon Stab^\dagger(S) \to NS(M_v(S, \tau))$. The map is independent of τ and the image of a generic stability condition τ' is in the ample cone of the birational model of $M_v(S, \tau)$ given by $M_v(S, \tau')$. Moreover, the image of ℓ contains big and movable divisors on $M_v(S, \tau)$. Therefore, every $\mathbb{Q}$-factorial K-trivial model of $M_v(S, \tau)$ which is isomorphic to it in codimension one arises as a moduli space $M_v(S, \tau')$ for some generic $\tau' \in Stab^\dagger(S)$.*

We will need a well known result in lattice theory, which goes under the name of Eichler's criterion. For any even lattice L, we can define the discriminant group $A_L := L^\vee/L$ and this inherits a quadratic form from L with values in $\mathbb{Q}/2\mathbb{Z}$. Any isometry of L has then an induced action on A_L, and the kernel of this map is the subgroup $\widetilde{O}(L)$. Moreover, to any element $v \in L$ we can define its divisibility $\operatorname{div}(v)$ as the positive generator of the ideal (v, L). This gives a natural map $L \to A_L$ sending v to $[v/\operatorname{div}(v)]$. The following can be found in [GHS10, Lemma 3.5].

Lemma 1.13. *Let L' be an even lattice and let $L = U^2 \oplus L'$. Let $v, w \in L$. Suppose in addition that*

- $v^2 = w^2$,
- $[v/\operatorname{div}(v)] = [w/\operatorname{div}(w)]$ *in A_L.*

Then there is an isometry in $O^+(L)$ sending w to v.

2. Lagrangian fibrations

Lemma 2.1. *Let $l \in L := U^3 \oplus E_8(-1)^2 \oplus A_2(-1)$ be a primitive element of square zero. Then $\operatorname{div}(l) = 1$ and there is a single orbit for the action of $O^+(L)$.*

Proof. The discriminant group of L is of three torsion, hence l can have only divisibility one or three. Inside $A_2(-1)$, any element s of divisibility three has the form $e_1 + 2e_2 + 3t$ or $-e_1 - 2e_2 + 3t$, where e_1 and e_2 are the standard generators with $e_1^2 = e_2^2 = -2$ and $\langle e_1, e_2 \rangle = 1$ and t is any element. It follows that, modulo 18, the square of such an element is -6. Therefore, a primitive element of divisibility 3 inside L is an element of the form $3w + as$ for some $w \in U^3 \oplus E_8(-1)^2$, a not divisible by three and $s \in A_2(-1)$ primitive of divisibility three. The square of such an element is congruent to $-6a^2$ modulo 18, which cannot be zero. Therefore l has divisibility one and, by Lemma 1.13, the action of $O^+(L)$ has a single orbit. $\square$

The above Lemma is the technical core of the section, as for any irreducible holomorphic symplectic manifold X, if $p \colon X \to \mathbb{P}^n$ is a lagrangian fibration, the divisor $p^*(\mathcal{O}(1))$ is nef, isotropic and, if nonreduced fibres are in codimension at least two, primitive (see [KV19]). In particular, if a divisor is induced by a lagrangian fibration on a different birational model of X, it will be isotropic and in the boundary of the birational Kähler cone. The following is a converse for manifolds of OG10-type:

Theorem 2.2. *Let X be an irreducible holomorphic symplectic manifold of OG10-type and let $\mathcal{O}(D) \in Pic(X)$ be a primitive non-trivial line bundle whose Beauville-Bogomolov square is 0. Assume that the class $[D]$ of $\mathcal{O}(D)$ belongs to the boundary of the birational Kähler cone of X.*

Then, there exists a smooth irreducible holomorphic symplectic manifold Y, a bimeromorphic map $\psi \colon Y \dashrightarrow X$ and a lagrangian fibration $p \colon Y \to \mathbb{P}^5$ such that $\mathcal{O}(D) = \psi^* p^*(\mathcal{O}(1))$. Moreover, the smooth fibres of p are principally polarized abelian fivefolds[2].

Proof. The proof of this theorem is completely analogous to [MR21, Theorem 7.2] for the case of sixfolds of O'Grady type, for the convenience of the reader we sketch it here. First of all, notice that we can always assume that $\mathcal{O}(D)$ is primitive, up to replacing it by its primitive submultiple, as the parallel transport of the two coincides. By the work of Matsushita [Mat17, Theorem 1.2] and Wieneck [Wie16, Theorem 1.1], the statement of the Theorem either holds for all deformations of the pair $(X, \mathcal{O}(D))$ where the parallel transport of $[D]$ belongs to the boundary of the birational Kähler cone, or never holds. By Lemma 2.1 and [Mar13, Section 5.3 and Lemma 5.17(ii)], the moduli space of pairs $(X, \mathcal{O}(D))$ with D primitive and isotropic is connected. Therefore, we only need to provide an example of a lagrangian fibration with principally polarized fibres, but such an example is well known and represented by moduli spaces $M_v(S, H)$ of torsion sheaves as in Example 1.4. Notice that a general fibre of the Fitting morphism is isomorphic to the jacobian of a genus five curve, therefore it is a principally polarized abelian variety of dimension 5. □

The above Theorem has an immediate consequence concerning the movable cone of a projective manifold of OG10-type. The following can be thought as a converse of [HT09, Theorem 7] for manifolds of OG10-type.

Corollary 2.3. *Let X be a projective manifold of OG10-type. Then the set of movable divisors of X is $\overline{\mathcal{BK}}_X \cap \mathrm{H}^{1,1}(X, \mathbb{Z})$.*

Proof. By [HT09, Theorem 7], the closure of the movable cone equals $\overline{\mathcal{BK}}_X$. By [HT09, Corollary 19], all elements of $\overline{\mathcal{BK}}_X \cap H^{1,1}(X, \mathbb{Z})$ with positive square are in the movable cone. By Theorem 2.2, also isotropic elements in this intersection are in the movable cone too. □

By using a result of Riess [Rie16, Theorem 4.2], we obtain as a corollary that the weak splitting property conjectured by Beauville [Bea07] holds when the manifold has a square zero divisor.

Corollary 2.4. *Let X be a projective irreducible holomorphic symplectic manifold of OG10-type and suppose that there exists an isotropic divisor class. Let $DCH(X) \subset CH_{\mathbb{Q}}(X)$ be the subalgebra generated by divisor classes. Then the restriction of the cycle class map*

$$cl_{|DCH(X)} \colon DCH(X) \to \mathrm{H}^*(X, \mathbb{Q})$$

is injective.

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[2]Notice that without a section the lagrangian fibration does not give a family of abelian varieties.

Proof. [Rie16, Theorem 4.2] proves that the weak splitting property holds for all manifolds X such that one of their birational model has a lagrangian fibration. By [Mar11, Section 6], a manifold with a square zero divisor has a square zero divisor in the boundary of the birational Kähler cone and from Theorem 2.2 it follows that X has a birational model with a lagrangian fibration. □

Remark 2.5. The same statement has been proved by Voisin in [Sac20, Appendix] for intermediate jacobian fibrations of very general cubic fourfolds. We remark that the ideas behind our proof and Voisin's one are essentially the same.

3. The birational Kähler cone

Proposition 3.1. *Let X be a manifold of OG10-type and let $D \in Pic(X)$ be a primitive divisor. Then a multiple of D is stably prime exceptional if and only if $D^2 = -2$ or $D^2 = -6$ and $\mathrm{div}(D) = 3$.*

Proof. If E is a stably prime exceptional divisor and D is a primitive divisor such that $E = aD$, Markman [Mar13] proves that the reflection $R_E = R_D$ which sends a divisor F to $F - 2\frac{(D,F)}{D^2}D$ is a monodromy operator, and in particular it is integral. Therefore, $2\mathrm{div}(D)/D^2$ is an integer. As $\mathrm{H}^2(X, \mathbb{Z}) \cong U^3 \oplus E_8(-1)^2 \oplus A_2(-1)$, $\mathrm{div}(D)$ can be either 1 or 3. This in turn implies $D^2 = -2$ in the first case and $D^2 = -6$ in the second case. Being a stably prime exceptional divisor is a property which is invariant under the Hodge monodromy group by [Mar11, Section 6], and elements of square -2 form a single monodromy orbit by Lemma 1.13 and Theorem 1.7, as do elements of square -6 and divisibility 3. Therefore, to prove our claim it suffices to produce an example of a manifold X with two prime exceptional divisors having degree -2 and degree -6, respectively.

Let S be a projective K3 surface and M_S the moduli space of semistable rank 2 sheaves with trivial first Chern class and second Chern class of degree 4. The locus B parametrising non-locally free sheaves is a (Weil) divisor. Take $X = \widetilde{M_S}$ to be O'Grady's symplectic desingularisation; then the strict transform $\widetilde{B}$ of B and the exceptional divisor $\widetilde{\Sigma}$ of the desingularisation are prime exceptional divisors with the right degrees and divisibilities, as written in Example 1.3, by [Rap08].

□

Theorem 3.2. *Let X be a manifold of OG10-type. Then, the birational Kähler cone of X is an open set containing a Kähler class inside one of the connected components of*

$$\mathcal{C}(X) \setminus \bigcup_{D^2 = -2 \ or \ (D^2 = -6 \ and \ \mathrm{div}(D) = 3)} D^\perp.$$

Let us explain what is the open subset in the statement. There is only one chamber in the decomposition above that contains the Kähler cone. This chamber is denoted fundamental exceptional chamber in [Mar11, Section 5] and its closure coincides with the closure of the birational Kähler cone. Since the birational Kähler cone is the union of the pullback of Kähler cones via all bimeromorphic morphisms, the latter is obtained from the former by removing more walls, corresponding to birational transformations (see Section 5).

Proof. By the discussion above, the birational Kähler cone is an open set in the fundamental exceptional chamber, which is the connected component of

$$\mathcal{C}(X) \setminus \bigcup_{D \text{ stably prime exceptional}} D^\perp,$$

containing a Kähler class. By Proposition 3.1, stably prime exceptional divisors are multiples of divisors of square -2 or of square -6 and divisibility 3, therefore the claim follows.   □

Example 3.3. If V is a very general cubic fourfold (in the sense of Hassett), denote by $\mathrm{IJ}(V)$ the (compactified and smooth) families of intermediate jacobians associated to V. They have been constructed in [LSV16], where it is shown that they are irreducible holomorphic symplectic manifolds of OG10-type with a natural lagrangian fibration. In Section 6 we recall some of their geometry. By [Ono20, Proposition 4.1], $\mathrm{Pic}(\mathrm{IJ}(V)) \cong U$, where U is the unimodular hyperbolic plane. By fixing standard square zero generators of $\mathrm{Pic}(\mathrm{IJ}(V)) = U = \langle e, f \rangle$, all square zero classes are $e, f, -e, -f$ and, up to a sign choice, positive ones are only e and f – in particular f can be chosen to be the class of the fibration. However, the class $e - f$ is negative on e and positive on f, therefore by Theorem 3.2 only one among e and f can be movable. In particular, the movable cone is the region between the walls generated by $e - f$ and f.

As a consequence one can see that, for very general V, the variety $\mathrm{IJ}(V)$ cannot be birational to its twisted version $\mathrm{IJ}(V)^t$ (see [Voi18]). In fact, Voisin proved in [Voi18] that $\mathrm{IJ}(V)$ and $\mathrm{IJ}^t(V)$ are not birational as lagrangian fibrations, therefore if they were birational there would be two square zero movable divisors in $\mathrm{Pic}(\mathrm{IJ}(V))$, which is excluded by the description of the movable cone given above.

We remark that the same conclusions were already obtained by Saccà with essentially the same arguments (see [Sac20, Corollary 3.9]).

4. EXAMPLES OF WALL DIVISORS

In this section we exhibit explicit examples of wall divisors. The following remark is fundamental for the computations. If X is an irreducible holomorphic symplectic manifold and $\mathrm{H}_2(X, \mathbb{Z})$ is the homology group of curve classes, then Poincaré duality and the non-degeneracy of the Beauville–Bogomolov–Fujiki form give the embedding

$$\mathrm{H}_2(X, \mathbb{Z}) \cong \mathrm{H}^2(X, \mathbb{Z})^* \hookrightarrow \mathrm{H}^2(X, \mathbb{Q}).$$

If $R \in \mathrm{H}_2(X, \mathbb{Z})$ is the class of an extremal curve of the Mori cone of curves, then by [Mon15, Lemma 1.4] an integral generator D of the line $\mathbb{Q}R \subset \mathrm{H}^2(X, \mathbb{Q})$ is the class of a wall divisor, and $D^\vee := D/\mathrm{div}(D) = R$. The strategy is then to look for lines describing certain Mukai flops and write down their classes in the group $\mathrm{H}^2(X, \mathbb{Q})$.

We refer to Example 1.3 and Example 1.4 for the notation and background.

4.1. **Example: the zero section.** Let S be a very general K3 surface of genus 2, that is $\mathrm{Pic}(S) = \mathbb{Z}H$ with $H^2 = 2$. We work with the moduli space $M_{(0,2H,-4)}$ and its symplectic desingularisation $\widetilde{M}_{(0,2H,-4)}$. The generic point of $M_{(0,2H,-4)}$ is of the form $i_* L$, where $i : C \to S$ is the embedding of a genus 5 curve such that $C \in |2H|$ and L is a degree 0 line bundle on C. In particular, the generic fibre of the Fitting support morphism $p : M_{(0,2H,-4)} \to |2H|$ is an abelian variety, so that

there is a zero section $s \colon |2H| \dashrightarrow M_{(0,2H,-4)}$. (Since the K3 surface is very general, one can see that the zero section is in fact regular.) We denote by Z_S its image.

We want to take a general line $l \subset Z_S$ and compute its class in $\mathrm{Pic}(\widetilde{M}_{(0,2H,-4)})_{\mathbb{Q}} = \langle a, b, \sigma \rangle$, where $a = (-1, H, 0)$, $b = (0, 0, 1)$ are the generators of $v_{\mathrm{alg}}^{\perp}$, and σ is the class of the exceptional divisor of the desingularisation. The class of the line l generates an extremal ray of the Mori cone of $\widetilde{M}_{(0,2H,-4)}$, so that the divisor D such that $l = D^{\vee}$ is a wall divisor.

We take as l the horizontal curve defined in [Ono20, Section 4.1.1, after Remark 4.7]; the following result is shown there.

Lemma 4.1 ([Ono20, Remark 4.10, Lemma 4.11]). *$l.a = -1$, $l.b = 1$ and $l.\sigma = 0$.*

As a consequence we get that $l = a - 3b$. In particular it is already integral, so that the associated wall divisor is $D = a - 3b$. Notice that D has degree -4 and divisibility 1.

Remark 4.2. We remark that since l is not contained in the singular locus, this example is just the pullback of [MZ16, Remark 8.5].

4.2. **Example:** $\mathbb{P}^5$. Let S be an elliptic K3 surface such that $\mathrm{Pic}(S) = \langle e, f \rangle$, where $e - f$ is the class of a section and f is the class of a fibre; in particular $e^2 = 0 = f^2$ and $(e, f) = 1$. The class $H = e + 3f$ is ample and generic (in the sense of Example 1.3); $\widetilde{M}_S(H)$ is the associated smooth O'Grady moduli space.

Consider the class $H_0 = e + 2f$; then H_0 is ample but not generic. In fact one can describe explicitly the singular locus of $M_S(H_0)$: a semistable sheaf F is singular in $M_S(H_0)$ if either F is strictly H-semistable or F fits in a short exact sequence

$$(1) \qquad\qquad 0 \longrightarrow L \longrightarrow F \longrightarrow L^{\vee} \longrightarrow 0,$$

where $L = e - 2f$. This follows directly from the proof of [OGr99, Lemma 1.1.5]. Notice that a sheaf F fitting in a sequence like (1) is locally free, and there is a $\mathbb{P}^5$ of such extensions. In particular, blowing up the locus $\Sigma(H_0)$ of strictly H-semistable sheaves produces a symplectic variety $\widetilde{M}_S(H_0)$ (which is still singular).

The same proof as in [MR21, Proposition 4.4 and Corollary 4.6] yields that any H-semistable sheaf remains H_0-semistable, so there is a regular contraction morphism $c \colon M_S(H) \to M_S(H_0)$. Moreover, this morphism lifts to a regular morphism between the corresponding blow-ups, i.e. there is a commutative diagram

$$
\begin{array}{ccc}
\widetilde{M}_S(H) & \xrightarrow{\ \tilde{c}\ } & \widetilde{M}_S(H_0) \\
\downarrow & & \downarrow \\
M_S(H) & \xrightarrow{\ c\ } & M_S(H_0),
\end{array}
$$

and $\tilde{c}$ is a flopping contraction. The extremal curve R associated to the contraction $\tilde{c}$ is any line inside the $\mathbb{P}^5$ of extensions of the form (1). We claim that the divisor $D \in \mathrm{Pic}(\widetilde{M}_S(H))$ such that $D^{\vee} = R$ is the class $e - 2f$, seen as a divisor class in $\mathrm{Pic}(\widetilde{M}_S(H))$ via the Mukai–Donaldson–Le Poitier morphism $\mathrm{Pic}(S) \to \mathrm{Pic}(M_S(H))$ (see beginning of [OGr99, Section 5]) composed with the pullback of the desingularisation map. In particular, $D = e - 2f$ is a wall divisor of degree -4.

First of all, we notice that the contracted $\mathbb{P}^5$ is contained in the locally free locus, so that the curve R is disjoint from both the exceptional divisor $\widetilde{\Sigma}$ and the divisor

$\widetilde{B}$ of non-locally free sheaves (cf. Example 1.3). In particular, we can suppose that $[R] \in \mathrm{H}^2(S, \mathbb{Z}) \cong \langle \widetilde{B}, \widetilde{\Sigma} \rangle^{\perp}$ (here we are implicitly using the Mukai–Donaldson–Le Poitier morphism again to identify classes on the K3 surface with the corresponding classes on the moduli space). On the other hand, such a class must be orthogonal to the ample class H_0 by construction. It follows that, up to a constant, $R = e - 2f$, and we are done.

4.3. **Example: $\mathbb{P}^3$-bundle.** In this section we work with the symplectic resolution $\widetilde{M}_v(S, H)$ of the moduli space $M_v(S, H)$, where (S, H) is a polarised K3 surface of genus 2 and $v = (0, 2H, 2)$. Moreover, we assume that (S, H) is very general, that is $\mathrm{Pic}(S) = \mathbb{Z}H$.

We start by defining a closed subvariety $Y \subset M_v(S, H)$ of codimension 3 such that Y is generically a $\mathbb{P}^3$-bundle over $\mathrm{Hilb}^2(S)$. Then we identify the class of the extremal ray corresponding to a line in a $\mathbb{P}^3$-fibre of Y.

Let $w = (1, 2H, 3)$ and notice that $M_w(S, H) \cong \mathrm{Hilb}^2(S)$. The moduli space $M_w(S, H)$ parametrises sheaves of the form $I_\xi(2)$, where $\xi \in \mathrm{Hilb}^2(S)$ and I_ξ is the corresponding sheaf of ideals. By a direct computation, when $\xi \in \mathrm{Hilb}^2(S)$ is general, we get that $h^0(I_\xi(2)) = 4$ and $h^1(I_\xi(2)) = h^2(I_\xi(2)) = 0$. If $s \in \mathbb{P}\,\mathrm{H}^0(I_\xi(2))$ is a section, then the sheaf F_s fitting in the short exact sequence

$$(2) \qquad\qquad 0 \longrightarrow \mathcal{O}_S \xrightarrow{s \cdot} I_\xi(2) \longrightarrow F_s \longrightarrow 0$$

belongs to the moduli space $M_v(S, H)$. The subvariety $Y \subset M_v(S, H)$ formed by sheaves arising as in (2) is by construction generically a $\mathbb{P}^3$-bundle over $M_w(S, H)$.

Let $\widetilde{M}_v(S, H)$ be the symplectic desingularisation of $M_v(S, H)$. We want to determine the class in $\mathrm{H}_2(\widetilde{M}_v(S, H), \mathbb{Z}) \subset \mathrm{H}^2(\widetilde{M}_v(S, H), \mathbb{Q})$ of a line $R \cong \mathbb{P}^1$ in a general fibre of Y. Then a primitive integral generator D of the line $\mathbb{Q}R$ is a wall divisor.

From now on we fix a very general point $\xi \in \mathrm{Hilb}^2(S)$; the support of ξ is composed by two reduced and disjoint points, i.e. $\mathrm{Supp}(\xi) = \{p, q\}$ and $p \neq q$. If $s \in \mathbb{P}\,\mathrm{H}^0(I_\xi(2))$ is a section, we denote by C_s the zero locus of s. With an abuse of notation, we identify s with C_s when no confusion arises.

Claim 4.3. There exists a pencil $L \subset \mathbb{P}\,\mathrm{H}^0(I_\xi(2))$ such that C_s is smooth for all but a finite number of sections $s_1, s_2, \cdots, s_k \in L$ and moreover:

- C_{s_1}, C_{s_2} and C_{s_3} have two smooth irreducible components; more precisely $C_{s_i} = C_{s_i,1} \cup C_{s_i,2}$ with $C_{s_i,j} \in |H|$ smooth.
- C_{s_i}, $i = 4, \cdots, k$, is irreducible.

Proof. We can think $\mathbb{P}\,\mathrm{H}^0(I_\xi(2))$ as the subset of $|2H|$ of curves passing through the points p and q, support of ξ. Recall that the K3 surface S is the double cover of $\mathbb{P}^2$ ramified along a smooth sextic curve Γ; the linear system $|2H|$ is then isomorphic to the linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|$ of conics in $\mathbb{P}^2$. A curve $C \in |2H|$ is smooth only if the corresponding conic in $|\mathcal{O}_{\mathbb{P}^2}(2)|$ is smooth and is not tangent to the sextic Γ. Moreover, the locus of singular conics in $|\mathcal{O}_{\mathbb{P}^2}(2)|$ is a degree 3 hypersurface.

Let $d \colon S \to \mathbb{P}^2$ be the cover. By the generality of ξ, the image $d(\xi)$ consists of two disjoint points and $\mathbb{P}\,\mathrm{H}^2(I_\xi(2))$ is isomorphic to $\mathbb{P}\,\mathrm{H}^0(I_{d(\xi)}(2)) \subset |\mathcal{O}_{\mathbb{P}^2}(2)|$. The locus of singular conics in $\mathbb{P}\,\mathrm{H}^0(I_{d(\xi)}(2))$ is then again a hypersurface of degree 3. It follows that a general pencil L' in $\mathbb{P}\,\mathrm{H}^0(I_{d(\xi)}(2))$ contains three singular members: these correspond to the three reducible curves in $L = f^*L'$. Now, let $\mathcal{C} \subset \Gamma \times$

$\mathbb{P}\,\mathrm{H}^0(I_{d(\xi)}(2))$ be the incidence variety of conics tangent to Γ at some point. The projection $\mathcal{C} \to \mathbb{P}\,\mathrm{H}^0(I_{d(\xi)}(2))$ is finite on its image. The projection $\mathcal{C} \to \Gamma$ has generic fibre of dimension 1. So the pencil L' contains only finitely many conics tangent to Γ and the claim follows. $\square$

Remark 4.4. The locus of reducible conics in $\mathbb{P}\,\mathrm{H}^0(I_{d(\xi)}(2))$ is the union $S_1 \cup S_2$ of two surfaces. The surface S_1 parametrises reducible conics where both the points in the support of $d(\xi)$ are on the same irreducible component. Since there exists a unique line passing through two points, it follows that S_1 is a linear surface. On the other hand, the surface S_2 parametrises reducible conics such that each irreducible component contains one point of the support of $d(\xi)$; S_2 has degree 2.

Let L be as above. In order to define the line $R \subset M_v(S, H)$ in a general $\mathbb{P}^3$-fibre of Y, we need to define a family $\mathcal{F}$ of sheaves on $L \times S$, flat over L, such that $\mathcal{F}_s \in M_v(S, H)$ for every $s \in L$.

Let π_L and π_S the projections from $L \times S$ to L and S, respectively. The following result is an adaptation to our case of [Per10, Section 2.2 and Appendix].

Lemma 4.5. *Let* $\mathrm{P} = \mathbb{P}\,\mathrm{H}^0(I_\xi(2))$. *There exists an injective morphism*

$$\varphi \colon \pi_S^* \mathcal{O}_S \otimes \pi_\mathrm{P}^* \mathcal{O}_\mathrm{P}(-1) \longrightarrow \pi_S^* I_\xi(2)$$

defining a sheaf $\mathcal{F}' := \mathrm{coker}(\varphi)$ *on* $\mathrm{P} \times S$, *flat over* P, *such that* $\mathcal{F}'_s$ *is the sheaf in* $M_v(S, H)$ *corresponding to the section* s *(cf. (2)), for every* $s \in \mathrm{P}$.

Proof. Viewing $\mathbb{P}\,\mathrm{H}^0(I_\xi(2))$ as a $\mathbb{P}^3$-bundle over a point $\star = \mathrm{Spec}(\mathbb{C})$, we can write

$$\mathrm{P} = \mathbb{P}\,\mathrm{H}^0(I_\xi(2)) = \mathbb{P}\left(o_*\mathcal{H}om(\mathcal{O}_S, I_\xi(2))\right),$$

where $o \colon S \to \star$ is the structure morphism of S. Denote by $p \colon \mathrm{P} \to \star$ the induced morphism. The universal property of $\mathbb{P}$-bundles yields a canonical injective morphism $f \colon \mathcal{O}_\mathrm{P}(-1) \to p^* o_* \mathcal{H}om(\mathcal{O}_S, I_\xi(2))$. By the commutativity of the square

$$
\begin{array}{ccc}
\mathrm{P} \times S & \xrightarrow{\;q_\mathrm{P}\;} & \mathrm{P} \\
\downarrow{\scriptstyle q_S} & & \downarrow{\scriptstyle p} \\
S & \xrightarrow{\;o\;} & \star
\end{array}
$$

we eventually get the isomorphism

$$p^* o_* \mathcal{H}om(\mathcal{O}_S, I_\xi(2)) \cong q_{\mathrm{P}\,*}\mathcal{H}om(q_S^*\mathcal{O}_S, q_S^* I_\xi(2)).$$

It follows that f defines a section $\phi \in \mathrm{H}^0(\mathrm{P} \times S, \mathcal{H}om(q_S^*\mathcal{O}_S q_\mathrm{P}^* \mathcal{O}_\mathrm{P}(-1), q_S^* I_\xi(2)))$.

By construction ϕ is defined fibrewise, and its restriction to each fibre is injective. It follows that ϕ is injective, so that $\mathcal{F} = \mathrm{coker}(\varphi)$ is flat over P, and that $\mathcal{F}_s$ is the sheaf associated to the section $s \in \mathrm{P}$. $\square$

Let $j \colon L \times S \to \mathrm{P} \times S$ be the inclusion and $\mathcal{F} := j^* \mathcal{F}'$ the restriction. Then $\mathcal{F}$ is a sheaf on $L \times S$, flat over L. The pair $(L, \mathcal{F})$ defines a line $R \subset M_v(S, H)$. Now, we want to understand the intersection of R with Σ, the singular locus of $M_v(S, H)$.

Lemma 4.6. *R intersects Σ in one point, corresponding to the unique reducible curve with the property that ξ is contained in only one irreducible component.*

Proof. Let $s \in \mathbb{P}\,\mathrm{H}^0(I_\xi(2))$ be a section, C_s the associated curve and F_s the associated sheaf. If C_s is irreducible, then F_s is stable. Assume then that $C_s = C_1 \cup C_2$ is reducible. The stability of F_s is checked by studying the sheaves $G_i := (F_s|_{C_i})/\mathrm{tors}$. Since F_s is defined by the short exact sequence (2), the sheaf G_i is defined by

$$0 \longrightarrow \mathcal{O}_S \longrightarrow I_{\xi \cap C_i}(1) \longrightarrow G_i \longrightarrow 0.$$

Using the same notation as in Remark 4.4, if $s \in S_1$, then without loss of generality we can suppose that $\xi \subset C_2$. This implies that $I_{\xi \cap C_1}(1) = \mathcal{O}_S(1)$ and we have a square

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_S & \xrightarrow{s\cdot} & \mathcal{O}_S(1) & \longrightarrow & i_{1*}\mathcal{O}_{C_1}(2) & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle id} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & \mathcal{O}_S & \xrightarrow{s\cdot} & I_\xi(2) & \longrightarrow & F_s & \longrightarrow & 0
\end{array}
$$

with injective vertical arrows. Notice that the dotted arrow is induced by the commutativity of the first square, which is in turn induced by the inclusion $\mathrm{H}^0(\mathcal{O}_S(1)) \subset \mathrm{H}^0(I_\xi(2))$ given by multiplication by the equation of C_2 (which is uniquely determined by ξ).

On the other hand, if $s \in S_2 \setminus (S_1 \cap S_2)$, then $I_{\xi \cap C_i}(1) = I_{p_i}(1)$ and one can directly check that neither of G_i can destabilise F_s.

Finally, since L is general, the three points s_1, s_2 and s_3 corresponding to the three reducible curves of Claim 4.3 decompose as $s_1 \in S_1$ and $s_2, s_3 \in S_2$. It follows that $R \cap \Sigma = \{F_{s_1}\}$ and the claim holds. $\qquad\square$

Let now $\widetilde{R}$ be the strict transform of R in $\widetilde{M}_v$, and $\pi \colon \widetilde{M}_v \to M_v$ the desingularisation morphism. We want to determine the coefficients of $\widetilde{R}$ in

$$\mathrm{Pic}(\widetilde{M}_v)_\mathbb{Q} = \mathrm{span}_\mathbb{Q}\{a, b, \sigma\},$$

where $a = (2, H, 0)$ and $b = (0, 0, 1)$ are the generators of $v_{\mathrm{alg}}^\perp$, and σ is the class of the exceptional divisors of π.

Lemma 4.7. $\widetilde{R}.a = 2$ *and* $\widetilde{R}.b = 1$.

Proof. By Remark 1.5, $\widetilde{R}.a = R.a$ and $\widetilde{R}.b = R.b$. The claim follows then from [HL10, Theorem 8.1.5] and the Grothendieck–Riemann–Roch formula, as explained in [Ono20, Lemma 4.11]. $\qquad\square$

Together with Lemma 4.6, this shows that

$$\widetilde{R} = -\frac{1}{2}a - \frac{3}{2}b - \frac{1}{6}\sigma = -\frac{1}{2}(a + 3b + \sigma) + \frac{1}{3}\sigma.$$

If we put $x = -\frac{1}{2}(a + 3b + \sigma)$, then we notice that $x^2 = -4$, so that x is integral. It follows that the divisor

$$D := 3x + \sigma$$

is a wall divisor such that $D^\vee = \widetilde{R}$. Notice that $D^2 = -24$ and $\mathrm{div}(D) = 3$.

Remark 4.8. The image of D inside $\mathrm{Pic}(M_v)$, obtained by contracting the exceptional divisor, is $-a - 3b$, which has square -10 and divisibility 2, giving an example of [MZ16, Theorem 5.3, item (SC)]. In fact, this example is an explicit geometric description of [MZ16, Example 8.6].

Remark 4.9. There is a birational isomorphism

$$\widetilde{M}_{(0,2H,2)} \cong \widetilde{M}_{(0,2H,-2)} \dashrightarrow \widetilde{M}_{(0,2H,-4)}$$

induced by the unique $\mathfrak{g}_2^1$ on the smooth curves in $|2H|$ (see Example 4.6). The first isomorphism is the one given by taking the tensor with $-H$. This birational morphism induces a morphism on the corresponding Picard groups

$$\varphi \colon \operatorname{Pic}(\widetilde{M}_{(0,2H,2)}) \longrightarrow \operatorname{Pic}(\widetilde{M}_{(0,2H,-4)})$$

that has been written down in coordinates in [Ono20, Section 4.1.2, Section 4.1.3]. Therefore one can explicitly see that $\varphi(x)$ coincides with the class of the wall divisor of Example 4.1. Notice that φ is a parallel transport operator (since it is induced by a birational isomorphism), therefore this implies that the class x corresponds to the class of a wall divisor that is deformation of the zero section in $\widetilde{M}_{(0,2H,-4)}$.

4.4. **Example: the O'Grady birational morphism.** Let S be a projective K3 surface with a polarisation H of degree 2. We consider the moduli space $M_{(0,2H,2)}$, whose generic member is of the form i_*L, where $i \colon C \to S$ is the closed embedding of a smooth curve $C \in |2H|$ and L is a degree 6 line bundle on C.

Following a construction of O'Grady in [OGr99, Section 4.1], we define a birational morphism

$$\phi \colon M_{(0,2H,2)} \dashrightarrow M_{(2,2H,0)}$$

in the following way. First of all, notice that if C is smooth and L is a line bundle of degree 6 on C, then $h^0(i_*L) \geq 2$. O'Grady defines then the open subset $\mathcal{J}^0 \subset M_{(0,2H,2)}$ consisting of sheaves of the form i_*L, where C is smooth and L is a globally generated line bundle of degree 6 such that $h^0(L) = 2$. He defines $\phi(i_*L)$ as the dual of the kernel of the surjection $\mathrm{H}^0(L) \otimes \mathcal{O}_S \to i_*L$. As remarked in the proof of [Ono20, Lemma 4.23], this birational morphism coincides with the birational morphism induced by the Fourier–Mukai transform with kernel the ideal sheaf of the diagonal in the product $S \times S$.

We denote by $\widetilde{\phi} \colon \widetilde{M}_{(0,2H,2)} \dashrightarrow \widetilde{M}_{(2,2H,0)}$ the birational morphism induced on the symplectic desingularisations.

The indeterminacy locus of ϕ (and of $\widetilde{\phi}$) is generically identified with the relative Brill–Noether locus $W = \mathcal{W}_2^6(|2H|)$ of line bundles L of degree 6 such that $h^0(L) = 3$. Notice that W has dimension 7. On the other hand, one can easily check that the general member of the $\mathbb{P}^3$-bundle $Y \subset M_{(0,2H,2)}$, defined before in Example 4.3, is of the form i_*L with $h^0(L) = 3$. It follows that $W = Y$ and so that $\widetilde{\phi}$ is the generalised Mukai flop around a (generic) $\mathbb{P}^3$-bundle.

4.5. **Example: $\mathbb{P}^3$-bundles over the locus of non-reduced curves.** In this example we work with the symplectic resolution $\widetilde{M}_{(0,2H,-2)}(S, H)$ of the moduli space $M_{(0,2H,-2)}(S, H)$, where (S, H) is a polarised K3 surface of genus 2; moreover, we assume that (S, H) is very general, that is $\operatorname{Pic}(S) = \mathbb{Z}H$. The moduli space $M_{(0,2H,-2)}$ comes with a lagrangian fibration structure $p \colon M_{(0,2H,-2)} \to |2H|$ that associates to each sheaf its Fitting support. Let $\Delta \subset |2H|$ be the locus of non-reduced curves, that is any curve in Δ is of the form $C = 2C'$, where $C' \in |H|$.

Now, let us consider the restriction

$$p \colon M_\Delta \longrightarrow \Delta,$$

where $M_\Delta \subset M_{(0,2H,-2)}$ is the sublocus of sheaves whose Fitting support is not reduced. It is known (see for example [DeRS19, Section 3.7]) that M_Δ has two irreducible components, denoted M_1 and M_2. The component M_1 parametrises sheaves whose schematic support is the reduced curve: these sheaves are of the form i_*G, where $i: C' \to S$ and G is a rank 2 torsion free sheaf of degree 0. The component M_2 contains an open subset $M_2^0 := M_2 \setminus (M_2 \cap M_1)$ parametrising sheaves whose schematic support is the non-reduced curve itself. We recall that M_2^0 consists of stable sheaves, that is it does not intersect the singular locus Σ of $M_{(0,2H,-2)}$ (cf. [Moz07, Lemma 3.1.7]).

In this section we consider the component M_1, which is a $\mathbb{P}^3$-bundle over a smooth moduli space of sheaves on S of dimension 4. We recall this strucure and compute the square and divisibility of the associated wall divisor.

As already remarked, a general sheaf in M_1 is of the form i_*G, where $i: C' \to S$ is the embedding of the reduced curve C' and G is a rank 2 vector bundle of degree 0. Therefore the determinant $\det G$ of G is a line bundle of degree 0 on C'. There is then a well-defined rational morphism

$$m_1: M_1 \dashrightarrow M_{(0,H,-1)},$$

whose general fibre $m_1^{-1}(C', L)$ is the moduli space of rank 2 semistable vector bundles G on C' such that $\det G = L$. By [NR69, Theorem 2] (cf. [DeRS19, Proposition 3.7.4]), it follows that $m_1^{-1}(C', L) \cong \mathbb{P}^3$. More precisely, if θ is the theta divisor of the jacobian surface $J_{C'}^1$, then $m_1^{-1}(C', L) \cong |2\theta|$. Moreover, Narasimhan and Ramanan also show that the locus in $m_1^{-1}(C', L)$ of strictly semistable sheaves is isomorphic to the Kummer surface that is the image of $J_{C'}^1$ in $|2\theta|$ (cf. [NR69, Proposition 6.3]). In the following we denote by $\tilde{m}_1: \widetilde{M_1} \to M_{(0,H,-1)}$ the strict transform of $m_1: M_1 \to M_{(0,H,-1)}$ via the desingularisation morphism $\pi: \widetilde{M}_{(0,2H,-2)} \to M_{(0,2H,-1)}$. This is a generic $\mathbb{P}^3$-bundle.

We denote by $\widetilde{R}$ a line inside a general fibre of $\tilde{m}_1$ and, as in the previous examples, we want to compute its class $\tilde{l} = [\widetilde{R}]$ in $\mathrm{Pic}(\widetilde{M}_{(0,2H,-2)})_\mathbb{Q} = \langle a, b, \sigma \rangle$. Here $a = (-2, H, 0)$ and $b = (0, 0, 1)$ are the generators of $v_{\mathrm{alg}}^\perp$, and σ is the class of the exceptional divisor of the desingularisation map π. The first remark is that $\widetilde{R}$ is obtained as the strict transform of a line R in a general fibre of m_1, whose class is denoted by $l = [R]$. The following intersections hold.

Lemma 4.10. $\tilde{l}.a = 2$, $\tilde{l}.b = 0$ and $l.\sigma = 4$

Proof. Let us start with the first two intersection numbers. By Remark 1.5, the projection formula reduces the problem to compute $l.a$ and $l.b$. We claim that $l = b$. In fact, since R is contained in a fibre of the Fitting map $p: M_{(0,2H,-2)} \to |2H|$, its class l must be a multiple of $b = p^*\mathcal{O}(1)$; and because both R and b are primitive and positive, the claim follows.

Finally, let Σ be the singular locus of $M_{(0,2H,-2)}$. We claim that $R \cap \Sigma$ consists of four points with multiplicity 1: this will finish the proof. In fact, we have already remarked that if $C' \in |H|$ is general and $L \in J_{C'}$ is a degree 0 line bundle, then $m_1^{-1}(C', L) \cap \Sigma$ is isomorphic to the Kummer surface in $\mathbb{P}^3$, hence it is a degree four hypersurface. Since R is a general curve in the $\mathbb{P}^3$, it intersects Σ in four points of multiplicity 1 as claimed. $\square$

As a corollary, we get that

$$\tilde{l} = b - \frac{2}{3}\sigma \in \mathrm{Pic}(\widetilde{M}_{(0,2H,-2)})_{\mathbb{Q}}.$$

Since $\tilde{l}$ generates an extremal ray in the Mori cone, the divisor D such that $D/\mathrm{div}(D) = \tilde{l}$ must be a wall divisor. It follows that

$$D = 3b - 2\sigma$$

is a wall divisor and it has degree -24 and divisibility 3.

Remark 4.11 (Structure of the irreducible component M_2). We thank the anonymous referee for pointing out this situation to us and for having corrected an erroneous statement made in a previous version of this paper.

We use the notation introduced at the beginning of this section. In particular we work on a non-reduced curve $C = 2C'$, where $C' \in |H|$, and the component M_2 contains the open subset $M_2^0 = M_2 \setminus (M_2 \cap M_1)$. A general point in M_2^0 corresponds to a sheaf of the form $i_* F$, where $i\colon C \to S$ is the inclusion and F is a line bundle on C. We recall that these sheaves are always stable. By [Moz07, Lemma 3.3.1], there is a short exact sequence

$$0 \longrightarrow E \otimes K_{C'}^{\vee} \longrightarrow F \longrightarrow E \longrightarrow 0,$$

where $K_{C'}$ is the canonical bundle of C' and E is the restriction of F to C'. Moreover, E is a line bundle of degree 1 on C'. There exists a well defined map $m_2^0\colon M_2^0 \to M_{(0,H,0)}$ that sends F to E ([Moz07, Lemma 3.3.3]). There is a short exact sequence ([Moz07, Corollary 3.2.2])

$$0 \to \mathrm{Ext}^1_{\mathcal{O}_{C'}}(E, E \otimes K_{C'}^{\vee}) \to \mathrm{Ext}^1_{\mathcal{O}_C}(E, E \otimes K_{C'}^{\vee}) \to \mathrm{End}_{\mathcal{O}_{C'}}(E \otimes K_{C'}^{\vee}) \to 0,$$

so that the fibre of m_2^0 over a point $(C', E) \in M_{(0,H,0)}$ is identified with the affine space

$$\mathbb{P}\,\mathrm{Ext}^1_{\mathcal{O}_C}(E, E \otimes K_{C'}^{\vee}) \setminus \mathbb{P}\,\mathrm{Ext}^1_{\mathcal{O}_{C'}}(E, E \otimes K_{C'}^{\vee}) \cong \mathrm{Ext}^1_{\mathcal{O}_{C'}}(E, E \otimes K_{C'}^{\vee}) \cong \mathrm{H}^1(K_{C'}^{\vee}) = \mathbb{C}^3.$$

It follows that $m_2^0\colon M_2^0 \to M_{(0,H,0)}$ is a generic $\mathbb{C}^3$-bundle.

Let us denote by $\hat{m}_2\colon \hat{M}_2 \to M_{(0,H,0)}$ the fibration whose general fibre is the projective space $\mathbb{P}\,\mathrm{Ext}^1_{\mathcal{O}_C}(E, E \otimes K_{C'}^{\vee}) = \mathbb{P}^3$. The component M_2 is then obtained from $\hat{M}_2$ by gluing the boundary of the fibres of $\hat{m}_2$ with M_1.

We claim that there are 16 fibres of $\hat{m}_2$ that glue to a given general fibre $m_1^{-1}(C', L)$ of m_1. Moreover the gluing is performed along a boundary $\mathbb{P}^2$ of the fixed $\mathbb{P}^3 = m_1^{-1}(C', L)$. In fact, the $\mathbb{P}^2$ must be the space of extensions of $\mathcal{O}_{C'}$-modules $\mathbb{P}\,\mathrm{Ext}^1_{\mathcal{O}_{C'}}(E, E \otimes K^{\vee})$, and if a sheaf belongs to such a $\mathbb{P}^2$, then it belongs to M_1 only if $2E \otimes K^{\vee} = L$. Since there are 16 points of 2-torsion on an abelian surface, the claim follows. Moreover each of these 16 planes must meet in the fixed $\mathbb{P}^3$, therefore M_2 cannot be normal.

4.6. **Example: the hyperelliptic birational map.** We keep the same notations as in the previous Section 4.5. So, in particular, we work with the moduli space $M_{(0,2H,-4)}$ and we denote by $p\colon M_{(0,2H,-4)} \to |2H|$ the Fitting support morphism. Since any smooth curve C in the linear system $|2H|$ is hyperelliptic, there exists a unique $\mathfrak{g}_2^1(C)$. Tensoring with this degree 2 line bundle defines a birational map

$$(3) \qquad\qquad \varphi\colon M_{(0,2H,-4)} \dashrightarrow M_{(0,2H,-2)}.$$

We denote by $\tilde{\varphi}\colon \widetilde{M}_{(0,2H,-4)} \dashrightarrow \widetilde{M}_{(0,2H,-2)}$ the birational map between the respective symplectic desingularisations.

These birational maps have already been considered in [Ono20, Section 4.1.2], where it is shown that φ does not preserve the singular loci of the respective moduli spaces.

In this section we want to understand the wall divisor associated to $\tilde{\varphi}$. We study the indeterminacy locus in terms of the Fitting support of the objects parametrised by $\widetilde{M}_{(0,2H,-4)}$ and $\widetilde{M}_{(0,2H,-2)}$. We start with the following technical result.

Proposition 4.12.   (1) $\tilde{\varphi}$ *is well defined on sheaves whose schematic support is an irreducible curve of arithmetic genus* 5.
  (2) $\tilde{\varphi}$ *is well defined on stable sheaves whose schematic support is a reducible and reduced curve of arithmetic genus* 5; φ *is not defined on sheaves whose support is a reducible and reduced curve of arithmetic genus* 5.
  (3) $\tilde{\varphi}$ *is not defined on stable sheaves whose schematic support is a double curve of arithmetic genus* 5.

Proof. Recall that C is the double cover of a conic in $\mathbb{P}^2$, ramified along a sextic curve; the pullback of a point on the conic defines a $\mathfrak{g}_2^1$ on C.

  (1) If the conic is smooth, there exists only one equivalence class of such a point, so the $\mathfrak{g}_2^1$ is uniquely determined. In this case, the curve C is irreducible (but possibly singular if the conic is tangent to the sextic curve) and the maps φ and $\tilde{\varphi}$ are well defined.

  (2) Suppose that $C = C_1 \cup C_2$ is reducible. We can take a pencil $\mathcal{C} \to \mathbb{P}^1$ of curves in $|2H|$ whose general member is smooth and whose central member $\mathcal{C}_0$ is the reducible curve C. (For example, the pencil in Claim 4.3.) After possibly a finite base change, we can suppose that the pencil has a bi-section Z, that is the pullback of a section of the corresponding pencil of conics. Denote by $j\colon \mathcal{C} \to S \times \mathbb{P}^1$ the natural embedding. Let now $\mathcal{F} = j_*\mathcal{L}$ be a flat family of semistable sheaves on S parametrised by $\mathbb{P}^1$, such that $\mathcal{F}_t$ is the pushforward of a degree 0 line bundle on $\mathcal{C}_t$. In particular $\mathcal{F}$ defines a curve in $M_{(0,2H,-4)}$.

  Using the bi-section Z on $\mathcal{C} \to \mathbb{P}^1$, we can then define the sheaf on $S \times \mathbb{P}^1$

  $$\mathcal{E} = j_*\mathcal{L}(Z)$$

  that is flat over $\mathbb{P}^1$. Moreover, for general t, the sheaf $\mathcal{E}_t = \mathcal{E}|_{S \times \{t\}}$ is stable and coincides with $\varphi(j_{t,*}\mathcal{L}_t)$ by construction. We need to investigate the stability of the central member $\mathcal{E}_0$.

  First of all, we make the following remark: Z_0 is contained in one irreducible component, and without loss of generality we can assume that it does not contain any node of C. Therefore the sheaf $\mathcal{E}_0$ is obtained by twisting the line bundle $\mathcal{L}_0$ with the $\mathfrak{g}_2^1$ of the corresponding irreducible component. We divide the analysis in two cases.

  Both C_1 and C_2 are smooth. Let $F = i_*G$ be a sheaf in $M_{(0,2H,-4)}$. Each C_j has a $\mathfrak{g}_2^1$, call it $\mathcal{O}_{C_j}(p_j + q_j)$ with $p_j, q_j \notin C_1 \cap C_2$, and we denote by $\varphi_j(F)$ the sheaf $F(p_j + q_j)$. Both $\varphi_j(F)$ belong to the moduli space $M_{(0,2H,-2)}$. We claim that $\varphi_1(F) = \varphi_2(F)$ if and only if F is stable. Moreover, in this case $\varphi(F)$ is strictly semistable.

Let us denote by G_j the torsion free part of the restriction of G to C_j; G_j is a line bundle on C_j of degree d_j. With an abuse of notation we keep calling i the inclusion of the curve C_j in S. Since F is semistable, we have that $d_j \geq -1$. There is a short exact sequence

$$0 \to F \to i_*G_1 \oplus i_*G_2 \to Q \to 0,$$

where Q is a torsion sheaf supported on $C_1 \cap C_2$. If F is stable, the only possibility is that Q has length 2 and $d_1 = d_2 = 0$. If the length of Q is 2 and F is strictly semistable, then either $d_1 = 1$ and $d_2 = -1$ or $d_1 = -1$ and $d_2 = 1$. If the length of Q is 1, then F is necessary strictly semistable and either $d_1 = 0$ and $d_2 = -1$ or $d_1 = -1$ and $d_2 = 0$. Finally if $Q = 0$, then F is polystable and $d_1 = d_2 = -1$.

Let us now study the semistability of $\varphi_1(F)$. As for F, we do that by studying the torsion free restrictions $\varphi_1(F)_1 = i_*(G_1(p_1 + q_1))$ and $\varphi_1(F)_2 = i_*(G_2)$, of degree (respectively) $d_1 + 2$ and d_2.

Suppose that F is stable, in which case $d_1 = d_2 = 0$. It follows that the S-equivalence class of $\varphi_1(F)$ is $[i_*G_1(p_1+q_1-n_1-n_2) \oplus i_*G_2] = [i_*G_1 \oplus i_*G_2]$, since $p_1 + q_1 - n_1 - n_2$ is linearly equivalent to zero on C_1. Here n_1 and n_2 are the nodes of C (possibly coinciding – recall that Q has length 2 in this case).

If F is strictly semistable, then we have three cases, depending on the length of Q. If the length of Q is 2, then either $d_1 + 2 = 1$ and $d_2 = 1$ or $d_1 + 2 = 3$ and $d_2 = -1$; it follows that $\varphi_1(F)$ is stable in the first case and unstable in the second case. If the length of Q is 1, then either $d_1 + 2 = 2$ and $d_2 = -1$ or $d_1 + 2 = 1$ and $d_2 = 0$; it follows that $\varphi_1(F)$ is unstable in the first case and strictly semistable in the second case. If $Q = 0$, then $d_1 + 2 = 1$ and $d_2 = -1$, so $\varphi_1(F)$ is unstable.

We can perform the same analysis for $\varphi_2(F)$ and confront the result with the previous case. For example, if F is stable, then $\varphi_2(F)$ is strictly semistable and its S-equivalence class is $[i_*G_1 \oplus i_*G_2(p_2 + q_2 - n_1 - n_2)] = [i_*G_1 \oplus i_*G_2]$, since $p_2 + q_2 - n_1 - n_2$ is linearly equivalent to zero on C_2. Again n_1 and n_2 are the nodes of C. On the other hand, if F is strictly semistable and Q has length 1, then $\varphi_2(F)$ is stable, but it is not isomorphic to $\varphi_1(F)$. Finally, there is a remaining case (when F is strictly semistable and Q has length 1) where both $\varphi_1(F)$ and $\varphi_2(F)$ are strictly semistable, but their S-equivalence classes are different.

At least one between C_1 and C_2 is singular. The main tool we used for the smooth case is the Riemann–Roch formula, which holds in the singular case for locally free sheaves (e.g [Har77, Exercise IV.1.9]). The same computations can be performed in the singular case using the following remark. Sheaves supported on a singular curve C are of the form i_*F, where F is torsion free of rank 1. We then have two cases, either F is locally free or F is the pushforward of a line bundle on a (partial) normalisation.

Let us conclude the proof. The analysis above ensures that the stability of the central sheaf $\mathcal{E}_0$ depends to Z and the stability of $\mathcal{F}_0$. In particular, we see that if $\mathcal{F}_0$ is stable, then $\mathcal{E}_0$ is strictly semistable, but its S-equivalence class does not depend on the choice of the bi-section Z, nor on the choice of the pencil $\mathcal{C}$. We see in a while that the morphism $\tilde{\varphi}$ can be defined in these points by putting $\tilde{\varphi}(\mathcal{F}_0) = \mathcal{E}_0$. On the other hand, if $\mathcal{F}_0$ is strictly

semistable, then we see that either $\mathcal{E}_0$ is unstable, or it depends on the choice of Z, so that $\tilde{\varphi}$ cannot be defined in such points.

If φ were regular at stable sheaves supported on a reducible and reduced curve, then $\mathcal{F}_0$ would be mapped to the S-equivalence class of $\mathcal{E}_0$ as constructed above, which would therefore not be injective. Hence it is not regular there. On the other hand, this issue does not occur for $\tilde{\varphi}$, as the different pencils $\mathcal{C} \to \mathbb{P}^1$ parametrize the one dimensional space of sheaves S-equivalent to $\mathcal{E}_0$, as the locus of reducible curves is a divisor in $|2H|$.

(3) Now the curve $C = 2C_0$ is not reduced and the reduced curve C_0 is either smooth or has at worst nodal or cusp singularities.

As in the point before, we study a general pencil of stable sheaves limiting to a general sheaf whose Fitting support is $C = 2C_0$. We take as a general point the sheaf $j_* \mathcal{O}_{2C_0}$, which can be checked being stable in $M_{(0,2H,-4)}$.

First of all, we consider a general pencil $\mathcal{Q} \to \mathbb{P}^1$ of conics in $\mathbb{P}^2$, whose central fibre $\mathcal{Q}_0 = 2L_0$ is a double line. (For a local toy example we can think at $V(ty - x^2) \subset \mathbb{A}^3$.) The total space $\mathcal{Q}$ is singular at the base locus of the pencil, which consists of two points (counted with multiplicity 2). In order to be able to consider a section, we pass to a double cover $\mathcal{Q}'$. Notice that now $\mathcal{Q}'$ is singular along L_0. By a local computation, one can see that the normalisation $\widetilde{\mathcal{Q}}$ of $\mathcal{Q}$ is smooth, and the natural projection $\widetilde{\mathcal{Q}} \to \mathbb{P}^1$ has a section. Notice that $\widetilde{\mathcal{Q}}_0 = L_0$.

Let us take the pullback via the ramified double cover $f \colon S \to \mathbb{P}^2$ associated to the degree 2 polarisation H of S. This gives a pencil $\widetilde{\mathcal{C}} \to \mathbb{P}^1$ with central fibre $\widetilde{C}_0 = C_0$ and having a bi-section Z. If $p \colon \widetilde{\mathcal{C}} \to \mathcal{C}$ is the normalisation morphism and $j \colon \mathcal{C} \to S \times \mathbb{P}^1$ is the natural embedding, then we consider the sheaf

$$\mathcal{E} = j_* p_* \mathcal{O}_{\widetilde{\mathcal{C}}}(Z),$$

which is flat over $\mathbb{P}^1$. The main remark is that for general $t \neq 0$, the fibre $\mathcal{E}_t = \varphi(j_{t*} \mathcal{O}_{\widetilde{C}_t})$, hence if $\tilde{\varphi}$ extends, it must hold $\tilde{\varphi}(j_{0*} \mathcal{O}_{2C_0}) = \mathcal{E}_0$.

We claim that $\mathcal{E}_0$ is strictly semistable. First of all, we remark that $Z_0 = K_{C_0}$ is the canonical divisor of $\widetilde{C}_0 = C_0$. Moreover, $\mathcal{E}_0 = j_{0*} p_{0*} \mathcal{O}_{C_0}(K_{C_0})$, where $p_{0*} \mathcal{O}_{C_0}(K_{C_0})$ is a rank 2 vector bundle on C_0 whose degree is 0. Since $h^0(C_0, p_{0*} \mathcal{O}_{C_0}(K_{C_0})) = 2$, any section defines an inclusion $\mathcal{O}_{C_0} \to p_{0*} \mathcal{O}_{C_0}(K_{C_0})$ destabilising it. Notice that the cokernel N of the inclusion is isomorphic to $\mathcal{O}_{C_0}$. Now, let $L \subset p_{0*} \mathcal{O}_{C_0}(K_{C_0})$ be any subsheaf of positive degree. Without loss of generality, we can suppose that L is a line bundle. Then the composition with the map $p_{0*} \mathcal{O}_{C_0}(K_{C_0}) \to N = \mathcal{O}_{C_0}$ gives a map $L \to N = \mathcal{O}_{C_0}$ that must be zero. It would follow that $L \subset \mathcal{O}_{C_0}$, which is absurd, and so $p_{0*} \mathcal{O}_{C_0}(K_{C_0})$ is semistable.

As before, if φ were regular at these points, then $\mathcal{F}_0$ would be mapped to the S-equivalence class of $\mathcal{E}_0$ as constructed above, which would therefore not be injective. On the other hand, this time $\tilde{\varphi}$ cannot extend either. In fact, the S-equivalence class of $\mathcal{E}_0$ is a $\mathbb{P}^1$ of extensions, while there is a $\mathbb{P}^3$ of directions where we can take the pencil limiting to $j_* \mathcal{O}_{2C_0}$. Therefore $\tilde{\varphi}$ would not be injective at these points again.

$\square$

In the following we denote by $M_\Delta \subset M_{(0,2H,-4)}$ the locus of sheaves whose Fitting support is a non-reduced curve, and with $\widetilde{M}_\Delta \subset \widetilde{M}_{(0,2H,-4)}$ its strict transform.

Remark 4.13. As in Section 4.5, it is known that M_Δ has two irreducible components, parametrising sheaves whose schematic support is a non-reduced curve of genus 5 or a reduced curve of genus 2, respectively. The locus parametrising sheaves supported on reduced curves is again a generic $\mathbb{P}^3$-bundle. On the other hand, as in Remark 4.11, one can see that the other component is not normal, but its normalisation is again a generic $\mathbb{P}^3$-bundle. Moreover, these two irreducible components intersects along a union of subvarieties isomorphic to $\mathbb{P}^2$. In particular M_Δ (and $\widetilde{M}_\Delta$) is uniruled and the class of the line ruling it is the same on both the irreducible components.

Corollary 4.14. *The indeterminacy locus of the map $\widetilde{\varphi}$ is $\widetilde{M}_\Delta$*

Proof. By Proposition 4.12, the indeterminacy locus of $\tilde{\varphi}$ is contained in $\widetilde{M}_\Delta \cup \widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is the exceptional divisor. We claim that it also contains the whole $\widetilde{M}_\Delta$. In fact, by item (3) of Proposition 4.12 it contains the irreducible component of $\widetilde{M}_\Delta$ corresponding to stable sheaves whose schematic support is the non-reduced curve. Since this locus is uniruled, and the line ruling it also rules the other component (cf. Remark 4.13), it follows that $\tilde{\varphi}$ cannot be defined on the whole $\widetilde{M}_\Delta$.

Now, since $\tilde{\varphi}$ is a birational morphism between irreducible holomorphic symplectic varieties, it is regular on the generic point of the exceptional divisor, and the corollary follows. □

Remark 4.15. If we denote by $\varphi' \colon M_{(0,2H,-2)} \to M_{(0,2H,0)}$ the analogous morphism, then the composition

$$\tilde{\varphi}' \circ \tilde{\varphi} \colon \widetilde{M}_{(0,2H,-4)} \to \widetilde{M}_{(0,2H,0)}$$

is regular and coincides with the morphism that sends any sheaf to its tensor with the polarisation H. In fact, if C is a smooth curve, then $2\mathfrak{g}_2^1(C) = H|_C$. Therefore, with an abuse of notation, we can write $\tilde{\varphi}^{-1} = \tilde{\varphi}' \otimes H^\vee$.

Now, if $U = \widetilde{M}_{(0,2H,-4)} \setminus \widetilde{M}_\Delta$, then $\tilde{\varphi}'$ must be regular on $\tilde{\varphi}(U) \subset \widetilde{M}_{(0,2H,-2)}$. Moreover, if there exists $V \supset \tilde{\varphi}(U)$ where $\tilde{\varphi}'$ is regular, then $\tilde{\varphi}'(V) \otimes H^\vee \supset U$, which is absurd. It follows that $\tilde{\varphi}'$ is regular exactly on $\tilde{\varphi}(U)$.

Since the morphisms $\tilde{\varphi}$ and $\tilde{\varphi}'$ are defined fibrewise (with respect to the lagrangian fibration induced by the Fitting morphism map), it follows that the indeterminacy locus of $\tilde{\varphi}'$, which is equal to the indeterminacy locus of $\tilde{\varphi}^{-1}$, coincides again with the locus of sheaves whose Fitting support is the non-reduced curve. As we have seen in Section 4.5, this locus has two irreducible components: the first one is a generic $\mathbb{P}^3$-bundle, while the second one has a normalisation that is generically a $\mathbb{P}^3$-bundle (see Remark 4.11). In particular, the wall divisor dual to the line ruling this locus is the divisor of divisibility 3 and square -24 described in Section 4.5.

5. THE KÄHLER CONE

Let S be a projective K3 surface, $v = 2w$ with $w^2 = 2$ and H is a v-generic polarisation. Consider the moduli space $M_v(S,H)$ and its desingularisation

$$\pi \colon \widetilde{M}_v(S,H) \to M_v(S,H),$$

and denote by $\widetilde{\Sigma}$ the exceptional divisor. Recall that

$$\pi^*\colon \mathrm{H}^2(M_v(S,H),\mathbb{Z}) \to \mathrm{H}^2(\widetilde{M}_v(S,H),\mathbb{Z})$$

is a Hodge isometric embedding whose orthogonal complement is generated by $\widetilde{\Sigma}$.

If D is an effective divisor on $\widetilde{M}_v(S,H)$, then $\pi(D) \subset M_v(S,H)$ is a Weil divisor. We put $E = \pi(D)$ or $E = 2\pi(D)$ if either $\pi(D)$ is Cartier or $2\pi(D)$ is Cartier (this is possible because $M_v(S,H)$ is either locally factorial or 2-factorial). We remark that E coincides with the lattice-theoretic projection of D into the sublattice $\widetilde{\Sigma}^\perp \cong \mathrm{H}^2(M_v(S,H),\mathbb{Z})$. More precisely $D = aE + k\widetilde{\Sigma}$, with $E \in \widetilde{\Sigma}^\perp \cong \mathrm{H}^2(M_v(S,H),\mathbb{Z})$ and $a, k \in \frac{1}{2}\mathbb{Z}$ such that $a + k \in \mathbb{Z}$ (see [PR14]). In general, if D is a divisor class in $\mathrm{H}^2(\widetilde{M}_v(S,H))$ (not necessary effective), we can still write $D = aE + k\widetilde{\Sigma}$, with E, a and k as before, and we refer to E as the orthogonal projection of D into $\mathrm{H}^2(M_v(S,H),\mathbb{Z}) \cong \widetilde{\Sigma}^\perp$.

Proposition 5.1. *In the setting above, suppose that $D \in \mathrm{H}^2(\widetilde{M}_v(S,H),\mathbb{Z})$ is a wall divisor. Then the corresponding $E \in \mathrm{H}^2(M_v(S,H),\mathbb{Z})$ is a multiple of one of the following:*

- *a divisor of square -2 and divisibility 1,*
- *a divisor of square -2 and divisibility 2,*
- *a divisor of square -4 and divisibility 1,*
- *a divisor of square -10 and divisibility 2.*

Proof. Let us first suppose that $D^\perp \cap \overline{\mathcal{BK}}_{\widetilde{M}_v(S,H)} \neq 0$. Consider the saturation T of the lattice generated by D and $\widetilde{\Sigma}$. This is a negative definite rank 2 lattice, and since $E \in T$, we have that $E^2 < 0$.

Now, since we have that $D^\perp \cap \overline{\mathcal{BK}}_{\widetilde{M}_v(S,H)} \neq 0$, there is a class l corresponding to an extremal curve R on a smooth and irreducible symplectic birational model of $\widetilde{M}_v(S,H)$, such that l lies in $T \otimes \mathbb{Q}$ and is not a multiple of $\widetilde{\Sigma}$. Here we are using the embedding $\mathrm{H}_2(\widetilde{M}_v(S,H),\mathbb{Z}) \subset \mathrm{H}^2(\widetilde{M}_v(S,H),\mathbb{Q})$ induced by the Beauville–Bogomolov–Fujiki form. If δ is the curve class dual to $\widetilde{\Sigma}$, then $T \otimes \mathbb{Q}$ is generated by δ and l. Moreover, up to replacing l by $-l$, we can suppose that l is effective (cf. [BHT15] and [Mon15]), so that both δ and l generate extremal rays of the Mori cone.

Since $\widetilde{M}_v(S,H)$ is projective, we can take a very general big and movable divisor P in $D^\perp \cap \widetilde{\Sigma}^\perp$ (the orthogonal complement to T has signature $(1, \rho - 3)$ in $\mathrm{Pic}(\widetilde{M}_v(S,H))$). Let $P' := P - \epsilon D - \eta\widetilde{\Sigma}$ for small ϵ and η: clearly, it is still a big divisor. Up to replacing P' with a multiple, we can suppose that P' is big and effective. We can also take P (and therefore also P' and the surface S) very general with respect to these properties, i.e. all negative divisors orthogonal to P are in T (and the Picard rank of S is one). Let us run the MMP for the pair $(\widetilde{M}_S, \mu P')$ (for μ small enough, the pair is klt, see [HT09, Remark 12]). By [LP16, Theorem 2.1], this MMP terminates and its termination does not depend on the order of the contractions. Therefore, as a first step we can contract the curve δ; this contraction is the morphism $\pi\colon \widetilde{M}_v(S,H) \to M_v(S,H)$. This produces the pair $(M_v(S,H), \mu\pi(P'))$, where $\pi(P') = P - \epsilon'E$ (here we are tacitly identifying P and $\pi(P)$ via the identification $\mathrm{H}^2(M_v(S,H),\mathbb{Z}) \cong \widetilde{\Sigma}^\perp$). Notice that E differs from $\pi(D)$ by a multiple, which is hidden in the constant ϵ'.

By construction, P is still orthogonal to the class l we associated to D at the beginning of the proof. Therefore we can run the MMP for the pair $(M_v(S, H), \mu P - \epsilon'' E)$, resulting in a divisorial contraction or in a flopping contraction. Birational modifications of these types have been studied and classified by Meachan and Zhang in [MZ16]. In particular, the centre of the modification E must be a multiple of one of the classes described in [MZ16, Proposition 5.2 and Theorem 5.3] (see Theorem 1.11).

If on the other hand we have $D^\perp \cap \overline{\mathcal{BK}}_{\widetilde{M}_v(S,H)} = 0$, by the Kawamata–Morrison cone conjecture for the Movable cone (see [Mar11, Section 6]) we can replace D with $f(D)$, where $f \in \mathrm{Mon}^2_{Hdg}(\widetilde{M}_v(S, H))$ is generated by monodromy reflections and the claim follows by applying the previous argument to the wall divisor $f(D)$.
$\square$

Remark 5.2. The statement of Proposition 5.1 actually holds for all very general K3 surfaces such that $\langle \widetilde{\Sigma}, D' \rangle$ is negative definite. Indeed, in this case there is a MMP contracting or flopping all the curves in this negative lattice, which in turn implies that the singular moduli space has some non generic stability conditions.

Proposition 5.3. *Let X be a manifold of OG10-type and let $D \in \mathrm{Pic}(X)$ be a wall divisor. Then one of the following is satisfied:*

(1) $D^2 = -2$ *and* $\mathrm{div}(D) = 1$,
(2) $D^2 = -4$ *and* $\mathrm{div}(D) = 1$,
(3) $D^2 = -6$ *and* $\mathrm{div}(D) = 3$,
(4) $D^2 = -24$ *and* $\mathrm{div}(D) = 3$.

Proof. Up to the monodromy action we can suppose that $X \cong \widetilde{M}_v(S, H)$ for a projective K3 surface S, a Mukai vector $v = 2w$ with $w^2 = 2$ and a v-generic polarization H. Let us explain this claim. First of all, if S is a sufficiently special K3 surface, then we can find an divisor D' on $\widetilde{M}_v(S, H)$ such that $D'^2 = D^2$ and $\mathrm{div} D' = \mathrm{div} D$. Now, if η is a marking of X and η' is a marking of $\widetilde{M}_v(S, H)$, then by Lemma 1.13 there exists an orientation preserving isometry sending $\eta(D)$ to $\eta(D')$. Since orientation preserving isometries are of monodromy type by Theorem 1.7, it follows that there exists a deformation of X to $\widetilde{M}_v(S, H)$ and a parallel transport operator sending D to D'. In particular, D' is a wall divisor with same degree and divisibility as D (see Theorem 1.10), and the claim follows.

Suppose that $\mathrm{div}(D) = 1$. Up to the monodromy action again, we can suppose that $(D, \widetilde{\Sigma}) = 0$ (cf. Lemma 1.13). By Proposition 5.1, this immediately implies $D^2 = -2$ or -4 as claimed.

Let us then suppose that $\mathrm{div}(D) = 3$. This time, up to the monodromy action, we can write $D = 3E + \widetilde{\Sigma}$ by Lemma 1.13, with $E \in \widetilde{\Sigma}^\perp$ primitive (and we can even take E inside $\mathrm{H}^2(S, \mathbb{Z}) \subset \mathrm{H}^2(M_v(S, H), \mathbb{Z})$). Proposition 5.1 now implies that one of the following holds: $E = 0$, $E^2 = -2$ or $E^2 = -4$. The first two cases correspond to the cases in our claim, let us exclude the third one.

Let us take a very general polarized K3 surface (S, H) of degree 2 and take the Mukai vector $v = (0, 2H, -2)$ (and H as the v-generic polarization). Let us take the two orthogonal generators $D_1 = (-2, H, 0)$ and $D_2 = (-2, H, -1)$ of $\mathrm{Pic}(M_v(S, H))$. Notice that D_2 has divisibility 2 in $\mathrm{H}^2(M_{(0,2H,-2)})$. We have $\mathrm{Pic}(\widetilde{M}_v(S, H)) = \langle D_1, \frac{D_2 + \widetilde{\Sigma}}{2}, \widetilde{\Sigma} \rangle$, where $\widetilde{\Sigma}$ is the exceptional divisor. The divisor class $W = 4D_1 +$

$5D_2$ has square -18 and $\mathrm{div}(W) = 2$ in $\mathrm{H}^2(M_v(S, H), \mathbb{Z})$. Notice that $\frac{W - \widetilde{\Sigma}}{2} \in$ $\mathrm{Pic}(\widetilde{M}_v(S, H))$. Now, suppose that divisors of square -42 and divisibility 3 are wall divisors. It would follow that $W' := 3\frac{W - \widetilde{\Sigma}}{2} + \widetilde{\Sigma}$ is a wall divisor such that $\langle W', \widetilde{\Sigma} \rangle$ is negative definite, hence the projection of W' inside $\widetilde{\Sigma}^\perp$ must be a multiple of one of the cases contained in Theorem 1.11 by Remark 5.2 and the proof of Proposition 5.1. However, this projection is clearly a multiple of W, which gives the desired contradiction. $\qquad\square$

Proposition 5.4. *Let X be a manifold of OG10-type and $D \in \mathrm{Pic}(X)$. Then D is a wall divisor if and only if one of the following holds:*

(1) *$D^2 = -2$ and $\mathrm{div}(D) = 1$,*
(2) *$D^2 = -6$ and $\mathrm{div}(D) = 3$,*
(3) *$D^2 = -4$ and $\mathrm{div}(D) = 1$,*
(4) *$D^2 = -24$ and $\mathrm{div}(D) = 3$.*

Moreover, there is a curve of class proportional to D covering a divisor in the first two cases, a codimension 5 rational subvariety in the third case and a codimension 3 subvariety in the last case.

Proof. By Proposition 5.3, the above cases are the only ones which can give wall divisors. By Proposition 3.1, the first two cases have a multiple which is stably prime exceptional, hence they are wall divisors and there is a curve ruling a divisor proportional to them by [Mar11, Section 5 and 6]. Notice that if a wall divisor has an associated curve covering a codimension k subvariety, any deformation of the curve (preserving its Hodge type) covers a codimension k subvariety (see [AV21, Theorem 1.6]). Moreover, the two subvarieties so obtained are always diffeomorphic. By Example 4.2 and Example 4.1, the third case is a wall divisor corresponding to a lagrangian $\mathbb{P}^5$. Finally the last case follows from Example 4.3. $\qquad\square$

Theorem 5.5. *Let X be a manifold of OG10-type. Then, the Kähler cone of X is one of the connected components of*

$$\mathcal{C}(X) \setminus \bigcup_{(0 > D^2 \geq -4)\ or\ (\mathrm{div}(D) = 3\ and\ 0 > D^2 \geq -24)} D^\perp.$$

Proof. Notice that if D is a divisor of non negative square, then $D^\perp \cap \mathcal{C}(X) = \emptyset$, so the relevant divisors for the claim are only those of negative square. By [Mon15], the Kähler cone is a connected component of

$$\mathcal{C}(X) \setminus \bigcup_{D \text{ wall divisor}} D^\perp.$$

By Proposition 5.4, wall divisors are divisors of square either -2 or -4 and divisors of divisibility 3 and square either -6 or -24, therefore the claim follows.

$\qquad\square$

6. An application to irreducible symplectic compactifications of intermediate jacobian fibrations

6.1. Generalities on intermediate jacobian fibrations and their irreducible symplectic compatifications.
Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold. The smooth linear sections $Y \subset V$ have a principally polarised intermediate Jacobian J_Y, and we denote by J_Y^k the torsor parametrising degree k cycles. Notice that,

up to adding multiples of the degree 3 hyperplane cycle h^2, and up to changing the sign of the cycles parametrised by J_Y^k, we can suppose that there are only two jacobians, namely $J_Y = J_Y^0$ and $J_Y^t = J_V^1$, up to a canonical isomorphism.

If $U \subset \mathbb{P} \operatorname{H}^0(V, \mathcal{O}_V(1))^*$ is the open subset parametrising smooth linear sections, then there are two fibrations $\mathcal{J}_U \to U$ and $\mathcal{J}_U^t \to U$ obtained by relativising the construction above. By [LSV16] and [Voi18], when V is general there exist smooth and symplectic compactifications

$$p_V \colon \operatorname{IJ}(V) \to \mathbb{P}^5 \qquad \text{and} \qquad p_V^t \colon \operatorname{IJ}^t(V) \to \mathbb{P}^5$$

of $\mathcal{J}_U \to U$ and $\mathcal{J}_U^1 \to U$, respectively. Both $\operatorname{IJ}(V)$ and $\operatorname{IJ}^t(V)$ are irreducible holomorphic symplectic manifolds of OG10-type, and p_V and p_V^t are lagrangian fibrations. More precisely, we want to remark that if $U_1 \subset \mathbb{P} \operatorname{H}^0(V, \mathcal{O}_V(1))^*$ is the open subset of linear sections with at worst one ordinary double point, then there is a natural partial compactification $\mathcal{J}_{U_1} \to U_1$ of $\mathcal{J}_U \to U$ and of which $\operatorname{IJ}(V) \to \mathbb{P}^5$ is a smooth and symplectic compactification (see for example [DM96] and [LSV16]). The same holds for the twisted case, and we denote by $\mathcal{J}_{U_1}^t \to U_1$ the corresponding partial compactification. The reason for introducing $\mathcal{J}_{U_1}$ and $\mathcal{J}_{U_1}^t$ is that the complements $\operatorname{IJ}(V) \setminus \mathcal{J}_{U_1}$ and $\operatorname{IJ}(V)^t \setminus \mathcal{J}_{U_1}^t$ have codimension bigger or equal to 2.

When the cubic fourfold becomes more special, Laza–Saccà–Voisin and Voisin's constructions do not necessarily produce a smooth compactification. We remark though that in [LSV16, Section 3.2] it is proved that the LSV compactification exists for a general Pfaffian cubic fourfold. Moreover, in this case the two varieties $\mathcal{J}_{U_1}$ and $\mathcal{J}_{U_1}^t$ are isomorphic (e.g. [Ono18, Example 4.3.6]), so that by construction ([Voi18]) also $\operatorname{IJ}^t(V)$ exists and is isomorphic to $\operatorname{IJ}(V)$.

In general, if $X \supset \mathcal{J}_{U_1}$ (resp. $X \supset \mathcal{J}_{U_1}^t$), we say that X is a *smooth and irreducible symplectic compactification* if X is an irreducible holomorphic symplectic manifold and there exists a regular morphism $X \to \mathbb{P}^5$ extending the natural fibration $\mathcal{J}_{U_1} \to U_1$ (resp. $\mathcal{J}_{U_1}^t \to U_1$). In particular we want that $X \to \mathbb{P}^5$ is a lagrangian fibration.

By [Sac20, Theorem 1.6] smooth and irreducible symplectic compactifications of $\mathcal{J}_{U_1}$ and $\mathcal{J}_{U_1}^t$ exist for every smooth cubic fourfold. The proof uses deep and recent results in the minimal model program theory. In particular, the existence of such compactifications is not constructive and a priori there may exist more compactifications of the same object (preserving the lagrangian structure). The main result of this section (see Theorem 6.6) deals with this unicity problem in the twisted case.

In the following we denote by X_V and X_V^t any smooth and irreducible symplectic compactification of $\mathcal{J}_{U_1}$ and $\mathcal{J}_{U_1}^t$. We remark that they are all irreducible holomorphic symplectic manifolds of OG10-type ([Sac20, Theorem 1.6]). Moreover, if X_V and X_V' are two such compactifications, then they are birational by construction.

We finish this introductory section by studying two distinguished divisor classes on the varieties X_V and X_V^t. Let $\mathcal{F}_V$ be the relative Fano variety of lines: it exists as a projective variety of dimension 7, fibred over $\mathbb{P}^5$. We notice that $\mathcal{F}$ is a generic $\mathbb{P}^3$-bundle over $F(V)$, the Fano variety of lines of V. Consider the rational morphism

$$(4) \qquad\qquad s^t \colon \mathcal{F}_V \times \mathcal{F}_V \dashrightarrow \mathcal{J}_{U_1}^t,$$

defined on the locus fibred over smooth linear sections by sending two disjoint lines l and l' to the (twisted) Abel–Jacobi invariant of their sum $l + l'$ (see [Voi02, Section 12.3.3] for the definition of the twisted Abel–Jacobi map via Deligne cohomology). By [Dru00, Théorème 1.4] (see also the proof of [Bea02, Corollary 6.4]), this map is birational, hence the closure in X_V^t of its image (with reduced scheme structure) defines a divisor $T_{X_V^t}$ in X_V^t, called relative twisted theta divisor.

Similarly, consider the rational morphism

(5) $$s\colon \mathcal{F}_V \times_V \mathcal{F}_V \dashrightarrow \mathcal{J}_{U_1},$$

defined on the locus fibred over smooth linear sections by sending two disjoint lines l and l' to the Abel–Jacobi invariant of the difference $l - l'$. By [CG72], the closure in X_V of its image (with reduced scheme structure) is a relative theta divisor, which we denote by T_{X_V}.

Finally, if $p_{X_V}\colon X_V \to \mathbb{P}^5$ and $p_{X_V^t}\colon X_V^t \to \mathbb{P}^5$ are the lagrangian fibrations, then we denote by $b_{X_V} = p_{X_V}^* \mathcal{O}_{\mathbb{P}^5}(1)$ and $b_{X_V^t} = p_{X_V^t}^* \mathcal{O}_{\mathbb{P}^5}(1)$ the respective classes.

Remark 6.1. Let $\mathcal{V} \to B$ be a family of smooth cubic fourfolds and let $q\colon \mathcal{X}_{\mathcal{V}} \to B$ be an associated family of compactified intermediate jacobians fibrations. Then the classes T_{X_V} and b_{X_V} extend to flat sections $T_{\mathcal{X}_{\mathcal{V}}}$ and $b_{\mathcal{X}_{\mathcal{V}}}$ of the local system $R^2 q_* \mathbb{Z}$ that remains of type $(1,1)$ on all the members of the family. The same holds for the twisted case.

Lemma 6.2. *Let V be a smooth cubic fourfold and X_V and X_V^t as above. Then*

$$P_{X_V} := \langle T_{X_V}, b_{X_V} \rangle = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$P_{X_V^t} := \langle T_{X_V^t}, b_{X_V^t} \rangle = \begin{pmatrix} -18 & 3 \\ 3 & 0 \end{pmatrix}.$$

Proof. We start by remarking that the special case in which V is general and $X_V = \mathrm{IJ}(V)$ is proved in [Ono20, Proposition 4.1], using [Sac20, Theorem 2] for the square of $T_{\mathrm{IJ}(V)}$.

If V is general and $X_V^t = \mathrm{IJ}^t(V)$, then we can deform V to a general Pfaffian cubic fourfold V_0. In this case we have already remarked that $\mathrm{IJ}^t(V) \cong \mathrm{IJ}(V)$; moreover, $T_{V_0}^t = 3 T_{V_0}$ (see proof of [Bea02, Corollary 6.4]), while $b_{V_0}^t = b_{V_0}$. Therefore, by Remark 6.1, the claim follows from the untwisted case[3].

Let now V be any smooth cubic fourfold and X_V any smooth and irreducible symplectic intermediate jacobian fibration. We can deform V to a general cubic fourfold V_0 such that the LSV variety $\mathrm{IJ}(V_0)$ exists. By Remark 6.1 again, the claim follows from the analogous claim for the variety $\mathrm{IJ}(V_0)$.

Finally, the case of varieties of the form X_V^t follows from the case $\mathrm{IJ}^t(V)$ as before. $\square$

6.2. **The second integral cohomology of X_V^t.** In this section we work with preferred varieties of the form X_V^t that are birational to a non-commutative moduli space of Bridgeland stable objects on the cubic fourfold V. We start by recalling which moduli space we consider.

_____

[3]The same construction already appeared in [Ono18, Section 4.5], but there was an initial step to prove a wrong result, namely the existence of a twisted theta divisor of divisibility 1.

Let V be a smooth cubic fourfold. We refer to Huybrechts' lecture notes [Huy19] for generalities on the derived category $\mathrm{D}^b(V)$ of V and the geometry of the corresponding K3 category $\mathcal{A}_V$. The Mukai lattice $\widetilde{\mathrm{H}}(\mathcal{A}_V, \mathbb{Z})$ contains a distinguished sublattice isometric to the A_2 lattice. This sublattice is generated by two classes λ_1 and λ_2 corresponding to the projection in $\mathcal{A}_V$ of the classes of the structure sheaves of a line and a point, respectively. In particular, $\lambda_i^2 = 2$ and $(\lambda_1, \lambda_2) = 1$ ([Huy19, Remark 1.17]). If $\lambda_0 = \lambda_1 + \lambda_2$, then Li, Pertusi and Zhao construct in [LPZ20] a Bridgeland stability condition τ such that the moduli space $M_{2\lambda_0}(V, \tau)$ of τ-semistable objects in $\mathcal{A}_V$ of class $2\lambda_0$ is a normal projective symplectic variety, and a symplectic desingularisation $\pi_V\colon \widetilde{M}_{2\lambda_0}(V, \tau)$ such that $\widetilde{M}_{2\lambda_0}(V, \tau)$ is an irreducible holomorphic symplectic manifold of OG10-type. Moreover, by [LPZ20, Theorem 1.3], $\widetilde{M}_{2\lambda_0}(V, \tau)$ is birational to X_V^t. In the following we abuse notation and simply write M_V for $M_{2\lambda_0}(V, \tau)$ and $\widetilde{M}_V$ for $\widetilde{M}_{2\lambda_0}(V, \tau)$.

Since M_V is a normal symplectic variety admitting an irreducible symplectic desingularisation and using a deformation argument as in [BLM+, Theorem 29.2] (see [LPZ20, Proposition 3.7]), one can prove the following statement.

Lemma 6.3 ([GGO])**.** $\mathrm{H}^2(M_V, \mathbb{Z})$ *has a pure Hodge structure of weight two induced by the injective pullback* $\pi_V^*\colon \mathrm{H}^2(M_V, \mathbb{Z}) \to \mathrm{H}^2(\widetilde{M}_V, \mathbb{Z})$. *Moreover, the Beauville–Bogomolov–Fujiki form on* $\mathrm{H}^2(\widetilde{M}_V, \mathbb{Z})$ *restricts to a non-degenerate bilinear form on* $\mathrm{H}^2(M_V, \mathbb{Z})$ *such that the latter is Hodge isometric to the lattice* $\lambda_0^\perp \subset \widetilde{\mathrm{H}}(\mathcal{A}_V, \mathbb{Z})$.

The Hodge isometry between $\lambda_0^\perp$ and $\mathrm{H}^2(M_V, \mathbb{Z})$ is constructed as in the commutative case by using a quasi-universal family $\mathcal{E}$ of objects parametrised by M_V. In particular, this produces a morphism

$$f_\mathcal{E}\colon \widetilde{\mathrm{H}}(\mathcal{A}_V, \mathbb{Z}) \longrightarrow \mathrm{H}^2(M_V, \mathbb{Z})$$

that restricts to the needed isometry on $\lambda_0^\perp$.

Proposition 6.4. *There is an isometric immersion*

$$\xi\colon \mathrm{H}^4(V, \mathbb{Z})_{\mathrm{prim}} \longrightarrow \mathrm{H}^2(\widetilde{M}_V, \mathbb{Z}).$$

whose image is orthogonal to a lattice isometric to $U(3)$, *where* U *is the unimodular hyperbolic plane.*

Proof. First of all, there is a Hodge inclusion $\mathrm{H}^4(V, \mathbb{Z})_{\mathrm{prim}} \subset \lambda_0^\perp$ (see [Huy19, Theorem 3.1]), so that the morphism ξ is obtained by composing the restriction of the morphism $f_\mathcal{E}$ above with the pullback by the desingularisation map $\pi_V\colon \widetilde{M}_V \to M_V$.

For the last claim, we start by remarking that the class of the exceptional divisor $\widetilde{\Sigma}_V$ of π_V belongs to $\mathrm{Im}(f_\xi)^\perp$ by construction. If α is a primitive generator of $\mathrm{H}^4(V, \mathbb{Z})_{\mathrm{prim}}^\perp$ in $\lambda_0^\perp$, then we denote by P the sublattice in $\mathrm{H}^2(\widetilde{M}_V, \mathbb{Z})$ generated by the image of α and the class of $\widetilde{\Sigma}_V$. Since both the construction of ξ and the desingularisation π_V are functorial in the cubic fourfold, the lattice P deforms to the very general cubic fourfold V. In this case $\mathrm{H}^4(V, \mathbb{Z})_{\mathrm{prim}} \cap \mathrm{H}^{2,2}(V, \mathbb{Z}) = 0$ and $\widetilde{M}_V$ is birational to $\mathrm{IJ}^t(V)$. It follows that $\mathrm{H}^2(\widetilde{M}_V, \mathbb{Z}) \cong \mathrm{H}^2(\mathrm{IJ}^t(V), \mathbb{Z})$ and the image of P under this Hodge isometry must be a sublattice of the lattice $P_{\mathrm{IJ}^t(V)}$. Since both P and $P_{\mathrm{IJ}^t(V)}$ have rank 2 and $P_{\mathrm{IJ}^t(V)} = U(3)$ by Lemma 6.2, the claim follows as soon as P contains a class of divisibility 3. On the other hand, the class of

$\widetilde{\Sigma}_V$ has divisibility 3 (and square -6): this can be seen, for example, by deforming the very general cubic fourfold to a cubic fourfold having an associated K3 surface (in the sense of Kuznetsov), so that $\widetilde{\Sigma}_V$ deforms to the exceptional divisor of a commutative moduli space. $\qquad\square$

Corollary 6.5. *Let V be a smooth cubic fourfold and X_V^t a smooth and irreducible symplectic intermediate jacobian fibration. Then the orthogonal complement $P_{W_V^t}^{\perp}$ in $\mathrm{H}^2(X_V^t, \mathbb{Z})$ is Hodge isometric to $\mathrm{H}^4(V, \mathbb{Z})_{\mathrm{prim}}$.*

Proof. By [LPZ20, Theorem 1.3], X_V^t is birational to $\widetilde{M}_V$ so that there is a Hodge isometry $\mathrm{H}^2(X_V^t, \mathbb{Z}) \cong \mathrm{H}^2(\widetilde{M}_V, \mathbb{Z})$. The claim follows from Proposition 6.4 and the last part of its proof. $\qquad\square$

6.3. **Uniqueness of the compactification.** Let V be a smooth cubic fourfold. As before we denote by $\mathcal{J}_{U_1}^t \to U_1$ the intermediate jacobian fibration over linear sections with at worst an ordinary double point. Let

$$p_{X_V^t} : X_V^t \longrightarrow \mathbb{P}^5$$

be a smooth and irreducible symplectic compactification of $\mathcal{J}_{U_1}^t \to U_1$, whose existence is guaranteed by [Sac20, Theorem 1.6]. The goal of this section is to investigate the uniqueness of such compactification, under one additional assumption on the fibres of $p_{X_V^t}$.

Theorem 6.6. *Let V be a smooth cubic fourfold and X_V^t a smooth and irreducible symplectic compactification as before. Assume that the fibres of $p_{X_V^t} : X_V^t \to \mathbb{P}^5$ are irreducible.*

Let $p_{Y_V^t} : Y_V^t \to \mathbb{P}^5$ be another smooth and irreducible symplectic compactification of $\mathcal{J}_{U_1}^t \to U_1$. Then $X_V^t \cong Y_V^t$ are isomorphic.

Proof. By hypothesis, since X_V^t and Y_V^t compactify the same space, they are birational. Therefore there is a Hodge isometry

$$\mathrm{H}^2(X_V^t, \mathbb{Z}) \cong \mathrm{H}^2(Y_V^t, \mathbb{Z})$$

that restricts to an isometry $P_{X_V^t} \cong P_{Y_V^t}$ (see Lemma 6.2 for the notation). More precisely, since the birational morphism between X_V^t and Y_V^t commutes with the lagrangian fibration by hypothesis, the classes $b_{X_V^t}$ and $b_{Y_V^t}$ are sent one in to the other; furthermore, since the twisted relative divisors $T_{X_V^t}$ and $T_{Y_V^t}$ are obtained by closure of the same divisor on $\mathcal{J}_{U_1}$, they are also sent one in to the other.

Now, notice that the divisor $b_{X_V^t}$ is nef and $T_{X_V^t}$ is relatively ample on U_1 by construction. Moreover, by [Voi18, Lemma 4.4] (see also item (1) of [Sac20, Proposition 3.1]), since the fibres of $p_{X_V^t} : X_V^t \to \mathbb{P}^5$ are irreducible by hypothesis, $T_{X_V^t}$ is relatively ample on the whole base $\mathbb{P}^5$. It follows that $T_{X_V^t} + cb_{X_V^t}$ is big and nef for $c \gg 0$.

Let us suppose that X_V^t is not isomorphic to Y_V^t. Then $T_{X_V^t} + cb_{X_V^t}$ cannot be ample, and moreover it does not intersect some curves in the indeterminacy locus of the birational morphism between X_V^t and Y_V^t. Therefore, by definition, there must exist a wall divisor orthogonal to $T_{X_V^t}$ and $b_{X_V^t}$. By Theorem 5.5, a wall divisor is a divisor satisfying one of the following:

(1) $D^2 = -2$ and $\mathrm{div}(D) = 1$,
(2) $D^2 = -6$ and $\mathrm{div}(D) = 3$,

(3) $D^2 = -24$ and $\mathrm{div}(D) = 3$,
(4) $D^2 = -4$ and $\mathrm{div}(D) = 1$.

Recall that $T_{X_V^t}$ and $b_{X_V^t}$ generate the lattice $P_{X_V^t}$ by definition, and that $P_{X_V^t}^\perp \cong$ $\mathrm{H}^4(V, \mathbb{Z})_{\mathrm{prim}}$ is a Hodge isometry by Corollary 6.5. Therefore

$$\mathrm{H}^4(V, \mathbb{Z})_{\mathrm{prim}} \cap \mathrm{H}^{2,2}(V, \mathbb{Z}) \neq 0$$

and so the cubic fourfold must be special in the sense of Hassett. More precisely, using the numerical characterisation in [Has00] of the divisors $\mathcal{C}_d$ parametrising special cubic fourfolds, the four cases above correspond, respectively, to the four situations below:

(1) chordal cubics (divisor $\mathcal{C}_2$);
(2) nodal cubics (divisor $\mathcal{C}_6$);
(3) cubics containing a plane (divisor $\mathcal{C}_8$);
(4) cubics containing a cubic scroll (divisor $\mathcal{C}_{12}$).

By [Bro18, Corollary 1.6], if the compactification X_V^t has irreducible fibres, then for every linear section $Y \subset V$ we have an equality $d(Y) = b_2(Y) - b_4(Y) = 0$. As already remarked in [Sac20, Remark 3.11], $d(Y) > 0$ if Y contains a plane; on the other hand, by [Dol16, Section 6], $d(Y) < 0$ if Y contains a rational cubic scroll. Since the first two cases in the list above correspond to singular cubics, these situations are all excluded and the theorem follows. $\qquad\square$

Remark 6.7. Notice that in the statement of Theorem 6.6 we do not assume that Y_V^t has irreducible fibres, but this must be true a posteriori.

6.4. **Proposition 5.3 revisited.** We conclude the paper with the following corollary of Theorem 6.6. The key argument of Proposition 5.3 was a lattice-theoretic argument which allowed us to conclude that divisors of square -42 and divisibility 3 are not wall divisors. Here we give a geometric proof of this claim. In [LSV16, Section 3.2], the authors prove that the intermediate jacobian construction works well for general Pfaffian cubic fourfolds and use this geometry to prove that their construction gives irreducible holomorphic symplectic manifolds deformation equivalent to O'Grady's tenfolds. This geometric construction is precisely what we will use.

Proposition 6.8. *Let X be a manifold of OG10-type and let $D \subset Pic(X)$ be a divisor such that $D^2 = -42$ and $\mathrm{div}(D) = 3$. Then D is not a wall divisor.*

Proof. As divisors with these discrete properties form a single monodromy orbit, it is enough to prove our claim on a well chosen X. Let V be a general Pfaffian cubic fourfold, and let $\mathrm{IJ}(V) \to \mathbb{P}^5$ be the compactification of its intermediate jacobian fibration. Recall that in this case $\mathrm{IJ}(V) \cong \mathrm{IJ}^t(V)$ (see beginning of Section 6.1). By [LSV16, Section 3.2, Theorem 4.20, Theorem 5.7], it is smooth. Therefore, the primitive cohomology of V embeds into $\mathrm{H}^2(\mathrm{IJ}(V), \mathbb{Z})$ by Lemma 6.4. By work of Hassett [Has00], $\mathrm{H}^{2,2}(V, \mathbb{Z}) = \langle h^2, d \rangle$ where $(h^2)^2 = 3$, $d^2 = 10$ and $(h, d) = 4$. It follows that $\mathrm{H}^{2,2}(V, \mathbb{Z})_{\mathrm{prim}} = \langle 3d - 4h \rangle$. Put $D = 3d - 4h$ and, by abuse of notation, let us denote with D also its image in $\mathrm{Pic}(\mathrm{IJ}(V))$ under the isomorphism in Lemma 6.4. Clearly, $D^2 = -42$ and $\mathrm{div}(D) = 3$. However, $D^\perp \subset \mathrm{Pic}(\mathrm{IJ}(V))$ is isometric to the hyperbolic plane $P_{\mathrm{IJ}(V)}$. As the isotropic class $b_{\mathrm{IJ}(V)}$ is nef and the relative theta divisor $T_{\mathrm{IJ}(V)}$ is relatively ample, it follows that $T_{\mathrm{IJ}(V)} + cb_{\mathrm{IJ}(V)}$ is big and nef for $c \gg 0$. Moreover, since there exists a unique compactification

for Pfaffian fourfolds (see Theorem 6.6), this class must be ample. However, D is orthogonal to this ample class, hence it cannot be a wall divisor. $\square$

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