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FANO VARIETIES OF K3 TYPE AND IHS MANIFOLDS

ENRICO FATIGHENTI AND GIOVANNI MONGARDI

ABSTRACT. We construct several new families of Fano varieties of K3 type. We give a geometrical explanation of the K3 structure and we link some of them to projective families of irreducible holomorphic symplectic manifolds.

1. INTRODUCTION

Fano varieties and Irreducible Holomorphic Symplectic manifolds (for short, IHS) are two of the most studied classes of varieties in algebraic geometry. They are very different in nature (for example, they have different Kodaira dimensions) and they are often studied using different tools. Fano varieties are at the core of birational geometry, while IHS manifolds (sometimes called hyperkähler when the context is more differential-geometric) can be considered as a higher dimensional analogue of K3 surfaces, with lattice theory as one of the most relevant operative tools.

One of the most important properties of Fano varieties is their *boundedness*: it is well known that in every dimension there exists a finite number of families of Fano varieties up to deformations. It is therefore natural to aim for a classification, but such a problem is currently out of reach. A complete answer is known when the dimension is up to three, see for example [IP99]. From dimension four onwards, only partial results and at best a few hundreds examples in each dimension are known.

On the contrary the main problem in the study of IHS is the lack of examples. Although no result of boundedness is known in general for IHS manifolds, the known deformation types include two series of examples found by Beauville for every even dimension and two sporadic examples in dimension 6 and 10, found by O’Grady. Even if we fix the deformation type and we look for *polarized* families (in analogy with the K3 surfaces case) the situation does not improve much: very few examples of projective families are known. A survey can be found for example in [Bea11].

The interplay between special classes of Fano varieties and IHS manifold is not new: an important example by Beauville and Donagi is the Fano variety of lines on a smooth cubic fourfold, which describes a maximal family of polarized IHS. We remark that this is not the unique IHS manifold that can be linked to a cubic fourfold, as the recent constructions of Lehn-Lehn-Sorger-van Straten, [LLSvS17] (an 8-fold of K3^[4]-type) and Laza-Saccà-Voisin, [LSV17] (example of OG10 manifold) highlight. The cubic fourfold is not the only Fano variety to which we can associate polarized families of IHS manifolds: this is indeed a common feature of a special subclass of Fano varieties, called *Fano varieties of K3 type* (FK3 for short) whose study is the central topic of this paper.

These Fano varieties of K3 type are, roughly speaking, Fano varieties whose Hodge theory contains one (or more) K3 structure as summand. The reader should refer to 1.2 for a precise definition. We are going to deal as well with Fano varieties with more than one K3 and in more generality CY structure, see for example 3.31, in the sense of definition 1.1.

Our main motivation for the study of the FK3 case is their relation with IHS manifolds. Indeed, a result of Kuznetsov and Markushevich in [KM09] shows that if \mathfrak{M} is a moduli space of stable or simple sheaves on X , then any form in $H^{n-q-2}(X, \Omega^{n-q})$ defines a closed 2-form in $H^0(\mathfrak{M}^{\text{smooth}}, \Omega^2)$. This is therefore a good starting point in the hunt for examples of IHS manifolds. In particular, let us mention the IHS manifolds linked to FK3 varieties e.g. to the Debarre-Voisin twentyfold hypersurface, or to a Gushel–Mukai fourfold, or to a section of a product of \mathbb{P}^3 , see [DV10], [DK18], [IM19].

Although FK3 are definitely easier to hunt than IHS manifolds, there are not many known examples in the literature. For example, as complete intersections in (weighted) projective spaces one finds

only the cubic fourfold, see [PS20]. More examples are found if one allows terminal and \mathbb{Q} -factorial singularities, see [FRZ19] but no new examples of IHS manifolds are produced anyway. In [FM18] we conjectured that even taking complete intersection in Grassmannian one does not get any new example other than a complete intersection with four linear hypersurfaces in the Grassmannian $\text{Gr}(2, 8)$ and the above mentioned examples.

This paper deals with the construction of examples of FK3 as zero locus of general global section of homogeneous vector bundles in Grassmannians or products of such. This is motivated by Kuchle's list, see [Ku95], of index 1 Fano fourfolds obtained in such a way, where few more interesting FK3 are found. Therefore the aim of this paper is twofold: construct new examples of Fano varieties of K3 type and construct examples of polarized families of IHS manifolds from our FK3. We can summarise our results as follows, cf. Thm 1.3 and 1.5.

Results of this paper. *We construct 23 new families of FK3 varieties as in Table 1, and we link some of them to projective families of IHS manifolds as in Table 2.*

We give now the key definitions and the general strategy of the paper.

1.1. Definitions and strategy. A Fano variety is a smooth projective variety X such that its anti-canonical bundle $-K_X$ is ample. For a Fano variety X the index ι_X is defined as the greatest positive integer that divides $-K_X$ as a class in the Picard group.

An IHS manifold Y is a compact Kahler manifold Y such that $\pi_1(Y) = \{*\}$ and $H^0(Y, \Omega_Y^2) \cong \mathbb{C} \cdot \sigma_Y$, with σ_Y everywhere non-degenerate. From these two conditions it follows that Y has even dimension and $K_Y \cong \mathcal{O}_Y$. We will only consider polarized (therefore projective) examples.

A Fano variety of CY type is a Fano variety with special Hodge-theoretical properties closely resembling those of a Calabi-Yau manifold.

Definition 1.1. *Let X be a smooth, projective n -dimensional Fano variety and j be a non-negative integer. The cohomology group $H^j(X, \mathbb{C}) \cong \bigoplus_{p+q=j} H^{p,q}(X)$ (with $j \geq k$) is said to be of k -Calabi-Yau (k -CY) type if*

- $h^{\frac{k+j}{2}, \frac{j-k}{2}} = 1$;
- $h^{p,q} = 0$, for all $p+q = j$, $p > \frac{k+j}{2}$.

X is said to be of k -CY type if there exists at least a positive j such that $H^j(X, \mathbb{C})$ is of k -CY. Similarly, X is said to be of (k_1, \dots, k_s) -CY type if the cohomology of X has different level CY structures in different weights.

Definition 1.2. *A smooth projective Fano variety X is a Fano variety of K3 type (FK3 for short) if it is a Fano variety of 2-CY type.*

Fano varieties of CY type were first introduced and studied by Iliev and Manivel in [IM15]. The authors focus on the case $k = 3$, adding moreover an extra condition on $H^1(T_X)$ (which we do not ask, since it would rule out already the cubic fourfold and many other interesting examples). They classify Fano varieties of 3-CY type that can be obtained by slicing homogeneous spaces with linear and quadratic equations. We remark that our definition is purely Hodge-theoretical, but there are deep links with the concept of CY subcategories, see for example [Kuz19]. In particular, constructing examples of Fano varieties of K3 and CY type might help in finding new playground for testing Kuznetsov's conjecture on rationality.

The main problem here is that in general translating the (Hodge-theoretical) requirement of being of K3 type into algebraic conditions is not easy. Using some tools that we developed in [FM18] we were anyway able to find some numerical condition useful to produce examples of FK3, see Construction Method 1. Unfortunately the conditions in 1 are still too general for replicating a classification-type argument. However, 1 has the advantage of highlighting the connection between FK3 and *central* Fano varieties, that is Fano varieties of Hodge level 0 for all cohomology group, see Subsection 2.2 for a definition. It would therefore be interesting to classify such varieties.

1.2. How we subdivide the examples. We first write down the list of examples that we have found. Later on in the paper we will explain the numerology behind our list, and give a detailed geometrical description of our examples. Our purpose is twofold. Indeed to a Fano variety of K3 type we want to associate (whenever possible) both a K3 category and an IHS manifold. For the definition of K3 or CY (sub) category we follow [Kuz19]. Before doing this, we need to prove first that the families of Fano varieties that we consider are of K3 type. This is done usually with either Riemann-Roch type computations as for example in Lemma 3.10 or using our Griffiths ring-type construction as in Proposition 3.13, or via a Borel-Bott-Weil computation, as in Proposition 3.27. In particular we divide our list into three distinct blocks. We say that a FK3 X is of *blow-up* type (**B**) if there exists a pair (Y, S) , with $S \subset Y$, Y Fano variety, S K3 surface such that $X \cong \text{Bl}_S Y$. Examples of this type are already included in Küchle's list [Ku95], called *c7* and *d3*. We say that a FK3 X is of *Mukai type* (**M**) if we can reduce systematically the study of its derived category to Mukai's classification of Fano threefolds. We say that a FK3 X is *sporadic* (**S**) if it does not fall in one of the two previous categories. We collect all our list of examples of FK3 in Table 1.

For FK3 of blow-up and Mukai type the question on the existence of a K3-subcategory admits always a positive answer. This is the content of Propositions 2.3, 2.5 and Theorem 2.4. However the question of existence of an IHS manifold linked to any FK3 is far from being answered. We give an example in Proposition 3.12. For the FK3 of sporadic type, we do not have any information a priori. For all of them the question on the existence of a K3-subcategory is open, and we have to cook up ad-hoc methods even to show that they are of K3 type (in the Hodge theoretical sense). Here as well there is no easy answer from the IHS viewpoint. A new construction is given for example in Proposition 3.17. Special attention must be placed upon examples (**S6**) and (**S7**). Indeed they are cut by irreducible vector bundles which are not linear. We observe as well the appearance of mixed structures of (2,3)-CY type. The last part of the paper is devoted to the study of these varieties. The results about IHS manifolds are collected in Table 2. We point out that we believe that to any of the example in Table 1 we will eventually be able to construct an example of polarized IHS manifolds. We added in both our tables two examples found independently by Iliev and Manivel in [IM19], while our work was still in the very early stage. These are the families **B1** and **S3**. Although they were already known we decided to include them anyway in our list, since they fit perfectly in our pattern.

We highlight now the main results and the structure of this paper.

1.3. Results and Structure. This paper is devoted to the construction of a meaningful set of examples of Fano varieties of K3 type. We mainly exploit our condition in 1, coming from a similar analysis to the one we carried out in [FM18]. Our main result can be summarised in

Theorem 1.3. *There exist 23 examples of families of Fano varieties of K3 type obtained as zero loci of general global section of homogeneous vector bundles over Grassmannians or products of such. These Fano varieties have dimension $4 \leq n \leq 20$, Picard rank $1 \leq \rho_X \leq 3$ and index $\frac{n-1}{2} \leq \iota_X \leq \frac{n}{2}$.*

See Table 1 for the list of these Fano varieties. For each of these Fano varieties, we first needed to prove that they are of K3 type. We either explain geometrically in a systematic way (whenever possible) the presence of a K3 structure (both from a Hodge-theoretical and derived category viewpoint) or we give an ad-hoc description for the sporadic cases. We point out that new examples may and will be discovered and analysed in a series of future works.

Some of the Fano varieties we analyse have new and interesting behaviours. We collect some of the results here.

Theorem 1.4. *There exist prime Fano varieties with multiple CY structures (see Proposition 3.29) and with mixed Calabi-Yau (2,3) structure, (see Proposition 3.31).*

To the best of our knowledge, these are the first examples known of prime Fano varieties with this property. The prime hypothesis eliminates the possibility for these CY structures to come from a blow-up, a projective bundle or other related constructions. We link some of these Fano varieties to projective families of IHS manifolds. Unfortunately, up to now we have only found new ways of

describing old examples, but we believe that a further extensive examination of our list could lead to new constructions. We collect our results here and in Table 2.

Theorem 1.5. *We show that the Hilbert square on a K3 surface of genus 8 is isomorphic to the zero locus of a certain bundle on $\mathrm{Gr}(4, 6) \times \mathrm{Gr}(2, 6)$, see Proposition 3.12. We show that the Debarre-Voisin IHS 4-folds are isomorphic to the space of special rational fourfolds on varieties of type $T_1(2, 10)$, see Proposition 3.30 and to the compactification of the space of $(\mathbb{P}^1)^3$ on a linear section of $S_2\mathrm{Gr}(3, 8)$, see Theorem 3.26.*

Plan of the paper. In **Section 2** we explain how our numerical condition creates the list and we explain some straightforward geometric tricks and a general strategy to attack these Fano varieties. In **Section 3** we perform a case-by-case analysis of the most interesting examples and we prove our main results, including the above theorem on IHS manifolds. We finish with some **Appendices**, where we describe three related cases we encountered: some extra Fano varieties of 3CY type, a trio of infinite series of Calabi–Yau varieties and a Fano variety with an unexpected lack of a K3 structure.

Notation. With \mathcal{R} and \mathcal{Q} we denote (respectively) the rank k tautological and the rank $n - k$ quotient bundle on the Grassmannian of subspaces $\mathrm{Gr}(k, n)$. We fix the convention that $\mathcal{O}_G(1) = \det(\mathcal{Q}) = \det(\mathcal{R}^\vee)$. We will use the shorthand $X_{\mathcal{F}} \subset G$ to denote the zero locus of a general global section of the vector bundle \mathcal{F} over G . We denote with

- $S_i \mathrm{Gr}(k, n)$ the i -th symplectic Grassmannian, or simply $S\mathrm{Gr}(k, n)$ if $i = 1$.
- $\mathrm{OGr}(k, n)$ the orthogonal Grassmannian and \mathbb{S}_n denotes one of the two connected components of $\mathrm{OGr}(n, 2n)$ in its spinor embedding.
- $T_1(k, n)$ denotes the subvariety of $\mathrm{Gr}(k, n)$ cut by the zero locus of a general three-form $\sigma \in \wedge^3 V_n^\vee$. Whenever the ambient Grassmannian is fixed and there is no danger of confusion, we will sometimes in proofs shorten $T_1(k, n)$ with T_1 to help the readability.

The notation $H_{\mathrm{van}}^n(X)$ (and similarly for the (p, q) part) will denote the vanishing subspace of the cohomology group, see [Voi02, 2.27] for a definition. If X and Y are smooth projective varieties we will use the shorthand $D^b(X) \hookrightarrow D^b(Y)$ to mean that one can construct a semiorthogonal decomposition for $D^b(Y)$ where $D^b(X)$ appears as one of the factors, up to a fully faithful functor. The notation S_g means a K3 surface of genus g . With Q_k we indicate the k -dimensional quadric hypersurface.

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2. THE QUEST FOR EXAMPLES

2.1. Tables of FK3 and IHS. In the tables below we collect the results about FK3 and IHS in a schematic way. We give a few extra data useful for better comprehension here, and in the rest of the paper.

- $S_i \mathrm{Gr}(k, n)$ can be characterised as $X_{(\wedge^2 \mathcal{R}^\vee)^{\oplus i}} \subset \mathrm{Gr}(k, n)$.

- In the table, $\text{HSGr}(3, 6)$ will denote a linear section of $\text{SGr}(3, 6)$.
- According to k , $\text{T}_1(k, n)$ can be represented as the zero locus of a general global section of a different vector bundle. For example the bundle is $\mathcal{Q}^\vee(1)$ for $k = 2$, $\mathcal{O}(1)$ if $k = 3$ and $\Lambda^3 \mathcal{R}^\vee$ when $k = 4$.
- $\text{T}_1(2, 7)$ is classically known in the literature as G_2 , and we will refer to it as such.
- We use $X_1 \subset G$ to denote a linear section of the variety G (and similarly for higher degree or multidegree). Whenever there might be ambiguity or we want to emphasize the choice of the linear subspace we might write X_H .

no.	$X \subset Y$	$\dim X$	ι_X	ρ_X	Comments
B1	$X_{(2,1,1)} \subset \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$	4	1	3	$X \cong \text{Bl}_{S_7}(\mathbb{P}^3 \times \mathbb{P}^1)$
B2	$X_{(2,1)} \subset \text{Gr}(2, 4) \times \mathbb{P}^1$	4	1	2	$X \cong \text{Bl}_{S_5} \text{Gr}(2, 4)$
M1	$X_{(1,1,1)} \subset \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$	8	3	3	$D^b(S_3) \hookrightarrow D^b(X)$ [IM19, Section 4]
M2	$X_{(1,1,1)} \subset Q_3 \times \mathbb{P}^2 \times \mathbb{P}^2$	6	2	3	$D^b(S_4) \hookrightarrow D^b(X)$
M3	$X_{(1,1)} \subset \text{Gr}(2, 5) \times Q_5$	10	4	2	$D^b(S_6) \hookrightarrow D^b(X)$
M4	$X_{(1,1)} \subset \text{SGr}(2, 5) \times Q_4$	8	3	2	$D^b(S_6) \hookrightarrow D^b(X)$
M5	$X_{(1,1)} \subset \text{S}_2\text{Gr}(2, 5) \times Q_3$	6	2	2	$D^b(S_6) \hookrightarrow D^b(X)$
M6	$X_{(1,1)} \subset \mathbb{S}_5 \times \mathbb{P}^7$	16	7	2	$D^b(S_7) \hookrightarrow D^b(X)$
M7	$X_{(1,1)} \subset \text{Gr}(2, 6) \times \mathbb{P}^5$	12	5	2	$D^b(S_8) \hookrightarrow D^b(X)$
M8	$X_{(1,1)} \subset \text{SGr}(2, 6) \times \mathbb{P}^4$	10	4	2	$D^b(S_8) \hookrightarrow D^b(X)$
M9	$X_{(1,1)} \subset \text{S}_2\text{Gr}(2, 6) \times \mathbb{P}^3$	8	3	2	$D^b(S_8) \hookrightarrow D^b(X)$
M10	$X_{(1,1)} \subset \text{SGr}(3, 6) \times \mathbb{P}^3$	8	3	2	$D^b(S_9) \hookrightarrow D^b(X)$
M11	$X_{(1,1)} \subset \text{HSGr}(3, 6) \times \mathbb{P}^2$	6	2	2	$D^b(S_9) \hookrightarrow D^b(X)$
M12	$X_{(1,1)} \subset G_2 \times \mathbb{P}^2$	6	2	2	$D^b(S_{10}) \hookrightarrow D^b(X)$
M13	$X_{(1,1)} \subset \text{Gr}(2, 8) \times \mathbb{P}^3$	14	1	2	$D^b(S_3) \hookrightarrow D^b(X)$
S1	$X_{1^4} \subset \text{Gr}(2, 8)$	8	4	1	$D^b(S_3) \hookrightarrow D^b(X)$
S2	$X_1 \subset \text{OGr}(3, 8)$	8	3	2	$D^b(S_7) \hookrightarrow D^b(X)$
S3	$X_1 \subset \text{SGr}(3, 9)$	14	6	1	[IM19, Section 5]
S4	$X_1 \subset \text{S}_2\text{Gr}(3, 8)$	8	3	1	
S5	$X_1 \subset \text{T}_1(2, 9)$	6	2	1	
S6	$\text{T}_1(2, 10)$	8	3	1	$3 \times \text{K3}$ structure
S7	$X_1 \subset \text{T}_1(2, 10)$	7	2	1	$2 \times \text{K3}$ structure, $1 \times 3\text{CY}$
S8	$X_{1^j} \subset \text{T}_1(k, 10)$			1	invariants depending by k, j

TABLE 1. Fano of K3 type with invariants

no.	$X \subset Y$	IHS Z	Comments
M1	$X_{(1,1,1)} \subset \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$	[IM19, Section 4]	$Z \cong \text{Hilb}^2 S_3$
M7	$X_{(1,1)} \subset \text{Gr}(2, 6) \times \mathbb{P}^5$	Prop.3.12	$Z \cong \text{Hilb}^2 S_8$
S2	$X_1 \subset \text{OGr}(3, 8)$	Prop.3.17	$Z \cong S_7$
S3	$X_1 \subset \text{SGr}(3, 9)$	[IM19, Section 5]	$Z \cong Z_{DV}$
S4	$X_1 \subset \text{S}_2\text{Gr}(3, 8)$	Thm. 3.26	$Z \cong Z_{DV}$
S6	$\text{T}_1(2, 10)$	Prop. 3.30	$Z \cong Z_{DV}$

TABLE 2. Projective families of IHS linked to FK3

2.2. What are we looking for? Many of the examples in the above table are obtained by chasing up the same numerology. Indeed from arguments similar to the one used in [FM18] one can come up with a numerical criterion (cf. [IM15] and [Kuz19] for comparison and similar criteria). For a smooth projective variety we define the *level* (lv) of $H^j(X, \mathbb{C})$ as the largest difference $|p - q|$ for which $H^{p,q}(X) \neq 0$, with $p + q = j$. It is obvious that $\text{lv}(H^j(X, \mathbb{C})) \leq \text{wt}(H^j(X, \mathbb{C})) \leq \dim X$. For a Fano variety by Kodaira vanishing the first inequality is always strict. For example, if X is a Fano of dimension n , then $\text{lv}(H^n(X, \mathbb{C})) \leq \dim X - 2$. Moreover we say that a variety X is *central* if all of its H^j have level zero, or equivalently if $h^{p,q}(X) = 0$ for $p \neq q$.

Method 1. *Let Y be a smooth projective Fano variety of dimension $2t + 1$ and index ι_Y . Assume that t divides ι_Y and that $\text{lv}(H^{2t+1}(Y)) \leq 1$. Then a generic $X \in |-\frac{1}{t}K_Y|$ is a Fano variety of K3 type, with the K3 type structure located in degree $2t$.*

Thanks to the above method, we can search for Fano varieties with the properties above. These are definitely satisfied when a Fano is a homogeneous (or quasi-homogeneous) variety. In particular we started by considering some varieties of this type, to see if any interesting example could be found.

In this case we can rewrite the condition 1 above and consider the positive integer m such that $\omega_G \cong \mathcal{O}_G(-m)$ and $D = \dim G$. The equations in 1 becomes

$$(1) \quad 2t + 1 = D \text{ and } at = m.$$

$\text{Gr}(k, k + l)$. For the Grassmannian $\text{Gr}(k, k + l)$ the dimension is $D = lk$ and the index equals $k + l$. First notice that D must be odd, and that we can suppose $l \neq 1$. The equations are $2t + 1 = kl$ and $at = k + l$, some a . Substituting we get $\frac{a(kl-1)}{2} = k + l$ and thus $akl = a + 2k + 2l$. Since $a \geq 1$ we have $kl \leq a + 2k + 2l$. It is easy to see that there are no solutions if $k \geq 5$, and for obvious reasons the cases $k = 2, 4$ are excluded. In the case case $k = 3$ substituting we get $l = \frac{a+6}{3a-2}$. This implies $a < 3$ for the previous number to be an integer. The case $a = 2$ gives an even dimensional Grassmannian, so we discard it. The case $a = 1$ corresponds to $G = \text{Gr}(3, 10)$. The associated FK3 is the Debarre-Voisin variety.

$\text{SGr}(k, k + l)$. The symplectic Grassmannian $\text{SGr}(k, k + l)$ has dimension $kl - \binom{k}{2}$ and index equal to $l + 1$. If we substitute this in the equation above and look for solutions we find as triple $(k, l, a) = (2, 3, 2), (3, 6, 1), (5, 3, 2), (10, 6, 1)$. However, if ω is a non-degenerate skew symmetric $(k + l) \times (k + l)$ matrix, there are no k -dimensional isotropic subspaces if $k > l$ and $k + l$ even. We can therefore discard the last two triples and we are left with $X_2 \subset \text{SGr}(2, 5)$ (Gushel-Mukai fourfold) and $X_1 \subset \text{SGr}(3, 9)$, which was already considered in [IM19, Section 6].

$\text{S}_2\text{Gr}(k, k + l)$. The bisymplectic Grassmannian $\text{S}_2\text{Gr}(k, k + l)$ has dimension $kl - k(k - 1)$ and index equal to $l - k + 2$. If we substitute in the equation above and look for solutions we find as triple $(k, l, a) = (3, 5, 1), (5, 5, 1)$. The second one can be identified with a (multi)-linear section of $(\mathbb{P}^1)^5$, see [Kuz15], the first one, an 8-fold linear section of $\text{S}_2\text{Gr}(3, 8)$ is new.

A similar computation can be done for the tri-symplectic Grassmannian $\text{S}_3\text{Gr}(k, k + l)$. This is relevant since two K3 surfaces by Mukai (genus 6 and genus 12) can be considered as (respectively) quadratic and linear section of it. However, no more examples have been found.

$\text{OGr}(k, k + l)$. The orthogonal Grassmannian $\text{OGr}(k, k + l)$ has dimension $kl - \binom{k+1}{2}$ and index $l - 1$ (with respect to the Plücker line bundle $\mathcal{O}_G(1)$, albeit non-irreducible in the Picard group). The only admissible triple is $(3, 5, 1)$. This is a linear section of the orthogonal Grassmannian $\text{OGr}(3, 8)$.

$Z_{Q^\vee(1)}$. This variety is the zero locus of a general global section of the bundle $Q^\vee(1)$ on $\text{Gr}(k, k + l)$. If $k = 2$, it is $\text{T}_1(k, k + l)$. It has dimension $l(k - 1)$ and index $k + 1$. There are two admissible triples, $(2, 7, 1), (6, 3, 1)$. However the second one can be identified with $X_1 \subset \text{SGr}(3, 9)$. The first one $X_1 \subset \text{T}_1(2, 9)$ is new. Notice that we find as well the generic K3 surface of genus 4 as $(2, 3, 3)$ since the zero locus of $Q^\vee(1)$ on $\text{Gr}(2, 5)$ is a quadric threefold. There are as well some FK3 obtained by $\text{T}_1(2, n)$. However, they do not fall in this pattern, and we will examine them separately.

Other varieties. We tried other bundles to produce varieties of K3 type, such as $\mathcal{R}^\vee(1)$ or the locus of $\mathrm{Sym}^2 \mathcal{R}^\vee \oplus \wedge^2 \mathcal{R}^\vee$ (the *orthosymplectic Grassmannian*). Even without no guarantee on the weight of the Hodge structure, our attempt was motivated by some examples in the list of Kuchle, see [Kuz15, Subsection 2.1]. However, we found no more new example.

Products. Products of projective spaces do produce a handful more of examples. One can easily see that no more than 5 projectives can be involved, by bounding t from above with the index of the smallest projective factor, and from below with the number of factors times the dimension of the smallest factor using (1). Here, the extremal case is $X_{(1^5)} \subset (\mathbb{P}^1)^5$. Other examples are $X_{(1,1,1)} \subset (\mathbb{P}^3)^3$, and $X_{(2,1,1)} \subset \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$. In the products of Grassmannians when $k > 1$, no further example is found. Indeed the index of a product of Grassmannians has index the $\mathrm{gcd}(k_i + l_i)$. Substituting in the equations, one first find that no more than two Grassmannians can be used, and only one of them can have $k > 1$. The possible cases are $X_{(1,1)} \subset \mathrm{Gr}(2, 6) \times \mathbb{P}^5$ and $X_{(2,1)} \subset \mathrm{Gr}(2, 4) \times \mathbb{P}^1$. Identical computations yield all the remaining cases.

2.2.1. *Remarks on our method.* We want to highlight with a couple of remarks how the method given in 1 could be turned into a Theorem.

Remark 2.1. *The condition in 1 is not necessary. In fact notable exceptions are (S1) (where the divisibility relation does not hold) and (S6), where there the decomposition in irreducibles of the bundle that cuts the variety has no linear factor (albeit the variety has the correct ratio between dimension and index), and moreover two K3 sub-Hodge structures are present, in degree 6 and 8.*

Remark 2.2. *As it is stated, the method 1 is not an exact result. To turn it into a Theorem one would need to impose specific cohomological vanishing in each case. Therefore what could be gained in accuracy would be lost in terms of general applicability. There could be ways of turning it into a statement or a conjecture. For example we could ask for Y to have a rectangular Lefschetz decomposition in the categorical sense. Or, whenever Y itself is cut by a section of an homogeneous vector bundle $\mathcal{F} = \bigoplus \mathcal{F}_i$ on $\mathrm{Gr}(k, n)$, we might ask that the slope $\mu(-\frac{1}{i}K_Y) > \mu(\mathcal{F}_i)$ for all i . However, for the purpose of the current paper, we prefer to leave it as is is, and we plan to formalise this statement in a future work.*

However not formal, Method 1 is a cheap and easy way to produce several candidates, which turn out to be all of the desired type. We believe that this condition will be useful as well in the future to produce new examples of Fano varieties of K3 and CY type.

2.2.2. *Further discussion and examples.* The list of FK3 in the tables has no presumption of being complete. The main problem is the condition on the level of Hodge theory of the ambient variety, which is quite hard to control. The first case to investigate is the one of complete intersections in homogeneous varieties. We conjectured in [FM18] that there are no more FK3 as complete intersection in $\mathrm{Gr}(k, n)$ other than the well-known cubic fourfold, the Gushel-Mukai fourfold, the Debarre-Voisin twentyfold hypersurface and a codimension four linear section of Grassmannian $\mathrm{Gr}(2, 8)$. We have not been able to prove this conjecture yet, but no counterexample has been found either. We tried as well hypersurfaces in other homogeneous varieties other than $\mathrm{Gr}(k, n)$, for example using the list of Konno in [Ko86], but none of them satisfied the above condition. For the complete intersections in homogeneous space, we do not have any reasonable conjecture. Atanas Iliev informed us that a FK3 variety can be obtained by taking a 6-codimensional linear section of the E_6 variety $\mathbb{O}\mathbb{P}^2$, but we have not pursued this direction yet. We did not include in this paper some of the well-known Fano varieties of K3 type, such as the cubic fourfolds, the Gushel-Mukai 4-folds and 6-folds, the Debarre-Voisin 20-fold, and the Kuchle 4-folds of K3 type (c_5 , c_7 and d_3 , following [Ku95]).

2.3. Geometric tools and tricks.

2.3.1. *A blow-up lemma.* We state here a blow up lemma. Although it merely descends from definitions, it is worth to recall it. It is worth to point out that a similar lemma is used in [Kuz15, Lemma 3.4 and Corollary 3.5].

Lemma 2.3. *Let $X = X_{(d,1)} \subset Z \times \mathbb{P}^1$. Then $X \cong Bl_S Z$, where S is the intersections of 2 divisors of degree d on Z .*

Proof. Let $\mathbb{P}^1 = \text{Proj}(\mathbb{C}[y_0, y_1])$ and $V^\vee \cong \mathbb{C}[y_0, y_1]_1$ (that is, homogeneous forms of degree 1). Denote by $W^\vee \cong H^0(\mathcal{O}_Z(d))$. X is given by definition by a choice of $\lambda \in W^\vee \otimes V^\vee$, or equivalently by a map (that we will still denote by λ) $\lambda : V \rightarrow W^\vee$. This map gives a 2-dimensional subspace of W^\vee , or equivalently a pencil of divisors in Z . The base locus of this pencil coincides with the S defined in the lemma. The (only) incidence equation for the blow up of Z in S is $y_0 f_d + y_1 g_d$ and this is of course the same equation defining X . \square

2.3.2. *Higher codimension case and Cayley trick(s).* The above blow-up lemma admits a higher-codimensional generalisation. Indeed, when X is the zero locus of a $(1, 1)$ divisor in $U \times \mathbb{P}^{r-1}$ (with the obvious generalisation if $\rho(U) > 1$) then X can be given either by an element of $W^\vee \otimes V_r^\vee$ or as a map

$$\lambda : V_r \rightarrow W^\vee.$$

If $r > 2$ we cannot identify X with any birational modification of the pair (U, S) , where S is the base locus of the above linear system. However X and S share a deep relation, known as the *Cayley trick*. More precisely the result is the following

Theorem 2.4 (Thm. 2.10 in [Orl06], Thm 2.4 in [KKLL17]). *Let $q : E \rightarrow U$ be a vector bundle of rank $r \geq 2$ over a smooth projective variety U and let $S = s^{-1}(0) \subset U$ denote the zero locus of a regular section $s \in H^0(U, E)$ such that $\dim S = \dim U - \text{rank } E$. Let $X = w^{-1}(0) \subset \mathbb{P}E^\vee$ be the zero locus of the section $w \in H^0(\mathbb{P}E^\vee, \mathcal{O}_{\mathbb{P}E^\vee}(1))$ determined by s under the natural isomorphism $H^0(U, E) \cong H^0(\mathbb{P}E^\vee, \mathcal{O}_{\mathbb{P}E^\vee}(1))$. Then we have the semiorthogonal decomposition*

$$D^b(X) = \langle q^* D^b(U), \dots, q^* D^b(U) \otimes_{\mathcal{O}_X} \mathcal{O}_X(r-2), D^b(S) \rangle.$$

When this happens, we will write $D^b(S) \hookrightarrow D^b(X)$. There is as well an (older) analogue Hodge-theoretic statement, cf. Prop. 4.3 in [Ko91], stating that the vanishing cohomologies of S and X are isomorphic up to a shift. When the hypotheses of the above Theorem are verified, this therefore proves at once that X is of K3 type. The Cayley trick can be generalised in the following way, using the formalism of Homological projective duality. For a concise survey of Homological projective duality, we refer to [Pe19, Section 1]. It is a generalization of classical projective duality, where the dual objects are a variety X with a map to a projective space $X \rightarrow \mathbb{P}(V)$ (which need not be an immersion) such that its derived category has a Lefschetz decomposition and a variety X^\vee with a map to the dual projective space $\mathbb{P}(V^\vee)$. The variety X^\vee , when it exists, can be thought of as a family of the non trivial subcategories of $X \times_{\mathbb{P}(V)} \mathbb{P}(L)$ as the hyperplane L varies.

Lemma 2.5. *Let Y_1 and Y_2 be a pair of varieties with Lefschetz decompositions and embedded in $\mathbb{P}(V)$. Let Z_H be the intersection of $Y_1 \times Y_2$ with a general $(1,1)$ -divisor H . Let f_H be the map that H naturally defines from $\mathbb{P}(V)$ to $\mathbb{P}(V^\vee)$. Let $X_H = Y_1 \cap f_H^{-1}(Y_2^\vee)$, where Y_2^\vee is the Homological Projective dual to Y_2 . Then $D(X_H) \hookrightarrow D(Z_H)$.*

Proof. Let $D(Y_2) = \langle A_0, A_1(1), \dots, A_m(m) \rangle$ be the given Lefschetz decomposition of Y_2 . The divisor H parametrizes, for every point of Y_1 , a hyperplane section of Y_2 , hence it defines a map $f_H : Y_1 \rightarrow \mathbb{P}(V^\vee)$. In this way, Z_H is identified with the pullback through f_H of the universal hyperplane section $\mathcal{Y}_2 \subset Y_2 \times \mathbb{P}(V^\vee)$. Now, by [Kuz14, Lemma 2.3] we have

$$D(\mathcal{Y}_2) = \langle D(Y_2^\vee), A_1(1) \boxtimes D(\mathbb{P}(V^\vee)), \dots, A_m(m) \boxtimes D(\mathbb{P}(V^\vee)) \rangle.$$

By applying base change [Kuz11, Thm 5.6] to the diagram

$$\begin{array}{ccc} Z_H & \longrightarrow & \mathcal{Y}_2 \\ \downarrow \iota & & \downarrow \pi_2 \\ Y_1 & \xrightarrow{f_H} & \mathbb{P}(V^\vee), \end{array}$$

we obtain:

$$D(Z_H) = \langle D(Y_2^\vee \times_{\mathbb{P}(V^\vee)} Y_1), A_1(1) \boxtimes D(Y_1), \dots, A_m(m) \boxtimes D(Y_1) \rangle.$$

And the variety in the first factor here is precisely $X_H = Y_2^\vee \times_{\mathbb{P}(V^\vee)} Y_1 = Y_1 \cap f_H^{-1}(Y_2)^\vee$. \square

3. CASE-BY-CASE ANALYSIS

3.1. Identifications. Before analysing in details the examples in our list, we want to eliminate some varieties that are well-known examples in disguise. We recall some results of Kuznetsov, that we conveniently bundle together. Recall that the variety $S_2\text{Gr}(k, n)$ is the *bisymplectic Grassmannian*. It can be thought either as the intersection of two symplectic Grassmannian $S\text{Gr}(k, n)$ inside $\text{Gr}(k, n)$ or as the zero locus over $\text{Gr}(k, n)$ of a general global section of the bundle $\Lambda^2 \mathcal{R}^\vee \oplus \Lambda^2 \mathcal{R}^\vee$. We will better describe this variety later in the paper.

Theorem 3.1 (Thm 3.1 and Cor. 3.5 in [Kuz15]). *The following hold:*

- *There is an isomorphism $S_2\text{Gr}(n, 2n) \cong (\mathbb{P}^1)^n$;*
- *The variety $X_{(1,1,1,1,1)} \subset (\mathbb{P}^1)^5$ is isomorphic to $W = \text{Bl}_S((\mathbb{P}^1)^4)$, where $S = S_{(1,1,1,1)^2}$ is a non-generic K3 surface of genus $g = 13$, given as the intersection of two divisors of multidegree $(1, 1, 1, 1)$.*

Some of the Fano of K3 type that we found in our search can be actually identified with the W above. For this reason they are not included in our main table. More precisely we have

Lemma 3.2. *Let W be the Fano of K3 type in [Kuz15] defined above. Then the following Fano of K3 type*

- $X_{(1,1,1,1,1)} \subset Q_2 \times Q_2 \times \mathbb{P}^1$;
- $X_{(1,1,1,1,1)} \subset S_2\text{Gr}(4, 8) \times \mathbb{P}^1$;
- $X_{(1,1,1,1,1)} \subset S_2\text{Gr}(3, 6) \times S_2\text{Gr}(2, 4)$;

are isomorphic to W .

Proof. The first case is obvious, since $Q_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$. For the other two cases, by definition and Kuznetsov's result $S_2\text{Gr}(n, 2n)$ coincides with $(\mathbb{P}^1)^n$. \square

There is one more identification between two descriptions.

Lemma 3.3. $X_{(1,1,1)} \subset \mathbb{S}_3 \times \mathbb{P}^1 \times \mathbb{P}^1 \cong X_{(2,1,1)} \subset \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. It follows from the well known identification $\mathbb{S}_3 \cong \mathbb{P}^3$, see for example [Kuz18, (2.2)]. The difference in the degree is explained since the line bundle giving the spinor embedding for \mathbb{S}_3 is the square root of one giving the Plücker embedding. \square

3.2. Blow-up and Mukai type. To prove that each of the variety of type M and B are of K3 type one can use the Cayley trick statement, as in Theorem 2.4. Indeed the (stronger) derived category statement implies the Hodge theoretical one, see [Add16] for a comparison of open conjectures regarding Hodge and derived invariants in our setting. Indeed this can be seen by writing down such a semiorthogonal decomposition as prescribed by 2.4 and then taking Hochschild homology. Alternatively one can use Riemann-Roch and standard exact sequences to compute the relevant Hodge numbers. We did these calculations as sanity checks for all our examples, however we believe it is neither worth nor interesting to list all of them, since they are quite similar. Therefore we will include

just one example, namely Proposition 3.6, where Theorem 2.4 does not apply in a straightforward way. For the families B1 and B2, Lemma 2.3 settles the matter.

In terms of construction of polarized families of IHS, we investigate another construction of the Hilbert scheme of points on a genus 8 K3 surface, see Proposition 3.12. We believe that each of the examples in our list of Fano could lead to similar constructions: this would be especially interesting, considering the lack of examples of polarized families of Hilbert schemes of points on K3 surfaces.

3.3. M3: a (different) computation in intersection theory. The variety M3 is $X_{(1,1)} \subset \text{Gr}(2, 5) \times Q_5$. It has dimension 10 and index 4. It is neither a blow up with a center in a K3 surface, nor we can apply the Cayley trick. However we can show that it is a Fano of K3 type using Proposition 2.5. Indeed we have

Lemma 3.4. *Let S_6 be a K3 surface of genus 6 in the Mukai model and X the variety M3 from the table. Then $D^b(S_6) \hookrightarrow D^b(X)$.*

Proof. It suffices to apply Lemma 2.5, since the Grassmannian $\text{Gr}(2, 5)$ is projectively self-dual. The intersection of the Grassmannian $\text{Gr}(2, 5)$ with a 5-dimensional quadric (or, equivalently, the intersection of $\text{Gr}(2, 5)$ with a quadric and 3 hyperplanes in its Plücker embedding) is a K3 surface of genus 6 and degree 10 by Mukai's classification. \square

The following Lemma allows us to pass from the derived categorical to the Hodge-theoretical statement.

Lemma 3.5. *X is of K3 type.*

Proof. By Hochschild-Konstant-Rosenberg isomorphism, cf. [Kuz16b, Theorem 1.16], for Y smooth and projective of dimension n and $\mathcal{T} := D^b(Y)$, one has

$$\text{HH}_\bullet(\mathcal{T}) := \bigoplus_{k=0}^n \text{HH}_k(\mathcal{T}) \cong \bigoplus_k \bigoplus_{p-q=k} H^{p,q}(Y) \cong H^\bullet(X, \mathbb{C}).$$

Moreover, by [Kuz09, Theorem 7.5], for any semiorthogonal decomposition $\mathcal{T} = \langle A_1, \dots, A_m \rangle$ one has $\text{HH}_\bullet(\mathcal{T}) \cong \bigoplus_i \text{HH}_\bullet(A_i)$. Moreover if E is an exceptional object, $\text{HH}_\bullet(E) \cong \text{HH}_0(E) \cong \mathbb{C}$. By Lemma 2.5, X (the Fano M3) admits a semiorthogonal decomposition whose first piece is $D^b(S_6)$, and the other objects are all exceptional. It follows that $\bigoplus_{k=0}^2 H^{p,q}(S_6) \subset \bigoplus_{k=0}^{10} H^{p,q}(X)$. Moreover each exceptional object contributes only to $H^{p,p}(X)$. It follows that X is of K3 type. \square

As an alternative method one could show that M3 is of K3 type using a lengthy (but rather standard) computation with long exact sequences and cohomological vanishings.

Proposition 3.6. *Let $X = X_{(1,1)} \subset \text{Gr}(2, 5) \times Q_5$. Then X is of K3 type.*

The proof of the above proposition can be split in two lemmas. The first one is a Chern class computation, the second one is essentially an application of Bott's theorem.

Lemma 3.7. *The topological Euler characteristic of X is $e(X) = 72$.*

Proof. This is a lengthy (but direct) exercise in intersection theory, and we will spare the details to the reader. Let us denote $Y = \text{Gr}(2, 5) \times Q_5$. Denote by $\alpha_1 = c_1(\mathcal{O}_Q(-1))$ and $\beta_1 = c_1(\mathcal{O}_G(-1))$. Denote by $\beta_2 = c_2(\mathcal{R})$. One has $H^4(\text{Gr}(2, 5), \mathbb{Z}) = \langle \beta_1^2, \beta_2 \rangle$. One easily compute $c(Q)$, $c(G)$ and $c(Y) = c(G)c(Q)$. In particular $c_{11}(Y) = -6\alpha_1^5\beta_1^6$ and

$$e(Y) = \int_Y -6\alpha_1^5\beta_1^6 = 60.$$

We then use the normal sequence associated to X

$$0 \rightarrow T_X \rightarrow TY|_X \rightarrow \mathcal{O}_X(1, 1) \rightarrow 0.$$

This implies $c(TY|_X) = c(X)(1 - \alpha_1 - \beta_1)$. We can compute recursively the Chern classes of X , with in particular

$$c_{10}(X) = (9\alpha_1^5\beta_1\beta_2^2 + 9\alpha_1^4\beta_1^2\beta_2^2)|_X.$$

To compute the restriction we evaluate against the class of X , and we have $c_{10}(X) \cdot X = 18\alpha_1^5\beta_1^2\beta_2^2$. Using the relation in $A(G)$ given by $2\beta_1^5 = 5\beta_1\beta_2^2$ we get

$$c_{10}(X) \cdot X = -\frac{2 \cdot 18}{5}\alpha_1^5\beta_1^6 = \frac{6}{5}e(Y) = 72.$$

□

Lemma 3.8. *For $0 \leq i \leq 3$ we have $h^{i,10-i}(X) = 0$. Moreover $h^{6,4}(X) = h^{4,6}(X) = 1$.*

Proof. As before let us denote $Y = \text{Gr}(2,5) \times Q_5$, and with $\mathcal{L} \cong \mathcal{O}_Y(1,1)$ (and its restriction to X as \mathcal{L}_X). We use the following two exact sequences

$$(2) \quad 0 \rightarrow \Omega_X^{k-1} \otimes \mathcal{L}^\vee \rightarrow \Omega_{Y|X}^k \rightarrow \Omega_X^k \rightarrow 0$$

and

$$(3) \quad 0 \rightarrow \Omega_Y^k \otimes \mathcal{L}^\vee \rightarrow \Omega_Y^k \rightarrow \Omega_{Y|X}^k \rightarrow 0,$$

possibly twisting by some positive multiple of \mathcal{L}^\vee when required. The computation is rather lengthy and technical, and we will skip most of the details. To find similar computations the reader can refer to, e.g., [FM18, Lemma 4.9]. For the results on the cohomological vanishings for both $\text{Gr}(2,5)$ and Q_5 one can consult for example [PW95], [Snow86]. The first vanishing $h^{0,10}(X)$ is obvious. Let us show the first non-obvious one, that is $h^{1,9}(X) = 0$. Consider the two sequences 2 and 3 above with $k = 1$. Using the Künneth formula one easily see that the cohomology of $\text{Gr}(2,5) \times Q_5$ is of level zero. Moreover from Kodaira vanishing and since $H^{10}(X, \mathcal{L}) \cong H^0(X, \mathcal{O}_X(-3, -3)) = 0$ one reduces to

$$0 \rightarrow H^9(\Omega_{Y|X}^1) \rightarrow H^9(\Omega_X^1) \rightarrow 0$$

and

$$0 \rightarrow H^9(\Omega_{Y|X}^1) \rightarrow H^{10}(\Omega_Y^1 \otimes \mathcal{L}^\vee) \rightarrow 0.$$

However, if we denote with π_1 (resp. π_2) the projection on $\text{Gr}(2,5)$ (resp. Q_5) we have $\Omega_Y^1 \cong \pi_1^*\Omega_{\text{Gr}(2,5)}^1 \oplus \pi_2^*\Omega_Q^1$, and from the Künneth formula for the box product and the well known vanishings for the twisted cohomologies of $\text{Gr}(2,5)$ and Q_5 we have

$$H^{10}(\Omega_Y^1 \otimes \mathcal{L}^\vee) \cong H^9(\Omega_{Y|X}^1) \cong H^9(\Omega_X^1) = 0.$$

For $h^{2,8}(X)$ we use the sequences 2 and 3 with $k = 2$ and $k = 1$ twisted by \mathcal{L}^\vee . Indeed one has from 2

$$0 \rightarrow H^8(\Omega_{Y|X}^2) \rightarrow H^8(\Omega_X^2) \rightarrow H^9(\Omega_X^1) \rightarrow H^9(\Omega_{Y|X}^2) \rightarrow 0.$$

The two external terms can be checked to be 0 using 3, again together with the Künneth formula and the usual vanishings (using the decomposition for Ω_Y^2). Using the twisted version of 2 and 3 we reduce to the isomorphism $H^8(\Omega_X^2) \cong H^{10}((\mathcal{L}_X^\vee)^{\otimes 2}) = 0$. The same argument works as well for $h^{3,7}(X) = 0$, where for $h^{4,6}(X)$ we get

$$H^6(\Omega_X^4) \cong H^{10}((\mathcal{L}_X^\vee)^{\otimes 4}) \cong H^0(\mathcal{O}_X) \cong \mathbb{C}.$$

□

The last Lemma is enough to prove that X is of K3 type. In fact by Lefschetz theorem on hyperplane section $h^{p,p}(X) = 0$ unless $p + q = 10$. By Künneth formula it is easy to compute directly these numbers. Moreover the two above Lemmas prove that the level of $H^{10}(X)$ is 2. The knowledge of $h^{4,6} = h^{6,4} = 1$ and the Euler characteristic from 3.7 allows us to explicitly compute all the Hodge numbers.

Corollary 3.9. *Suppose $p + q \neq 10$. The only non-zero Hodge numbers $h^{p,q}$ of X are*

$$h^{0,0} = h^{10,10} = 1, \quad h^{1,1} = h^{9,9} = 2, \quad h^{2,2} = h^{8,8} = 4, \quad h^{3,3} = h^{7,7} = 6, \quad h^{4,4} = h^{6,6} = 8.$$

For $p + q = 10$ the only non-zero Hodge numbers are

$$h^{6,4} = h^{4,6} = 1, \quad h^{5,5} = 28,$$

with moreover the dimension of the vanishing cohomology subspace $h_{\text{van}}^{5,5} = 19$.

3.4. From M7 to $S_8^{[2]}$. The 12-fold X_{M7} is given by the zero locus of a (1,1) section on $\text{Gr}(2,6) \times \mathbb{P}^5$. Let $S_8 = \text{Gr}(2,6) \cap H_1 \cap \dots \cap H_6$. Then S_8 is a general K3 surface of genus 8 in Mukai's model. From the Cayley trick argument one has that $D^b(S_8) \hookrightarrow D^b(X_{M7})$. On the Hodge-theoretical level indeed we have:

Lemma 3.10. *Let X_{M7} as above. Then X_{M7} is of K3 type with $h^{6,6} = 31$ and the vanishing subspace $h_{\text{van}}^{6,6} = 19$.*

Proof. Since $\text{Gr}(2,6) \times \mathbb{P}^5$ is a central variety, it is enough to compute the Euler characteristics $\chi(\Omega^i)$ for $i = 5, 6$. This can be done for example via Riemann-Roch or using Macaulay2. \square

As expected, we can associate to X_{M7} an IHS, which is linked to the genus 8 K3 surface. To do this, let Z be given by the zero locus of a general global section of the bundle $\Lambda^2 \mathcal{R}_{4,6}^\vee \otimes \mathcal{R}_{2,6}^\vee$ on $\text{Gr}(4,6) \times \text{Gr}(2,6)$. We have the following proposition.

Proposition 3.11. *Z is an IHS fourfold.*

Proof. Recall the formula for the first Chern class of a product $c_1(\Lambda^2 \mathcal{R}_{4,6}^\vee \otimes \mathcal{R}_{2,6}^\vee) = \text{rk}(\mathcal{R}_{2,6}^\vee) \cdot c_1(\Lambda^2 \mathcal{R}_{4,6}^\vee) + \text{rk}(\Lambda^2 \mathcal{R}_{4,6}^\vee) \cdot c_1(\mathcal{R}_{2,6}^\vee)$. By adjunction it follows that for a general section Z is a smooth fourfold with $c_1 = 0$. We compute now its holomorphic Euler characteristic $\chi(\mathcal{O}_Z)$. This can be done for example via a Riemann-Roch computation, since

$$\chi(\mathcal{O}_Z) = \frac{c_2^2 - c_4}{720}.$$

We will use a Macaulay2 code in order to speed up the calculation.

```
loadPackage "Schubert2"
k1=2, l1=4, k2=4, l2=2;
G26=flagBundle({k1,l1}, VariableNames=>{r1,q1});
(R1,Q1)=G26.Bundles;
V=abstractSheaf(G26, Rank=>6);
G46=flagBundle({k2,l2}, V, VariableNames=>{r2,q2});
(R2,Q2)=G46.Bundles;
p=G46.StructureMap;
R1G46=p^(dual R1);
F=R1G46**exteriorPower_2 dual R2;
Z=sectionZeroLocus F;
chi(OO_Z);
```

Running the previous code one verifies $\chi(\mathcal{O}_Z) = 3$. In particular the statement follows by simply applying Beauville-Bogomolov decomposition theorem. \square

The deformation type of Z can be shown to be the expected one as follows.

Proposition 3.12. *Z is isomorphic to $\text{Hilb}^2(S_8)$.*

Proof. Let $h \in \Lambda^2 V_6^* \otimes V_6^*$ defining X_{M7} . As above, we can consider h as a morphism

$$h : V_6 \rightarrow \bigwedge^2 V_6^*.$$

A point in $\text{Hilb}^2(S_8)$ is therefore given by a pair (u_1, u_2) , $u_i \in \Lambda^2 V_6$ on both of which h vanishes. Consider $W \subset \Lambda^2 V_6$ spanned by u_1, u_2 . Consider further the restricted morphism $\bar{h}^t : W \rightarrow V_6^\vee$. This has rank 2, and we can take $P = \text{Im}(\bar{h}^t)$. By construction h vanishes on the pair $(W, P) \in \text{Gr}(4, 6) \times \text{Gr}(2, 6)$, thus defining a point in Z . From this construction, it is clear that W determines P . Moreover, the map we constructed inside $\text{Gr}(4, 6)$ can be seen as the same map (after duality) which associates to $\text{Hilb}^2(S_8)$ a line in the pfaffian cubic fourfold, hence it is an isomorphism. \square

We point out the similarities between this construction and [KPS18, Proposition B.6.3]. Here it is proved how the variety of lines (resp. conics) of a smooth cubic threefold (resp. a generic Fano threefold of genus 8) is isomorphic to a section of the bundle $\Lambda^2 \mathcal{R}_{4,5}^\vee \otimes \mathcal{R}_{2,5}^\vee$ over $\text{Gr}(4, 5) \times \text{Gr}(2, 5)$. In turn, their proof can be modified to give an alternative proof of 3.12.

3.5. Sporadic examples. This subset of the list is the most interesting one. For each one of these Fano already proving that they are of K3 type requires an ad-hoc strategy. Our most interesting results comes indeed from this section: indeed we reinterpret the Debarre-Voisin IHS fourfold as moduli space of relevant objects on a Fano of K3 type in two different ways, as in Theorem 3.26 and Proposition 3.30. Moreover we produce the first examples of a Fano with multiple K3 structures, cf. Proposition 3.29 and with a mixed (2, 3) CY structure, cf. Proposition 3.31. We also give geometrical descriptions of many of the examples we consider, since we believe them to be a rich and beautiful sources of geometries.

3.6. S1: four codimensional linear section of $\text{Gr}(2, 8)$. We already considered this example in our previous work [FM18, Proposition 5.2], therefore we will not spend too much time on it. It is described in a surprisingly simple way as a codimensional 4 linear section of the Grassmannian $\text{Gr}(2, 8)$.

Proposition 3.13. *Let $X_{1,1,1,1} \subset \text{Gr}(2, 8)$ be given by a generic section of $\mathcal{O}_G(1)^{\oplus 4}$. Then X is an 8-fold of K3 type, with $h_{\text{van}}^{4,4}(X) = 19$.*

We remark that there is another FK3 closely related to S1. This is $X_{(1,1)} \subset \text{Gr}(2, 8) \times \mathbb{P}^3$. In our main table this is listed as M13. We chose this notation since, although there is no K3 in the Mukai model related, it shares many similarities with the other Fano in the M group. In particular one can apply directly Cayley trick to prove that this Fano is of K3 type.

As already remarked in our previous work the conjectural homological projective dual of $X_{1,1,1,1} \subset \text{Gr}(2, 8)$ is quartic K3 surface $S_3 \subset \mathbb{P}^3$. An embedding of the derived category of the quartic K3 inside the derived category of the above linear section is proved in [ST18], Thm 2.8.

We already conjectured in [FM18, Conjecture 5.3] that this complete intersection in $\text{Gr}(k, n)$ should be the only FK3 obtained in this way. We repeat the conjecture here for completeness.

Conjecture 3.14. *Let $X = X_{d_1, \dots, d_c} \subset \text{Gr}(k, n)$ be a Fano smooth complete intersection of even dimension. Then X is not of K3 type unless*

$$(\{d_i\}, k, n) = (\{3\}, 1, 6), (\{2, 1\}, 2, 5), (\{1, 1, 1, 1\}, 2, 8), (\{1\}, 3, 10).$$

3.7. S2: a K3 of genus 7 from $\text{OGr}(3, 8)$. This sporadic example is a linear section $X = \text{OGr}(3, 8) \cap H$ of the orthogonal Grassmannian $\text{OGr}(3, 8)$. It is worth to spend few words on the ambient variety. In general the orthogonal Grassmannian $\text{OGr}(n-1, 2n)$ behaves differently from $\text{OGr}(k, 2n)$, which for $k \neq n-1$ is a prime Fano variety. Indeed $\text{OGr}(n-1, 2n)$ can be realised as a \mathbb{P}^{n-1} bundle over each of the two connected components \mathbb{S}_n^1 and \mathbb{S}_n^2 of the maximal orthogonal Grassmannian $\text{OGr}(n, 2n)$ in the spinor embedding. In particular the Picard group of $\text{OGr}(n-1, 2n)$ has rank 2 with the Plücker line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{S}^1}(1) \boxtimes \mathcal{O}_{\mathbb{S}^2}(1)$ is very ample. $\text{OGr}(n-1, 2n)$ is non-degenerate in the Plücker embedding, and

$$H^0(\text{OGr}(n-1, 2n), \mathcal{L}) \cong \bigwedge^{n-1} V_{2n}^\vee.$$

With $X = X_1 \subset \text{OGr}(3, 8)$ in the introductory table we mean the zero locus of a generic global section of \mathcal{L} . Such X is an 8-fold of index $\iota = 3$. Since it is a linear section of a central variety, to compute its

Hodge numbers it suffices to compute the Euler characteristics $\chi(\Omega_X^i)$, together with the knowledge of the cohomology of $\text{OGr}(3, 8)$. A full computation by the means of Borel-Bott-Weil theorem, can be found in the PhD thesis of the first author, in Appendix A. We recall here the result.

Lemma 3.15 (cf. [Fa17], Proposition A.1.1). *X is a Fano 8-fold of K3 type with $h^{4,4}(X) = 24$, and its vanishing subspace of rank 19.*

We explain now a link between this 8-fold X and a genus 7 K3 surface. Recall from the work of Mukai that a such a generic K3 surface can be obtained by cutting \mathbb{S}_{10} with 8 hyperplanes. Here we use a different description for the K3 surface. Let $X \subset \text{OGr}(3, 8)$ defined by $V(\sigma)$, $\sigma \in H^0(\mathcal{L})$. Let \mathbb{S}_8 be (one of the two connected component of) the orthogonal Grassmannian $\text{OGr}(4, 8)$, denote with \mathcal{R} the restriction of the tautological bundle. Since σ can be seen as an element in $H^0(\mathbb{S}_8, \wedge^3 \mathcal{R}^\vee)$, which is as well isomorphic to $\wedge^3 V_8^\vee$ by Borel-Bott-Weil, we can denote by $S = V(\sigma) \subset \mathbb{S}_8$. It is easy to check that S is a K3 of genus 7 (notice that \mathbb{S}_8 is nothing but a 6-dimensional quadric hypersurface in disguise, either using triality or checking dimension and invariants). Such S is responsible for the interesting part of the derived category (and therefore the Hodge theory of X). Indeed we quote the following result of Ito-Miura-Okawa-Ueda. Denote π the restriction of the projection p from X to (one of the two) \mathbb{S}_8 .

Lemma 3.16 (Lemma 2.1 in [IMOU20]). *The morphism π is a \mathbb{P}^2 -bundle over $\mathbb{S}_8 \setminus S$ and a \mathbb{P}^3 -bundle over S , locally trivial in the Zariski topology.*

In turn we can use an adapted version of Orlov's blow-up formula to this case. This is indeed a generalisation of the Cayley trick. We borrow this result from [BFM19, Proposition 47]. For this reason, the proof will be omitted here.

First, in the notation above, denote by $\iota : S \subset \mathbb{S}_8$. The above Lemma is equivalent to the following commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{j} & X \\ p \downarrow & & \downarrow \pi \\ S & \xrightarrow{\iota} & \mathbb{S}_8, \end{array}$$

with F a smooth projective subvariety, $j : F \subset X$ of codimension $d = 4 + 2 - 3 = 3$ and a locally free sheaf \mathcal{F} of rank 4 on S such that $p : F \simeq \mathbb{P}_S(\mathcal{F}) \rightarrow S$. We denote by $\mathcal{O}_F(H)$ the relative ample bundle of p and we assume that there is a line bundle $\mathcal{O}_X(H)$ such that $\mathcal{O}_X(H)|_F \simeq \mathcal{O}_F(H)$ and that there is a vector bundle \mathcal{E} of rank d on X such that F is the zero locus of a general section of $\pi^* \mathcal{E} \otimes \mathcal{O}_X(-H)$.

We define the functors $\Phi_l : D^b(S) \rightarrow D^b(F)$ by the formula $\Phi_l(A) = j_*(p^* A \otimes \mathcal{O}(lH))$.

Proposition 3.17. *In the configuration above, Φ_l is fully faithful for any integer l , and there is a semiorthogonal decomposition:*

$$D^b(X) = \langle \Phi_{-1} D^b(S), \pi^* D^b(\mathbb{S}_8), \dots, \pi^* D^b(\mathbb{S}_8) \otimes \mathcal{O}_X(2H) \rangle$$

3.8. S4: bisymplectic Grassmannian $\text{S}_2\text{Gr}(3, 8)$ and Debarre-Voisin IHS. The variety $\text{S}_2\text{Gr}(k, n)$ is given by the vanishing of a global section of the bundle $\wedge^2 \mathcal{R}^\vee \oplus \wedge^2 \mathcal{R}^\vee$ on the Grassmannian $\text{Gr}(k, n)$. Equivalently, given a pencil $\lambda : \mathbb{C}^2 \rightarrow \wedge^2 V^\vee$ it parametrises those k -dimensional subspaces which are isotropic for all skew-forms in the pencil. This variety is studied by Kuznetsov in [Kuz15] in the case $k = n/2$ and by Benedetti in [Be18b] with a strong emphasis in the case $k = 2$. Let us recall some key facts of the construction. Assume that $n = 2m$ is even. To a general pencil λ are canonically associated m degenerate skew-forms $\{\lambda_1, \dots, \lambda_m\}$, given by the intersection between the line $L_\lambda \subset \mathbb{P}(\wedge^2 V^\vee)$ and the (Pfaffian) discriminant hypersurface D , corresponding to degenerate skew-forms. Denote by K_i the kernel of λ_i . The smoothness of S_2Gr is equivalent to the λ_i being distinct, and moreover we can decompose $V = K_1 \oplus \dots \oplus K_m$ as a direct sum.

Kuznetsov gives as well the canonical form for the pencil, expressing the two generators (up to dividing by 2) as

$$\omega_1 = x_{1,2} + x_{3,4} + \dots + x_{n-1,n}, \quad \omega_2 = a_1 x_{1,2} + a_2 x_{3,4} + \dots + a_m x_{n-1,n},$$

with the a_i pairwise distinct, and $x_{i,j} := x_i \wedge x_j$. This way, we can identify $K_1 := \langle e_1, e_2 \rangle$, $K_2 := \langle e_3, e_4 \rangle$ and so on. When $m = k$ one has $S_2\text{Gr}(k, 2k) \cong (\mathbb{P}^1)^k$, see the Theorem 3.1. When $m \neq k$ however we do not have such a nice description as a product. For $k = 2$ for example $S_2\text{Gr}(2, n)$ is an intersection of $\text{Gr}(2, n)$ with a linear subspace of codimension 2.

Let us now focus on the case $S_2\text{Gr}(3, 8)$. We compute first the cohomology of a linear section of $S_2\text{Gr}(3, 8)$. We first prove an auxiliary lemma.

Lemma 3.18. $S_2\text{Gr}(k, n)$ is a central variety, with $e(S_2\text{Gr}(k, n)) = 2^k \binom{n}{k}$.

Proof. There are many ways of proving this statement. One could for example use Borel-Bott-Weil, or a more conceptual argument as follows. In [Be18b, Proposition 2.10] it is proved that there is a torus $T \cong (\mathbb{C}^*)^n$ acting on $S_2\text{Gr}(k, n)$ with the fixed locus constituted only by $2^k \binom{n}{k}$ points. This implies, thanks to [CS79, Theorem 2] that the $S_2\text{Gr}(k, n)$ is a central variety, with $2^k \binom{n}{k}$ being its topological Euler characteristic. \square

Proposition 3.19. A linear section $X_1 = V(\sigma_1) \subset S_2\text{Gr}(3, 8)$ is of $K3$ type.

Proof. We already proved in Lemma 3.18 that $S_2\text{Gr}(3, 8)$ is a central variety. Lefschetz theorem on hyperplane section enables us to describe the cohomology of X except all the Hodge groups $h^{p,q}(X)$ with $p + q = 8$. We can determine these dimensions by computing the Euler characteristics of $\chi(\Omega_X^i)$. The latter can be computed via a direct but lengthy computation, and computer algebra systems as Macaulay2 can speed up everything. One has in particular

$$\begin{aligned} \chi(\Omega_X^1) &= \chi(\Omega_{S_2\text{Gr}(3,8)}^1) = 1 \\ \chi(\Omega_X^2) &= \chi(\Omega_{S_2\text{Gr}(3,8)}^2) = 2 \\ \chi(\Omega_X^3) &= \chi(\Omega_{S_2\text{Gr}(3,8)}^3) + 1 = 7 \\ \chi(\Omega_X^4) &= 26. \end{aligned}$$

\square

This gives as well all the Hodge numbers. We collect them in the next corollary for the reader's convenience.

Corollary 3.20. The only non-zero Hodge numbers $h^{p,q}$ of $S_2\text{Gr}(3, 8)$ are

$$h^{0,0} = h^{9,9} = 1, \quad h^{1,1} = h^{8,8} = 1, \quad h^{2,2} = h^{7,7} = 2, \quad h^{3,3} = h^{6,6} = 6, \quad h^{4,4} = h^{5,5} = 6.$$

Corollary 3.21. Suppose $p + q \neq 8$. The only non-zero Hodge numbers $h^{p,q}$ of X are

$$h^{0,0} = h^{8,8} = 1, \quad h^{1,1} = h^{7,7} = 1, \quad h^{2,2} = h^{6,6} = 2, \quad h^{3,3} = h^{5,5} = 6.$$

For $p + q = 8$ the only non-zero Hodge numbers are

$$h^{3,5} = h^{5,3} = 1, \quad h^{4,4} = 26,$$

with moreover $h_{\text{van}}^{4,4} = 20$.

We want now to associate to a Fano X of type S4 an IHS Z . To do this, at first notice that $S_2\text{Gr}(3, 8)$ is degenerate in the Plücker embedding in $\mathbb{P}(\wedge^3 V_8)$. It lies indeed in $\mathbb{P}(U)$, where

$$U := \ker(\varphi : \bigwedge^3 V_8 \xrightarrow{(\lrcorner_1, \lrcorner_2)} V_8 \oplus V_8),$$

where \lrcorner_i denotes the contraction with the 2-skew form ω_i . Equivalently, we have that $S_2\text{Gr}(3, 8)$ is defined by a general $\sigma_1 \in U^\vee$.

Consider now the Grassmannian $\text{Gr}(6, 8)$. Denote by $\bar{\omega}_i$ the contraction with the restriction of the two form $\omega_i|_W$ to a 6-space W . For the generic $[W] \in \text{Gr}(6, 8)$ the map

$$\bar{\varphi} : \bigwedge^3 W \xrightarrow{(\bar{\omega}_1, \bar{\omega}_2)} W \oplus W$$

remains surjective, since the rank of $\omega_1|_W$ and $\omega_2|_W$ is still maximal. However, when W is such that every element of the pencil restricted to such W has rank 4, then the above map is not surjective anymore. As a special example, one can take a subspace given by $x_1 = x_3 = 0$. Then for example the vector (e_5, de_5) , $d \neq 1$ is not in the image of $\bar{\varphi}$. To identify in general the locus D where $\bar{\varphi}$ is not surjective let us write (in the notation above) $V_8 = K_1 \oplus K_2 \oplus K_3 \oplus K_4$. We can then describe D as

$$D := \{W_6 \subset V_8 \mid \dim(W_6 \cap K_i) \geq 1, \forall i\}.$$

D is therefore isomorphic to a $\text{Gr}(2, 4)$ -bundle over $(\mathbb{P}^1)^4 \cong \text{S}_2\text{Gr}(4, 8)$, where the four dimensional space is the quotient of V by the intersections $W_6 \cap K_i$ (as the forms have rank 4, we cannot have $K_i \subset W_6$). Over D we have a cokernel sheaf \mathcal{G} of rank 4 on its support, given by the Kernel of the rank 4 map $W \rightarrow W^* \oplus W^*$. Summing up, we have the following result

Proposition 3.22. *On $\text{Gr}(6, 8)$ there is an exact sequence of sheaves*

$$0 \rightarrow F \rightarrow \bigwedge^3 \mathcal{R} \rightarrow \mathcal{R} \oplus \mathcal{R} \rightarrow \mathcal{G} \rightarrow 0.$$

Corollary 3.23. *F^\vee is a globally generated vector bundle of rank 8 and $H^0(F^\vee) = U^\vee$.*

Proof. Dually, there is a surjective morphism of sheaves $\bigwedge^3 \mathcal{R}^\vee \rightarrow F^\vee$, which is surjective on stalks. Hence, global sections of F^\vee which are images of global sections of $\bigwedge^3 \mathcal{R}^\vee$ are sufficient to generate stalks, so that F^\vee is globally generated. \square

Moreover, since \mathcal{G} is a torsion sheaf supported in codimension 4 we have the following corollary.

Corollary 3.24. *$c_1(F^\vee) = 8h$, where h is the class of the Plücker hyperplane.*

Proposition 3.25. *Let $Z \subset \text{Gr}(6, 8)$ defined by the zero locus of a general global section of the vector bundle F^\vee . Then Z is a fourfold with canonical class $\omega_Z \cong \mathcal{O}_Z$.*

Theorem 3.26. *Let Z as above, and let $Z_{DV} \subset \text{Gr}(6, 10)$ the Debarre-Voisin IHS. Then Z is isomorphic to Z_{DV} . Moreover, Z can be interpreted as (the compactification of) the space of $\text{S}_2\text{Gr}(3, 6) \cong (\mathbb{P}^1)^3$ inside $X_1 \subset \text{S}_2\text{Gr}(3, 8)$.*

Proof. With a non canonical choice of a two-space $\langle v, w \rangle = V_2 \subset V_{10}$, the three form ω defining Z_{DV} can be written as $\omega = \omega_8 + v^\vee \wedge \sigma_1 + w^\vee \wedge \sigma_2$, where ω_8 is a three form on an eight dimensional vector space V_8 and σ_i are two forms on the same space. The natural projection from $\mathbb{P}(V_{10})$ to $\mathbb{P}(V_8)$ induces a rational map π from $\text{Gr}(6, 10)$ to $\text{Gr}(6, 8)$. For this map, there are three kinds of six-spaces:

- Type 0 Six spaces which do not intersect the fixed two space V_2 .
- Type 1 Six spaces meeting the fixed V_2 in a line U_1 .
- Type 2 Six spaces containing the fixed V_2 .

By a dimension count and the genericity assumption on Z_{DV} , spaces of type 2 do not occur inside Z_{DV} . Spaces of type 1 are given by the Schubert cycle $\sigma_{3,0^5}(V_2)$, and inside Z_{DV} this is a curve of degree 132, as computed by Macaulay2, which is smooth since one can check that the Schubert cycle we use to obtain it is smooth as well (see e.g. [HP18, Section 2.2]). The blow up of Z_{DV} along this curve maps into a subvariety of $\text{Gr}(6, 8)$ given by six spaces where the three form ω_8 is $\sigma_1 \wedge t_1^\vee + \sigma_2 \wedge t_2^\vee$ for some vectors t_1, t_2 of the six space itself. Thus, the image of Z_{DV} is precisely the variety Z for the forms $\omega_8, \sigma_1, \sigma_2$. The local picture in the exceptional divisor is given by sending a six plane $U_1 \subset U_6$ to the set of all possible six planes in V_8 containing U_6/U_1 , which is a \mathbb{P}^2 . The image $\pi(U_6)$ of a six space $U_6 \in Z_{DV}$ contains three spaces parametrized by $X_1 \subset \text{S}_2\text{Gr}(3, 8)$ where the form ω_8 restricts to zero, hence also the two forms σ_1, σ_2 are zero. That is, a point of Z parametrizes a copy $\text{S}_2\text{Gr}(3, 6) \cong (\mathbb{P}^1)^3$

Corollary 3.28. *Let $X = T_1(2, 9) \cap H$ be a linear section of $T_1(2, 9)$. This is a Fano of K3 type with Hodge diamond*

$$\begin{array}{cccccccc}
 & & & & & & & 1 \\
 & & & & & & 0 & 0 \\
 & & & & & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 2 & 0 & 0 \\
 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
 0 & 0 & 0 & 1 & 22 & 1 & 0 & 0 \\
 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\
 & & 0 & 0 & 2 & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 1 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & 1
 \end{array}$$

The vanishing subspace is $h_{\text{van}}^{3,3}(X) = 20$. The holomorphic Euler characteristics for X are $\chi(\Omega_X^1) = -1$, $\chi(\Omega_X^2) = 1$, $\chi(\Omega_{T_1}^3) = -18$. Moreover the topological Euler characteristic $e_{\text{top}}(X) = 24$.

Proof. The Hodge numbers for X follow from those of $T_1(2, 9)$ together with the computations of $\chi(\Omega^i)$, which can be easily done a priori via Riemann-Roch and the help of computer algebra. \square

3.9.1. *Borel-Bott-Weil computation for $T_1(2, 9)$.* Borel-Bott-Weil theorem is a powerful tool for computing cohomologies of vector bundles on homogeneous spaces. Together with some well-known sequences it is often sufficient to compute Hodge numbers for varieties cut by general global sections of homogeneous vector bundles. Although rather long and involved, the procedure is mostly algorithmic. We will include the general setup (skipping most details for the sake of readability) in order to give the reader a toolbox for further computations. We refer to [Wey03] for the full picture on the subject.

General BBW strategy. Let $\text{Gr}(k, n)$ be the Grassmannian of k -dimensional subspaces of V_n . Consider two dominant weights $\alpha = (\alpha_1, \dots, \alpha_{n-k})$ and $\beta = (\beta_1, \dots, \beta_k)$ for the Schur functors Σ applied to \mathcal{Q} and \mathcal{R} and their concatenation $\gamma = (\gamma_1, \dots, \gamma_n)$. Let δ be the decreasing sequence $\delta = (n-1, \dots, 0)$ and consider $\gamma + \delta$. Write $\text{sort}(\gamma + \delta)$ for the sequence obtained by arranging the entries of $\gamma + \delta$ in non-increasing order, and define $\tilde{\gamma} = \text{sort}(\gamma + \delta) - \delta$. If $\gamma + \delta$ has repeated entries, then

$$H^i(\text{Gr}(k, n), \Sigma_\alpha \mathcal{Q} \otimes \Sigma_\beta \mathcal{R}) = 0$$

for all $i \geq 0$. Otherwise, writing l for the *number of disorders*, that is the number of pairs (i, j) with $1 \leq i < j \leq n$ and $\gamma_i - i < \gamma_j - j$ we have

$$H^l(\text{Gr}(k, n), \Sigma_\alpha \mathcal{Q} \otimes \Sigma_\beta \mathcal{R}) = \Sigma_{\tilde{\gamma}} V$$

and $H^i(\text{Gr}(k, n), \Sigma_\alpha \mathcal{Q} \otimes \Sigma_\beta \mathcal{R}) = 0$ for $i \neq l$. Let now $Z \subset \text{Gr}(k, n)$ be a variety which is the zero locus of a general section of a rank r globally generated vector bundle F^\vee . We have the Koszul complex for Z , which is indeed a resolution

$$(4) \quad 0 \rightarrow \det(F) \rightarrow \bigwedge^{r-1} F \rightarrow \dots \rightarrow F \rightarrow \mathcal{O}_G \rightarrow \mathcal{O}_Z \rightarrow 0.$$

If H is another globally generated vector bundle on $\text{Gr}(k, n)$ we can tensor the above sequence by H : we have the spectral sequence

$$\mathbf{E}_1^{-q,p} = H^p(\text{Gr}(k, n), H \otimes \bigwedge^q F) \Rightarrow H^{p-q}(Z, H|_Z),$$

if moreover both F and H are homogeneous we can compute all terms on the left by BBW formula. We can now compute the Hodge numbers for our X . Notice that the F in the Koszul complex above is the dual of bundle we start with. In this case it will be $\mathcal{Q}(-1)$.

The Hodge numbers $h^{1,i}(T_1(2,9))$. We apply the above formula together with the conormal sequence, which since $N_{T_1/\text{Gr}}^\vee \cong F$ becomes

$$0 \rightarrow F|_{T_1} \rightarrow \Omega_G^1|_{T_1} \rightarrow \Omega_{T_1}^1 \rightarrow 0.$$

We can compute the cohomologies of the first two bundles using the above strategy. $F|_X$ turns out to be acyclic, whereas the only non-zero cohomology of $\Omega_G^1|_{T_1}$ is $H^1(\Omega_G^1|_{T_1}) \cong H^1(\Omega_G^1) \cong \mathbb{C}$. It follows that the Hodge numbers $h^{1,i}(T_1) = 0$, $i \neq 1$ and $h^{1,1}(T_1) = 1$.

The Hodge numbers $h^{2,i}(T_1(2,9))$. In order to compute these other Hodge numbers we need to rise the conormal sequence to the second exterior power, that is

$$0 \rightarrow \text{Sym}^2 F|_{T_1} \rightarrow (F \otimes \Omega_G^1)|_{T_1} \rightarrow \Omega_G^2|_{T_1} \rightarrow \Omega_{T_1}^2 \rightarrow 0.$$

$\text{Sym}^2 F \otimes \wedge^i F$ is acyclic for $i \neq 7$. This can be checked using first the Littlewood-Richardson formula to determine the irreducible decomposition of each of these bundles, and then applying several iteration of the BBW formula. For $i = 7$ it is $\Sigma_{3,16} \mathcal{Q} \otimes \Sigma_{9,9} \mathcal{R}$ that has $H^{12}(\text{Sym}^2 F \otimes \wedge^7 F) \cong \mathbb{C}$ (and therefore $H^5(\text{Sym}^2 F|_{T_1}) \cong \mathbb{C}$). The bundle $\Omega^1 \otimes F \otimes \wedge^i F$ is acyclic for all i . The bundle $\Omega^2 \otimes \wedge^i F$ is not acyclic for $i = 0$ (and $H^2(\Omega_G^2|_{T_1}) \cong \mathbb{C}^2$) and for $i = 3$. Indeed in the case $i = 3$ its decomposition in irreducibles contains the summand $\Sigma_{3,3,3,2,2,1,1} \mathcal{Q} \otimes \Sigma_{7,5} \mathcal{R}$. This gives $H^6(\Omega^2 \otimes \wedge^3 F) = \mathbb{C}$. Putting all these data together one obtains $H^2(\Omega_{T_1}^2) = H^3(\Omega_{T_1}^2) \cong \mathbb{C}^2$ with the other Hodge numbers $h^{2,i} = 0$.

The Hodge numbers $h^{3,i}(T_1(2,9))$. By Riemann-Roch one gets $\chi(\Omega_{T_1}^3) = 2$. Thanks to the knowledge of $h^{i,3}(T_1)$ for $i \neq 3, 4$, this implies $h^{3,3}(T_1) = h^{4,3}(T_1)$. We use the third power of the conormal sequence, namely

$$0 \rightarrow \text{Sym}^3 F|_{T_1} \rightarrow (\Omega^1 \otimes \text{Sym}^2 F)|_{T_1} \rightarrow (\Omega^2 \otimes F)|_{T_1} \rightarrow \Omega_G^3|_{T_1} \rightarrow \Omega_{T_1}^3 \rightarrow 0.$$

One strategy is to split the sequence above in three short ones, namely

$$(5) \quad 0 \rightarrow \text{Sym}^3 F|_{T_1} \rightarrow (\Omega^1 \otimes \text{Sym}^2 F)|_{T_1} \rightarrow J_2 \rightarrow 0,$$

$$(6) \quad 0 \rightarrow J_2 \rightarrow (\Omega^2 \otimes F)|_{T_1} \rightarrow J_1 \rightarrow 0,$$

$$(7) \quad 0 \rightarrow J_1 \rightarrow \Omega_G^3|_{T_1} \rightarrow \Omega_{T_1}^3 \rightarrow 0.$$

The only cohomological contributions come from

- (a) $H^{12}(\text{Sym}^3 F \otimes \wedge^6 F) = \mathbb{C}^{81} \cong \text{End}(V_9) \cong \mathfrak{gl}(V_9)$;
- (b) $H^{12}(\text{Sym}^3 F \otimes \wedge^7 F) = \mathbb{C}^{84} \cong \wedge^3 V_9$;
- (c) $H^{13}(\Omega^1 \otimes \text{Sym}^2 F \otimes \wedge^7 F) = \mathbb{C} \cong H^6((\Omega^1 \otimes \text{Sym}^2 F)|_{T_1})$;
- (d) $H^6(\Omega^2 \otimes F \otimes \wedge^2 F) = \mathbb{C} \cong H^4((\Omega^2 \otimes F)|_{T_1})$;
- (e) $H^{10}(\Omega^2 \otimes F \otimes \wedge^5 F) = \mathbb{C} \cong H^5((\Omega^2 \otimes F)|_{T_1})$;
- (f) $H^3(\Omega^3) = \mathbb{C}^2 \cong H^3(\Omega_G^3|_{T_1})$;
- (g) $H^7(\Omega^3 \otimes \wedge^3 F) = \mathbb{C} \cong H^4(\Omega_G^3|_{T_1})$;
- (h) $H^{11}(\Omega^3 \otimes \wedge^6 F) = \mathbb{C} \cong H^5(\Omega_G^3|_{T_1})$.

Except in the case of (a) and (b) one can compute immediately the cohomology of the restriction of the bundles to T_1 . The only non obvious case is given by the exact sequence

$$0 \rightarrow H^5(\text{Sym}^3 F|_{T_1}) \rightarrow \bigwedge^3 V \xrightarrow{\phi_f} \text{End}(V_9) \rightarrow H^6(\text{Sym}^3 F|_{T_1}) \rightarrow 0.$$

The situation is analogous to ([KMM10], Appendix B). Indeed the dual of the map ϕ_f is the map $\varphi_f : \text{End}(V_9) \rightarrow \wedge^3 V_9^\vee$ mapping $u \mapsto u(f)$, where f is the defining section for T_1 and u is the Lie action. This is because one can do the same computation in family, use the $GL(V)$ equivariance to ensure that φ_f depends linearly on f . Since up to a scalar there is a unique equivariant map from $\wedge^3 V^\vee$ to $\text{Hom}(\text{End}(V), \wedge^3 V^\vee)$ we can conclude. Therefore for general f the map φ_f is injective (this can be verified for example using the general form for f given in ([DFMR17], 4.14) with sufficiently

general coefficients and therefore ϕ_f is surjective as required.

If we plug in these cohomological informations in the long exact sequence associate to the sequence (7) we get several non-zero cohomology groups. In particular the final groups in this sequence are

$$\dots \rightarrow \mathbb{C} \xrightarrow{\epsilon} H^4(\Omega_{T_1}^3) \xrightarrow{\mu} \mathbb{C}^2 \xrightarrow{\nu} \mathbb{C} \rightarrow 0$$

Therefore $h^{3,3}(T_1) = h^{3,4}(T_1) = \dim(\ker \mu) + \dim(\text{Im } \mu)$ and by standard properties of long exact sequences $h^{3,3}(T_1) = h^{3,4} \leq 2$. On the other hand by Hard Lefschetz $h^{3,3}(T_1) = h^{3,4} \geq 2$. This concludes the proof of the Proposition.

3.9.2. Geometry of $T_1(2, 9)$ and X . This rather atypical (for our setting) Hodge structure for $T_1(2, 9)$ has a geometrical explanation.

First consider a linear section $X_H \subset \text{Gr}(3, 9)$. It is a Fano 17-fold of index 8. One can compute that its central Hodge structure has level 1, with the same numerology of a genus 2 curve. Consider the configuration in the diagram below. The map $p : \text{Fl}(2, 3, 9) \rightarrow \text{Gr}(3, 9)$ is a \mathbb{P}^2 bundle, given by the choice of $V_2 \subset V_3$. It remains as well a \mathbb{P}^2 bundle if we restrict p to X_{p^*H} , $\bar{p} : X_{p^*H} \rightarrow X_H$. The Hodge structure of $X_H \subset \text{Gr}(3, 9)$ is therefore repeated three times in X_{p^*H} . Consider as well the projection ϕ from $\text{Fl}(2, 3, 9) \cong \mathbb{P}_{\text{Gr}(2,9)}(\mathcal{Q}(-1))$ to $\text{Gr}(2, 9)$, that is a \mathbb{P}^6 -bundle. Restricting ϕ to X_{p^*H} this gives a \mathbb{P}^5 bundle generically on $\text{Gr}(2, 9)$, that degenerates to a \mathbb{P}^6 on the zero locus Z_H of a section of the dual of $\mathcal{Q}(-1)$, that is $T_1(2, 9)$.

$$(8) \quad \begin{array}{ccccc} & & F & & X_{p^*H} \hookrightarrow \text{Fl}(2, 3, 9) \\ & \swarrow & & \searrow & \downarrow \bar{p} \\ & & & & X_H \hookrightarrow \text{Gr}(3, 9) \\ & \swarrow & \phi & \searrow & \\ Z_H \hookrightarrow \text{Gr}(2, 9) & & & & \end{array}$$

One can prove that the Hodge structure of $T_1(2, 9)$ can be pushed down from X_{p^*H} , which in turn can be calculated from $X_H \subset \text{Gr}(3, 9)$. This can be considered as an alternative (and a bit more geometrical) proof of Proposition 3.27. The precise details of this construction and extension to the derived category case appeared in [BFM19]. In particular a similar argument, albeit in a more complicated version, can be used to derive directly Corollary 3.28 and geometrically explain the K3 structure. We do not produce here a result interpreting some moduli space on X as an IHS: however we expect a similar result to Proposition 3.30 to hold here as well.

3.10. S6 and its three K3 structures. This sporadic Fano has some interesting features. First of all, unlike all our other examples, it is not a section of another Fano by the zero locus of a line bundle. Then it is a Fano of K3 type in two different ways.

The variety $T_1(2, 10)$ is the zero locus of a general global section of the bundle $\mathcal{Q}^\vee(1)$ on the Grassmannian $\text{Gr}(2, 10)$. As in the previous case **S5** we have

$$H^0(\text{Gr}(2, 10), \mathcal{Q}^\vee(1)) \cong \bigwedge^3 V_{10}^\vee,$$

therefore $T_1(2, 10)$ is given by the locus of two-spaces in a 10-dimensional space which are annihilated by a 3-form. It is straightforward to check that $T_1(2, 10)$ is a Fano 8-fold of index $\iota = 3$. We compute first its Hodge numbers

Proposition 3.29. *The Hodge numbers of $T_1(2, 10)$ are*

Moreover we denote by $T_{1,\omega}(2, 10)$ a distinguished element of the family defined by a specified 3-form ω .

Proposition 3.30. *The Debarre-Voisin fourfold F_ω is birational to the moduli space (contained in the Hilbert scheme) of fourfolds $Z_{\mathcal{O}(1)^4}$ contained in the variety $T_{1,\omega}(2, 10)$.*

Proof. Let W be a general point in the Debarre-Voisin fourfold given by a general three form ω . Let us consider the subscheme of $T_{1,\omega}(2, 10)$ given by all two spaces contained inside W . This does not coincide with the full Grassmannian $\text{Gr}(2, 6)$, as the condition $\omega(W) = 0$ does not imply $\omega \lrcorner \Lambda^2 U = 0$ for all $U \subset W$ two-spaces. Notice that this is not the case if one considers three spaces contained in W , that is the construction of the Debarre-Voisin IHS fourfold as a moduli space of $\text{Gr}(3, 6)$ contained in the respective twentyfold.

On $\text{Gr}(k, 10)$ for all k we have a sequence $0 \rightarrow \mathcal{R} \rightarrow V_{10} \otimes \mathcal{O} \rightarrow (V_{10} \otimes \mathcal{O})/\mathcal{R} \rightarrow 0$ which dually gives a sequence $0 \rightarrow \mathcal{R}^\perp \rightarrow V_{10}^\vee \otimes \mathcal{O} \rightarrow \mathcal{R}^\vee \rightarrow 0$. This gives a filtration of $\Lambda^3 V_{10}^\vee \otimes \mathcal{O}$ with factors $\Lambda^3 \mathcal{R}^\perp$, $\Lambda^2 \mathcal{R}^\perp \otimes \mathcal{R}^\vee$, $\mathcal{R}^\perp \otimes \Lambda^2 \mathcal{R}^\vee$ and $\Lambda^3 \mathcal{R}^\vee$. The three-form ω is a section of the last factor $\Lambda^3 \mathcal{R}^\vee$ on $\text{Gr}(6, 10)$. On the zero locus of such a section, this lift to a section of $\mathcal{R}^\perp \otimes \Lambda^2 \mathcal{R}^\vee$, which corresponds to a map $V_{10}/W \rightarrow \Lambda^2 W^\vee$. The image of such a map is a four dimensional space H_4 of two forms on W , for every six space W in the Debarre-Voisin twentyfold given by ω .

Let $U \subset W$ be a point of $T_{1,\omega}(2, 10)$. The space U is isotropic for all two forms in H_4 , indeed if this were not the case we would have a two form $\sigma \in H_4$ such that $\sigma|_U$ is non degenerate and, by how forms in H_4 are obtained, this would imply $\omega \lrcorner \Lambda^2 U \neq 0$. On the contrary, in an appropriate basis, it is not difficult to show that $\omega \lrcorner \Lambda^2 U = 0$ is implied by $\sigma(U) = 0$ for all $\sigma \in H_4$. Thus, the scheme of subspaces $U \subset W$ with fixed W is parametrized by a fourfold $Z_{\mathcal{O}(1)^4} \subset \text{Gr}(2, W)$, which a Fano fourfold of index two, rational by [Fei12, Thm. 2.2.1], with central cohomology $(h^{1,1}, h^{2,2}) = (1, 8)$. This gives a rational map between the Debarre-Voisin fourfold and the space of $Z_{\mathcal{O}(1)^4}$ contained in $T_1(2, 10)$ (and in a fixed $\text{Gr}(2, 6)$). As by changing the point of the Debarre Voisin fourfold we change the ambient Grassmannian $\text{Gr}(2, 6)$, it is clear that such a map is generically injective, hence birational. \square

3.11. S7: a mixed (2,3) CY structure.

A curious yet interesting thing happens when we take a linear section X_H of the above $T_1(2, 10)$. Indeed by Lefschetz's hyperplane section theorem we know that the K3 structure of $T_1(2, 10)$ in H^6 and H^8 must transfer to its linear section: what is most interesting is that the H^7 presents as well a Calabi-Yau structure of level three. To the best of our knowledge, this is the first example of a prime variety that has 2 different examples of CY-structure, of course in different weights. The precise result is

Proposition 3.31. *The Hodge numbers of a linear section $X_H \subset T_1(2, 10)$ are*

$$\begin{array}{ccccccccccc}
 & & & & & & & & & & 1 \\
 & & & & & & & & & & 0 & 0 \\
 & & & & & & & & & & 0 & 1 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & & & & & & & & & & 0 & 0 & 1 & 44 & 44 & 1 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 1 & 22 & 1 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 1 & & & & & & & & 1
 \end{array}$$

The above proposition can be proved with a Borel-Bott-Weil computation similar to the ones above. We will not add further details here in order to preserve the readability of the current paper. We will indeed give a sketch of a geometrical explanation of why such numbers appear.

Indeed as an expert reader might notice, the 3CY structure in our X_H has the same dimension of the 3CY structure appearing in the H^{23} of a linear section $X_1 \subset \text{Gr}(3, 11)$. We will give now an explanation on why and how this 3CY structure gets projected from such varieties to our $X_H \subset \text{T}_1(2, 10)$. This will be only sketched, since the details (in a more general context) can be found in [BFM19, Theorem 3]. The first steps are the following lemmata.

Lemma 3.32. *A linear section $X_1 \subset \text{Gr}(3, 11)$ is a Fano 23-fold of 3CY type. Indeed its non-zero Hodge numbers of weight 23 are $(h^{10,13}, h^{11,12}, h^{12,11}, h^{13,10}) = (1, 44, 44, 1)$.*

This lemma can easily be proved, for example using our results in [FM18]. We notice that such a variety is of 3CY even in the (stronger) categorical sense, see [Kuz19, 4.5]. The orthogonal complement to the Calabi-Yau category is generated by 150 exceptional objects. The following Lemma is less obvious

Lemma 3.33. *A linear section $Y_1 \subset \text{SGr}(3, 10)$ is a Fano 17-fold of 3CY type. Indeed its non-zero Hodge numbers of weight 17 are $(h^{7,10}, h^{8,9}, h^{9,8}, h^{10,7}) = (1, 44, 44, 1)$.*

This Lemma can be proven for example with similar calculations to Corollary 3.21, since we already know that the symplectic Grassmannian $\text{SGr}(k, n)$ is a central variety. However one can prove that the above statement is more than merely a coincidence of Hodge numbers. Indeed, one can show the existence of a fully faithful functor $\Phi : D^b(Y_1) \rightarrow D^b(X_1)$ and a semiorthogonal decomposition of $D^b(X_1)$ with $\Phi D^b(Y_1)$ as first component, together with several exceptional objects. This obviously proves the Hodge-theoretical statement as well. This in turn explains the 3CY structure in $X_H \subset \text{T}_1(2, 10)$. Indeed it is possible to write a diagram like the one for $\text{T}_1(2, 9)$ in (8), appropriately modified; in particular we have to pass through the symplectic partial flag $\text{SFL}(2, 3, 9)$. The construction is more involved, but it is enough to explain that this mixed (2,3) Calabi-Yau structure ultimately comes from a hyperplane section of (respectively) $\text{Gr}(3, 10)$ and $\text{Gr}(3, 11)$. An interesting problem is therefore to look for other examples of varieties with mixed CY structure that are not induced by these constructions tricks outlined in [BFM19].

3.12. S8: other K3 structures as $X_L \subset \text{T}_1(k, 10)$. A similar construction can be applied to $\text{T}_1(4, 10)$, $\text{T}_1(5, 10)$ and their linear sections. Indeed both of them will inherit several K3 type structures as in 8. As an example, in the case of $\text{T}_1(4, 10)$ the diagram will be

$$(9) \quad \begin{array}{ccccc} & & F & & X_{\pi^*H} \hookrightarrow \text{Fl}(3, 4, 10) \\ & \swarrow & & \searrow & \downarrow p \\ & Z_H \hookrightarrow \text{Gr}(4, 10) & & & X_H \hookrightarrow \text{Gr}(3, 10) \\ & \swarrow \phi & \swarrow \phi & & \downarrow p \end{array}$$

The map p is a \mathbb{P}^6 bundle, whereas ϕ is generically a \mathbb{P}^2 bundle specialising to a \mathbb{P}^3 bundle over Z_H . This suggests that $\text{T}_1(4, 10)$ should have 7xK3 type structure, and a Borel-Bott-Weil calculation confirms this. A similar construction, albeit more complicated can be performed as well for $\text{T}_1(5, 10)$, where the fibers of the map on the right hand side of the diagram are $\text{Gr}(2, 7)$. Moreover on the left side of the diagram there are three type of fibers, corresponding to (generically) smooth hyperplane sections of $\text{Gr}(2, 5)$, singular sections in codimension 3 and the whole of $\text{Gr}(2, 5)$ in codimension 10. The linear sections of both $\text{T}_1(4, 10)$ and $\text{T}_1(5, 10)$ inherit a structure of K3 type by Lefschetz theorem (depending of course by the dimension of the linear subspace). It is interesting to notice that codimensional 1 and 2 linear sections will be of mixed CY type, with a similar argument to the one of the previous section.

Finally, we remark that $\text{T}_1(6, 10)$ and $\text{T}_1(1, 10)$ admit structures of K3 type: the first type of variety is nothing but the IHS fourfold of Debarre-Voisin, while the second one can be used to construct the

We point out that $X_H \subset \mathrm{SO}(3, 8)$, $X_2 \subset \mathrm{SGr}(4, 7)$ and $X_2 \subset \mathrm{S}_2\mathrm{Gr}(2, 6)$ are particularly interesting as Fano of 3-CY type, since they are of dimension 5 (the minimal possible) and therefore relevant for testing a modified version of Kuznetsov's conjecture on rationality and derived categories [Kuz10, Conjecture 1.1]. The original conjecture states that a cubic fourfold is rational if and only if it contains the derived category of a K3, and its generalization is that a Fano n -fold is rational if and only if its derived category can be obtained by derived categories of $n - 2$ folds.

We collect in the next table the 3-CY structure mentioned in the above discussion. We mention that in analogy with the K3 case, $\mathrm{T}_1(2, 11)$ can be considered as well as an example of $3 \times \mathrm{CY}$ structure. Taking other k and appropriate number of linear sections is possible as well to produce examples of *mixed* $(3, j)$ -CY structure. However we do not include them in the following table. Of course more examples could be found by considering products and such as in the FK3 case, but we decided to not consider them here in order to keep the length of this paper within an acceptable limit. We of course do not include the examples already considered in [IM15].

Type	dim.	ι_X	$h^{n-1/2, n+1/2}$
$X_1 \subset \mathrm{OGr}(3, 9)$	11	4	49
$X_1 \subset \mathrm{OGr}(4, 9)$	9	3	70
$X_1 \subset \mathrm{SGr}(3, 10)$	17	7	44
$X_1 \subset \mathrm{SGr}(4, 9)$	13	5	45
$X_1 \subset \mathrm{S}_2\mathrm{Gr}(3, 9)$	11	4	44
$X_1 \subset \mathrm{S}_2\mathrm{Gr}(4, 9)$	7	2	45
$X_2 \subset \mathrm{S}_2\mathrm{Gr}(2, 6)$	5	2	67
$X_H \subset \mathrm{SO}(3, 8)$	5	1	45
$X_2 \subset \mathrm{SGr}(4, 7)$	5	2	72

TABLE 3. 3CY structure in OGr, SGr, S₂Gr and SO

APPENDIX B: INFINITE CY SERIES

During our search we identified some interesting class of varieties. Even if they are not directly related to the search of Fano manifolds of k -CY type, we decided to include some of them in this appendix. We identified some interesting infinite families of varieties (of every dimension) with trivial canonical bundle obtained using the same bundles in different Grassmannians. We checked that these varieties are actually Calabi-Yau for low dimension (up to 6). We expect them to be always like this. We describe now these series of varieties, according to the type of bundles involved.

$$\begin{aligned} \mathcal{A}(k, l) &:= \mathcal{Q}(1) \oplus \bigwedge^2 \mathcal{R}^\vee \text{ on } \mathrm{Gr}(k, k+l); \\ \mathcal{B}(k, l) &:= \mathcal{Q}^\vee(1) \oplus \mathrm{Sym}^2 \mathcal{R}^\vee \text{ on } \mathrm{Gr}(k, k+l); \\ \mathcal{C}(k, k+1) &:= \mathrm{Sym}^2 \mathcal{R}^\vee \oplus \bigwedge^2 \mathcal{R}^\vee \oplus \mathcal{O}(1) \text{ on } \mathrm{Gr}(k, 2k+1). \end{aligned}$$

We will denote with capital letters a variety defined as the zeroes of a general section of the corresponding bundle. Thus, $A(k, l)$ has dimension $l(k-1) - \binom{k}{2}$, $B(k, l)$ has dimension $l(k-1) - \binom{k+1}{2}$ and $C(k)$ has dimension $k-1$. Notice that $A(k, l)$ can naturally be seen as $Z_{\mathcal{Q}(1)} \subset \mathrm{SGr}(k, k+l)$, $B(k, l)$ as $Z_{\mathcal{Q}^\vee(1)} \subset \mathrm{OGr}(k, k+l)$ and $C(k, k+1)$ is a linear section of the ortho-symplectic Grassmannian. In particular, as in [Kuz16] in the case of bisymplectic Grassmannian, one can prove that

$$C(k, k+1) \cong X_{(2, \dots, 2)} \subset (\mathbb{P}^1)^k.$$

When $k=2$, $A(2, l)$ is indeed a deformation of a complete intersection given by $(\mathcal{O}(1))^{l+3}$ on $\mathrm{Gr}(2, l+3)$. Indeed, first notice that on $\mathrm{Gr}(2, l+3)$ we have $(\mathcal{O}(1))^{l+3} \cong \mathcal{Q}(1) \oplus \mathcal{R}(1) \cong \mathcal{Q}(1) \oplus \mathcal{R}^\vee$. Then notice that the zero locus of a general global section of $\mathcal{Q}(1) \oplus \mathcal{R}^\vee$ on $\mathrm{Gr}(2, l+3)$ is isomorphic to the

zero locus of a general global section of $\mathcal{Q}(1) \oplus \mathcal{O}(1)$ on $\mathrm{Gr}(2, l+2)$. This follows from the standard fact that $\mathrm{Gr}(k, n) \supset Z_{\mathcal{R}^\vee} \cong \mathrm{Gr}(k, n-1)$ and under the previous isomorphism $\mathcal{Q}(1)_{k,n}$ projects to $\mathcal{Q}(1)_{k,n-1} \oplus \mathcal{O}(1)$.

For dimensions $d = 2, 3, 4$ we refer to [Be18], [IIM19]. For $d = 5, 6$ the Calabi-Yaus in the series A and B are listed below. We do not include $B(5, 5)$, since it can be seen as a deformation of the double spinor variety studied by Manivel in [Man19]. In *loc.cit.* Manivel computed the Hodge numbers of this family and its locally completeness, cf. Proposition 3.1 and Proposition 3.3. Since $C(k, k+1)$ is indeed a well-known class of varieties in disguise, we will not compute the Hodge numbers for the first values of the series. In the following list of invariants we do not include either trivially known Hodge numbers such as $h^{0,n}$. Moreover the number not listed are always 0.

dim.	Type	$h^{1,1}$	$h^{2,2}$	$h^{4,1}$	$h^{3,2}$
5	$A(2, 6)$	1	2	163	1784
5	$A(3, 4)$	1	2	148	1619
5	$B(4, 5)$	1	2	165	1806

TABLE 4. First values of infinite series for fivefolds

dim.	Type	$h^{1,1}$	$h^{2,2}$	$h^{5,1}$	$h^{4,2}$	$h^{3,3}$
6	$A(2, 7)$	1	2	251	5202	14004
6	$A(4, 4)$	1	1	251	5181	13960
6	$B(2, 9)$	1	2	120	2254	6274
6	$B(3, 6)$	1	2	125	2380	6596

TABLE 5. First values of infinite series for sixfolds

An interesting question, which however falls beyond the scope of this paper, is to investigate whether the varieties constructed in this way are generic in moduli, that is whether all of their deformations are embedded in the same Grassmannian. This can be done by a direct computation of $h^1(TG|_X)$ using Koszul complex and Borel-Bott-Weil theorem, however these calculations are quite demanding in each specific case, and a general argument is out of reach.

APPENDIX C: UNEXPECTED LACK OF K3 STRUCTURE

The numerical condition in (1) restricted most of our search to vector bundles in which one of the irreducible summand is linear. One can of course try to rearrange this condition in order to eliminate the constraint. Indeed this is geometrically meaningful, as for example $T_1(2, 10)$ shows (it is a zero locus of an indecomposable bundle that is non-linear, with slope $\mu = c_1(E)/r(E) = 7/8$). It is possible, and we plan to do so, to fully investigate this case.

During a preliminary search we found this example, the zero locus $X_{\mathcal{R}^\vee(1)} \subset \mathrm{Gr}(2, 6)$. It is a sixfold of index 3, defined by a bundle of slope $\mu = 3/2$, satisfying all our preliminary numerical condition. Although it is not of K3 type, it is rather curious, and we decided to add it anyway. Indeed its Hodge numbers are

$$\begin{array}{ccccccc}
& & & & 1 & & \\
& & & & 0 & & 0 \\
& & & 0 & 1 & & 0 \\
& & & 0 & 0 & 0 & 0 \\
& & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & 22 & 0 & 0 & 0 \\
& & 0 & 0 & 0 & 0 & 0 & 0 \\
& & 0 & 0 & 2 & 0 & 0 & \\
& & & 0 & 0 & 0 & & \\
& & & 0 & 1 & 0 & & \\
& & & 0 & 0 & & & \\
& & & & 1 & & &
\end{array}$$

The absence of the 1 in $h^{4,2}$ is explained by a Borel-Bott-Weil computation, since an inconvenient cancellation in the spectral sequence occurs. It is possible that some higher-dimensional analogue of this *false positive* may occur, although we expect this to be quite an exception and not the general rule.

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