

Stable integral simplicial volume of 3-manifolds

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ABSTRACT

We show that non-elliptic prime 3-manifolds satisfy integral approximation for the simplicial volume, that is, that their simplicial volume equals the stable integral simplicial volume. The proof makes use of integral foliated simplicial volume and tools from ergodic theory.

1. Introduction

The *simplicial volume* of an oriented compact n -manifold M (possibly with non-empty boundary) over a normed ring R is defined by

$$\|M, \partial M\|_R := \inf \left\{ \sum_{j=1}^m |a_j| \left| \sum_{j=1}^m a_j \cdot \sigma_j \in C_n(M; R) \text{ is a relative } R\text{-fundamental cycle of } (M, \partial M) \right. \right\},$$

which is an algebraic version of (stable) complexity of manifolds. The classical case is $\|M, \partial M\| := \|M, \partial M\|_{\mathbb{R}}$, introduced by Gromov [18, 25, 31] in the context of hyperbolic geometry and the study of topological properties of (minimal) volume.

1.1. The approximation problem for simplicial volume

If M admits enough finite coverings (that is, if $\pi_1(M)$ is residually finite), it makes sense to consider the *stable integral simplicial volume*

$$\|M, \partial M\|_{\mathbb{Z}}^{\infty} := \inf \left\{ \frac{\|W, \partial W\|_{\mathbb{Z}}}{d} \mid d \in \mathbb{N}, W \text{ a } d\text{-sheeted covering of } M \right\}.$$

In the closed case, stable integral simplicial volume gives an upper bound for L^2 -Betti numbers [19, p. 305][34], logarithmic torsion growth of homology [13, Theorem 1.6; 33], and the rank gradient [26].

As for Betti numbers, ranks of fundamental groups, or logarithmic torsion of homology, one can ask which (typically aspherical) manifolds M satisfy integral approximation for simplicial volume, that is, $\|M, \partial M\| = \|M, \partial M\|_{\mathbb{Z}}^{\infty}$.

The main goal of this paper is to show that non-elliptic prime 3-manifolds satisfy integral approximation for simplicial volume (Theorem 1) and that reducible 3-manifolds in general do not (Section 1.3), thereby answering the approximation question in the 3-dimensional case [9, Question 1.3].

The following classes of manifolds were already known to satisfy integral approximation for simplicial volume: closed surfaces of positive genus [18, p. 9], closed hyperbolic 3-manifolds [13,

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Theorem 1.7], closed aspherical manifolds with amenable residually finite fundamental group [13, Theorem 1.10], compact manifolds where S^1 acts ‘non-trivially’ [7, 8], as well as graph manifolds not covered by S^3 [9].

In contrast, approximation fails uniformly for higher dimensional hyperbolic manifolds [11, Theorem 2.1] and it fails for closed manifolds with non-abelian free fundamental group [13, Remark 3.9].

1.2. Main approximation result

More precisely, we have the following positive result, which includes all closed aspherical 3-manifolds.

THEOREM 1 (integral approximation for simplicial volume of non-elliptic prime 3-manifolds). *Let M be an oriented compact connected 3-manifold with empty or toroidal boundary. If M is prime and not covered by S^3 , then*

$$\|M, \partial M\|_{\mathbb{Z}}^{\infty} = \|M, \partial M\| = \frac{\text{hypvol}(M)}{v_3}.$$

Here, v_3 is the volume of a (whence every) ideal regular tetrahedron in \mathbb{H}^3 , and $\text{hypvol}(M)$ is the total volume of the hyperbolic pieces in the JSJ decomposition of M (see Definition 6.3). The equality $\|M, \partial M\| = \text{hypvol}(M)/v_3$ follows from the work of Soma [35], which also holds in the non-prime case.

1.3. Non-approximation results

In the non-prime case, not all closed 3-manifolds (with infinite fundamental group) satisfy integral approximation for simplicial volume; similar to previously known non-approximation results via the first L^2 -Betti number [13, Remark 3.9], we obtain (see Section 7 for the proofs):

THEOREM 2. *Let $d \in \mathbb{N}_{\geq 3}$, let $m, n \in \mathbb{N}$, and let $M_1, \dots, M_m, N_1, \dots, N_n$ be oriented closed connected d -manifolds with the following properties.*

- (1) *We have $\|M_j\| > 0$ for all $j \in \{1, \dots, m\}$ as well as $\|N_k\| = 0$ for all $k \in \{1, \dots, n\}$.*
- (2) *Moreover, $m + n - 1 - \sum_{k=1}^n 1/|\pi_1(N_k)| > \sum_{j=1}^m \|M_j\|$ (with the convention that $1/\infty := 0$).*

Then the connected sum $M := M_1 \# \dots \# M_m \# N_1 \# \dots \# N_n$ does not satisfy integral approximation for simplicial volume, that is, we have $\|M\| < \|M\|_{\mathbb{Z}}^{\infty}$.

COROLLARY 3. *Let N be an oriented closed connected hyperbolic 3-manifold and let $k > \text{vol}(N)/v_3$. Then the oriented closed connected 3-manifold $M := N \# \#^k(S^1)^3$ satisfies $\|M\| < \|M\|_{\mathbb{Z}}^{\infty}$.*

In the case of closed 3-manifolds with vanishing simplicial volume, we have a complete characterisation of approximability:

COROLLARY 4. *Let M be an oriented closed connected 3-manifold with $\|M\| = 0$. Then the following are equivalent.*

- (1) *The simplicial volume of M satisfies integral approximation, that is, $\|M\|_{\mathbb{Z}}^{\infty} = \|M\|$.*
- (2) *The manifold M is prime and has infinite fundamental group or M is homeomorphic to $\mathbb{R}P^3 \# \mathbb{R}P^3$.*

1.4. Strategy of proof of Theorem 1

Clearly, we have $\|M, \partial M\|_{\mathbb{Z}}^{\infty} \geq \|M, \partial M\|$ (by the degree estimate [18, p. 8]). Therefore, in combination with Soma's computation [35], in the situation of Theorem 1, we have

$$\|M, \partial M\|_{\mathbb{Z}}^{\infty} \geq \|M, \partial M\| = \frac{\text{hypvol}(M)}{v_3}.$$

Thus, it suffices to show the converse estimate

$$\|M, \partial M\|_{\mathbb{Z}}^{\infty} \leq \frac{\text{hypvol}(M)}{v_3}.$$

As in Soma's computation of the classical simplicial volume of 3-manifolds, we use the JSJ decomposition and hyperbolisation to cut irreducible manifolds M along tori into pieces W that are hyperbolic or Seifert-fibred, and not covered by S^3 (the additional case $M \cong S^1 \times S^2$ being also Seifert-fibred).

If a piece W is Seifert-fibred (or, more generally, a non-elliptic graph manifold), then it is known that $\|W, \partial W\|_{\mathbb{Z}}^{\infty} = 0 = \text{hypvol}(W)/v_3$ [9; 27, Section 8].

Therefore, two main challenges remain:

- the hyperbolic case with toroidal boundary; and
- subadditivity with respect to glueings along tori.

To this end, it is convenient to rewrite stable integral simplicial volume as parametrised simplicial volume with respect to the canonical action on the profinite completion of the fundamental group(oid) (Section 2.3):

$$\|W, \partial W\|_{\mathbb{Z}}^{\infty} = |W, \partial W|_{\pi_1(W)}.$$

For subadditivity with respect to glueings along tori, we need control over the size of the boundaries of the relative fundamental cycles. In order to avoid a technically demanding equivalence theorem, we proceed similarly to Soma's work with parametrised relative simplicial volume $|W, \partial W|_{\partial}^{\pi_1(W)}$ with boundary control (Section 2.4). Then, as in the case of graph manifolds [9], we can use the uniform boundary condition on tori in the parametrised setting [10], to establish subadditivity (Section 6.2). A subtle point is that, during the glueing step, we also need to stay in control of the parameter spaces; at this point, we will use profinite properties of JSJ decompositions (Section 6.3).

Finally, we need to show for hyperbolic pieces W that $|W, \partial W|_{\partial}^{\pi_1(W)} = \text{vol}(W^{\circ})/v_3$. This generalisation of the closed case [13, Theorem 1.7] will take up a large part of the paper. More specifically, we will proceed in two steps:

First, using a suitably adapted smearing process and the approximation results in the closed case, we establish the following proportionality in the open case (Section 4):

THEOREM 5. *Let M be an oriented complete connected finite-volume hyperbolic 3-manifold (without boundary). Then*

$$|M|_{\text{lf}} = \frac{\text{vol}(M)}{v_3}.$$

Second, we relate this locally finite version to the parametrised simplicial volume with boundary control of the ambient compact manifold W : We have $|M|_{\text{lf}} \geq |W, \partial W|_{\partial}$ (Section 3.3). Combining the fact that $\pi_1(W)$ satisfies Property EMD* from ergodic theory (Proposition 5.2) with monotonicity of boundary-controlled integral foliated simplicial volume with respect to weak containment of parameter spaces (Proposition A.1), we then obtain (Section 5.2):

COROLLARY 6. *Let W be an oriented compact connected hyperbolic 3-manifold with empty or toroidal boundary and let $M := W^\circ$. Then*

$$|W, \partial W|_{\partial}^{\widehat{\pi_1(W)}} = \frac{\text{vol}(M)}{v_3}.$$

Overview of this article

We recall basic terminology related to integral foliated/parametrised simplicial volume in Section 2. In Section 3, we generalise this setup to the case of open manifolds. We then prove proportionality for complete hyperbolic 3-manifolds of finite volume: The locally finite case is established in Section 4; the relative case is derived in Section 5. The JSJ glueing argument and the proof of the main theorem (Theorem 1) are explained in Section 6.2. Finally, in Section 7, we prove the non-approximation results.

2. Integral foliated simplicial volume: the compact case

We recall basic terminology related to integral foliated simplicial volume in the compact case; the open case will be covered in Section 3. Integral foliated simplicial volume of a compact manifold M is a variation of simplicial volume with local coefficients in integer-valued L^∞ -functions on a probability space with a $\pi_1(M)$ -action. Therefore, we first briefly review normed local coefficients.

2.1. Normed local coefficients

We now focus on local coefficient systems that carry a compatible norm. Our main example will be spaces of essentially bounded functions (with the L^1 -norm) on standard Borel probability spaces.

DEFINITION 2.1 (normed local coefficient system). Let W be a topological space. A *normed local coefficient system on W* is a functor from the fundamental groupoid $\Pi(W)$ of W to the category of normed abelian groups and norm non-increasing group homomorphisms.

Given a k -simplex $\sigma \in \text{map}(\Delta^k, W)$ in a topological space W , the l -dimensional face of σ spanned by the vertices $\sigma(e_{i_0}), \dots, \sigma(e_{i_l})$ (with $0 \leq i_0 < \dots < i_l \leq k$) will be denoted by $\sigma[i_0, \dots, i_l]$.

DEFINITION 2.2 (chain complex with local coefficients). Let W be a topological space and let L be a normed local coefficient system on W . Then the *chain complex of W with local coefficients in L* is given by

$$C_k(W; L) := \bigoplus_{\sigma \in \text{map}(\Delta^k, W)} L(\sigma[0])$$

for each $k \in \mathbb{N}$, with boundary operators

$$C_k(W; L) \rightarrow C_{k-1}(W; L)$$

$$a \cdot \sigma \mapsto L(\sigma[0, 1])(a) \cdot \partial_0 \sigma + \sum_{i=1}^k (-1)^i \cdot a \cdot \partial_i \sigma.$$

We equip it with the ℓ^1 -norm $|\cdot|_{1,L}$ induced by the norm on L .

If $V \subseteq W$ is a subspace, we write $C_*^W(V; L)$ for all chains in $C_*(W; L)$ that are supported in V . We define the *chain complex of W relative to V with local coefficient system L* by

$$C_*(W, V; L) := C_*(W; L)/C_*^W(V; L)$$

and endow it with the quotient norm of $|\cdot|_{1,L}$.

DEFINITION 2.3 (homology with local coefficients). Let W be a topological space, let L be a normed local coefficient system on W and let $V \subseteq W$ be a subspace. We define the *k th homology group with local coefficient system L* by

$$H_k(W, V; L) := H_k(C_*(W, V; L))$$

and write $\|\cdot\|_{1,L}$ for the induced ℓ^1 -seminorm on homology.

DEFINITION 2.4 (standard G -space). Let G be a groupoid. A *standard G -space* is a contravariant functor from G to the category of all standard Borel probability spaces and probability measure-preserving transformations.

DEFINITION 2.5 (associated normed local coefficient system). Let G be a groupoid and let α be a standard G -space. Then the *associated normed local coefficient system $L^\infty(\alpha; \mathbb{Z})$ on G* is the post-composition of α with the (contravariant) dualising functor $L^\infty(-, \mathbb{Z})$. In other words,

$$L^\infty(\alpha; \mathbb{Z})(x) := L^\infty(\alpha(x); \mathbb{Z})$$

for all objects x in G (equipped with the L^1 -norm), and

$$\begin{aligned} L^\infty(\alpha; \mathbb{Z})(g): L^\infty(\alpha(x); \mathbb{Z}) &\rightarrow L^\infty(\alpha(y); \mathbb{Z}) \\ f &\mapsto f \circ \alpha(g) \end{aligned}$$

for all morphisms $g: x \rightarrow y$ in G .

REMARK 2.6 (from groups to groupoids). Local coefficient systems are quite similar to the more conventional twisted coefficients. In the setting of local (rather than twisted) coefficients, the role of the fundamental group is played by the fundamental groupoid, which is convenient because it spares us from caring about basepoints, and from working at the level of the universal cover.

Given a topological space W and a standard $\Pi(W)$ -space α , we obtain, for each choice of basepoint $x_0 \in W$, a canonical standard $\pi_1(W, x_0)$ -space by restriction of α to x_0 . That is, we let $\pi_1(W, x_0)$ act on the Borel probability space $\alpha(x_0)$ by $\gamma \cdot x := \alpha(\gamma)(x)$, where $\gamma \in \pi_1(W, x_0)$ and $x \in \alpha(x_0)$.

Conversely, if W is path-connected, then each standard $\pi_1(W, x_0)$ -space X can be extended to a standard $\Pi(W)$ -space α by first choosing, for each point $p \in W$, a path γ_p (up to homotopy class relative to endpoints) from x_0 to p , and then setting:

- at every point $p \in W$, we put $\alpha(p) := X$;
- for each morphism $(\gamma: p \rightarrow q)$ in $\Pi(W)$, we set $h_\gamma := \gamma_p * \gamma * \gamma_q^{-1} \in \pi_1(W, x_0)$ and

$$\alpha(\gamma): X \rightarrow X$$

$$x \mapsto h_\gamma \cdot x.$$

It is easily verified that this makes α a contravariant functor, whose restriction to x_0 , as explained previously, recovers the left $\pi_1(W, x_0)$ -action on X . This is ultimately a reflection of

the fact that when $\pi_1(W, x_0)$ is regarded as a one-object sub-category of $\Pi(W)$, the inclusion $\pi_1(W, x_0) \hookrightarrow \Pi(W)$ is an equivalence of categories.

We should also mention that the choice of path classes $(\gamma_p: x_0 \rightarrow p)_{p \in W}$ is immaterial, as picking a different collection $(\gamma'_p: x_0 \rightarrow p)_{p \in W}$ leads to the construction of a contravariant functor α' that is isomorphic to α , in the category-theoretical sense. Indeed, it is easy to verify that the maps

$$X \rightarrow X$$

$$x \mapsto (\gamma'_p * \gamma_p^{-1}) \cdot x,$$

over all $p \in W$, assemble to a natural transformation $\alpha \rightarrow \alpha'$, which is clearly invertible.

Similarly, a normed local coefficient system L on a topological space W can be restricted to a chosen basepoint $x_0 \in W$, yielding a normed *right* $\pi_1(W, x_0)$ -module $L(x_0)$. And conversely, a right $\pi_1(W, x_0)$ -action on a normed abelian group A can be extended to a normed local coefficient system L that is constantly A on objects, by choosing paths $\gamma_p: x_0 \rightarrow p$ as before, and setting, for each $(\gamma: p \rightarrow q)$ in $\Pi(W)$,

$$L(\gamma): A \rightarrow A$$

$$a \mapsto a \cdot (\gamma_p * \gamma * \gamma_q^{-1}).$$

It is straightforward to check that homology with twisted coefficients in a normed right $\pi_1(W, x_0)$ -module A is isomorphic to homology with local coefficients in any normed local coefficient system obtained as an extension of A .

We also remark that all these constructions are compatible with the dualising procedure introduced in Definition 2.5. Indeed, the construction of the associated normed local coefficient system to a standard $\Pi(W)$ -space fits into the following commutative diagram:

Standard (left)	$\xrightarrow{\text{extend}}$	Standard	$\xrightarrow{\text{restrict}}$	Standard (left)
$\pi_1(W, x_0)$ -spaces		$\Pi(W)$ -spaces		$\pi_1(W, x_0)$ -spaces
$\downarrow L^\infty(-, \mathbb{Z})$		$\downarrow L^\infty(-, \mathbb{Z})$		$\downarrow L^\infty(-, \mathbb{Z})$
Normed right	$\xrightarrow{\text{extend}}$	Normed local	$\xrightarrow{\text{restrict}}$	Normed right
$\pi_1(W, x_0)$ -modules		coefficient systems		$\pi_1(W, x_0)$ -modules

2.2. Integral foliated simplicial volume

Integral foliated simplicial volume relaxes the integrality of coefficients by allowing for integer-valued coefficient functions on a probability space.

DEFINITION 2.7 (parametrised fundamental class with local coefficients). Let M be an oriented compact connected n -manifold and let α be a standard $\Pi(M)$ -space. Then the α -*parametrised fundamental class* $[M, \partial M]^\alpha$ of M is defined to be the image of the integral fundamental class $[M, \partial M]$ of M under the change of coefficient map

$$H_n(M, \partial M; \mathbb{Z}) \rightarrow H_n(M, \partial M; L^\infty(\alpha; \mathbb{Z}))$$

induced by the inclusion of \mathbb{Z} into $L^\infty(\alpha; \mathbb{Z})$ as constant functions.

DEFINITION 2.8 (integral foliated simplicial volume). Let M be an oriented compact connected manifold and let α be a standard $\Pi(M)$ -space. Then the α -*parametrised simplicial volume of M* is defined by

$$|M, \partial M|^\alpha := \|[M, \partial M]^\alpha\|_{1, L^\infty(\alpha; \mathbb{Z})}.$$

Moreover, the *integral foliated simplicial volume* $|M, \partial M|$ of M is the infimum of all parametrised simplicial volumes, where the infimum is taken over all standard $\Pi(M)$ -spaces.

If M is closed, we also write $|M|^\alpha$ and $|M|$ for the corresponding quantities.

PROPOSITION 2.9 (comparison with ordinary simplicial volume [19; 34, Theorem 5.35]). *Let W be an oriented compact connected manifold and let α be a standard $\Pi(M)$ -space. Then integration of the coefficients shows that*

$$\|W, \partial W\| \leq |W, \partial W|^\alpha.$$

In particular, $\|W, \partial W\| \leq |W, \partial W|$.

2.3. The profinite completion

The link between stable integral simplicial volume and the parametrised simplicial volume and ergodic theory is given by the profinite completion of the fundamental group.

DEFINITION 2.10 (profinite completion of a group). Let Γ be a countable group and consider the inverse system of finite-index normal subgroups of Γ together with their inclusion homomorphisms. The inverse limit

$$\widehat{\Gamma} := \varprojlim_{\substack{\Lambda \triangleleft \Gamma \\ \text{f.i.}}} \Gamma/\Lambda$$

of the corresponding group quotients is called the *profinite completion* of Γ . The left translation action and the normalised counting measures on the quotients Γ/Λ turn $\widehat{\Gamma}$ into a standard Γ -space.

Moreover, group homomorphisms induce canonical maps on profinite completions, making this definition functorial [32, Lemma 3.2.3].

If Γ is a countable group, then a straightforward computation shows that the standard Γ -space $\widehat{\Gamma}$ is (essentially) free if and only if Γ is residually finite.

DEFINITION 2.11 (profinite completion of the fundamental groupoid). Let W be a path-connected topological space. The standard $\Pi(W)$ -space $\widehat{\Pi(W)}$ is defined as follows.

- For each $p \in W$, we set $\widehat{\Pi(W)}(p) := \widehat{\pi_1(W, p)}$.
- Given a morphism $\gamma: p \rightarrow q$ in $\Pi(W)$, we take $\widehat{\Pi(W)}(\gamma): \widehat{\pi_1(W, q)} \rightarrow \widehat{\pi_1(W, p)}$ to be the map induced on profinite completions by

$$\begin{aligned} \pi_1(W, q) &\rightarrow \pi_1(W, p) \\ \alpha &\mapsto \gamma * \alpha * \gamma^{-1}. \end{aligned}$$

If $x_0 \in W$, then the standard $\Pi(W)$ -space $\widehat{\Pi(W)}$ is isomorphic to the ‘extension’ construction from Remark 2.6, applied to the standard $\pi_1(W, x_0)$ -space $\widehat{\pi_1(W, x_0)}$.

PROPOSITION 2.12 (stable integral simplicial volume via profinite completion). *Let W be an oriented compact connected manifold. Then*

$$|W, \partial W|^{\widehat{\Pi(W)}} = \|W, \partial W\|_{\mathbb{Z}}^\infty.$$

Proof. The proof for the closed case (with twisted coefficients) [27, Remark 6.7][13, Theorem 2.6] can be adapted to the relative case (with local coefficients) in a straightforward manner. \square

2.4. Adding boundary control

When proving additivity estimates for simplicial volumes under glueings, one needs additional control on the boundary. We will use the following version of integral foliated simplicial volume, similar to the relative simplicial volume considered by Thurston [36, Chapter 6.5]:

DEFINITION 2.13 (integral foliated simplicial volume with boundary control). Let W be an oriented compact connected manifold and let α be a standard $\Pi(W)$ -space. Then the *controlled α -parametrised simplicial volume of W* is defined by

$$|W, \partial W|_{\partial}^{\alpha} := \sup_{\varepsilon \in \mathbb{R}_{>0}} \inf \{ |c|_1 \mid c \in C_n(W; L^{\infty}(\alpha; \mathbb{Z})) \text{ relative } \alpha\text{-fundamental cycle of } W, |\partial c|_1 \leq \varepsilon \}$$

(with the convention that $\inf \emptyset = +\infty$).

Taking the infimum over all α , we obtain $|W, \partial W|_{\partial}$.

It follows from this definition that if $|\partial W| > 0$, then $|W, \partial W|_{\partial} = +\infty$, and we also clearly have $|W, \partial W| \leq |W, \partial W|_{\partial}$. Whether the converse inequality holds is a more subtle question, because it involves a simultaneous optimisation problem. We show in Corollary 5.3 that if W is a compact hyperbolic 3-manifold with toroidal boundary, then this is the case. We do have that vanishing of $|W, \partial W|$ implies vanishing of $|W, \partial W|_{\partial}$:

LEMMA 2.14 (vanishing of integral foliated simplicial volume transfers to boundary-controlled setting). *Let W be an oriented compact connected n -manifold and α a standard $\Pi(W)$ -space. If $|W, \partial W|^{\alpha} = 0$, then also $|W, \partial W|_{\partial}^{\alpha} = 0$.*

Proof. Let $\varepsilon, \varepsilon_{\partial} > 0$. Since $|W, \partial W|^{\alpha} = 0$, there exists a relative α -fundamental cycle $c \in C_n(W; L^{\infty}(\alpha; \mathbb{Z}))$ of W with $|c|_1 < \min(\varepsilon, \frac{\varepsilon_{\partial}}{n+1})$. It follows that $|c|_1 < \varepsilon$ and $|\partial c|_1 \leq (n+1)|c|_1 < \varepsilon_{\partial}$, giving the desired boundary control. \square

In certain situations, it is helpful to restrict to ergodic parameter spaces. Here, a standard $\Pi(W)$ -space α is *ergodic* if the $\pi_1(W, x)$ -space $\alpha(x)$ is ergodic in the classical sense for one (hence every) $x \in W$.

PROPOSITION 2.15 (ergodic parameters suffice). *Let W be an oriented compact connected manifold. Then for each $\varepsilon \in \mathbb{R}_{>0}$, there exists an ergodic standard $\Pi(W)$ -space α with*

$$|W, \partial W|_{\partial}^{\alpha} \leq |W, \partial W|_{\partial} + \varepsilon.$$

Proof. As in the closed case (with twisted coefficients) [27, Proposition 4.17], this can be shown via an ergodic decomposition argument — we only need to take the coefficients of the boundary contribution into account. \square

3. Integral foliated simplicial volume: the non-compact case

We now extend the definition of integral foliated simplicial volume to the non-compact case.

3.1. Basic definitions

In the non-compact case, we will replace singular chains by locally finite singular chains (while keeping normed local coefficients).

DEFINITION 3.1 (locally finite chains, homology with local coefficients). Let W be a topological space and let L be a normed local coefficient system on W . A (possibly infinite) chain $\sum_{\sigma \in \text{map}(\Delta^n, W)} a_\sigma \cdot \sigma$ with $a_\sigma \in L(\sigma[0])$ is called *locally finite* if every compact subset of W intersects only finitely many singular simplices σ in W with $a_\sigma \neq 0$. We write $C_*^{\text{lf}}(W; L)$ for the chain modules of all locally finite chains with local coefficients in L and extend the boundary operator from Definition 2.2 to locally finite chains.

Let $V \subset W$ be a subspace. We write $C_*^{\text{lf}, W}(V; L)$ for the subcomplex of all chains in $C_*^{\text{lf}}(W; L)$ that are supported in V and define

$$C_*^{\text{lf}}(W, V; L) := C_*^{\text{lf}}(W; L) / C_*^{\text{lf}, W}(V; L).$$

We obtain a chain complex and write $|\cdot|_{1, L}$ for the ℓ^1 -norm induced by L .

Moreover, we define the k th *locally finite homology group with local coefficient system L* by

$$H_k^{\text{lf}}(W, V; L) := H_k(C_*^{\text{lf}}(W, V; L))$$

for the corresponding homology and $\|\cdot\|_{1, L}$ for the (potentially infinite) seminorm induced on homology by $|\cdot|_{1, L}$.

Strictly speaking, locally finite chains are functions on the space of singular simplices; however, the suggestive notation as ‘formal sums’ has proved to be efficient in the classical case of locally finite homology. It should be noted that the boundary operator is indeed well defined (the local finiteness condition takes care of this).

Note that if L is a functor which associates to any object of the fundamental groupoid $\Pi(W)$ the normed abelian group \mathbb{R} and to any morphism the identity on \mathbb{R} , we recover the classical definition of locally finite homology with real coefficients. Then, one can define the locally finite simplicial volume of an oriented connected n -manifold M without boundary, denoted by $\|M\|_{\text{lf}}$, as the ℓ^1 -seminorm of its real locally finite fundamental class. This is thoroughly discussed in the literature [14, 18, 24].

DEFINITION 3.2 (locally finite fundamental class with local coefficients). Let M be an oriented connected n -manifold without boundary and let α be a standard $\Pi(M)$ -space. Then the α -parametrised locally finite fundamental class $[M]_{\text{lf}}^\alpha$ of M is defined to be the image of the integral locally finite fundamental class $[M]_{\text{lf}}$ of M under the change of coefficients map

$$H_n^{\text{lf}}(M; \mathbb{Z}) \rightarrow H_n^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z}))$$

induced by the inclusion of \mathbb{Z} into $L^\infty(\alpha; \mathbb{Z})$ as constant functions.

DEFINITION 3.3 (locally finite integral foliated simplicial volume). Let M be an oriented connected manifold without boundary and let α be a standard $\Pi(M)$ -space. Then the α -parametrised locally finite simplicial volume of M is given by

$$|M|_{\text{lf}}^\alpha := \|[M]_{\text{lf}}^\alpha\|_{1, L^\infty(\alpha; \mathbb{Z})}.$$

The *locally finite integral foliated simplicial volume* $|M|_{\text{lf}}$ of M is defined to be the infimum over all parametrised locally finite simplicial volumes of M .

If M is an oriented closed connected manifold, then $|M|^\alpha = |M|_{\text{lf}}^\alpha$ for all standard $\Pi(M)$ -spaces α , because every locally finite chain on a compact space is an ordinary chain (and vice versa).

3.2. Integration

Integrating parametrised locally finite chains over their coefficients leads to the following comparison between parametrised locally finite simplicial volume and ordinary locally finite simplicial volume.

PROPOSITION 3.4 (integration of coefficients). *Let M be an oriented connected manifold without boundary and let α be a standard $\Pi(M)$ -space. Then*

$$I_{\mathbb{R}}: C_*^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z})) \rightarrow C_*^{\text{lf}}(M; \mathbb{R})$$

$$\sum_{\sigma \in \text{map}(\Delta^*, M)} f_\sigma \cdot \sigma \mapsto \sum_{\sigma \in \text{map}(\Delta^*, M)} \left(\int_{\alpha(\sigma[0])} f_\sigma \right) \cdot \sigma$$

is a well-defined chain map that maps α -parametrised fundamental cycles to locally finite \mathbb{R} -fundamental cycles. In particular,

$$\|M\|_{\text{lf}} \leq \|M\|_{\text{lf}}^\alpha$$

and so

$$\|M\|_{\text{lf}} \leq |M|_{\text{lf}}.$$

Proof. This can be proved in the same way as in the closed case [27, Proposition 4.6; 34, Remark 5.23]. □

Integrating both the coefficients and the simplices (over the volume form) provides a useful criterion to detect fundamental cycles. Integration over simplices requires some regularity on the simplices (we will use smoothness) as well as global bounds to ensure convergence (we will use a global Lipschitz bound).

DEFINITION 3.5 (integration of a simplex). Let M be an oriented Riemannian n -manifold. For a smooth singular n -simplex $\sigma: \Delta^n \rightarrow M$, we define the *integration* of σ over M to be

$$\langle \text{dvol}_M, \sigma \rangle := \int_{\Delta^n} \sigma^* \text{dvol}_M.$$

DEFINITION 3.6 (Lipschitz chain). Let M be a Riemannian manifold, let α be a standard $\Pi(M)$ -space, and let $c \in C_*^{\text{lf}}(M, L^\infty(\alpha; \mathbb{Z}))$ be an α -parametrised locally finite chain. Let $\text{Lip}(c) \in [0, \infty]$ denote the supremum of the (possibly infinite) Lipschitz constants over the simplices in c (that occur with non-zero coefficient). We say that c is *Lipschitz* if $\text{Lip}(c) < \infty$.

An important class of Lipschitz simplices are geodesic simplices in hyperbolic spaces:

DEFINITION 3.7 (geodesic simplex). Given two points x and y in \mathbb{H}^n , we define $[x, y]: [0, 1] \rightarrow \mathbb{H}^n$ to be the constant-speed parametrisation of the unique geodesic segment of \mathbb{H}^n joining x to y . The standard n -simplex Δ^n is given by the following set: $\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}_{\geq 0}^{n+1} \mid \sum_{i=0}^n x_i = 1\}$. We identify Δ^{n-1} as the subset of Δ^n given by those points whose last coordinate is zero. A *geodesic simplex* $\sigma: \Delta^n \rightarrow \mathbb{H}^n$ with vertices x_0, \dots, x_n , often denoted by $\text{straight}(x_0, \dots, x_n)$, is the map defined inductively as follows:

$$\sigma((1-t)s + t(0, \dots, 0, 1)) := [\sigma(s), x_n](t),$$

where $s \in \Delta^{n-1}$ and $t \in [0, 1]$.

We say that a singular chain in a hyperbolic n -manifold M is *geodesic* (or *straight*) if each simplex with non-zero coefficient is the composition of a geodesic simplex with the universal covering projection $\mathbb{H}^n \rightarrow M$.

REMARK 3.8 (geodesic simplices are Lipschitz). All geodesic simplices $\Delta^k \rightarrow \mathbb{H}^n$ are smooth, by smoothness of the exponential map. Moreover, geodesic simplices are Lipschitz maps, with Lipschitz constant depending only on their diameter [28, Proposition 2.4, Remark 2.5]. This shows that chains supported on geodesic simplices of uniformly bounded diameter are Lipschitz.

REMARK 3.9 (geodesic straightening). Every singular cycle in a hyperbolic manifold M is canonically homologous to a geodesic one with (at most) the same ℓ^1 -norm [3, Lemma C.4.3].

DEFINITION 3.10 (double integration). Let M be an oriented Riemannian n -manifold without boundary, let α be a standard $\Pi(M)$ -space, and let $c = \sum_{\sigma \in \text{map}(\Delta^n, M)} f_\sigma \cdot \sigma \in C_n^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z}))$ be an α -parametrised locally finite chain that is Lipschitz, supported on smooth simplices, and such that $|c|_1 < \infty$. We define the *double integration* of c over M by

$$\langle \text{dvol}_M, c \rangle := \langle \text{dvol}_M, I_{\mathbb{R}}(c) \rangle = \sum_{\sigma \in \text{map}(\Delta^n, M)} \left(\int_{\alpha(\sigma[0])} f_\sigma \right) \cdot \langle \text{dvol}_M, \sigma \rangle.$$

The hypotheses on c ensure that the sum on the right-hand side converges absolutely.

We will now come to the recognition of fundamental cycles through integration; for simplicity, we restrict to the case of tame manifolds and ergodic parameter spaces.

DEFINITION 3.11 (tame manifold). Let M be a connected non-compact manifold without boundary. We say that M is *tame* if it is homeomorphic to the interior of a connected compact manifold with boundary.

In order to prove a characterisation of α -parametrised locally finite fundamental cycles of a tame n -manifold M in terms of the double integration map, we need to compute the top-dimensional homology $H_n^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z}))$.

PROPOSITION 3.12 (top locally finite homology of tame manifolds with local coefficients). *Let M be a tame oriented connected n -manifold and let α be an ergodic standard $\Pi(M)$ -space. Then the map $H_n^{\text{lf}}(M; \mathbb{Z}) \rightarrow H_n^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z}))$ induced by the inclusion of constant functions is an isomorphism, and so*

$$H_n^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z})) \cong \mathbb{Z},$$

generated by the α -parametrised locally finite fundamental class $[M]_{\text{lf}}^\alpha$.

Proof. Let W be the closure of M . The topological collar theorem [5] shows that $M \cong W \cup_{\partial W} (\partial W \times [0, +\infty))$. Via this identification, we may consider the compact subspaces $K_r := W \cup_{\partial W} (\partial W \times [0, r])$ of M ; clearly, the family $(K_r)_{r \in \mathbb{N}}$ is cofinal in the directed set of all compact subspaces of M . Therefore, we can write the locally finite chain complex as the inverse limit

$$C_n^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z})) \cong \varprojlim_{r \rightarrow \infty} C_n^{\text{lf}}(M, M \setminus K_r; L^\infty(\alpha; \mathbb{Z})).$$

It is easy to check that the directed system $(C_n^{\text{lf}}(M, M \setminus K_r; L^\infty(\alpha; \mathbb{Z}))_{r \in \mathbb{N}}$ satisfies the Mittag-Leffler condition [38, Definition 3.5.6]. Moreover, for each $r \in \mathbb{N}$, we have

$$\begin{aligned} H_{n+1}^{\text{lf}}(M, M \setminus K_r; L^\infty(\alpha; \mathbb{Z})) &\cong H_{n+1}(K_r, \partial K_r; L^\infty(\alpha; \mathbb{Z})) \\ &\cong H_{n+1}(W, \partial W; L^\infty(\alpha; \mathbb{Z})) \cong 0. \end{aligned}$$

Thus, the \lim^1 -term in the short exact sequence computing the homology of a limit of chain complexes [38, Theorem 3.5.8] vanishes, and we see that the inclusions $(M, \emptyset) \hookrightarrow (M, M \setminus K_r)$ induce an isomorphism

$$H_n^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z})) \cong \varprojlim_{r \rightarrow \infty} H_n^{\text{lf}}(M, M \setminus K_r; L^\infty(\alpha; \mathbb{Z})).$$

Since for $r \in \mathbb{N}$ the pairs $(M, M \setminus K_r)$ are homotopy equivalent to $(W, \partial W)$ in a compatible way (namely, by collapsing $M \setminus W$ onto ∂W), the inverse system $(H_n^{\text{lf}}(M, M \setminus K_r; L^\infty(\alpha; \mathbb{Z}))_{r \in \mathbb{N}}$ is isomorphic to the constant system $H_n(W, \partial W; L^\infty(\alpha; \mathbb{Z}))$, and we have an isomorphism

$$H_n^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z})) \cong H_n(W, \partial W; L^\infty(\alpha; \mathbb{Z}))$$

induced by the collapse map $M \rightarrow W$.

Repeating the argument in the setting of constant \mathbb{Z} -coefficients yields a similar isomorphism, reducing our claim to the proof that the lower map in the following commutative diagram is an isomorphism:

$$\begin{array}{ccc} H_n^{\text{lf}}(M; \mathbb{Z}) & \longrightarrow & H_n^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z})) \\ \downarrow \cong & & \downarrow \cong \\ H_n(W, \partial W; \mathbb{Z}) & \longrightarrow & H_n(W, \partial W; L^\infty(\alpha; \mathbb{Z})) \end{array} .$$

Using Poincaré duality (with local coefficients), this can be further translated into a question about 0th cohomology:

$$\begin{array}{ccc} H_n(W, \partial W; \mathbb{Z}) & \longrightarrow & H_n(W, \partial W; L^\infty(\alpha; \mathbb{Z})) \\ \cong \uparrow_{\text{PD}} & & \cong \uparrow_{\text{PD}} \\ H^0(W; \mathbb{Z}) & \longrightarrow & H^0(W; L^\infty(\alpha; \mathbb{Z})) \end{array} .$$

Because α is ergodic, we know that for every $x_0 \in W$ the fixed module $L^\infty(\alpha(x_0); \mathbb{Z})^{\pi_1(W, x_0)}$ consists only of the constant functions. In other words, the inclusion of \mathbb{Z} into $L^\infty(\alpha(x_0); \mathbb{Z})^{\pi_1(W, x_0)}$ as constant functions induces an isomorphism on 0th twisted cohomology, which translates to the lower map in the previous diagram being an isomorphism. \square

Finally, we can prove our criterion for α -parametrised locally finite fundamental cycles of tame manifolds.

PROPOSITION 3.13 (recognising fundamental cycles through integration). *Let M be a tame Riemannian n -manifold and let α be an ergodic standard $\Pi(M)$ -space. Let $c \in C_n^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z}))$ be a smooth Lipschitz α -parametrised locally finite cycle with $|c|_1 < +\infty$. Then, the following are equivalent.*

- (1) *The chain c is an α -parametrised locally finite fundamental cycle.*
- (2) $\langle \text{dvol}_M, c \rangle = \text{vol}(M)$.
- (3) $|\langle \text{dvol}_M, c \rangle - \text{vol}(M)| < \text{vol}(M)$.

Proof. Because $I_{\mathbb{R}}$ is a norm non-increasing chain map, the ℓ^1 -norm of $I_{\mathbb{R}}(c)$ is finite. Let $k \in \mathbb{R}$ be the number that satisfies

$$[I_{\mathbb{R}}(c)] = k \cdot [M]_{\text{lf}}^{\mathbb{R}},$$

where $[M]_{\text{lf}}^{\mathbb{R}} \in H_n^{\text{lf}}(M; \mathbb{R})$ is the real fundamental class of M . Then [28, (proof of) Proposition 4.4]

$$\langle \text{dvol}_M, c \rangle = \langle \text{dvol}_M, I_{\mathbb{R}}(c) \rangle = k \cdot \text{vol}(M).$$

Moreover, by Proposition 3.12, the canonical homomorphism $H_n^{\text{lf}}(M; \mathbb{Z}) \rightarrow H_n^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z}))$ is an isomorphism and both modules are isomorphic to \mathbb{Z} , and the canonical homomorphism $H_n^{\text{lf}}(M; \mathbb{Z}) \rightarrow H_n^{\text{lf}}(M; \mathbb{R})$ corresponds to the inclusion $\mathbb{Z} \rightarrow \mathbb{R}$. Therefore, we obtain that k is an integer and that

$$[c] = k \cdot [M]_{\text{lf}}^\alpha \in H_n^{\text{lf}}(M; L^\infty(\alpha; \mathbb{Z})).$$

Now the equivalence of the three statements easily follows. \square

3.3. Gaining boundary control

One benefit of studying the integral foliated locally finite simplicial volume of tame manifolds is that it gives an upper bound on relative integral foliated simplicial volume of their closure, including boundary control. This fact will play a crucial role in deriving the main theorem (Theorem 1) from the finite-volume hyperbolic case.

PROPOSITION 3.14. *Let W be an oriented compact connected n -manifold, let $M := W^\circ$, and let α be a standard $\Pi(M)$ -space. Then*

$$|W, \partial W|_\partial^\alpha \leq |M|_{\text{lf}}^\alpha.$$

Proof. If $|M|_{\text{lf}}^\alpha = \infty$, there is nothing to show.

Otherwise, we prove that for every α -parametrised locally finite fundamental cycle c of M with $|c|_1 < \infty$, and for every $\varepsilon > 0$, there exists an α -parametrised relative fundamental cycle $c'(\varepsilon)$ of W such that

$$|c'(\varepsilon)|_1 \leq |c|_1 \quad \text{and} \quad |\partial c'(\varepsilon)|_1 \leq \varepsilon.$$

The proof follows the one for ordinary simplicial volume [23, Proposition 5.12]. Since M is the interior of W , we can identify $M \cong W \cup_{\partial W} (\partial W \times [0, +\infty))$ as in the proof of Proposition 3.12. Then, we can consider the exhaustion of M by compact sets given by $K_r := W \cup_{\partial W} (\partial W \times [0, r])$ for all $r \in \mathbb{N}$. Let us write $c_r \in C_n(M; L^\infty(\alpha; \mathbb{Z}))$ for the (finite) chain obtained from c by setting to 0 the coefficients of the simplices that do not intersect K_r . By construction, we have a monotone increasing sequence $(|c_r|_1)_{r \in \mathbb{N}}$ such that

$$\lim_{r \rightarrow +\infty} |c_r|_1 = |c|_1.$$

Therefore, since $|c|_1 < \infty$, it follows that $\lim_{r \rightarrow +\infty} |c|_1 - |c_r|_1 = 0$, and so for each $\varepsilon \in \mathbb{R}_{>0}$, taking $R \in \mathbb{N}$ sufficiently large yields

$$|c|_1 - |c_R|_1 \leq \frac{\varepsilon}{n+1}.$$

Since we have $|c|_1 - |c_R|_1 = |c - c_R|_1$, we get the following estimate on the ℓ^1 -norm of the boundary of c_R :

$$|\partial c_R|_1 = |\partial(c - c_R)|_1 \leq (n+1) \cdot |c - c_R|_1 \leq (n+1) \cdot \frac{\varepsilon}{n+1} = \varepsilon.$$

We now claim that the desired α -parametrised relative fundamental cycle $c'(\varepsilon)$ of W can be obtained by taking the pushforward of c_R along the collapse map $M \rightarrow W$. Indeed, this new α -parametrised chain $c'(\varepsilon) \in C_n(W; L^\infty(\alpha; \mathbb{Z}))$ satisfies

$$|c'(\varepsilon)|_1 \leq |c_R|_1 \quad \text{and} \quad |\partial c'(\varepsilon)|_1 \leq |\partial c_R|_1.$$

Performing the construction of restriction and projection to a locally finite *integral* fundamental cycle z of M and to a corresponding boundary ∂b to c for large enough r leads to a relative fundamental cycle z' of W [23, Theorem 5.4, Proposition 5.12] that differs by a boundary $\partial b'$ from $c'(\varepsilon)$. Hence, $c'(\varepsilon)$ is a relative α -parametrised fundamental cycle of W . This finishes the proof. \square

4. A proportionality principle for finite-volume hyperbolic 3-manifolds

In this section, we will show that the locally finite integral foliated simplicial volume satisfies a proportionality principle for complete finite-volume hyperbolic 3-manifolds.

THEOREM 5. *Let M be an oriented complete connected finite-volume hyperbolic 3-manifold (without boundary). Then*

$$|M|_{\text{lf}} = \frac{\text{vol}(M)}{v_3},$$

where v_3 is the volume of any regular ideal tetrahedron in hyperbolic 3-space. \square

As a corollary of this result, we will be able to compute the precise value of the $\widehat{\pi_1(M)}$ -parametrised integral simplicial volume of all compact hyperbolic 3-manifolds M with toroidal boundary (Section 5).

The main step of the proof of the proportionality Theorem 5 consists in the following result, whose proof is discussed in detail in Sections 4.1–4.4.

THEOREM 4.1. *Let $n \in \mathbb{N}$, let N be an oriented closed connected hyperbolic n -manifold, and let M be an oriented complete connected hyperbolic n -manifold of finite volume. Then*

$$\frac{\|N\|_{\mathbb{Z}}}{\text{vol}(N)} \geq \frac{|M|_{\text{lf}}}{\text{vol}(M)}.$$

Moreover, we need the following classical computations:

REMARK 4.2 (proportionality for classical simplicial volume). If M is an oriented closed connected hyperbolic n -manifold, then Gromov and Thurston established the proportionality [3, Theorem C.4.2; 18, 36]

$$\|M\| = \frac{\text{vol}(M)}{v_n}.$$

Furthermore, this proportionality also holds for

- the locally finite simplicial volume in the finite-volume case; and
- the relative simplicial volume (also with boundary control) in the compact case with toroidal boundary.

By now, many proofs of these extensions can be found in the literature – some of them deriving the relative case from the finite-volume case, and some of them deriving the finite-volume case from the relative case. A concise exposition is, for instance, given by Fujiwara and Manning [16, Appendix A].

We are now ready to prove the proportionality Theorem 5.

Proof of Theorem 5. We first prove that

$$|M|_{\text{lf}} \geq \frac{\text{vol}(M)}{v_3}.$$

By Proposition 3.4, we have the following inequality between locally finite integral foliated simplicial volume and the classical locally finite simplicial volume:

$$|M|_{\text{lf}} \geq \|M\|_{\text{lf}}.$$

Therefore, the proportionality principle for complete finite-volume hyperbolic manifolds (Remark 4.2) allows us to conclude that

$$|M|_{\text{lf}} \geq \|M\|_{\text{lf}} = \frac{\text{vol}(M)}{v_3}.$$

We now prove the opposite inequality. The stable integral simplicial volume of every closed hyperbolic 3-manifold Z agrees with its classical simplicial volume [13, Theorem 1.7]. This shows that

$$\|Z\|_{\mathbb{Z}}^{\infty} = \|Z\| = \frac{\text{vol}(Z)}{v_3},$$

where the last equality comes from the proportionality principle for closed hyperbolic manifolds (Remark 4.2). The crucial step of the proof is now Theorem 4.1. Indeed, since the volume is multiplicative with respect to finite coverings, we have

$$\frac{\|Z\|_{\mathbb{Z}}^{\infty}}{\text{vol}(Z)} = \inf \left\{ \frac{\|N\|_{\mathbb{Z}}}{\text{vol}(N)} \mid N \rightarrow Z \text{ is a finite covering} \right\}.$$

Applying Theorem 4.1, we can conclude that

$$\frac{1}{v_3} = \frac{\|Z\|_{\mathbb{Z}}^{\infty}}{\text{vol}(Z)} \geq \frac{|M|_{\text{lf}}}{\text{vol}(M)}.$$

This finishes the proof. \square

REMARK 4.3. It is worth noting that Theorem 5 cannot hold in higher dimensions. Indeed, in the closed case, in dimension $n \geq 4$, there is a uniform gap between integral foliated simplicial volume and simplicial volume for hyperbolic n -manifolds [13, Theorem 1.8].

4.1. Setup for the construction of the smearing map

We now come to the preparations for the proof of Theorem 4.1. In order to prove this weak form of proportionality, we will stick to a simplistic, discrete version of smearing (which will be enough for our purposes), based on the following notion of meshes and straightening.

DEFINITION 4.4 (decorated mesh of a Riemannian manifold). Let M be a Riemannian manifold without boundary. A *mesh* $\mathcal{P} = (P_i)_{i \in I}$ is a locally finite partition of M into (at most) countably many Borel sets obtained from a locally finite triangulation of M . Moreover, we ask that the universal covering projection be trivial over each P_i .

A *decorated mesh* of M is a pair (\mathcal{P}, T) , where \mathcal{P} is a mesh of M and $T = (t_i)_{i \in I}$ is a countable subset of M such that $t_i \in P_i$ for each $i \in I$.

We define the *size* of a (decorated) mesh \mathcal{P} to be the supremum of all the diameters of the Borel sets appearing in \mathcal{P} .

Setup 4.5. Let $n \in \mathbb{N}$, let N be an oriented *closed* connected hyperbolic n -manifold (with fundamental group Λ and universal covering projection $\pi_N: \mathbb{H}^n \rightarrow N$), and let M be an oriented connected *complete finite-volume* hyperbolic n -manifold (with fundamental group $\Gamma := \pi_1(M, x_0)$ and universal covering projection $\pi_M: \mathbb{H}^n \rightarrow M$). Moreover, let $G := \text{Isom}^+(\mathbb{H}^n)$ be the group of orientation-preserving isometries of \mathbb{H}^n .

We first choose an integral fundamental cycle c of N , which by Remark 3.9 we may assume is geodesic. We denote by Δ the maximal diameter of lifts to \mathbb{H}^n of simplices with non-zero coefficient in c .

We construct meshes on M and \mathbb{H}^n as follows.

- We choose a mesh \mathcal{P} of M whose size is smaller than the global injectivity radius of N .
- By the triviality condition in the definition of the mesh \mathcal{P} , we can choose, for each P_i , a homeomorphic lift $\tilde{P}_i \subset \mathbb{H}^n$. We write $\tilde{\mathcal{P}} := (\tilde{P}_i)_{i \in I}$. The union $D := \bigcup_{i \in I} \tilde{P}_i$ is then a Borel fundamental domain for the deck transformation action of Γ on \mathbb{H}^n . Then $\Gamma \cdot \tilde{\mathcal{P}}$ is a mesh of \mathbb{H}^n with size smaller than the global injectivity radius of N .
- We now impose one further restriction on the size of the Borel sets in \mathcal{P} (and $\Gamma \cdot \tilde{\mathcal{P}}$), which can be attained through a suitable subdivision. For each $i \in I$, let K_i be the closure of the union of all Borel sets in $\Gamma \cdot \tilde{\mathcal{P}}$ that intersect the Δ -neighbourhood of \tilde{P}_i . Now consider the volume function

$$\begin{aligned} \text{vol}: (\mathbb{H}^n)^{n+1} &\rightarrow \mathbb{R} \\ (x_0, \dots, x_n) &\mapsto \text{vol}_{\mathbb{H}^n}(\text{straight}(x_0, \dots, x_n)). \end{aligned}$$

Since $K_i^{n+1} \subset (\mathbb{H}^n)^{n+1}$ is compact, and vol is continuous with respect to the maximum metric d_∞ on $(\mathbb{H}^n)^{n+1}$, the restriction of vol to K_i^{n+1} is uniformly continuous. In particular, there exists $\delta_i > 0$ such that whenever \underline{x} and \underline{y} in K_i^{n+1} satisfy $d_\infty(\underline{x}, \underline{y}) \leq \delta_i$, we have

$$|\text{vol}(\underline{x}) - \text{vol}(\underline{y})| < \frac{\text{vol}(N)}{2 \cdot |c|_1}. \tag{1}$$

The fact that $\Gamma \cdot \tilde{\mathcal{P}}$ is locally finite implies that for each $j \in I$, the Borel set \tilde{P}_j is contained in at most finitely many Γ -translates of subsets K_i . Taking ε_j to be the (strictly positive) minimum among the respective δ_i , we refine the mesh \mathcal{P} by (locally finitely) subdividing each P_j into Borel sets of diameter less than ε_j .

We will use the following straightening procedure.

- For $k \in \mathbb{N}$, we write $\tilde{S}_k \subset \text{map}(\Delta^k, \mathbb{H}^n)$ for the set of geodesic simplices in \mathbb{H}^n whose vertices all lie in $\Gamma \cdot \tilde{T}$ and whose 0-vertex lies in \tilde{T} . For the corresponding set of simplices on M , we use the notation $S_k := \{\pi_M \circ \tau \mid \tau \in \tilde{S}_k\}$. For $\varrho \in S_k$, we designate by $\tilde{\varrho}$ the unique π_M -lift of ϱ in \tilde{S}_k .
- For $k \in \mathbb{N}$, we consider the function

$$\text{snap}: \text{map}(\Delta^k, \mathbb{H}^n) \rightarrow \Gamma \cdot \tilde{S}_k$$

that ‘snaps’ the vertices of $\sigma: \Delta^k \rightarrow \mathbb{H}^n$ to the grid $\Gamma \cdot \tilde{T}$ and then takes the geodesic simplex associated with this tuple of vertices. More precisely, let Q_0, \dots, Q_k be the Borel sets in the mesh $\Gamma \cdot \tilde{\mathcal{P}}$ containing the vertices of σ , and let q_0, \dots, q_k the corresponding points in $\Gamma \cdot \tilde{T}$. Then, the map snap sends σ to the geodesic simplex in $\Gamma \cdot \tilde{S}_k$ spanned by the vertices q_0, \dots, q_k .

We consider the standard Γ -space Z_M on M given by:

- the probability space G/Λ (with the normalised Haar measure $\bar{\mu}$); and
- the Γ -action by left translation on G .

Here, we endow G with the (unimodular) Haar measure μ satisfying the normalisation $\mu(X_M) = 1$, where $X_M \subset G$ is a Borel fundamental domain of the right Λ -action on G .

Moreover, we will use the following parameter space on M : For each point $p \in M$, we choose a path c_p in M from the basepoint $x_0 \in M$ to p , with the property that each c_p lifts to a path in \mathbb{H}^n having both endpoints in the same Γ -translate of the fundamental domain D . We then extend the standard Γ -space Z_M to a standard $\Pi(M)$ -space α_M using this choice of paths (Remark 2.6). Then α_M is ergodic by the Moore ergodicity theorem [40, Theorem 4.10.2]. We will also make use of the associated normed local coefficient system $L^\infty(\alpha_M; \mathbb{Z})$ on M (Definition 2.5).

4.2. A useful Borel map

The following maps allow to smear simplices of N over \mathbb{H}^n and then to associate the correct ‘weight’ to them on M .

DEFINITION 4.6 (a useful Borel map). In the situation of Setup 4.5, for $k \in \mathbb{N}$, for every singular simplex $\sigma: \Delta^k \rightarrow N$ and for every simplex $\tau \in \Gamma \cdot \tilde{S}_k$, we define the function $f_{\sigma, \tau} \in L^\infty(Z_M, \mathbb{Z})$ as follows: We choose a π_N -lift $\tilde{\sigma}$ of σ and then, we set

$$f_{\sigma, \tau}(x\Lambda) := \#\{\lambda \in \Lambda \mid \text{snap}(x\lambda\tilde{\sigma}) = \tau\}.$$

LEMMA 4.7. *In the situation of Setup 4.5, the map $f_{\sigma, \tau}$ is a well-defined essentially bounded Borel map $Z_M \rightarrow \mathbb{Z}$.*

Proof. First of all, we prove that the definition of $f_{\sigma, \tau}$ does not depend on the choice of lift $\tilde{\sigma}$ of σ . Indeed, for a different lift $\lambda\tilde{\sigma}$ (where $\lambda \in \Lambda$), we have a bijection

$$\begin{aligned} \{\mu \in \Lambda \mid \text{snap}(x\mu\tilde{\sigma}) = \tau\} &\rightarrow \{\mu \in \Lambda \mid \text{snap}(x\mu\lambda\tilde{\sigma}) = \tau\} \\ \mu &\mapsto \mu\lambda^{-1}. \end{aligned}$$

Similarly, independence of the choice of x over, say, $x\lambda$ (with $\lambda \in \Lambda$) as a coset representative for $x\Lambda$ follows from the bijection

$$\begin{aligned} \{\mu \in \Lambda \mid \text{snap}(x\mu\tilde{\sigma}) = \tau\} &\rightarrow \{\mu \in \Lambda \mid \text{snap}(x\lambda\mu\tilde{\sigma}) = \tau\} \\ \mu &\mapsto \lambda^{-1}\mu. \end{aligned}$$

This shows that $f_{\sigma, \tau}$ is a well-defined function (to $\mathbb{N} \cup \{\infty\}$).

To see that the function $f_{\sigma, \tau}$ lives in $L^\infty(Z_M; \mathbb{Z})$, we first show that it is essentially bounded. In fact, we prove more: it only takes values in the set $\{0, 1\}$. Indeed, given $x \in G$, we consider the translated decorated mesh $(x^{-1} \cdot \Gamma \cdot \tilde{\mathcal{P}}, x^{-1} \cdot \Gamma \cdot \tilde{T})$ of \mathbb{H}^n . The size of this mesh is still less than the global injectivity radius of N . Therefore, any two distinct Λ -translates of a simplex in \mathbb{H}^n cannot have the vertices in the same Borel sets of the mesh $x^{-1} \cdot \Gamma \cdot \tilde{\mathcal{P}}$. This means that no two such translates can have the vertices close to the ones of $x^{-1}\tau$. Looking again at the original mesh $\Gamma \cdot \tilde{\mathcal{P}}$, this means that for each simplex $\tilde{\sigma} \in \mathbb{H}^n$, there is at most one $\lambda \in \Lambda$ such that $\text{snap}(x\lambda\tilde{\sigma}) = \tau$.

To see that $f_{\sigma, \tau}$ is a Borel-measurable function, observe that by definition, we can express $f_{\sigma, \tau}$ as

$$x\Lambda \mapsto \begin{cases} 1 & \text{if there exists } \lambda \in \Lambda \text{ with } \text{snap}(x\lambda\tilde{\sigma}) = \tau \\ 0 & \text{otherwise} \end{cases}.$$

The function $f_{\sigma, \tau}$ is thus the characteristic function of a subset $A \subseteq G/\Lambda$, which we claim is Borel.

To this end, we express A differently: consider the (continuous, hence measurable) map

$$\begin{aligned} \psi: G \times \Lambda \times (\mathbb{H}^n)^{k+1} &\rightarrow (\mathbb{H}^n)^{k+1} \\ (x, \lambda, \alpha) &\mapsto x\lambda \cdot \alpha, \end{aligned}$$

where G acts diagonally on the right-hand side. Note that a tuple $\alpha \in (\mathbb{H}^n)^{k+1}$ uniquely determines a geodesic simplex in \mathbb{H}^n , and vice versa, so we now incorporate this identification into the notation. Let Q_0, \dots, Q_k be the Borel subsets in the mesh $\Gamma \cdot (\tilde{P}_i)_{i \in I}$ containing, respectively, the vertices $\tau[0], \dots, \tau[k]$ of τ . Then it is easy to check that A is the set obtained by projecting

$$(G \times \Lambda \times \{\tilde{\sigma}\}) \cap \psi^{-1}(Q_0 \times \dots \times Q_k)$$

via the composition $G \times \Lambda \times \{\tilde{\sigma}\} \rightarrow G \rightarrow G/\Lambda$.

This construction of A starts with the intersection of a Borel set and a closed set, which is thus Borel. Hence, we are only left to show that the two projection maps that follow take Borel sets to Borel sets. Let us denote by $\iota_\lambda: G \rightarrow G \times \Lambda \times \{\tilde{\sigma}\}$ the inclusion of G at the $(\lambda, \tilde{\sigma})$ -slice. Since the projection $G \times \Lambda \times \{\tilde{\sigma}\} \rightarrow G$ takes each Borel subset B to the countable union $\bigcup_{\lambda \in \Lambda} \iota_\lambda^{-1}(B)$, it sends Borel sets to Borel sets. Finally, if B is a Borel subset of G , then for the quotient map $p: G \rightarrow G/\Lambda$ we see that $p^{-1}(p(B)) = \bigcup_{\lambda \in \Lambda} \lambda \cdot B$ is Borel, so $p(B)$ is Borel. \square

4.3. The smearing map and its properties

DEFINITION 4.8 (the parametrised discrete smearing map). In the situation of Setup 4.5, for $k \in \mathbb{N}$, we define the *parametrised discrete smearing map in degree k* by

$$\begin{aligned} \varphi_k: C_k(N; \mathbb{Z}) &\rightarrow C_k(M; L^\infty(\alpha_M, \mathbb{Z})) \\ \sigma &\mapsto \sum_{\varrho \in S_k} f_{\sigma, \tilde{\varrho}} \cdot \varrho. \end{aligned}$$

LEMMA 4.9. In the situation of Setup 4.5, we have:

- (1) for each $k \in \mathbb{N}$, the map φ_k is well-defined;
- (2) the sequence $(\varphi_k)_{k \in \mathbb{N}}$ is a chain map;
- (3) the image of φ_* consists of smooth chains.

Proof. Ad 1. We have to show that φ_k sends integral finite chains to locally finite chains with local coefficients, and it is clearly enough to show that this is true for each simplex σ on N . So let $K \subset M$ be a compact subset and let $\varrho \in S_k$ be a simplex with non-zero coefficient in $\varphi_k(\sigma)$ that intersects K . We will see that these conditions on ϱ restrict it to a finite number of possibilities.

Since there is an isometric translate of $\tilde{\sigma}$ that snaps to $\tilde{\varrho}$, by the triangle inequality, we see that $\tilde{\varrho}$ must have diameter at most

$$d := \text{diam}(\tilde{\sigma}) + 2 \cdot \text{size}(\mathcal{P}),$$

and because the universal covering projection map π_M is distance-non-increasing, the same upper bound is valid for $\text{diam}(\varrho)$.

It follows that $\varrho[0]$ is in the d -neighbourhood of K , and as \mathcal{P} is locally finite, there are only finitely many $P_1, \dots, P_f \in \mathcal{P}$ intersecting this neighbourhood. Hence, $\varrho[0]$ lies in one among $\tilde{P}_1, \dots, \tilde{P}_f \in \tilde{\mathcal{P}}$. The image of $\tilde{\varrho}$ must therefore be contained in a d -neighbourhood of the union $\bigcup_{j=1}^f \tilde{P}_j$. This is a bounded subset of \mathbb{H}^n , which again intersects only finitely many Borel sets Q_1, \dots, Q_m of the locally finite mesh $\Gamma \cdot \tilde{\mathcal{P}}$. Denoting by t_1, \dots, t_m the corresponding

points of the decorated structure $\Gamma \cdot \tilde{T}$, we see that $\tilde{\varrho}$ is constrained to have its vertices in the set $\{t_1, \dots, t_m\}$. As a $\tilde{\varrho}$ is geodesic, and geodesic simplices are entirely determined by their (ordered) vertices, we conclude there are only finitely many possibilities for $\tilde{\varrho}$, and thus the same is true of ϱ .

Ad 2. We now prove that φ_k is a chain map, by showing that it is compatible with each face map ∂_j . In fact, we shall present only the computation of the most delicate case $j = 0$, the remaining ones being easily recovered by suppressing all occurrences of the correction term γ_ϱ . By Definition 2.2, we have, for each $\sigma: \Delta^k \rightarrow N$,

$$\partial_0(\varphi_k(\sigma)) = \sum_{\varrho \in S_k} (f_{\sigma, \tilde{\varrho}} \cdot \gamma_\varrho) \cdot \partial_0 \varrho,$$

where γ_ϱ denotes the element $c_{\varrho[0]} * \varrho[0, 1] * c_{\varrho[1]}^{-1} \in \Gamma$. It is straightforward to check that each $\gamma \in \Gamma$ acts on $f_{\sigma, \tau}$ by $f_{\sigma, \tau} \cdot \gamma = f_{\sigma, \gamma^{-1}\tau}$, so the previous formula can be rewritten as

$$\partial_0(\varphi_k(\sigma)) = \sum_{\varrho \in S_k} f_{\sigma, \gamma_\varrho^{-1} \tilde{\varrho}} \cdot \partial_0 \varrho.$$

Expressing this summation in terms of the $(k - 1)$ -simplices yields

$$\partial_0(\varphi_k(\sigma)) = \sum_{\tau \in S_{k-1}} \left(\sum_{\substack{\varrho \in S_k \\ \partial_0 \varrho = \tau}} f_{\sigma, \gamma_\varrho^{-1} \tilde{\varrho}} \right) \cdot \tau.$$

We now claim that for each $\tau \in S_{k-1}$, the inner summation simplifies as follows:

$$\sum_{\substack{\varrho \in S_k \\ \partial_0 \varrho = \tau}} f_{\sigma, \gamma_\varrho^{-1} \tilde{\varrho}} = f_{\partial_0 \sigma, \tilde{\tau}}. \tag{2}$$

After this step is justified, the proof will be complete, since

$$\sum_{\tau \in S_{k-1}} f_{\partial_0 \sigma, \tilde{\tau}} \cdot \tau = \varphi_{k-1}(\partial_0 \sigma).$$

We now justify Formula (2). The function on the left-hand side assigns to each coset $x\Lambda \in \Gamma/\Lambda$ the value $\sum_{\substack{\varrho \in S_k \\ \partial_0 \varrho = \tau}} \#\{\lambda \in \Lambda \mid \text{snap}(x\lambda\tilde{\sigma}) = \gamma_\varrho^{-1} \tilde{\varrho}\}$, which is the cardinality of the disjoint union

$$A := \coprod_{\substack{\varrho \in S_k \\ \partial_0 \varrho = \tau}} \{\lambda \in \Lambda \mid \text{snap}(x\lambda\tilde{\sigma}) = \gamma_\varrho^{-1} \tilde{\varrho}\}.$$

On the other hand, the right-hand side of Formula (2) assigns to $x\Lambda$ the cardinality of

$$\begin{aligned} B &:= \{\lambda \in \Lambda \mid \text{snap}(x\lambda\partial_0 \tilde{\sigma}) = \tilde{\tau}\} \\ &= \{\lambda \in \Lambda \mid \partial_0 \text{snap}(x\lambda\tilde{\sigma}) = \tilde{\tau}\}. \end{aligned}$$

Since each of $c_{\varrho[0]}, c_{\varrho[1]}$ lifts to a path with both endpoints in the same Γ -translate of D , and for each $\varrho \in S_k$ the simplex $\tilde{\varrho}$ has its 0-vertex in D , it follows that γ_ϱ^{-1} acts by bringing the 1-vertex of $\tilde{\varrho}$ to D . In other words, if $\partial_0 \varrho = \tau$, then $\partial_0(\gamma_\varrho^{-1} \tilde{\varrho}) = \tilde{\tau}$.

Having made this observation, it is straightforward to check that the map from A to B taking $(\lambda, \varrho) \mapsto \lambda$ is a well-defined bijection, with inverse $\lambda \mapsto (\lambda, \pi_M \circ \text{snap}(x\lambda\tilde{\sigma}))$.

Ad 3. As the chains in the image of φ_* consist of geodesic simplices, they are also smooth (Remark 3.8). □

PROPOSITION 4.10. *In the situation of Setup 4.5, the map $H_n(\varphi_*)$ sends the integral fundamental class $[N]$ of N to the α_M -parametrised locally finite fundamental class $[M]_{\text{lf}}^{\alpha_M}$ of M .*

In order to prove Proposition 4.10, we first establish an upper bound for the norm of the chain map φ , whence also for the norm of $H_n(\varphi_*)$.

LEMMA 4.11. *In the situation of Setup 4.5, we have, for all $k \in \mathbb{N}$:*

(1) *if $\sigma \in \text{map}(\Delta^k, N)$, then*

$$\sum_{\varrho \in S_k} \int_{G/\Lambda} f_{\sigma, \varrho} \, d\bar{\mu} = \sum_{\varrho \in S_k} \int_{G/\Lambda} |f_{\sigma, \varrho}| \, d\bar{\mu} = \frac{\text{vol}(M)}{\text{vol}(N)};$$

(2) *in particular, $\|\varphi_k\| \leq \text{vol}(M)/\text{vol}(N)$.*

Proof. Since we are dealing with the ℓ^1 -norms on the chain complexes, it suffices to prove the first part. Let $\sigma \in \text{map}(\Delta^k, N)$ and let $\tilde{\sigma}$ be a π_N -lift of σ . We set $t_0 := \tilde{\sigma}[0] \in \mathbb{H}^n$ and $X_N := \{x \in G \mid x(t_0) \in D\}$. Then X_N is a Borel fundamental domain for the left action of Γ on G and (because the Haar measure on G pushes forward, under the canonical projection $G \rightarrow \mathbb{H}^n$, to a scalar multiple of the hyperbolic volume)

$$\mu(X_N) = \frac{\mu(X_N)}{\mu(X_M)} = \frac{\text{vol}(M)}{\text{vol}(N)}.$$

Moreover, we have

$$\begin{aligned} |\varphi_k(\sigma)|_1 &= \sum_{\varrho \in S_k} \int_{G/\Lambda} |f_{\sigma, \varrho}| \, d\bar{\mu} \\ &= \int_{G/\Lambda} (x\Lambda \mapsto \#\{\lambda \in \Lambda \mid x\lambda\tilde{\sigma}[0] \in D\}) \, d\bar{\mu} \\ &= \int_{X_M} (x \mapsto \#\{\lambda \in \Lambda \mid x\lambda \in X_N\}) \, d\mu \\ &= \int_{X_M} \sum_{\lambda \in \Lambda} \chi_{X_N \cdot \lambda^{-1}} \, d\mu \\ &= \sum_{\lambda \in \Lambda} \mu(X_M \cap X_N \cdot \lambda^{-1}) \\ &= \sum_{\lambda \in \Lambda} \mu(X_M \cdot \lambda \cap X_N) \\ &= \mu(X_N) = \frac{\text{vol}(M)}{\text{vol}(N)}. \end{aligned}$$

Because the functions $f_{\sigma, \varrho}$ all are non-negative (by construction), we also have $\sum_{\varrho \in S_k} \int_{G/\Lambda} f_{\sigma, \varrho} \, d\bar{\mu} = |\varphi_k(\sigma)|_1 = \text{vol}(M)/\text{vol}(N)$. This finishes the proof. \square

Proof of Proposition 4.10. We will prove that $H_n(\varphi_*)([N])$ is an α -parametrised locally finite fundamental class via double integration. Let c be the geodesic fundamental cycle of N appearing in Setup 4.5. We aim to show that $\varphi_n(c)$ satisfies condition (3) of Proposition 3.13. However, in order to apply this proposition, we first have to check that $\varphi_n(c)$ is Lipschitz and

of finite ℓ^1 -norm. Since φ_n has bounded norm by Lemma 4.11 and $|c|_1 < +\infty$, also $\varphi_n(c)$ has finite ℓ^1 -norm. In order to show that $\varphi_n(c)$ is Lipschitz, let us call L the size of the mesh \mathcal{P} . Then, all pairs of vertices of the geodesic simplices lying in \tilde{S}_n that contribute to $\varphi_n(c)$ have distance at most $\Delta + 2L$. Therefore, they all have diameter bounded by a constant depending only on Δ and L . Therefore, by Remark 3.8, we know that also their Lipschitz constants are all bounded by a uniform constant depending only on Δ and L . This condition clearly reflects to the simplices of S_n . This proves that $\varphi_n(c)$ is Lipschitz.

We are now ready to apply double integration to $\varphi_n(c)$. Writing $c = \sum_{j=1}^t a_j \sigma_j$, we have, on the one hand,

$$\langle \text{dvol}_M, I_{\mathbb{R}}(\varphi_n(c)) \rangle = \sum_{j=1}^t a_j \cdot \left(\sum_{\varrho \in S_n} \left(\int_{G/\Lambda} f_{\sigma_j, \tilde{\varrho}} d\bar{\mu} \right) \cdot \int_{\Delta^n} \varrho^* \text{dvol}_M \right),$$

where $\bar{\mu}$ denotes the normalised Haar measure described in Setup 4.5; on the other hand, because c is a fundamental cycle of N , we have

$$\text{vol}(M) = \frac{\text{vol}(M)}{\text{vol}(N)} \cdot \left(\sum_{j=1}^t a_j \int_{\Delta^n} \tilde{\sigma}_j^* \text{dvol}_{\mathbb{H}^n} \right).$$

Therefore, using Lemma 4.11.1, we obtain

$$\begin{aligned} & \left| \langle \text{dvol}_M, I_{\mathbb{R}}(\varphi_n(c)) \rangle - \text{vol}(M) \right| \\ &= \left| \sum_{j=1}^t a_j \sum_{\varrho \in S_n} \left(\int_{G/\Lambda} f_{\sigma_j, \tilde{\varrho}} d\bar{\mu} \right) \cdot \left(\int_{\Delta^n} \varrho^* \text{dvol}_M - \int_{\Delta^n} \tilde{\sigma}_j^* \text{dvol}_{\mathbb{H}^n} \right) \right| \\ &= \left| \sum_{j=1}^t a_j \sum_{\varrho \in S_n} \left(\int_{G/\Lambda} f_{\sigma_j, \tilde{\varrho}} d\bar{\mu} \right) \cdot \left(\int_{\Delta^n} \tilde{\varrho}^* \text{dvol}_{\mathbb{H}^n} - \int_{\Delta^n} \tilde{\sigma}_j^* \text{dvol}_{\mathbb{H}^n} \right) \right| \\ &\leq \sum_{j=1}^t |a_j| \sum_{\varrho \in S_n} \left(\int_{G/\Lambda} |f_{\sigma_j, \tilde{\varrho}}| d\bar{\mu} \right) \cdot \left| \int_{\Delta^n} \tilde{\varrho}^* \text{dvol}_{\mathbb{H}^n} - \int_{\Delta^n} \tilde{\sigma}_j^* \text{dvol}_{\mathbb{H}^n} \right|. \end{aligned}$$

Now observe that whenever $\tilde{\sigma}_j$ is not a G -translate of some simplex that snaps to $\tilde{\varrho}$, the function $f_{\sigma_j, \tilde{\varrho}}$ is identically 0, and so the summands corresponding to such j, ϱ vanish. For the remaining summands j, ϱ , we see that the volume estimate given by Formula (1) from Setup 4.5 applies to $\tilde{\sigma}_j, \tilde{\varrho}$. The previous expression is thus bounded above by

$$\begin{aligned} & \sum_{j=1}^t |a_j| \sum_{\varrho \in S_n} \left(\int_{G/\Lambda} |f_{\sigma_j, \tilde{\varrho}}| d\bar{\mu} \right) \cdot \frac{\text{vol}(N)}{2 \cdot |c|_1} \\ & \leq \left(\sum_{j=1}^t |a_j| \right) \cdot \frac{\text{vol}(M)}{\text{vol}(N)} \cdot \frac{\text{vol}(N)}{2 \cdot |c|_1} = \frac{\text{vol}(M)}{2}. \end{aligned}$$

We have thus verified condition (3) from Proposition 3.13, which finishes the proof. □

4.4. Proof of Theorem 4.1

The proof of Theorem 4.1 now follows easily from the previous construction.

Proof of Theorem 4.1. Let $n \in \mathbb{N}$. The hypotheses on N and M allow us to assume the situation of Setup 4.5. Thus, we have the chain map

$$\varphi_* : C_*(N; \mathbb{Z}) \rightarrow C_*(M; L^\infty(\alpha_M, \mathbb{Z}))$$

described in Section 4.3. Let $c \in C_n(N; \mathbb{Z})$ be an integral fundamental cycle of N . As proved in Propositions 4.10 and 4.11 (2), we have that $\varphi_n(c) \in C_n(M; L^\infty(\alpha_M, \mathbb{Z}))$ is an α_M -parametrised locally finite fundamental cycle whose ℓ^1 -norm is bounded as follows:

$$|\varphi_n(c)|_1 \leq \frac{\text{vol}(M)}{\text{vol}(N)} \cdot |c|_1.$$

By taking the infimum on both sides, we get the claim

$$\frac{\|N\|_{\mathbb{Z}}}{\text{vol}(N)} \geq \frac{|M|_{\text{lf}}}{\text{vol}(M)}. \quad \square$$

5. A proportionality principle for hyperbolic 3-manifolds with toroidal boundary

From Theorem 5, we can now derive the corresponding relative result with boundary control:

COROLLARY 6. *Let W be an oriented compact connected hyperbolic 3-manifold with empty or toroidal boundary and let $M := W^\circ$. Then*

$$|W, \partial W|_{\partial}^{\widehat{\Pi}(W)} = \frac{\text{vol}(M)}{v_3}. \quad \square$$

As in the corresponding result for the closed case, the proof will use input from ergodic theory, more specifically of the ergodic theoretic properties of the profinite completion of fundamental groups of hyperbolic 3-manifolds (Section 5.1).

5.1. The profinite completion of hyperbolic 3-manifold groups

The dependency of parametrised simplicial volume on the chosen parameter space seems to be a difficult problem (similar to the fixed price problem for cost of groups). In some special cases, it is known that the profinite completion of the fundamental group is the ‘best’ parameter space. This can be formalised in terms of *weak containment* of parameter spaces and the fact that parametrised simplicial volume is monotone with respect to weak containment of parameter spaces. These technical definitions and statements have been deferred to the Appendix.

DEFINITION 5.1 (Property EMD*). An infinite countable group Γ satisfies EMD* if every ergodic standard Γ -space is weakly contained in the profinite completion $\widehat{\Gamma}$ of Γ .

In the following, we prove that hyperbolic 3-manifolds with empty or toroidal boundary satisfy EMD*. In the closed case, this has been already noticed by Kechris [22, p. 487] and Bowen and Tucker-Drob [4, p. 212] and proved in detail by Frigerio, Löh, Pagliantini and Sauer [13, Proposition 3.10].

PROPOSITION 5.2. *Let W be an oriented compact connected hyperbolic 3-manifold with empty or toroidal boundary. Then $\pi_1(W)$ satisfies EMD*.*

Proof. By Agol’s virtual fibre theorem [1, 12], there is a finite covering N of W that fibres over S^1 . Then, $\pi_1(N)$ is a semidirect product of Λ and \mathbb{Z} , where Λ is the fundamental group of an oriented compact surface. For residually finite groups, the property EMD* is equivalent to the property MD [37, Theorem 1.4], which is a property related to profinite completions introduced by Kechris [22, p. 464]. Surface groups satisfy MD [4, Theorem 1.4] and also free groups satisfy MD [22, Theorem 1], also Λ satisfies MD. By taking Λ as normal subgroup of $\pi_1(N)$, we apply MD-inheritance [4, Theorem 1.4] to see that $\pi_1(N)$ satisfies MD and as a residually finite group also satisfies EMD*. For residually finite groups, the property EMD*

is preserved under passing from a finite index subgroup to the ambient group and therefore, $\pi_1(W)$ satisfies EMD*. \square

5.2. Proof of Corollary 6

After these preparations, we can derive Corollary 6 from Theorem 5:

Proof of Corollary 6. On the one hand, we have (Proposition 2.9 and Remark 4.2)

$$|W, \partial W|_{\partial}^{\widehat{\pi_1(W)}} \geq \|W, \partial W\| = \frac{\text{vol}(M)}{v_3}.$$

On the other hand, we can argue as follows: The proportionality principle for finite-volume hyperbolic 3-manifolds (Theorem 5) and Proposition 3.14 show that

$$|W, \partial W|_{\partial} \leq |M|_{\text{lf}} = \frac{\text{vol}(M)}{v_3}.$$

Moreover, $\pi_1(W)$ satisfies EMD* (Proposition 5.2) and $|W, \partial W|_{\partial}$ can be computed via ergodic spaces (Proposition 2.15). Therefore, monotonicity of parametrised simplicial volume with respect to weak containment (Proposition A.1) shows that

$$|W, \partial W|_{\partial}^{\widehat{\pi_1(W)}} \leq |W, \partial W|_{\partial} \leq \frac{\text{vol}(M)}{v_3}. \quad \square$$

5.3. Summary of the relative case

We conclude this section by showing that all the integral foliated variations of the simplicial volume of compact hyperbolic 3-manifolds with toroidal boundary agree. Moreover, they will also provide the same value as the ordinary simplicial volume and the *ideal simplicial volume*. Recall that the ideal simplicial volume $\|\cdot\|_{\mathcal{I}}$ is a homotopy invariant of compact manifolds [15]. It differs from the standard simplicial volume by allowing the presence of *ideal simplices* in representatives of the fundamental class.

COROLLARY 5.3. *Let W be an oriented compact connected hyperbolic 3-manifold with empty or toroidal boundary. If M denotes the interior of W , then all the following versions of relative simplicial volume equal $\frac{\text{vol}(M)}{v_3}$:*

$$|W, \partial W|_{\partial}^{\widehat{\pi_1(W)}}, |W, \partial W|_{\partial}, |W, \partial W|, \|W, \partial W\|, \|W, \partial W\|_{\mathcal{I}}.$$

Proof. The interior M is a complete finite-volume hyperbolic manifold. Recall by Proposition 2.9 and Remark 4.2 that

$$|W, \partial W| \geq \|W, \partial W\| = \frac{\text{vol}(M)}{v_3}.$$

Thus, applying Proposition 3.14 and Theorem 5, it follows that

$$\frac{\text{vol}(M)}{v_3} = |M|_{\text{lf}} \geq |W, \partial W|_{\partial} \geq |W, \partial W| \geq \frac{\text{vol}(M)}{v_3}.$$

The proportionality principle for $|W, \partial W|_{\partial}^{\widehat{\pi_1(W)}}$ is the content of Corollary 6. For the ideal simplicial volume $\|W, \partial W\|_{\mathcal{I}}$, it has been shown before by Frigerio and the third author [15]. \square

6. *The glueing step*

Our computation of parametrised simplicial volume of complete finite-volume hyperbolic 3-manifolds includes boundary control (Corollary 6). This allows us to prove the following upper bound.

THEOREM 6.1. *Let M be an oriented compact connected 3-manifold with empty or toroidal boundary. If M is prime and not covered by S^3 , then*

$$\|M, \partial M\|_{\mathbb{Z}}^{\infty} = |M, \partial M|^{\widehat{\Pi}(M)} \leq \frac{\text{hypvol}(M)}{v_3}.$$

The proof is based on the JSJ decomposition. We recall terminology and notation around the JSJ decomposition in Section 6.1. The fundamental glueing estimates are established in Section 6.2, specifics of the 3-manifold situation are discussed in Section 6.3, and the proof of Theorem 6.1 is given in Section 6.4. Finally, we use Theorem 6.1 to prove our main Theorem 1 in Section 6.5.

6.1. *The JSJ decomposition*

In this section, we state one of the key ingredients allowing us to assemble Corollary 6 (which pertains to *hyperbolic* 3-manifolds of finite volume) and the computations for *Seifert-fibred* manifolds not covered by S^3 [9; 27, Section 8] into a statement about *prime* manifolds not covered by S^3 . Note that along this section we will only work with irreducible manifolds. Indeed, our argument does not apply to $S^1 \times S^2$. However, it is known that $S^1 \times S^2$ satisfies the integral approximation [27, 34].

Most of the results presented here are by now classical, and can be found, for example, in Aschenbrenner–Friedl–Wilton’s compendium [2] and Martelli’s book [30].

We first remind the reader of some standard terminology: *Seifert-fibred* manifolds are compact 3-manifolds that admit a certain type of decomposition into disjoint circles [2]. An oriented compact connected 3-manifold M is *atoroidal* if every π_1 -injective continuous map $S^1 \times S^1 \rightarrow M$ can be homotoped into ∂M .

We can now state the JSJ decomposition Theorem.

THEOREM 6.2 (JSJ decomposition [2, Theorem 1.6.1]). *Let M be an oriented compact connected 3-manifold with empty or toroidal boundary. If M is irreducible, then there is a (possibly empty) collection of disjointly embedded tori $T_1, \dots, T_m \subset M$, such that each piece obtained by cutting M along $\bigcup_i T_i$ is atoroidal or Seifert-fibred. Up to isotopy, there is a unique such collection with minimal number of tori.*

In keeping with tradition, we have stated the JSJ decomposition theorem for irreducible manifolds, although it also applies to $S^1 \times S^2$, the only prime manifold (without spherical boundary components) that is not irreducible. Indeed, $S^1 \times S^2$ is atoroidal and Seifert-fibred.

The atoroidal pieces in the JSJ decomposition theorem are, as we will now explain, suited to the methods developed throughout Section 5. Indeed, the Hyperbolisation Theorem [2, Theorem 1.7.5] ensures that every piece that is not Seifert-fibred is either hyperbolic, or has finite fundamental group. In our situation, we can, however, rule out the latter possibility, because every piece with finite fundamental group would have to be closed (and in particular, the only piece in the JSJ decomposition) and hence, by the Elliptisation Theorem [2, Theorem 1.7.3], covered by S^3 . We are excluding such manifolds from our main results by hypothesis.

DEFINITION 6.3 (hypvol). Let M be an oriented compact connected 3-manifold with empty or toroidal boundary. If M is irreducible, we denote by $\text{hypvol}(M)$ the sum of the volumes

of the hyperbolic pieces in the JSJ decomposition of M . We extend this definition to prime manifolds by setting $\text{hypvol}(S^1 \times S^2) := 0$.

6.2. Basic glueing estimates

As in the case of vanishing parametrised simplicial volume [9, Propositions 4.4 and 4.5], we can prove the following glueing estimates for glueings along tori. At this point, it is essential that we have control over the boundary contributions.

PROPOSITION 6.4 (glueing estimate). *Let W be an oriented compact connected n -manifold with $n \geq 2$, and let α be an essentially free standard $\Pi(W)$ -space. Let $T \subset W$ be an embedded $(n-1)$ -torus that separates W into two pieces W_1, W_2 . For each $i \in \{1, 2\}$, assume the inclusion $T \hookrightarrow W_i$ as a boundary-component is π_1 -injective and denote by α_i the restriction of α to W_i . Then*

$$|W, \partial W|_{\partial}^{\alpha} \leq |W_1, \partial W_1|_{\partial}^{\alpha_1} + |W_2, \partial W_2|_{\partial}^{\alpha_2}.$$

Proof. We proceed as in the proof with vanishing parametrised simplicial volume [9, proof of Proposition 4.4]. Let $\varepsilon, \varepsilon_{\partial} > 0$. For each $i \in \{1, 2\}$, let $c_i \in C_n(W_i; L^{\infty}(\alpha_i; \mathbb{Z}))$ be a relative fundamental cycle of $(W_i, \partial W_i)$ satisfying

$$|c_i|_1 \leq |W_i, \partial W_i|_{\partial}^{\alpha_i} + \varepsilon \quad \text{and} \quad |\partial c_i|_1 \leq \varepsilon_{\partial}.$$

Denoting by α_0 the restriction of α to T , we see that $c_0 := (\partial c_1)|_T + (\partial c_2)|_T$ is a null-homologous cycle in $C_{n-1}(T; L^{\infty}(\alpha_0; \mathbb{Z}))$ with $|c_0|_1 \leq |\partial c_1|_1 + |\partial c_2|_1 \leq 2 \cdot \varepsilon_{\partial}$.

As α_0 is essentially free, there is an $(n-1)$ -UBC constant $K > 0$ for $C_*(T; L^{\infty}(\alpha_0; \mathbb{Z}))$ [9, Proposition 4.1][10, Theorem 1.3]. This means there is a chain $b \in C_n(T; L^{\infty}(\alpha_0; \mathbb{Z}))$ with

$$\partial b = c_0 \quad \text{and} \quad |b|_1 \leq K \cdot |c_0|_1 \leq K \cdot 2 \cdot \varepsilon_{\partial}.$$

Then the local criterion shows that $c := c_1 + c_2 - b$ is a relative α -parametrised fundamental cycle of $(W, \partial W)$ [9, Proposition 3.13]. Moreover,

$$\begin{aligned} |c|_1 &\leq |W_1, \partial W_1|_{\partial}^{\alpha_1} + |W_2, \partial W_2|_{\partial}^{\alpha_2} + 2 \cdot \varepsilon + K \cdot 2 \cdot \varepsilon_{\partial} \quad \text{and} \\ |\partial c|_1 &\leq |\partial c_1|_1 + |\partial c_2|_1 \leq 2 \cdot \varepsilon_{\partial}. \end{aligned}$$

Taking first $\varepsilon \rightarrow 0$ and then $\varepsilon_{\partial} \rightarrow 0$, proves the claim. \square

PROPOSITION 6.5 (self-glueing estimate). *Let W be an oriented compact connected manifold of dimension $n \geq 2$, let $T_1, T_2 \subseteq \partial W$ be two different π_1 -injective components of ∂W that are homeomorphic to a torus, and let $f: T_1 \rightarrow T_2$ be an orientation-reversing homeomorphism. We consider the glued manifold $W' := W/(T_1 \sim_f T_2)$ and an essentially free standard $\Pi(W)$ -space α as well as the induced standard $\Pi(W')$ -space α' on W' . Then*

$$|W', \partial W'|_{\partial}^{\alpha'} \leq |W, \partial W|_{\partial}^{\alpha}.$$

Proof. We can argue in the same way as in the proof of Proposition 6.4. \square

6.3. Profinite completions in dimension 3

Since the glueing results of the previous section involve restrictions of parameter spaces, we will need to understand the effect of restriction on the parameter spaces associated with the profinite completion.

PROPOSITION 6.6. *Let M be an irreducible oriented compact connected 3-manifold with empty or toroidal boundary, and let W be a piece of the JSJ decomposition of M .*

(1) *If W is Seifert-fibred and not covered by S^3 , then*

$$|W, \partial W|_{\partial}^{\widehat{\Pi(M)}} = 0.$$

(2) *If W is hyperbolic, then*

$$|W, \partial W|_{\partial}^{\widehat{\Pi(M)}} \leq |W, \partial W|_{\partial}^{\widehat{\Pi(W)}}.$$

Here, the occurrences of $\widehat{\Pi(M)}$ are to be interpreted as the restrictions of these standard $\Pi(M)$ -spaces to $\Pi(W)$.

The proof relies on the following two facts:

LEMMA 6.7. *Let M be an irreducible oriented compact connected 3-manifold with empty or toroidal boundary, let W be a piece of the JSJ decomposition of M , and choose a basepoint x_0 for W . Then the map $\pi_1(\widehat{W}, x_0) \rightarrow \pi_1(\widehat{M}, x_0)$ induced by the inclusion $W \rightarrow M$ embeds $\pi_1(\widehat{W}, x_0)$ as a closed subgroup of $\pi_1(\widehat{M}, x_0)$.*

Proof. This statement is contained in the stronger fact that the profinite topology on $\pi_1(M)$ is *efficient* with respect to the graph-of-groups decomposition induced by the JSJ decomposition. We will not make further use of these technical notions, so we direct the reader to the original paper of Wilton and Zalesskii for the precise definitions and proofs [39]. \square

LEMMA 6.8 [17, Example 12]. *Let G be a locally compact second-countable group and $H \leq G$ a closed subgroup, both equipped with the left Haar measures μ_G, μ_H , respectively. Then, as an H -probability space, G is isomorphic to the product of H with a measured space carrying a trivial H -action.*

Proof. In the language of Gheysens and Monod [17, Example 12], this lemma is expressed as the statement that G is an *amplification* of H . \square

We will also make use of the notion of *weak containment* of standard G -spaces (for G a countable group or a groupoid with countable automorphism groups), and its relationship to integral foliated simplicial volume (with boundary control), as explained in the Appendix. The main result is Proposition A.1, which may be treated as a black box during the proof of Proposition 6.6.

Proof of Proposition 6.6. Whether we are in situation (1) or (2), Lemma 6.7 ensures that, for any choice of basepoint (which we now suppress from the notation), $\pi_1(\widehat{W})$ sits as a closed subgroup of $\pi_1(\widehat{M})$. Applying Lemma 6.8 and restricting along the canonical map $\pi_1(W) \rightarrow \pi_1(\widehat{W})$ yields an isomorphism of standard $\pi_1(W)$ -spaces

$$\widehat{\pi_1(M)} \cong \widehat{\pi_1(W)} \times \alpha,$$

where α is some standard $\pi_1(W)$ -space with trivial action.

It is now easy to see from Definition A.2 that this implies we have a weak containment of standard $\pi_1(W)$ -spaces $\widehat{\pi_1(W)} \prec \widehat{\pi_1(M)}$, which extends to the level of groupoids:

$$\widehat{\Pi(W)} \prec \widehat{\Pi(M)}.$$

Applying now Proposition A.1 immediately yields (2), and reduces (1) to the proof that Seifert-fibred spaces W that are not covered by S^3 satisfy $|W, \partial W|_{\partial}^{\widehat{\Pi(W)}} = 0$. Such manifolds are encompassed by earlier work [9; 27, Section 8], whence $\|W, \partial W\|_{\mathbb{Z}}^{\infty} = 0$. Now, Proposition 2.12 tells us that $|W, \partial W|^{\widehat{\Pi(W)}} = \|W, \partial W\|_{\mathbb{Z}}^{\infty}$, and by Lemma 2.14 this vanishing transfers to $|W, \partial W|_{\partial}^{\widehat{\Pi(W)}}$, finishing the proof. \square

6.4. Proof of Theorem 6.1

We only need to combine our previous considerations. If M satisfies the hypotheses in Theorem 6.1 and is irreducible, we have

$$\begin{aligned} \|M, \partial M\|_{\mathbb{Z}}^{\infty} &= |M, \partial M|^{\widehat{\Pi(M)}} && \text{(Proposition 2.12)} \\ &\leq |M, \partial M|_{\partial}^{\widehat{\Pi(M)}} \\ &\leq \sum_{W \text{ JSJ piece of } M} |W, \partial W|_{\partial}^{\widehat{\Pi(M)}} && \text{(Propositions 6.4, 6.5)} \\ &\leq \sum_{W \text{ hyperbolic piece of } M} |W, \partial W|_{\partial}^{\widehat{\Pi(W)}} && \text{(Proposition 6.6)} \\ &= \sum_{W \text{ hyperbolic piece of } M} \frac{\text{vol}(W^{\circ})}{v_3} && \text{(Corollary 6)} \\ &= \frac{\text{hypvol}(M)}{v_3}. \end{aligned}$$

Propositions 6.4 and 6.5 can be applied, because fundamental groups of compact 3-manifolds are residually finite [20] (whence the action on the profinite completion is essentially free) and the JSJ pieces have π_1 -injective boundary consisting of tori.

The additional case $M \cong S^1 \times S^2$ can, for instance, be treated via self-coverings of S^1 and Proposition 2.12 [9, 27 34].

6.5. Proof of Theorem 1

Finally, we can prove the main theorem, Theorem 1: On the one hand, it is well known (Section 1.4) that

$$\|M, \partial M\|_{\mathbb{Z}}^{\infty} \geq \|M, \partial M\| = \frac{\text{hypvol}(M)}{v_3}.$$

On the other hand, Theorem 6.1 gives us the converse estimate

$$\|M, \partial M\|_{\mathbb{Z}}^{\infty} \leq \frac{\text{hypvol}(M)}{v_3},$$

which finishes the proof.

7. Proofs of the non-approximation results

We use the first L^2 -Betti number to establish the non-approximation results stated in the introduction.

THEOREM 2. *Let $d \in \mathbb{N}_{\geq 3}$, let $m, n \in \mathbb{N}$, and let $M_1, \dots, M_m, N_1, \dots, N_n$ be oriented closed connected d -manifolds with the following properties.*

- (1) We have $\|M_j\| > 0$ for all $j \in \{1, \dots, m\}$ as well as $\|N_k\| = 0$ for all $k \in \{1, \dots, n\}$.
- (2) Moreover, $m + n - 1 - \sum_{k=1}^n 1/|\pi_1(N_k)| > \sum_{j=1}^m \|M_j\|$ (with the convention that $1/\infty := 0$).

Then the connected sum $M := M_1 \# \dots \# M_m \# N_1 \# \dots \# N_n$ does not satisfy integral approximation for simplicial volume, that is, we have $\|M\| < \|M\|_{\mathbb{Z}}^{\infty}$.

Proof. In dimension $d \geq 3$, simplicial volume is additive under connected sums [6; 18, p. 10]. Therefore,

$$\|M\| = \sum_{j=1}^m \|M_j\| + \sum_{k=1}^n \|N_k\| = \sum_{j=1}^m \|M_j\|.$$

On the other hand, we know that the first L^2 -Betti number of M satisfies $b_1^{(2)}(M) \leq \|M\|_{\mathbb{Z}}^{\infty}$ [34, Corollary 5.6] (the same proof in fact also gives the improved constant 1). Therefore, it suffices to show that $b_1^{(2)}(M) > \sum_{j=1}^m \|M_j\|$. The connected sum formula for L^2 -Betti numbers [29, Theorem 1.35] yields

$$\begin{aligned} b_1^{(2)}(M) &= m + n - 1 + \sum_{j=1}^m (b_1^{(2)}(M_j) - b_0^{(2)}(M_j)) \\ &\quad + \sum_{k=1}^n (b_1^{(2)}(N_k) - b_0^{(2)}(N_k)). \end{aligned}$$

For connected manifolds X , we have $b_0^{(2)}(X) = 1/|\pi_1(X)|$. Because $\|M_j\| > 0$ and $d > 0$, the fundamental group $\pi_1(M_j)$ is infinite [18, p. 39f]. Therefore,

$$\begin{aligned} b_1^{(2)}(M) &\geq m + n - 1 + \sum_{j=1}^m (b_1^{(2)}(M_j) - 0) + \sum_{k=1}^n \left(b_1^{(2)}(N_k) - \frac{1}{|\pi_1(N_k)|} \right) \\ &\geq m + n - 1 - \sum_{k=1}^n \frac{1}{|\pi_1(N_k)|} \\ &> \sum_{j=1}^m \|M_j\|. \end{aligned}$$

Therefore, $\|M\| < b_1^{(2)}(M) \leq \|M\|_{\mathbb{Z}}^{\infty}$, as claimed. □

COROLLARY 3. *Let N be an oriented closed connected hyperbolic 3-manifold and let $k > \text{vol}(N)/v_3$. Then the oriented closed connected 3-manifold $M := N \# \#^k(S^1)^3$ satisfies $\|M\| < \|M\|_{\mathbb{Z}}^{\infty}$.*

Proof. We only need to verify that the hypotheses of Theorem 2 are satisfied: We know $\|N\| = \text{vol}(N)/v_3$ (Remark 4.2) and $\|(S^1)^3\| = 0$ [18, p. 8]. Moreover, $\pi_1((S^1)^3) \cong \mathbb{Z}^3$ is infinite, so $1/|\pi_1((S^1)^3)| = 0$.

COROLLARY 4. *Let M be an oriented closed connected 3-manifold with $\|M\| = 0$. Then the following are equivalent.*

- (1) The simplicial volume of M satisfies integral approximation, that is, $\|M\|_{\mathbb{Z}}^{\infty} = \|M\|$.
- (2) The manifold M is prime and has infinite fundamental group or M is homeomorphic to $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Proof. Ad 1 \implies 2. For the contraposition, we consider the case that $M = N_1 \# \dots \# N_n$ is a non-trivial prime decomposition of M , that is, $n \geq 2$ and none of the N_k is homeomorphic to S^3 . Then $0 = \|M\| = \sum_{k=1}^n \|N_k\|$ [6; 18, p. 10] and so $\|N_k\| = 0$ for all $k \in \{1, \dots, n\}$.

Because of $N_k \not\cong S^3$, we have $|\pi_1(N_k)| \geq 2$ by the Poincaré Conjecture [2, Corollary 1.7.4]. We now distinguish the following cases.

- If $n > 2$, then

$$n - 1 - \sum_{k=1}^n \frac{1}{|\pi_1(N_k)|} \geq n - 1 - \frac{n}{2} = \frac{n}{2} - 1 > 0.$$

Therefore, from Theorem 2, we obtain $\|M\| < \|M\|_{\mathbb{Z}}^{\infty}$.

- If $n = 2$ and $|\pi_1(N_1)| > 2$ or $|\pi_1(N_2)| > 2$, then again

$$n - 1 - \sum_{k=1}^n \frac{1}{|\pi_1(N_k)|} > 2 - 1 - 1 = 0$$

and, by Theorem 2, $\|M\| < \|M\|_{\mathbb{Z}}^{\infty}$.

- If $n = 2$ and $|\pi_1(N_1)| = 2$ and $|\pi_1(N_2)| = 2$, then $\pi_1(N_1) \cong \mathbb{Z}/2 \cong \pi_1(N_2)$. By the Elliptisation Theorem [2, Theorem 1.7.3], N_1 and N_2 are both spherical and thus homeomorphic to the quotient of S^3 by a subgroup of $\mathrm{SO}(4)$ of order 2, which must be $\{\pm \mathrm{Id}\}$, and hence N_1 and N_2 are homeomorphic to $\mathbb{R}P^3$.

Ad 2 \implies 1. If M is prime with infinite fundamental group and $\|M\| = 0$, then M must be a graph manifold (with infinite fundamental group) [35]. Therefore, we obtain [9]

$$\|M\| = \|M\|_{\mathbb{Z}}^{\infty}.$$

Moreover, also $\mathbb{R}P^3 \# \mathbb{R}P^3$ satisfies $\|\mathbb{R}P^3 \# \mathbb{R}P^3\| = 0 = \|\mathbb{R}P^3 \# \mathbb{R}P^3\|_{\mathbb{Z}}^{\infty}$ (because this manifold admits a non-trivial self-covering) [27, Section 8]. \square

Appendix A. Weak containment

Parametrised simplicial volume of closed manifolds satisfies monotonicity with respect to weak containment of parameter spaces [13, Theorem 3.3]. This property admits a straightforward generalisation to the relative case (including boundary control).

PROPOSITION A.1 (monotonicity with boundary control). *Let W be an oriented compact connected manifold (with possibly non-empty boundary) and infinite fundamental group. Let α and β be essentially free standard $\Pi(W)$ -spaces with $\alpha \prec \beta$. Then*

$$|W, \partial W|_{\partial}^{\beta} \leq |W, \partial W|_{\partial}^{\alpha}.$$

For the sake of completeness, we carry out the transformation from the closed case to the relative case in detail.

Let us first recall basics on weak containment. Roughly speaking, $\alpha \prec \beta$ means that every finite relation between Borel sets and groupoid morphisms in α can be simulated in β with arbitrary precision.

DEFINITION A.2 (weak containment).

- Let Γ be a countable group and let $\alpha: \Gamma \curvearrowright (X, \mu)$ and $\beta: \Gamma \curvearrowright (Y, \nu)$ be standard Γ -spaces. Then α is *weakly contained* in β (in symbols: $\alpha \prec \beta$) if the following holds: For all $\varepsilon \in \mathbb{R}_{>0}$,

all $m \in \mathbb{N}$, all Borel sets $A_1, \dots, A_m \subset X$, and all finite subsets $F \subset \Gamma$, there exist Borel sets $B_1, \dots, B_m \subset Y$ such that

$$\forall_{j \in \{1, \dots, m\}} \quad \forall_{g \in F} \quad |\mu(g^\alpha(A_j) \cap A_j) - \nu(g^\beta(B_j) \cap B_j)| < \varepsilon.$$

- Let G be a connected groupoid with countable automorphism groups and let α and β be standard G -spaces. Then α is *weakly contained* in β (in symbols: $\alpha \prec \beta$) if the following holds: For one (whence every) object x_0 and $\Gamma := \text{Aut}_G x_0$, the standard Γ -space $\alpha(x_0)$ is weakly contained in the Γ -space $\beta(x_0)$.

REMARK A.3 (an alternative characterisation of weak containment). Let G be a connected groupoid with countable automorphism groups and let α and β be standard G -spaces. Moreover, let x_0 be an object of G and $\Gamma := \text{Aut}_G x_0$ as well as $(X, \mu) := \alpha(x_0)$. Then $\alpha \prec \beta$ if and only if $\alpha(x_0)$ lies in the closure of

$$\{\xi \in A(\Gamma, X, \mu) \mid \xi \cong_\Gamma \beta(x_0)\}$$

with respect to the weak topology on the space $A(\Gamma, X, \mu)$ of all μ -preserving Borel actions of Γ on (X, μ) [21, Proposition 10.1].

Proof of Proposition A.1. The proof is a straightforward adaption of the proof in the closed case [13, Theorem 3.3]; we only need to convert the proof from twisted to local coefficients and add boundary control.

Let $x_0 \in W$ and $\Gamma := \pi_1(W, x_0)$. Without loss of generality, we may assume that α is induced from a standard Γ -space $\alpha_0: \Gamma \curvearrowright (X, \mu)$ at x_0 and that β is induced from a standard Γ -space β_0 at x_0 (Remark 2.6).

Let $n := \dim W$, let $\varepsilon_\partial, \varepsilon \in \mathbb{R}_{>0}$, and let $c \in C_n(W; L^\infty(\alpha; \mathbb{Z}))$ be an α -parametrised relative fundamental cycle with

$$|\partial c|_1^\alpha \leq \frac{1}{2} \cdot \varepsilon_\partial.$$

It then clearly suffices to show the following claim.

(C) There exists a standard $\Pi(W)$ -space ξ with $\xi \cong \beta$ and a ξ -parametrised relative fundamental cycle $c' \in C_n(W; L^\infty(\xi; \mathbb{Z}))$ with

$$|c'|_1^\xi \leq |c|_1^\alpha + \varepsilon \quad \text{and} \quad |\partial c'|_1^{\text{res } \xi} \leq \varepsilon_\partial.$$

To establish claim (C), let $z \in C_n(W; \mathbb{Z})$ be a relative fundamental cycle. Because c is a relative fundamental cycle, there exist chains $b \in C_{n+1}(W; L^\infty(\alpha; \mathbb{Z}))$ and $w \in C_n(\partial W; L^\infty(\text{res } \alpha; \mathbb{Z}))$ with

$$c = z + \partial b + w \quad \text{in } C_n(W; L^\infty(\alpha; \mathbb{Z}));$$

more explicitly, we write

$$b = \sum_{\tau \in T} f_\tau \cdot \tau \quad \text{and} \quad w = \sum_{\varrho \in R} g_\varrho \cdot \varrho$$

with finite sets $T \subset \text{map}(\Delta^{n+1}, W)$, $R \subset \text{map}(\Delta^n, \partial W)$ and bounded measurable functions $(f_\tau)_{\tau \in T}$, and $(g_\varrho)_{\varrho \in R}$ on (X, μ) .

We choose a finite Borel partition $X = A_1 \sqcup \dots \sqcup A_m$ of X that refines the finite set

$$\{f_\tau^{-1}(k) \mid \tau \in T, k \in \mathbb{Z}\} \cup \{g_\varrho^{-1}(k) \mid \varrho \in R, k \in \mathbb{Z}\}.$$

Moreover, we set

$$\delta := \frac{1}{m} \cdot \min\left(\frac{\varepsilon}{\sum_{\tau \in T} \|f_\tau\|_\infty}, \frac{\varepsilon_\partial}{2 \cdot \sum_{\varrho \in R} \|g_\varrho\|_\infty}\right),$$

$$F := \{\tau[0, 1] \mid \tau \in T\} \cup \{\varrho[0, 1] \mid \varrho \in R\},$$

$$F_0 := \{h_f^{-1} \mid f \in F\} \subset \Gamma$$

(the construction of the h_f is explained in Remark 2.6). Because α is weakly contained in β , there exists a standard Γ -space $\xi_0 \in A(\Gamma, X, \mu)$ such that

$$\forall_{j \in \{1, \dots, m\}} \quad \forall_{h \in F_0^{-1}} \quad \mu(h^{\alpha_0}(A_j) \triangle h^{\xi_0}(A_j)) < \delta$$

and $\xi_0 \cong_\Gamma \beta_0$ (Remark A.3). Let ξ be the standard $\Pi(W)$ -space associated with ξ_0 (Remark 2.6) and let

$$c' := z + \partial b + w \quad \text{in } C_*(W; L^\infty(\xi; \mathbb{Z})).$$

By construction, c' is a relative ξ -parametrised fundamental cycle of $(W, \partial W)$ and $\xi \cong \beta$.

We now show that c' satisfies the norm estimates postulated in (C). We have

$$\begin{aligned} \left| |c|_1^\alpha - |c'|_1^\xi \right| &= \left| \left| z + \sum_{j=1}^{n+1} \sum_{\tau \in T} (-1)^j \cdot f_\tau \cdot \partial_j \tau + \sum_{\tau \in T} \alpha(\tau[0, 1])(f_\tau) \cdot \partial_0 \tau + w \right|_1^\alpha \right. \\ &\quad \left. - \left| z + \sum_{j=1}^{n+1} \sum_{\tau \in T} (-1)^j \cdot f_\tau \cdot \partial_j \tau + \sum_{\tau \in T} \xi(\tau[0, 1])(f_\tau) \cdot \partial_0 \tau + w \right|_1^\xi \right| \\ &\leq \left| z + \sum_{j=1}^{n+1} \sum_{\tau \in T} (-1)^j \cdot f_\tau \cdot \partial_j \tau + \sum_{\tau \in T} \alpha(\tau[0, 1])(f_\tau) \cdot \partial_0 \tau + w \right. \\ &\quad \left. - \left(z + \sum_{j=1}^{n+1} \sum_{\tau \in T} (-1)^j \cdot f_\tau \cdot \partial_j \tau + \sum_{\tau \in T} \xi(\tau[0, 1])(f_\tau) \cdot \partial_0 \tau + w \right) \right|_1^{(X, \mu)} \\ &\leq \sum_{\tau \in T} \left\| \alpha(\tau[0, 1])(f_\tau) - \xi(\tau[0, 1])(f_\tau) \right\|_1. \end{aligned}$$

For each $\tau \in T$, we write $f_\tau = \sum_{j=1}^m a_{\tau,j} \cdot \chi_{A_j}$ with $a_1, \dots, a_m \in \mathbb{Z}$. Then

$$\begin{aligned} \left\| \alpha(\tau[0, 1])(f_\tau) - \xi(\tau[0, 1])(f_\tau) \right\|_1 &\leq \sum_{j=1}^m |a_j| \cdot \mu(h_{\tau[0,1]}^{-1} \alpha_0(A_j) \triangle h_{\tau[0,1]}^{-1} \xi_0(A_j)) \\ &\leq m \cdot \|f_\tau\|_\infty \cdot \delta. \end{aligned}$$

Hence,

$$\left| |c|_1^\alpha - |c'|_1^\xi \right| \leq \sum_{\tau \in T} m \cdot \|f_\tau\|_\infty \cdot \delta \leq \varepsilon.$$

Similarly, we can handle ∂c and $\partial c'$. We have

$$\begin{aligned} \partial c &= \partial z + \partial \partial b + \partial w \quad \text{in } C_{n-1}(\partial W; L^\infty(\text{res } \alpha; \mathbb{Z})) \\ \partial c' &= \partial z + \partial \partial b + \partial w \quad \text{in } C_{n-1}(\partial W; L^\infty(\text{res } \xi; \mathbb{Z})). \end{aligned}$$

Moreover, $\partial \partial b = 0$ in both cases and ∂z does not depend on the parameter space (as z has constant coefficients). The same type of calculations as above shows that (where we simplify

notation by writing ξ instead of $\text{res } \xi$)

$$\begin{aligned} \left| |\partial c|_1^\alpha - |\partial c|_1^\xi \right| &\leq \sum_{\varrho \in R} \left\| \alpha(\varrho[0, 1])(g_\varrho) - \xi(\varrho[0, 1])(g_\varrho) \right\|_1 \\ &\leq \sum_{\varrho \in R} m \cdot \|g_\varrho\|_\infty \cdot \delta \\ &\leq \frac{1}{2} \cdot \varepsilon_\partial. \end{aligned}$$

Hence,

$$|\partial c|_1^\xi \leq |\partial c|_1^\alpha + \frac{1}{2} \cdot \varepsilon_\partial \leq \varepsilon_\partial.$$

This finishes the proof of claim (C). \square

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