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IDEAL SIMPLICIAL VOLUME OF MANIFOLDS WITH BOUNDARY

R. FRIGERIO AND M. MORASCHINI

ABSTRACT. We define the ideal simplicial volume for compact manifolds with boundary. Roughly speaking, the ideal simplicial volume of a manifold M measures the minimal size of possibly ideal triangulations of M "with real coefficients", thus providing a variation of the ordinary simplicial volume defined by Gromov in 1982, the main difference being that ideal simplices are now allowed to appear in representatives of the fundamental class.

We show that the ideal simplicial volume is bounded above by the ordinary simplicial volume, and that it vanishes if and only if the ordinary simplicial volume does. We show that, for manifolds with amenable boundary, the ideal simplicial volume coincides with the classical one, whereas for hyperbolic manifolds with geodesic boundary it can be strictly smaller. We compute the ideal simplicial volume of an infinite family of hyperbolic 3-manifolds with geodesic boundary, for which the exact value of the classical simplicial volume is not known, and we exhibit examples where the ideal simplicial volume provides sharper bounds on mapping degrees than the classical simplicial volume.

INTRODUCTION

The simplicial volume is an invariant of manifolds introduced by Gromov in his seminal paper [Gro82]. If M is a connected, compact, oriented manifold with (possibly empty) boundary, then the simplicial volume of M is the infimum of the sums of the absolute values of the coefficients over all singular chains representing the real fundamental cycle of M (see Subsection 2.1). We will denote the simplicial volume of M by the symbol $\|M\|$, both when M is closed and when M has non-empty boundary.

If τ is a triangulation of M, then a suitable algebraic sum of the simplices of τ provides a fundamental cycle for M. Hence the simplicial volume is bounded above by the minimal number of simplices in any triangulation of M, and it can be thought as the minimal size of triangulations of M "with real coefficients" (even if this analogy is very loose: fundamental cycles of manifolds can contain simplices which are very far from being embedded, for example).

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When dealing with manifolds with boundary, it is often useful to work with *ideal* triangulations rather than with traditional ones (we refer the reader to Section 2 for the precise definition of ideal triangulation). Indeed, ideal triangulations are usually more manageable than classical ones, and in many cases they are much more economical: for example, the smallest cusped hyperbolic 3-manifold (which was constructed by Gieseking in 1912, and shown to have the smallest volume among non-compact hyperbolic 3-manifolds by Adams [Ada87]) can be triangulated using only one ideal simplex, and the figure-eight knot complement, which doubly covers the Gieseking manifold, admits an ideal triangulation with 2 ideal tetrahedra.

In this paper we introduce and study the notion of *ideal simplicial volume*. To this aim, we first introduce a homology theory for manifolds with boundary, called *marked homology*, in which ideal singular simplices are allowed. We then show that marked homology is isomorphic to the singular homology of the manifold relative to its boundary. This allows us to define a fundamental class in the marked context, which will be called *ideal fundamental class*. We then define the ideal simplicial volume $\|M\|_{\mathcal{I}}$ of M as the infimum of the ℓ^1-norms of the representatives of the ideal fundamental class. We refer the reader to Section 2 for the precise definition. A crucial fact is that the canonical isomorphism between marked homology and relative homology is not isometric: as discussed in Remark 2.3, a seemingly reasonable definition of the ideal simplicial volume in the context of relative homology would not lead to an interesting theory.

Fundamental properties of the ideal simplicial volume. For every compact manifold with boundary M, let $c(M)$ denote the *complexity* of M, i.e. the minimal number of top-dimensional simplices in any ideal triangulation of M. Just as the simplicial volume of M is bounded from above by the number of top-dimensional simplices in a triangulation of M (see e.g. [FFM12, Proposition 1.1]), the ideal complexity provides an upper bound for the ideal simplicial volume:

Theorem 1. *Let M be a compact manifold with boundary. Then*

$$\|M\|_{\mathcal{I}} \leq c(M) \ .$$

We also show that the ideal simplicial volume may be exploited to bound the degree of maps between manifolds:

Theorem 2. *Let $f\colon (M, \partial M) \to (N, \partial N)$ be a map of pairs between compact, connected and oriented manifolds of the same dimension. Then*

$$\|M\|_{\mathcal{I}} \geq |\deg(f)| \cdot \|N\|_{\mathcal{I}} \ .$$

In particular, the ideal simplicial volume is a homotopy invariant of manifolds with boundary (where homotopies are understood to be homotopies of pairs).

The ideal simplicial volume vanishes if and only if the ordinary simplicial volume does:

Theorem 3. *There exists a constant K_n only depending on $n \in \mathbb{N}$ such that, for every n-dimensional compact manifold M, the following inequalities hold:*

$$\|M\|_{\mathcal{I}} \leq \|M\| \leq K_n \cdot \|M\|_{\mathcal{I}}.$$

In particular, $\|M\|_{\mathcal{I}} = 0$ if and only if $\|M\| = 0$.

Manifolds with amenable boundary. As Gromov himself pointed out in his seminal paper [Gro82], in order to compute the simplicial volume it is often useful to exploit the dual theory of bounded cohomology. One of the peculiar features of (singular) bounded cohomology is that it vanishes on spaces with amenable fundamental group. Via some elementary duality results, this implies in turn that amenable spaces are somewhat invisible when considering their simplicial volume: for example, closed manifolds with amenable fundamental group have vanishing simplicial volume, and (under some mild additional hypothesis) the simplicial volume of manifolds with boundary is additive with respect to gluings along boundary components with amenable fundamental group. In Section 4 we introduce the dual theory to marked homology and, building on results from [BBF$^+$14, Fri17], we exploit duality to deduce the following:

Theorem 4. *Let M be an n-dimensional compact manifold, and suppose that the fundamental group of every boundary component of M is amenable. Then*

$$\|M\|_{\mathcal{I}} = \|M\| \ .$$

It is proved in [Löh07, KK15] that, under the assumptions of Theorem 4, the simplicial volume $\|M\|$ of M coincides with the simplicial volume of the open manifold $\mathrm{int}(M) = M \setminus \partial M$ (which is defined in terms of the locally finite homology of $M \setminus \partial M$ [Gro82]), as well as with the Lipschitz simplicial volume of $\mathrm{int}(M)$ (see [Gro82, LS09] for the definition). Therefore, for manifolds whose boundary components have amenable fundamental group, all these invariants also coincide with the ideal simplicial volume $\|M\|_{\mathcal{I}}$ of M.

Let M be a complete finite-volume hyperbolic n-manifold. As usual, we will denote by M also the natural compactification of M, which is a compact manifold whose boundary components admit a flat Riemannian structure. A celebrated result of Gromov and Thurston shows that the simplicial volume of M is equal to the ratio $\mathrm{vol}(M)/v_n$, where $\mathrm{vol}(M)$ is the Riemannian volume of M and v_n is the volume of a regular ideal geodesic simplex in hyperbolic n-space $\mathbb{H}^n$ (all such simplices are isometric to each other). Since flat manifolds have virtually abelian (hence, amenable) fundamental group, Theorem 4 implies the following:

Corollary 5. *Let M be (the natural compactification of) a complete finite-volume hyperbolic n-manifold. Then*

$$\|M\|_{\mathcal{I}} = \|M\| = \frac{\mathrm{vol}(M)}{v_n} \ .$$

In fact, Theorem 4 applies to a much bigger class of complete finite-volume manifolds. Let M be an open complete manifold with finite volume and pinched negative curvature. As above we still denote by M its natural compactification. It is well known that the fundamental group of each boundary component of M is virtually nilpotent, hence amenable (see e.g. [BGS85] or [Bel16, Ex. 19.1]). Thus Theorem 4 implies the following:

Corollary 6. *Let M be (the natural compactification of) an open complete finite-volume negatively pinched n-manifold. Then*

$$\|M\|_{\mathcal{I}} = \|M\|.$$

Hyperbolic manifolds with geodesic boundary. The computation of the simplicial volume of manifolds whose boundary components have *non*-amenable fundamental groups is a very challenging task. Indeed, the only exact values of the simplicial volume of such manifolds are known for 3-dimensional handlebodies (and, more in general, for manifolds obtained by attaching 1-handles to Seifert manifolds) and for the product of a surface with the closed interval [BFP15]. In particular, the exact value of the ordinary simplicial volume is not known for any hyperbolic 3-manifold with geodesic boundary (and the only available estimates seem quite far from being sharp, see [BFP17]). On the contrary, in Section 5 we compute the exact value of the *ideal* simplicial volume for an infinite family of hyperbolic 3-manifolds with geodesic boundary (see Theorem 9 below).

In the closed case, Gromov's and Thurston's strategy to obtain lower bounds on the classical simplicial volume of hyperbolic manifolds is based on two facts: fundamental cycles may be represented by linear combinations of geodesic simplices, and the volume of geodesic simplices in hyperbolic space is uniformly bounded. When dealing with manifolds with geodesic boundary, the most natural building blocks turn out to be the so-called (partially) truncated simplices. By mimicking Gromov's and Thurston's strategy, here we exploit upper bounds on the volume of truncated simplices to obtain lower bounds on the ideal simplicial volume of hyperbolic manifolds with geodesic boundary.

If M is a compact hyperbolic manifold with geodesic boundary, then the *smallest return length* $\ell(M)$ of M is the length of the shortest path with both endpoints on ∂M which intersects ∂M orthogonally at each of its endpoints. Equivalently, it is the smallest distance between distinct boundary components of the universal covering of M.

Theorem 7. *Let M be a compact n-dimensional hyperbolic manifold with geodesic boundary. Then*

$$\|M\|_{\mathcal{I}} \geq \frac{\mathrm{vol}(M)}{V^n_{\ell(M)}} \ ,$$

where V^n_ℓ is the supremum of the volumes of n-dimensional fully truncated simplices whose edge lengths are not smaller than ℓ (see Definition 5.3).

It is well known (see for example [Ush06]) that, in dimension 3, the supremum of the volumes of all truncated tetrahedra is given by the volume $v_8 \approx 3.664$ of the regular ideal octahedron, so Theorem 7 implies the following:

Corollary 8. *Let M be a compact hyperbolic 3-manifold with geodesic boundary. Then*

$$\|M\|_{\mathcal{I}} \geq \frac{\mathrm{vol}(M)}{v_8} \ .$$

It has been recently proved that, if ℓ is sufficiently small, then the constant V^3_ℓ coincides with the volume of the regular truncated tetrahedron of edge length ℓ [FM]. Together with Theorem 7, this fact allows us to compute the exact value of the ideal simplicial volume of an infinite family of hyperbolic 3-manifolds with geodesic boundary.

For every $g \geq 2$ let $\overline{\mathcal{M}}_g$ be the set of hyperbolic 3-manifolds M with connected geodesic boundary such that $\chi(\partial M) = 2 - 2g$ (so ∂M, if orientable, is the closed orientable surface of genus g). Recall that for every 3-manifold with boundary M the equality $\chi(\partial M) = 2\chi(M)$ holds, and in particular $\chi(\partial M)$ is even. Therefore, the union $\bigcup_{g\geq 2} \overline{\mathcal{M}}_g$ coincides with the set of hyperbolic 3-manifolds with connected geodesic boundary.

For every $g \geq 2$ we denote by $\mathcal{M}_g$ the set of 3-manifolds M with boundary that admit an ideal triangulation by g tetrahedra and have Euler characteristic $\chi(M) = 1 - g$. Every element of $\mathcal{M}_g$ has connected boundary and supports a hyperbolic structure with geodesic boundary (which is unique by Mostow rigidity), hence $\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$ (see [FMP03]). Furthermore, Miyamoto proved in [Miy94] that elements of $\mathcal{M}_g$ are exactly the ones having the smallest volume among the elements of $\overline{\mathcal{M}}_g$. In particular, $\mathcal{M}_g$ is nonempty for every $g \geq 2$. The eight elements of $\mathcal{M}_2$ are exactly the smallest hyperbolic manifolds with nonempty geodesic boundary [KM91, Miy94]. Some estimates on the ordinary simplicial volume of elements in $\mathcal{M}_g$ can be found in [BFP17]. Here we prove the following:

Theorem 9. *Let $M \in \overline{\mathcal{M}}_g$. Then*

$$\|M\|_{\mathcal{I}} \geq g$$

and

$$\|M\|_{\mathcal{I}} = g$$

if and only if $M \in \mathcal{M}_g$.

Corollary 10. *The hyperbolic 3-manifolds with geodesic boundary having the smallest ideal simplicial volume are exactly the elements of $\mathcal{M}_2$.*

As a consequence of the previous corollary, among hyperbolic 3-manifolds with geodesic boundary, the minimum of the Riemannian volume is attained exactly at those manifolds which also realize the minimum of the ideal simplicial volume. The same result with ideal simplicial volume replaced by ordinary simplicial volume was conjectured in [BFP15].

A direct consequence of Theorems 2 and 9 is the following:

Corollary 11. *Take elements $M \in \mathcal{M}_g$ and $M' \in \mathcal{M}_{g'}$, where $g \geq g'$, and let*

$$f\colon (M, \partial M) \to (M', \partial M')$$

be a map of pairs. Then

$$\deg(f) \leq \frac{g}{g'} \ .$$

As discussed in Subsection 5.7, in some cases the bound provided by Corollary 11 is sharp. Moreover, it is strictly sharper than the bounds one can obtain by exploiting the ordinary simplicial volume to study the restriction of f to ∂M or the extension of f to the double of M.

Plan of the paper. In Section 1 we define marked spaces and marked homology. Moreover, we prove some fundamental results about marked homology that are needed for the definition of the ideal simplicial volume. In Section 2 we define the ideal simplicial volume and we prove Theorem 1 (in Subsection 2.3), Theorem 2 (in Subsection 2.5) and Theorem 3 (in Subsection 2.4). We also introduce marked bounded cohomology, and establish an elementary but fundamental duality result which will be exploited in the proofs of Theorems 4 and 7.

In Section 3 we introduce the universal covering of marked spaces, while Section 4 is devoted to the proof of Theorem 4. Finally, in Section 5 we focus on hyperbolic manifolds, and we prove Theorem 7, Theorem 9 and Corollary 11.

1. Marked homology of marked spaces

Before defining the ideal simplicial volume of manifolds with boundary we need to introduce and develop the theory of marked homology for marked spaces. If M is a manifold with boundary, one can consider the quotient space X obtained by separately collapsing the connected components of ∂M. Such a space X provides the motivating example for our theory, and the ideal simplicial volume of M will be defined in Section 2 just as the ℓ^1-seminorm of the real fundamental class in the marked homology of X. This section is mainly devoted to collecting the fundamental properties of marked homology, and to preparing the ground for the precise definition of the fundamental class in the context of marked homology.

We begin by introducing a slight generalization of the notion of topological cone.

Definition 1.1. Let B be a topological space and let b be a point of B. We say that B is a *quasicone* with apex b if the following condition holds: there exists a homotopy

$$H\colon B \times [0,1] \to B$$

between the identity and the constant map at b with the following additional properties: $H(b,t) = b$ for every $t \in [0,1]$ (i.e. the homotopy is relative to $\{b\}$), and for every $x \in B \setminus \{b\}$ the path $H(x,\cdot)\colon [0,1) \to B$ does not pass through b. In other words, we ask that the homotopy H is such that $H(x,0) = x$ for every $x \in B$, and

$$H^{-1}(\{b\}) = (\{b\} \times [0,1]) \cup (B \times \{1\}) \ .$$

A topological cone is obviously a quasicone, and in order to define the ideal simplicial volume of compact manifolds with boundary it would be sufficient to deal with usual cones. Nevertheless, when constructing coverings of marked spaces it will be useful to work in a slightly more general context.

The following definition singles out the fundamental objects we will be dealing with.

Definition 1.2. A *marked space* (X, B) is a topological pair satisfying the following properties:

(1) B is closed in X.
(2) For every $b \in B$ there exists a closed neighbourhood F_b of b in X such that F_b is a quasicone with apex b. Moreover, the F_b, $b \in B$, may be chosen to be pairwise disjoint.

As a consequence of the definition, the subset B is discrete. Moreover, if

$$F_B = \bigcup_{b \in B} F_b \ ,$$

then the F_b, $b \in B$, are exactly the path connected components of F_B.

Definition 1.3. A map $f\colon (X,B) \to (X',B')$ between marked spaces is *admissible* if and only if it is continuous and such that $f^{-1}(B') = B$.

It is immediate to check that there exists a well-defined category having marked spaces as objects and admissible maps as morphisms. The notion of homotopic maps admits an obvious admissible version:

Definition 1.4. Let $f, g\colon (X,B) \to (X',B')$ be admissible maps between marked spaces. An admissible homotopy between f and g is an ordinary homotopy $H\colon X \times [0,1] \to X'$ between f and g such that the map $H(\cdot,t)\colon X \to X'$ is admissible for every $t \in [0,1]$ or, equivalently, such that $H^{-1}(B') = B \times [0,1]$. Observe that, since B' is discrete, for every $b \in B$ the map $t \mapsto H(t,b)$ is constant: in particular, $f(b) = g(b)$ for every $b \in B$.

1.1. **The marked space associated to a manifold with boundary.** Let $(M, \partial M)$ be an n-manifold with boundary. We associate to $(M, \partial M)$ the marked space (X, B) which is defined as follows: X is the topological quotient obtained from M by separately collapsing every connected component of ∂M, while $B \subseteq X$ is the subset of X given by the classes associated to the components of ∂M. Using that the boundary of M admits a collar in M, it is immediate to check that (X, B) is indeed a marked space in the sense of Definition 1.2. We will refer to the pair (X, B) as the marked space associated to M, and the quotient map $p\colon (M, \partial M) \to (X, B)$ will be called the *natural projection*.

1.2. **Marked homology.** As explained in the introduction, the ideal simplicial volume measures the ℓ^1-seminorm of the ideal fundamental class, which in turn provides a representative of the fundamental class possibly containing (partially) ideal singular simplices. The following definition gives a precise meaning to the notion of partially ideal singular simplices.

Definition 1.5. Let (X, B) be a marked space. A singular simplex $\sigma\colon \Delta^n \to X$ is *admissible* if $\sigma^{-1}(B)$ is a subcomplex of Δ^n, i.e. a (possibly empty) union of (not necessarily proper) faces of Δ^n. For example, every constant singular simplex is admissible, and any bijective parametrization $\sigma\colon \Delta^n \to X$ of a top-dimensional simplex in an ideal triangulation of a manifold with associated marked space (X, B) is admissible, since $\sigma^{-1}(B)$ is equal to the set of vertices of Δ^n.

Let now $\mathcal{R}$ be a ring with unity. It is immediate to check that the restriction of an admissible singular simplex to any of its faces is still admissible (where we understand that each face of Δ^n is identified with Δ^{n-1} via the unique affine isomorphism which preserves the order of the vertices). This readily implies that admissible simplices define a subcomplex $\widehat{C}^{\mathcal{M}}_*(X, B; \mathcal{R})$ of the usual singular complex $C_*(X; \mathcal{R})$. Of course, the complex $\widehat{C}^{\mathcal{M}}_*(X, B; \mathcal{R})$ contains the subcomplex $C_*(B; \mathcal{R})$ (which, given that B is discrete, in each degree n only consists of the free $\mathcal{R}$-module over the constant n-simplices with values in B).

We are now ready to introduce the definition of the chain complex that computes the marked homology of marked spaces.

Definition 1.6. Let (X, B) be a marked space. The *marked chain complex* of (X, B) over the coefficient ring $\mathcal{R}$ is the quotient complex

$$C^{\mathcal{M}}_*(X, B; \mathcal{R}) = \widehat{C}^{\mathcal{M}}_*(X, B; \mathcal{R}) / C_*(B; \mathcal{R}) \ ,$$

endowed with the differential induced by the usual differential on $C_*(X; \mathcal{R})$.

The *marked homology* of (X, B) (with coefficients in $\mathcal{R}$) is the homology of the marked chain complex $C^{\mathcal{M}}_*(X, B; \mathcal{R})$, and will be denoted by $H^{\mathcal{M}}_*(X, B; \mathcal{R})$.

The composition of an admissible singular simplex with an admissible map is still an admissible singular simplex. Using this one can easily check

that marked homology indeed provides a functor on the category of marked spaces. We will see later that it is in fact a homotopy functor, i.e. that admissibly homotopic maps induce the same morphism on marked homology.

Let $i_*\colon C_*^{\mathcal{M}}(X,B;\mathcal{R}) \to C_*(X,B;\mathcal{R})$ be the inclusion of marked chains into ordinary relative chains. The main result of this section is the following:

Theorem 1.7. *For every $n \in \mathbb{N}$, the inclusion*

$$i_n\colon C_n^{\mathcal{M}}(X,B;\mathcal{R}) \to C_n(X,B;\mathcal{R})$$

induces an isomorphism

$$H_n(i_n)\colon H_n^{\mathcal{M}}(X,B;\mathcal{R}) \to H_n(X,B;\mathcal{R})$$

between the marked homology and the ordinary relative homology of the pair (X,B) with coefficients in $\mathcal{R}$.

This section will be mainly devoted to the proof of Theorem 1.7, which will be concluded in Subsection 1.6.

1.3. **The homotopy invariance of marked homology.** Recall that an admissible homotopy between admissible maps $f,g\colon (X,B) \to (X',B')$ is an ordinary homotopy $H\colon X \times [0,1] \to X'$ between f and g such that $H^{-1}(B') = B \times [0,1]$.

As anticipated above, marked homology is a homotopy functor:

Theorem 1.8. *Let $f,g\colon (X,B) \to (X',B')$ be admissibly homotopic admissible maps. Then the induced homomorphisms*

$$H_n^{\mathcal{M}}(f), H_n^{\mathcal{M}}(g)\colon H_n^{\mathcal{M}}(X,B;\mathcal{R}) \to H_n^{\mathcal{M}}(X',B';\mathcal{R})$$

coincide for every $n \in \mathbb{N}$.

Proof. The proof is a slight modification of the one for ordinary singular homology (see for instance [Hat02, Thm. 2.10]). As in the classical case, we subdivide the prism $\Delta^n \times I$ into $(n+1)$-dimensional simplices as follows. For every $i = 0,\dots,n$ we set $v_i = (e_i,0)$, $w_i = (e_i,1)$, where e_i is the i-th vertex of Δ^n. We then denote by $\sigma_i\colon \Delta^{n+1} \to \Delta^n \times I$ the affine isomorphism sending the vertices of Δ^{n+1} to the vertices $v_0,\cdots,v_i,w_i,\cdots,w_n$ of $\Delta^n \times I$.

Let $H\colon X \times I \to X'$ be an admissible homotopy between f and g. Then the usual homotopy operator $T_n\colon C_n(X;\mathcal{R}) \to C_{n+1}(X';\mathcal{R})$ is the unique $\mathcal{R}$-linear map such that, for every singular simplex $\sigma\colon \Delta^n \to X$,

$$T_n(\sigma) = \sum_{i=0}^{n} (-1)^i H \circ (\sigma \times \mathbb{1}) \circ \sigma_i \ .$$

It is clear that T_n sends $C_n(B;\mathcal{R})$ to $C_{n+1}(B';\mathcal{R})$, so in order to conclude the proof it is sufficient to show that $T_n(\sigma) \in \widehat{C}_{n+1}^{\mathcal{M}}(X',B;\mathcal{R})$ provided that $\sigma\colon \Delta^n \to X$ is admissible.

Let us denote by $K_i \subseteq \Delta^n \times I$ the image of the affine embedding $\sigma_i\colon \Delta^{n+1} \to \Delta^n \times I$, and observe that $\Delta^n \times I$ admits a structure of simplicial complex whose top-dimensional simplices are exactly the K_i. Let $S = \sigma^{-1}(B) \subseteq \Delta^n$.

Since σ is admissible, the subset S is a subcomplex of Δ^n. From the very definition of admissible homotopy it follows that $(H\circ(\sigma\times I))^{-1}(B') = S\times I$, so the pair

$$\left(\Delta^{n+1}, (H\circ(\sigma\times \mathbb{1})\circ\sigma_i)^{-1}(B')\right)$$

is affinely isomorphic (via σ_i) to the pair $(K_i, (S\times I)\cap K_i)$. But $S\times I$ is a subcomplex of $\Delta^n\times I$ (with respect to the structure of simplicial complex described above), so $(S\times I)\cap K_i$ is a subcomplex of K_i, and this implies that the singular simplex $H\circ(\sigma\times\mathbb{1})\circ\sigma_i\colon \Delta^{n+1}\to X'$ is admissible. We have thus proved that the homotopy operator T_* induces a well-defined homotopy operator

$$T'_*\colon C^{\mathcal{M}}_*(X,B;\mathcal{R}) \to C^{\mathcal{M}}_{*+1}(X',B';\mathcal{R})\ .$$

Now the conclusion follows from the very same argument for ordinary singular homology (see e.g. [Hat02, Thm. 2.10]). □

An admissible map $f\colon (X,B)\to(X',B')$ is an *admissible homotopy equivalence* if there exists an admissible map $g\colon (X',B')\to(X,B)$ such that $g\circ f$ and $f\circ g$ are admissibly homotopic to the identity maps $Id_{(X,B)}$ and $Id_{(X',B')}$, respectively. If this is the case, then f restricts to a bijection between B and B'. Moreover, Theorem 1.8 immediately implies the following:

Corollary 1.9. *Let $f\colon (X,B)\to(X',B')$ be an admissible homotopy equivalence. Then the induced map $f_*\colon H_*(X,B;\mathcal{R})\to H_*(X',B';\mathcal{R})$ is an isomorphism in every degree.*

1.4. **Marked homology of quasicones.** An important ingredient in our proof of Theorem 1.7 is the vanishing of marked homology of quasicones.

Proposition 1.10. *Let F be a quasicone with apex $b\in F$. Then*

$$H^{\mathcal{M}}_n(F,\{b\};\mathcal{R}) = 0$$

for every $n\geq 0$ and every coefficient ring $\mathcal{R}$.

Proof. Unfortunately, in order to prove the proposition we are not allowed to apply the homotopy invariance of marked homology to the contracting homotopy which retracts F onto b: indeed, such a homotopy is not admissible, and even the constant map $F\to F$ sending every point of the cone to b is not admissible. Nevertheless, the obvious algebraic contracting homotopy for singular chains on F takes admissible chains to admissible chains, thus yielding the desired vanishing of marked homology for quasicones.

Let us now give some details. Recall that by definition of quasicone there exists a homotopy

$$H\colon F\times[0,1]\to F$$

between the identity and the constant map at b which satisfies

$$H^{-1}(b) = (\{b\}\times I)\cup(F\times\{1\})\ .$$

Let $\sigma_i\colon \Delta^{n+1}\to\Delta^n\times I$, $i=0,\dots,n$, be the affine parametrizations of the top-dimensional simplices of the decomposition of $\Delta^n\times I$ described

in the proof of Theorem 1.8. We consider as before the usual homotopy operator $T_n\colon C_n(F;\mathcal{R}) \to C_{n+1}(F;\mathcal{R})$ such that, for every singular simplex $\sigma\colon \Delta^n \to F$,

$$T_n(\sigma) = \sum_{i=0}^{n} (-1)^i H \circ (\sigma \times \mathbb{1}) \circ \sigma_i \ .$$

Of course T_n sends $C_n(\{b\};\mathcal{R})$ to $C_{n+1}(\{b\};\mathcal{R})$, and we need to check that $T_n(\sigma) \in \widehat{C}^{\mathcal{M}}_{n+1}(F,\{b\};\mathcal{R})$ provided that $\sigma\colon \Delta^n \to F$ is admissible.

Let $K_i \subseteq \Delta^n \times I$ be the image of the affine embedding $\sigma_i\colon \Delta^{n+1} \to \Delta^n \times I$, and recall that $\Delta^n \times I$ admits a structure of simplicial complex whose top-dimensional simplices are exactly the K_i. Let $S = \sigma^{-1}(B) \subseteq \Delta^n$. Since σ is admissible, the subset S is a subcomplex of Δ^n. From the fact that $H^{-1}(\{b\}) = (\{b\} \times I) \cup (F \times \{1\})$ it follows that $(H \circ (\sigma \times I))^{-1}(\{b\}) = (F \times I) \cup (\Delta^n \times \{1\})$, so the pair

$$\left(\Delta^{n+1}, (H \circ (\sigma \times \mathbb{1}) \circ \sigma_i)^{-1}(\{b\})\right)$$

is affinely isomorphic (via σ_i) to the pair $(K_i, ((F \times I) \cup (\Delta^n \times \{1\})) \cap K_i)$. This implies that the singular simplex $H \circ (\sigma \times \mathbb{1}) \circ \sigma_i\colon \Delta^{n+1} \to X'$ is admissible. We have thus proved that the homotopy operator T_* induces a well-defined homotopy operator

$$T'_*\colon C^{\mathcal{M}}_*(F,\{b\};\mathcal{R}) \to C^{\mathcal{M}}_{*+1}(F,\{b\};\mathcal{R})$$

between the identity (since $H(\cdot,0)$ is the identity of F) and the zero map (since $H(\cdot,1)$ is the constant map at b, and simplices supported in $\{b\}$ are null in the marked chain complex). This concludes the proof. □

1.5. **Small Marked Homology.** A key fact in the proof of the Excision Property (or of the Mayer-Vietoris sequence) for ordinary singular homology is that singular homology may be computed by simplices which are supported in the elements of an open cover. In this subsection we show that the same property holds for marked homology.

Definition 1.11. Let (X,B) be a marked space and let $\mathcal{U}$ be a family of subsets of X such that

$$X \subseteq \bigcup_{U \in \mathcal{U}} \mathrm{int}(U) \ ,$$

where $\mathrm{int}(U)$ denotes the biggest open set contained in U. We say that a(n admissible) singular simplex $\sigma\colon \Delta^n \to X$ is *$\mathcal{U}$-small* if its image lies entirely in some $U \in \mathcal{U}$. We denote by $C^{\mathcal{U}}_*(X,B;\mathcal{R})$ (resp. $C^{\mathcal{U},\mathcal{M}}_*(X,B;\mathcal{R})$) the submodule of $C_*(X,B;\mathcal{R})$ (resp. of $C^{\mathcal{M}}_*(X,B;\mathcal{R})$) generated by the classes of $\mathcal{U}$-small simplices (resp. $\mathcal{U}$-small admissible simplices).

Then $C^{\mathcal{U}}_*(X,B;\mathcal{R})$ (resp. $C^{\mathcal{U},\mathcal{M}}_*(X,B;\mathcal{R})$) is a subcomplex of $C_*(X,B;\mathcal{R})$ (resp. of $C^{\mathcal{M}}_*(X,B;\mathcal{R})$), whose homology is denoted by $H^{\mathcal{U}}_*(X,B;\mathcal{R})$ (resp. $H^{\mathcal{U},\mathcal{M}}_*(X,B;\mathcal{R})$).

The following theorem extends a fundamental result about ordinary singular homology to marked homology:

Theorem 1.12. *Let $s_*\colon C_n^{\mathcal{U},\mathcal{M}}(X,B;\mathcal{R}) \to C_n^{\mathcal{M}}(X,B;\mathcal{R})$ be the obvious inclusion. Then, there exists a chain map $\rho_*\colon C_*^{\mathcal{M}}(X,B;\mathcal{R}) \to C_*^{\mathcal{U},\mathcal{M}}(X,B;\mathcal{R})$ such that the compositions $s_* \circ \rho_*$ and $\rho_* \circ s_*$ are both chain homotopic to the identity.*

Proof. The proof for ordinary singular homology (see e.g. [Hat02, Prop. 2.21]) works also in the context of marked homology. Here below we focus our attention on some subtleties that arise in the context of marked homology.

Let $S_*\colon C_*(X;\mathcal{R}) \to C_*(X;\mathcal{R})$ be the usual barycentric subdivision operator. We would like to prove that S_* induces a well-defined operator $S_*^{\mathcal{M}}\colon C_*^{\mathcal{M}}(X,B;\mathcal{R}) \to C_*^{\mathcal{M}}(X,B;\mathcal{R})$ on marked chains. The fact that S_* sends chains supported in B to chains supported in B is obvious, so we need to show that $S_*(\sigma)$ is an admissible chain provided that the singular simplex σ is admissible. To this aim, observe that a singular simplex $\sigma\colon \Delta^n \to X$ is admissible if and only if the following condition holds: let $p \in \Delta^n$ and let $D(p) \subseteq \Delta^n$ be the unique open face of Δ^n containing p (such a face may have any dimension between 0, when p is a vertex of Δ^n, and n, when p lies in the interior of Δ^n); if $\sigma(p) \in B$, then $\sigma(\overline{D(p)}) \subset B$.

Suppose now that $\sigma\colon \Delta^n \to X$ is admissible, let $K \subseteq \Delta^n$ be a geometric n-simplex appearing in the barycentric decomposition of Δ^n, and let $\tau\colon \Delta^n \to K$ be an affine parametrization of K. In order to show that $S_*(\sigma)$ is an admissible chain it is sufficient to prove that $\sigma \circ \tau$ is an admissible singular simplex. However, let $p \in \Delta^n$ be such that $\sigma(\tau(p)) \in B$. Let $D(p)$ (resp. $D(\tau(p))$) be the unique open face of Δ^n containing p (resp. $\tau(p)$), and observe that $\tau(\overline{D(p)}) \subseteq \overline{D(\tau(p))}$. Thus, if $\sigma(\tau(p)) \in B$, then by admissibility of σ we have $\sigma(\overline{D(\tau(p))}) \subseteq B$, hence $\sigma(\tau(\overline{D(p)})) \subseteq B$. This proves that $\sigma \circ \tau$ is admissible, thus showing that the operator $S_*^{\mathcal{M}}$ is indeed well defined.

In order to prove the theorem we also need to show that the operator $S_*^{\mathcal{M}}$ is homotopic to the identity. To this aim, let $T_*\colon C_*(X;\mathcal{R}) \to C_{*+1}(X;\mathcal{R})$ be the standard homotopy between S_* and the identity of $C_*(X;\mathcal{R})$ (as defined e.g. in [Hat02, Prop. 2.21]). As usual we need to prove that T_* preserves admissible chains. To this aim we first describe the geometric meaning of T_*. We inductively triangulate the prism $\Delta^n \times I$ by taking the cone of the whole triangulated boundary $(\Delta^n \times \{0\}) \bigcup (\partial\Delta^n \times I)$ with respect to the barycenter of $\Delta^n \times \{1\}$. We also fix an arbitrary top-dimensional simplex K of the triangulation of $\Delta^n \times I$ just described, and an affine parametrization $\tau\colon \Delta^{n+1} \to K$ of K.

Let now $\sigma\colon \Delta^n \to X$ be an admissible simplex, and let $\pi\colon \Delta^n \times I \to \Delta^n$ be the projection onto the first factor. In order to prove that T_* preserves

admissible chains, it is sufficient to show that the singular simplex

$$\sigma \circ \pi \circ \tau \colon \Delta^{n+1} \to X$$

is admissible. As before, for every $p \in \Delta^{n+1}$ we let $D(p)$ be the smallest open face of Δ^{n+1} containing p. Moreover, for every $q \in \Delta^n \times I$ we define $C(q)$ as the smallest open cell containing q in the product cell structure of $\Delta^n \times I$ (where we understand that Δ^n and I are endowed with the cellular structure induced by their simplicial structure). Then it is easy to check that $\tau(\overline{D(p)}) \subseteq \overline{C(\tau(p))}$. As a consequence, if $\sigma(\pi(\tau(p))) \in B$, then by admissibility of σ we have $\sigma(\overline{D(\pi(\tau(p)))}) \subseteq B$, hence $\sigma(\pi(\overline{C(\tau(p))})) \subseteq B$. But this implies in turn that $\sigma(\pi(\tau(\overline{D(p)}))) \subseteq B$, i.e. that $\sigma \circ \tau$ is admissible.

We have thus proved that the homotopy operator T_* induces an operator $T_*^{\mathcal{M}} \colon C_*^{\mathcal{M}}(X, B; \mathcal{R}) \to C_{*+1}^{\mathcal{M}}(X, B; \mathcal{R})$ which realizes a homotopy between $S_*^{\mathcal{M}}$ and the identity of $C_*^{\mathcal{M}}(X, B; \mathcal{R})$. Now the conclusion follows from the very same arguments described in [Hat02, Prop. 2.21] for ordinary singular homology. □

Corollary 1.13. *The inclusion $s_* \colon C_n^{\mathcal{U},\mathcal{M}}(X, B; \mathcal{R}) \to C_n^{\mathcal{M}}(X, B; \mathcal{R})$ induces an isomorphism*

$$H_n^{\mathcal{M}}(s_n) \colon H_n^{\mathcal{U},\mathcal{M}}(X, B; \mathcal{R}) \to H_n^{\mathcal{M}}(X, B; \mathcal{R})$$

in every degree $n \in \mathbb{N}$.

We are now ready to provide the proof of Theorem 1.7, which states that the inclusion

$$i_n \colon C_n^{\mathcal{M}}(X, B; \mathcal{R}) \to C_n(X, B; \mathcal{R})$$

induces an isomorphism

$$H_n(i_n) \colon H_n^{\mathcal{M}}(X, B; \mathcal{R}) \to H_n(X, B; \mathcal{R})$$

for every $n \in \mathbb{N}$.

1.6. **Proof of Theorem 1.7.** Let (X, B) be a marked space, and let

$$F_B = \bigcup_{b \in B} F_b$$

be the union of disjoint quasiconical closed neighbourhoods of the points in b. We also fix the cover $\mathcal{U} = \{F_B, X \setminus B\}$ of X, and we observe that the interiors of the elements of $\mathcal{U}$ still cover the whole of X. The marked chain complex $C_*^{\mathcal{U},\mathcal{M}}(F_B, B; \mathcal{R})$ is naturally a subcomplex of $C_*^{\mathcal{U},\mathcal{M}}(X, B; \mathcal{R})$, so we can consider the short exact sequence of complexes

$$C_*^{\mathcal{U},\mathcal{M}}(F_B, B; \mathcal{R}) \longrightarrow C_*^{\mathcal{U},\mathcal{M}}(X, B; \mathcal{R}) \xrightarrow{(\pi_E)_*} E_* \ ,$$

where

$$E_* = C_*^{\mathcal{U},\mathcal{M}}(X, B; \mathcal{R}) / C_*^{\mathcal{U},\mathcal{M}}(F_B, B; \mathcal{R}) \ .$$

Recall now from Proposition 1.10 that the marked homology of the pair (F_B, B) vanishes in every degree. Therefore, by looking at the long exact

sequence associated to the short exact sequence above we conclude that the quotient map induces an isomorphism

$$H_n(\pi_E)\colon H_n^{\mathcal{U},\mathcal{M}}(X,B;\mathcal{R}) \to H_n(E_*)$$

for every $n \in \mathbb{N}$.

The usual relative chain complex $C_*(F_B, B;\mathcal{R})$ is naturally a subcomplex of $C_*^{\mathcal{U}}(X,B;\mathcal{R})$, so we can consider the quotient map

$$(\pi_D)_*\colon C_*^{\mathcal{U}}(X,B;\mathcal{R}) \to D_* \;=\; C_*^{\mathcal{U}}(X,B;\mathcal{R})/C_*(F_B,B;\mathcal{R})\ .$$

Since B is a strong deformation retract of F_B we have $H_n(F_B,B;\mathcal{R}) = 0$ for every $n \in \mathbb{N}$, so arguing as above we deduce that the induced map

$$H_n(\pi_D)\colon H_n^{\mathcal{U}}(X,B;\mathcal{R}) \to H_n(D_*)$$

is an isomorphism for every $n \in \mathbb{N}$.

Observe that the set of (the classes of) singular simplices which are supported in $X \setminus B$ but not entirely contained in $F_B \setminus B$ provides a basis both of E_* and of D_*, so the inclusion $i_*^{\mathcal{U}}\colon C_*^{\mathcal{U},\mathcal{M}}(X,B;\mathcal{R}) \hookrightarrow C_*^{\mathcal{U}}(X,B;\mathcal{R})$ induces an isomorphism

$$\alpha_*\colon E_* \to D_*\ .$$

Since $H_n(\pi_E)$, $H_n(\pi_D)$, $H_n(\alpha)$ are all isomorphisms, from the commutative diagram

$$\begin{array}{ccc} H_n^{\mathcal{U},\mathcal{M}}(X,B;\mathcal{R}) & \xrightarrow{H_n(\pi_E)} & H_n(E_*) \\ \downarrow{\scriptstyle H_n(i_*^{\mathcal{U}})} & & \downarrow{\scriptstyle H_n(\alpha)} \\ H_n^{\mathcal{U}}(X,B;\mathcal{R}) & \xrightarrow[H_n(\pi_D)]{} & H_n(D_*) \end{array}$$

we deduce that the map

$$H_n(i_*^{\mathcal{U}})\colon H_n^{\mathcal{U},\mathcal{M}}(X,B;\mathcal{R}) \to H_n^{\mathcal{U}}(X,B;\mathcal{R})$$

is an isomorphism for every $n \in \mathbb{N}$.

Let us now fix $n \in \mathbb{N}$ and consider the commutative diagram

$$\begin{array}{ccc} H_n^{\mathcal{U},\mathcal{M}}(X,B;\mathcal{R}) & \longrightarrow & H_n^{\mathcal{M}}(X,B;\mathcal{R}) \\ \downarrow{\scriptstyle H_n(i_*^{\mathcal{U}})} & & \downarrow{\scriptstyle H_n(i_*)} \\ H_n^{\mathcal{U}}(X,B;\mathcal{R}) & \longrightarrow & H_n(X,B;\mathcal{R})\ , \end{array}$$

where the horizontal arrows are induced by the inclusions of small chains into generic chains. Theorem 1.12 and [Hat02, Prop. 2.21] ensure that the horizontal arrows are isomorphism, and we have just proved that also the map $H_n(i_*^{\mathcal{U}})$ is an isomorphism. It follows that the map $H_n(i_*)$ is also an isomorphism, and this concludes the proof of Theorem 1.7.

2. Ideal simplicial volume

In order to define the ideal simplicial volume we need to introduce an ℓ^1-seminorm on marked homology. Henceforth, unless otherwise stated, we will deal only with chains and classes with real coefficients. Therefore, when this does not create ambiguities, we will omit to specify the coefficients we are working with. The reader will understand that, unless otherwise stated, all the homology (and cohomology) modules will have real coefficients.

2.1. **ℓ^1-(semi)norms and simplicial volumes.** Let us first recall the definition of the ℓ^1-(semi)norm on ordinary singular homology. If (X, Y) is a topological pair, then for every singular chain $c = \sum_{i=1}^{k} a_i\sigma_i \in C_n(X, Y)$ written in reduced form (i.e. such that $\sigma_i \neq \sigma_j$ if $i \neq j$ and no σ_i is supported in $Y \subseteq X$) we set

$$\|c\|_1 = \sum_{i=1}^{k} |a_i| \ \in \mathbb{R} \ .$$

The norm $\|\cdot\|_1$ restricts to a norm on the space of relative cycles $Z_i(X, Y)$, which defines in turn a quotient seminorm (still denoted by $\|\cdot\|_1$) on the homology module $H_i(X, Y)$. If M is a compact oriented n-manifold with boundary, then $H_n(M, \partial M) \cong \mathbb{R}$ admits a preferred generator, called *real fundamental class*, which is the image via the change of coefficient map of the generator of $H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$ corresponding to the orientation of M. If $[M, \partial M] \in H_n(M, \partial M)$ denotes the real fundamental class of M, then the simplicial volume $\|M\|$ of M is defined by

$$\|M\| = \|[M, \partial M]\|_1 \ .$$

If (X, B) is a marked space, then the ℓ^1-norm on the relative chain complex $C_*(X, B)$ restricts to an ℓ^1-norm on $C_*^{\mathcal{M}}(X, B)$, which induces in turn a seminorm on the homology modules $H_*^{\mathcal{M}}(X, B)$. We will denote these (semi)norms again with the symbol $\|\cdot\|_1$.

Let now $(M, \partial M)$ be a compact oriented n-manifold with boundary, and denote by (X, B) the associated marked space as defined in Subsection 1.1, and by $p\colon M \to X$ the natural projection. Since ∂M is a strong deformation retract of an open neighboourhood of ∂M, the quotient map $p'\colon (M, \partial M) \to (M/\partial M, [\partial M])$ induces an isomorphism

$$H_k(p')\colon H_k(M, \partial M) \to H_k(M/\partial M, [\partial M])$$

for every $k \in \mathbb{N}$ (see e.g. [Hat02, Prop. 2.22]). For the same reason, also the obvious quotient map $q\colon (X, B) \to (M/\partial M, [\partial M])$ induces isomorphisms on relative homology in every degree, so from the commutative diagram

$$\begin{array}{ccc} H_k(M, \partial M) & \xrightarrow{\quad H_k(p') \quad} & H_k(M/\partial M, [\partial M]) \\ & {\scriptstyle H_k(p)} \searrow \quad \nearrow {\scriptstyle H_k(q)} & \\ & H_k(X, B) & \end{array}$$

we can deduce that also the map $H_k(p)\colon H_k(M,\partial M) \to H_k(X,B)$ is an isomorphism in every degreee $k \in \mathbb{N}$. By composing this map with the inverse of the isomorphism $H_k(i_*)\colon H_k^{\mathcal{M}}(X,B) \to H_k(X,B)$ (see Theorem 1.7), we thus obtain, for every $k \in \mathbb{N}$, the isomorphism

$$\psi_k\colon H_k(M,\partial M) \to H_k^{\mathcal{M}}(X,B)\ .$$

Definition 2.1. The *ideal fundamental class* of M is the element

$$[M,\partial M]^{\mathcal{M}} = \psi_n([M,\partial M])\ \in\ H_n^{\mathcal{M}}(X,B)\ ,$$

and the *ideal simplicial volume* of M is defined by

$$\|M\|_{\mathcal{I}} = \|[M,\partial M]^{\mathcal{M}}\|_1\ .$$

Observe that both the classical and the ideal simplicial volume of an oriented manifold do not depend on its orientation and that it is straightforward to extend the definition also to nonorientable or disconnected manifolds: if M is connected and nonorientable, then its simplicial volumes are equal to one half of the corresponding simplicial volumes of its orientable double covering, and the simplicial volumes of any manifold is the sum of the corresponding simplicial volumes of its connected components.

Henceforth, every manifold will be assumed to be compact, connected and oriented.

Remark 2.2. Let us describe a characterization of ideal fundamental cycles that will prove useful in the sequel.

Let (Z,Z') be a topological pair such that $Z\setminus Z'$ is an oriented n-manifold, and let α be a relative cycle in $C_n(Z,Z')$. Recall that for every $x \in Z \setminus Z'$ one may define the local degree of α at x as the real number $d(\alpha)(x)$ corresponding to the element defined by α in $H_n(Z,Z\setminus\{x\})$ via the canonical identification between $H_n(Z,Z\setminus\{x\})$ and $\mathbb{R}$ induced by the orientation of $Z \setminus Z'$. In fact, the number $d(\alpha)(x)$ is independent of $x \in Z \setminus Z'$.

Let now M be an n-dimensional compact oriented manifold with associated marked space (X,B). By definition, a marked cycle $z \in C_n^{\mathcal{M}}(X,B)$ is an ideal fundamental cycle if and only if, when considered as a cycle in the relative chain module $C_n(X,B)$, it defines the same homology class as the image via the natural projection of the classical fundamental class of M. Now the fundamental class of M may be characterized as the unique class $\alpha \in H_n(M,\partial M)$ having local degree equal to 1 at every point $p \in M \setminus \partial M$. As a consequence, z is an ideal fundamental cycle if and only if it has local degree equal to 1 at every point $x \in X \setminus B$.

Remark 2.3. One may wonder why we do not define the ideal simplicial volume just by taking the ℓ^1-seminorm of the image of $[M,\partial M]$ in the relative homology module $H_n(X,B)$, where (X,B) is the marked space associated to M. The following example shows that this choice would not lead to any meaningful invariant. Let M be the (natural compactification) of the complement of a hyperbolic knot in the 3-sphere S^3. Since the fundamental group of ∂M is abelian, hence amenable, we have $\|M\|_{\mathcal{I}} = \|M\| > 0$ (see

Corollary 5). On the other hand, it is not difficult to show that X is simply connected (indeed, the Wirtinger presentation of $\pi_1(M)$ shows that $\pi_1(M)$ is generated by meridian loops, all of which are killed in X). As a consequence, the bounded cohomology (with real coefficients) of the pair (X, B) vanishes in every positive degree [Gro82], and by a standard duality argument this implies in turn that the ℓ^1-seminorm of any element in $H_3(X, B)$ vanishes [Löh08].

Another possible definition of ideal simplicial volume would arise from defining admissible simplices as those singular simplices $\sigma\colon \Delta^n \to X$ such that $\sigma^{-1}(B)$ coincides with the set of vertices of Δ^n. On the one hand, this choice would lead to ideal fundamental cycles which are closer in spirit to ideal triangulations. On the other hand, with this choice the whole theory would be much more complicated: for example, showing that admissibly homotopic maps induce the same morphism on marked homology would be much more difficult; the simplices obtained by subdividing an admissible one would not be admissible; and even the fact that the ordinary simplicial volume bounds from above the ideal one, if true, would not be obvious at all. However, by Proposition 2.9 the ideal simplicial volume can be computed by looking only at ideal fundamental cycles whose simplices have all their vertices in B.

2.2. **(Marked) bounded cohomology and duality.** As anticipated in the introduction, in order to compute the (marked) simplicial volume it is often useful to switch from the study of singular chains to the dual theory of (bounded) singular cochains.

Recall that the chain modules $C_i(M, \partial M)$ and $C_i^{\mathcal{M}}(X, B)$ are endowed with ℓ^1-norms, both denoted by the symbol $\|\cdot\|_1$. We denote by $C_b^i(M, \partial M)$ (resp. $C_{\mathcal{M}}^i(X, B)$) the topological dual of $C_i(M, \partial M)$ (resp. of $C_i^{\mathcal{M}}(X, B)$), i.e. the space of bounded linear functionals on $C_i(M, \partial M)$ (resp. on $C_i^{\mathcal{M}}(X, B)$), endowed with the operator norm $\|\cdot\|_\infty$ dual to $\|\cdot\|_1$.

In the case of ordinary (relative) singular (co)chains, the modules $C_b^*(M, \partial M)$ define a subcomplex of the ordinary singular chain complex $C^i(M, \partial M)$, and are called the *bounded cochains* modules of the pair $(M, \partial M)$. The ℓ^∞-norm of a cochain $\alpha \in C_b^i(M, \partial M)$ is equal to the supremum of the values taken by α on single singular i-simplices with values in M (and not supported in ∂M). The cohomology of the complex $C_b^*(M, \partial M)$ is the *bounded cohomology* of the pair $(M, \partial M)$, and it is denoted by $H_b^*(M, \partial M)$. The ℓ^∞-norm on each $C_b^i(M, \partial M)$ restricts to the subspace of cocycles, and induces a seminorm (still denoted by $\|\cdot\|_\infty$) on $H_b^i(M, \partial M)$, for every $i \in \mathbb{N}$.

An analogous characterization holds also for the ℓ^∞-norm on $C_{\mathcal{M}}^i(X, B)$: indeed, for every $\varphi \in C_{\mathcal{M}}^i(X, B)$,

$$\|\varphi\|_\infty = \sup\{|\varphi(\sigma)| \,|\, \sigma \text{ admissible singular } i\text{-simplex not supported in } B\} \ .$$

Just as in the ordinary case, the boundary map $\partial_i\colon C_i^{\mathcal{M}}(X, B) \to C_{i-1}^{\mathcal{M}}(X, B)$ is bounded with respect to the ℓ^1-norm, hence it induces a bounded dual

map $\delta^{i-1}\colon C^{i-1}_{\mathcal{M}}(X,B) \to C^i_{\mathcal{M}}(X,B)$, which endows $(C^*_{\mathcal{M}}(X,B),\delta^*)$ with the structure of a normed complex. We denote by $H^*_{\mathcal{M}}(X,B)$ the cohomology of the complex $(C^*_{\mathcal{M}}(X,B),\delta^*)$, and we endow each $H^i_{\mathcal{M}}(X,B)$, $i \in \mathbb{N}$, with the seminorm induced by $\|\cdot\|_\infty$, which will still be denoted by $\|\cdot\|_\infty$.

The obvious pairings between $C^i_b(M,\partial M)$ and $C_i(M,\partial M)$ and between $C^i_{\mathcal{M}}(X,B)$ and $C_i^{\mathcal{M}}(X,B)$ induce pairings

$$\langle\cdot,\cdot\rangle\colon H^i_b(M,\partial M)\times H_i(M,\partial M)\to\mathbb{R}\,,$$
$$\langle\cdot,\cdot\rangle\colon H^i_{\mathcal{M}}(X,B)\times H_i^{\mathcal{M}}(X,B)\to\mathbb{R}\,.$$

The following duality result is proved e.g. in [Löh08], and will be used in the proofs of Theorem 4 and 7.

Proposition 2.4. *Let $\alpha \in H_i^{\mathcal{M}}(X,B)$. Then*

$$\|\alpha\|_1 = \max\{\langle\varphi,\alpha\rangle \mid \varphi\in H^i_{\mathcal{M}}(X,B),\ \|\varphi\|_\infty\le 1\}\ .$$

2.3. **Ideal simplicial volume vs. complexity.** Recall that a Δ-complex is a topological space obtained by gluing a family of copies of the standard simplex along affine diffeomorphisms of some of their faces. Therefore, Δ-complexes provide a mild generalization of (geometric realizations of) simplicial complexes, the only differences being that simplices in Δ-complexes need not be embedded (since identifications between pairs of faces of the same simplex are allowed), and that distinct simplices in Δ-complexes may share more than one face.

Definition 2.5. Let $(M,\partial M)$ be a compact n-manifold with boundary with associated marked space (X,B). An *ideal triangulation* of M is a realization of (X,B) as a Δ-complex whose set of vertices is equal to B. The *complexity* $c(M)$ of M is defined as the minimal number of n-dimensional simplices in any ideal triangulation of M.

Let us now prove that the complexity bounds the ideal simplicial volume from above. Let M be an n-manifold with boundary with $c(M)=m$, and let (X,B) be the marked space associated to M. By definition of complexity, there exists an ideal triangulation of M with m top-dimensional simplices, i.e. a realization of X as a Δ-complex with the following properties: X is obtained by gluing m copies $\Delta^n_1,\dots,\Delta^n_m$ of the standard simplex Δ^n; the set of vertices of the resulting Δ-complex is equal to B. Recall that M is oriented, so for every $j=1,\dots,m$ we can fix an orientation-preserving affine identification $\sigma_j\colon\Delta^n\to\Delta^n_j$. It is not quite true that the sum of the σ_j defines a marked cycle. Nevertheless, in order to obtain a cycle out of the σ_j it is sufficient to alternate them as follows.

Let $\mathfrak{S}_{n+1}$ be the group of permutations of the set $\{0,\dots,n\}$, and denote by $e_0,\dots,e_n$ the vertices of the standard simplex Δ^n. For every $\tau\in\mathfrak{S}_{n+1}$ we denote by $\overline{\tau}$ the unique affine automorphism $\overline{\tau}\colon\Delta^n\to\Delta^n$ such that $\overline{\tau}(e_i)=e_{\tau(i)}$ for every $i=0,\dots,n$. Observe now that for each $j=0,\dots,m$

and every $\tau \in \mathfrak{S}_{n+1}$ the set $(\sigma_j \circ \overline{\tau})^{-1}(B)$ is equal to the set of vertices of Δ^n, so $\sigma_j \circ \overline{\tau}$ is admissible. We may thus define the marked chain

$$z = \frac{1}{(n+1)!} \sum_{j=1}^{m} \sum_{\tau \in \mathfrak{S}_{n+1}} \varepsilon(\tau) \sigma_j \circ \overline{\tau} \ \in \ C_n^{\mathcal{M}}(X, B) \ ,$$

where $\varepsilon(\tau) \in \{1, -1\}$ denotes the sign of τ.

It is now easy to check that z is indeed a cycle. Moreover, the local degree of z at any point in $X \setminus B$ is equal to one, so Remark 2.2 implies that the class $[z]^{\mathcal{M}}$ of z in $H_n^{\mathcal{M}}(X, B)$ is the ideal fundamental class of M. Finally, we have $\|z\|_1 = m$, hence

$$\|M\|_{\mathcal{I}} = \|[M, \partial M]^{\mathcal{M}}\|_1 \leq \|z\|_1 = m = c(M) \ .$$

This concludes the proof of Theorem 1.

2.4. **Ideal simplicial volume vs. classical simplicial volume.** Let $\sigma \colon \Delta^i \to M$ be a singular simplex, and recall that $p \colon (M, \partial M) \to (X, B)$ is the natural projection. We say that σ is *admissible* if $p \circ \sigma$ is admissible as a simplex with values in the marked space (X, B), i.e. if $\sigma^{-1}(\partial M)$ is a subcomplex of Δ^i. Moreover, a chain $c \in C_*(M, \partial M)$ is admissible if it is (the class of) a linear combination of admissible simplices. The following lemma provides the key step in the proof of the inequality $\|M\|_{\mathcal{I}} \leq \|M\|$, and will be useful also in the proof of Theorem 2.

Lemma 2.6. *Let M be a manifold with boundary, and let A' be a closed subspace of a perfectly normal topological space A. Let also $f \colon (A, A') \to (M, \partial M)$ be a map of pairs. Then f is homotopic (as a map of pairs) to a map $g \colon (A, A') \to (M, \partial M)$ such that $g^{-1}(\partial M) = A'$.*

Proof. A classical result in general topology ensures that ∂M admits a closed collar $\mathcal{C}$ in M. More precisely, there exists a subset $\mathcal{C} \subset M$ containing ∂M such that the pair $(\mathcal{C}, \partial M)$ is homeomorphic to the pair $(\partial M \times [0,1], \partial M \times \{0\})$. Henceforth we fix an identification $\mathcal{C} \cong \partial M \times [0,1]$ induced by such a homeomorphism.

Since f is a map of pairs, we already know that $f^{-1}(\partial M) \supseteq A'$, and we need to perturb f in order to push out of ∂M all points not belonging to A'. Since A is perfectly normal, there exists a continuous map $h \colon A \to [0,1]$ such that $h^{-1}(0) = A'$. Moreover, for every $x \in f^{-1}(\mathcal{C})$ we have

$$f(x) = (m(x), d(x)) \in \partial M \times [0,1] \ ,$$

where $m \colon f^{-1}(\mathcal{C}) \to \partial M$ and $d \colon f^{-1}(\mathcal{C}) \to [0,1]$ are continuous functions. Let us now define a map $H \colon A \times [0,1] \to M$ as follows. If $x \in f^{-1}(\overline{M \setminus \mathcal{C}})$, then $H(x,t) = f(x)$ for every $t \in [0,1]$. If $x \in f^{-1}(\mathcal{C})$, then

$$H(x,t) = (m(x), \min\{d(x) + t h(x), 1\}) \ .$$

First observe that, if $x \in f^{-1}(\overline{M \setminus \mathcal{C}}) \cap f^{-1}(\mathcal{C})$, then necessarily $f(x) = (m(x), 1)$, and this readily implies that the map H is well defined, hence continuous. Moreover, if $x \in A'$, then $d(x) = h(x) = 0$, so $H(x,t) =$

$f(x) \in \partial M$ for every $t \in [0,1]$, i.e. the maps $f = H(\cdot,0)$ and $g = H(\cdot,1)$ are homotopic as map of pairs. Finally, we have $g(x) \in \partial M$ if and only if $f(x) \in \mathcal{C}$ and $d(x) = h(x) = 0$, i.e. if and only if $x \in A'$, as desired. □

We are now ready to prove that the ideal simplicial volume is bounded from above by the classical simplicial volume:

Theorem 2.7. *Let $(M, \partial M)$ be an orientable compact manifold with boundary. Then*

$$\|M\|_{\mathcal{I}} \leq \|M\| \ .$$

Proof. Let $\varepsilon > 0$ be given, and let $z \in C_n(M, \partial M)$ be a fundamental cycle such that $\|z\|_1 \leq \|M\| + \varepsilon$. We would like to modify z into an admissible fundamental cycle without increasing its ℓ^1-norm. To this aim, we briefly discuss a well-known geometric description of cycles in singular homology, also described in [Fri17, Section 13.2]. We refer the reader to [Löh16, Section 5.1] for an alternative approach to this construction.

Let $z = \sum_{i=1}^{k} a_i \overline{\sigma}_i$, and assume that $\overline{\sigma}_i \neq \overline{\sigma}_j$ for $i \neq j$. One can construct a Δ-complex P associated to z by gluing k distinct copies $\Delta^n_1, \dots, \Delta^n_k$ of the standard n-simplex Δ^n as follows. For every i we fix an identification between Δ^n_i and Δ^n, so that we may consider σ_i as defined on Δ^n_i. For every $i = 1, \dots, k$, $j = 0, \dots, n$, we denote by F^i_j the j-th face of Δ^n_i, and by $\partial^i_j \colon \Delta^{n-1} \to F^i_j \subseteq \Delta^n_i$ the usual face inclusion. We say that the faces F^i_j and $F^{i'}_{j'}$ are *equivalent* if $\overline{\sigma}_i|_{F^i_j} = \overline{\sigma}_{i'}|_{F^{i'}_{j'}}$, or, more formally, if $\partial^i_j \circ \overline{\sigma}_i = \partial^{i'}_{j'} \circ \overline{\sigma}_{i'}$. We now define a Δ-complex P as follows. The simplices of P are $\Delta^n_1, \dots, \Delta^n_k$, and, if F^j_i, $F^{j'}_{i'}$ are equivalent, then we identify them via the affine diffeomorphism $\partial^{j'}_{i'} \circ (\partial^j_i)^{-1} \colon F^j_i \to F^{j'}_{i'}$. By construction, the maps $\overline{\sigma}_1, \dots, \overline{\sigma}_k$ glue up to a well-defined continuous map $f \colon P \to M$. We also define the boundary ∂P of P as the subcomplex of P given by all the $(n-1)$-faces of P which are sent by f entirely into ∂M. By definition, the map f is a map of pairs from $(P, \partial P)$ to $(M, \partial M)$.

For every $i = 1, \dots, k$, let $\hat{\sigma}_i \colon \Delta^n \to P$ be the singular simplex obtained by composing the identification $\Delta^n \cong \Delta^n_i$ with the quotient map with values in P, and let us consider the singular chain $z_P = \sum_{i=1}^{k} a_i \hat{\sigma}_i \in C_n(P)$. By construction, z_P is a relative cycle in $C_n(P, \partial P)$, and the push-forward of z_P via f is equal to z. Moreover, $\|z_P\|_1 = \|z\|_1$.

By applying Lemma 2.6 to the map $f \colon (P, \partial P) \to (M, \partial M)$ we now obtain a map $g \colon (P, \partial P) \to (M, \partial M)$ such that $g^{-1}(\partial M) = \partial P$. We now set

$$z' = g_*(z_P) \ \in \ C_*(M, \partial M) \ .$$

Since g is homotopic to f as a map of pairs, the chain z' is a relative fundamental cycle for M. Moreover, we have

$$\|z'\|_1 = \|g_*(z_P)\|_1 \leq \|z_P\|_1 = \|z\|_1 \leq \|M\| + \varepsilon \ .$$

Finally, if $\sigma'_i\colon \Delta^n \to M$ is a singular simplex appearing in z', then $\sigma^{-1}(\partial M)$ is a (possibly empty) union of $(n-1)$-faces of Δ^n, hence it is admissible.

Let $p\colon (M, \partial M) \to (X, B)$ be the natural projection of M onto its associated marked space. Since z' is admissible, we have $p_*(z') \in C^{\mathcal{M}}(X, B)$. Moreover, since z' is a relative fundamental cycle for M, the chain $p_*(z')$ is an ideal fundamental cycle for M. Thus

$$\|M\|_{\mathcal{I}} = \|[p_*(z')]\|_1 \leq \|p_*(z')\|_1 \leq \|z'\|_1 \leq \|M\| + \varepsilon \ .$$

Since ε is arbitrary, this concludes the proof. □

In order to conclude the proof of Theorem 3 we now need to show the following:

Proposition 2.8. *There exists a constant $K_n > 0$ such that*

$$\|M\| \leq K_n \cdot \|M\|_{\mathcal{I}}$$

for every n-dimensional manifold M.

Proof. Let (X, B) be the marked space associated to M, let $\pi\colon M \to X$ be the natural projection, and let $z^{\mathcal{M}} = \sum_{i=1}^k a_i \sigma_i \in C_*^{\mathcal{M}}(X, B)$ be an ideal fundamental cycle for M. We are going to truncate the singular simplices σ_i to obtain a realization of the relative fundamental class of M via singular polytopes. We will then triangulate these polytopes to get a classical relative fundamental cycle, at the cost of the multiplicative constant K_n mentioned in the statement.

Let us begin with a general definition. Let Q be an m-dimensional polytope (i.e. the convex hull of a finite number of points which span an m-dimensional affine space in some Euclidean space), $m \leq n$. We inductively define the notion of *fundamental cycle* for Q as follows. If $m = 0$, then Q is a point, and we set $z_Q = \sigma \in C_0(Q)$, where σ is the constant singular simplex at Q. If $0 < m \leq n$, we say that a chain $z_Q \in C_m(Q)$ is a fundamental cycle for Q if the following conditions hold: the class of z_Q in $H_m(Q, \partial Q)$ is indeed a relative fundamental cycle for $(Q, \partial Q) \cong (D^m, \partial D^m)$; if $F_1, \dots, F_k$ are the facets of Q (i.e. the codimension-1 faces of Q), then $\partial z_Q = \sum_{i=1}^k c_i$, where c_i is a fundamental cycle for F_i for every $i = 1, \dots, k$.

We now fix $0 < \varepsilon < 1/(n+1)$ and, for every $0 \leq m \leq n$, we define a family $\Omega(m, \varepsilon)$ of oriented polytopes which are obtained by truncating the standard simplex

$$\Delta^m = \{(t_0, \dots, t_m) \in \mathbb{R}^{m+1} \mid \sum_{i=0}^m t_i = 1\}$$

around some of its faces. Namely, we say that a polytope Q belongs to $\Omega(m, \varepsilon)$ if there exists a subcomplex $K \subseteq \Delta^m$ such that

$$Q = \{(t_0, \dots, t_m) \in \Delta^m \mid \sum_{i \in A} t_i \leq 1 - \varepsilon \text{ whenever } A \text{ is the set of indices corresponding to a face in } K\} \ .$$

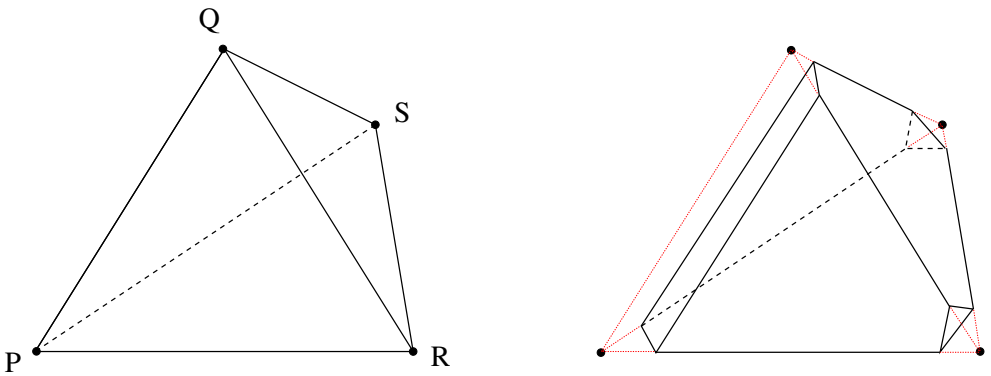

Figure 1. On the left, a standard simplex. On the right, the truncated simplex obtained by removing neighbourhoods of the edge PQ and of the vertices R and S.

In other words, Q is obtained by removing from Δ^m a neighbourhood of K (see Fig. 1). We endow Q with the orientation induced by Δ^m, and every facet of Q will also be oriented as (a subset of) the boundary of Q. The condition $\varepsilon < 1/(n+1)$ ensures that Q has nonempty interior unless $K = \Delta^m$. If Q lies in $\Omega(m, \varepsilon)$, then every facet of Q is affinely isomorphic to an element of $\Omega(m-1, \varepsilon)$.

We now claim that for every $0 \leq m \leq n$ and for every $Q \in \Omega(m, \varepsilon)$ there exists a fundamental cycle z_Q for Q such that the following conditions hold:

(1) $\|z_Q\|_1 \leq K_m$, where K_m is a constant only depending on m;
(2) let $F_1, \dots, F_k$ be the facets of Q and let $\partial z_Q = \sum_{i=1}^k c_i$, where c_i is a fundamental cycle for F_i for every $i = 1, \dots, k$. If $\varphi\colon F_i \to Q'$ is any orientation-preserving affine isomorphism between F_i and an element $Q' \in \Omega(m-1, \varepsilon)$, then $\varphi_*(c_i) = z_{Q'}$;
(3) if $\varphi\colon Q \to Q$ is an affine isomorphism, then $\varphi_*(z_Q) = \tau(\varphi) z_Q$, where $\tau(\varphi) = \pm 1$ is positive if φ is orientation-preserving, and negative otherwise.

The existence of the claimed family of fundamental cycles may be proved by induction as follows. If $m = 0$, then either $Q = \emptyset$ or Q is a point, and in both cases the conclusion easily follows. Suppose now that the desired fundamental cycles have been constructed for every $i = 0, \dots, m-1$, and let $Q \in \Omega(m, \varepsilon)$. For every facet F of Q we have an orientation-preserving affine isomorphism $\varphi\colon F \to Q'$ for some $Q' \in \Omega(m-1, \varepsilon)$, and we set $z_F = \varphi_*^{-1}(z_F) \in C_{m-1}(F) \subseteq C_m(Q)$. By (3) (applied to the polytope Q') the chain z_F does not depend on the chosen affine isomorphism. Let us now consider the cycle $z_\partial = \sum_F z_F$. Using (2) and (3) it is immediate to check that $\partial z_\partial = 0$, and this readily implies that z_∂ is a classical fundamental

cycle for ∂Q. We can then fill z_∂ thus obtaining a chain $z'_Q \in C_m(Q)$ such that $\partial z'_Q = z_\partial$. If G is the group of the affine isomorphisms of Q into itself, we then set $z_Q = (1/|G|)\sum_{g\in G}\tau(g)g_*(z'_Q)$. Using that each z_F satisfies (3) we easily see that $\partial z_Q = \partial z'_Q$, and this readily implies that z_Q satisfies (2) and (3). In order to get (1), we only need to observe that $\Omega(m,\varepsilon)$ is finite, so we may set $K_m = \max\{\|z_Q\|_1 \,|\, Q \in \Omega(m,\varepsilon)\}$. Indeed, if $\varepsilon' \neq \varepsilon$, then there exists an obvious bijection between $\Omega(m,\varepsilon)$ and $\Omega(m,\varepsilon')$, and fundamental cycles for elements of $\Omega(m,\varepsilon')$ may be chosen to have exactly the same ℓ^1-norm as the ones for the corresponding elements of $\Omega(m,\varepsilon)$. This concludes the proof of the claim.

We can now proceed with the proof of the proposition. Let $z^{\mathcal{M}} = \sum_{i=1}^s a_i\sigma_i \in C^{\mathcal{M}}(X,B)$ be an ideal fundamental cycle for M. For every $b \in B$ denote by F_b a closed quasiconical neighbourhood of b such that $X \setminus \bigsqcup_{b\in B}\operatorname{int}(F_b)$ is homeomorphic to M. We also set $F_B = \bigsqcup_{b\in B} F_b$ and we denote by ∂F_B the topological boundary of F_B in X, so that $(X\setminus\operatorname{int}(F_B), \partial F_B)$ is homeomorphic to $(M,\partial M)$.

By continuity, there exists $\varepsilon_0 > 0$ such that the following condition holds: for every $i = 1,\dots,s$, if $K_i = \sigma_i^{-1}(B)$, then $\sigma_i(\Delta^n\setminus\operatorname{int}(Q_i)) \subseteq \operatorname{int}(F_B)$, where $Q_i \in \Omega(n,\varepsilon_0)$ is obtained as above by removing from Δ^n a neighbourhood of K_i. For every $i = 1,\dots,s$ let now z_{Q_i} be the fundamental cycle for Q_i defined above, and set

$$z_i = (\sigma_i)_*(z_{Q_i}) \in C_n(X\setminus B)\,, \quad z = \sum_{i=1}^s a_i z_i\ .$$

Using that $z^{\mathcal{M}}$ is a marked cycle and the properties of the z_{Q_i} it is not difficult to show that $\partial z \in C_n(F_B \setminus B)$. In particular, z is a relative cycle in $C_n(X\setminus B, F_B\setminus B)$. By retracting each $F_b\setminus\{b\}$ onto ∂F_b we can construct a homotopy equivalence of pairs

$$r\colon\ (X\setminus B, F_B\setminus B) \to (X\setminus\operatorname{int}(F_B), \partial F_B) \cong (M,\partial M)\ .$$

We claim that the chain $r_*(z)$ is a relative fundamental cycle for M. Indeed, $z^{\mathcal{M}}$ is an ideal fundamental cycle, so its image in $H_n(X,B)$ has local degree equal to one at every point $x \in X\setminus B$ (see Remark 2.2). Using this and the fact that every z_{Q_i} is a relative fundamental cycle for Q_i, we deduce that also z has local degree equal to one at every point $x \in X\setminus F_B$. Then, the same holds true for $r_*(z)$, and this shows that $r_*(z)$ is a fundamental class for M. Therefore,

$$\begin{aligned}\|M\| \le \|r_*(z)\|_1 \le \|z\|_1 &= \left\|\sum_{i=1}^s a_i(\sigma_i)_*(z_{Q_i})\right\|_1 \\ &\le \sum_{i=1}^s |a_i|\cdot\|z_{Q_i}\|_1 \le K_n\sum_{i=1}^s|a_i| = K_n\cdot\|z^{\mathcal{M}}\|_1\end{aligned}$$

By taking the infimum over all possible ideal fundamental cycles we get

$$\|M\| \leq K_n \|M\|_{\mathcal{I}} \ ,$$

which concludes the proof. □

By exploiting the techinques introduced in the proof of Theorem 2.7 we may show that the ℓ^1-norm of a marked homology class can be computed by looking only at marked chains whose simplices have all their vertices in B:

Proposition 2.9. *Let (X, B) be a marked space such that $H_0(X, B) = 0$ (i.e. B intersects every path connected component of X), and let $i \in \mathbb{N}$. Then, for every $\alpha \in H_i^{\mathcal{M}}(X, B)$ and every $\varepsilon > 0$ there exists a marked cycle $z \in C_i^{\mathcal{M}}(X, B)$ satisfying the following properties:*

(1) $[z] = \alpha$ in $H_i^{\mathcal{M}}(X, B)$ and $\|c\|_1 \leq \|\alpha\|_1 + \varepsilon$;
(2) all the vertices of every singular simplex appearing in z lie in B.

Proof. Let $z \in C_i^{\mathcal{M}}(X, B)$ be an admissible cycle. In order to prove the proposition, it is sufficient to show that z is homologous to an admissible cycle $z' \in C_i^{\mathcal{M}}(X, B)$ such that $\|z'\|_1 \leq \|z\|_1$ and all the vertices of every singular simplex appearing in z' lie in B.

Let $z = \sum_{j=1}^{k} a_j \sigma_j$. We exploit the notation introduced in the proof of Theorem 2.7 (with $(M, \partial M)$ replaced by (X, B)), and we denote by P the Δ-complex associated to z (which is obtained by gluing k copies of the standard simplex Δ^i), and by $f \colon P \to X$ the continuous map obtained by gluing the maps $\sigma_1, \cdots, \sigma_k$. Moreover, we denote by $\hat{\sigma}_j \colon \Delta^i \to P$ the characteristic map of the j-th copy of Δ^i in P (see the proof of Theorem 2.7), and we set $z_P = \sum_{j=1}^{k} a_j \hat{\sigma}_j \in C_i(P)$.

By construction, z_P is a relative cycle in $C_i(P, f^{-1}(B))$, and $z = f_*(z_P)$. Moreover, since every σ_j is admissible, the subset $f^{-1}(B) \subseteq P$ is a subcomplex of P.

We now aim to construct a homotopy of pairs $H \colon (P, f^{-1}(B)) \times I \to (X, B)$ between f and a map g such that every singular simplex appearing in $z' = g_*(z_P)$ is admissible and has all its vertices in B. We will then have $[z'] = [z]$ in $H_i(X, B)$, hence in $H_i^{\mathcal{M}}(X, B)$ thanks to Theorem 1.7. Then the proposition will follow from the inequality

$$\|z'\|_1 = \|g_*(z_P)\|_1 \leq \|z_P\|_1 = \|z\|_1 \ .$$

Let $v \in P^0 \setminus f^{-1}(B)$ be a vertex of P which is not sent to B by f. Since $H_0(X, B) = 0$, there exists a continuous path $\gamma_v \colon [0, 1] \to X$ joining v with a point in B. Moreover, since every point of B has a quasiconical neighbourhood in X, we can safely assume that $\gamma_v^{-1}(B) = \{1\}$, i.e. the path γ_v hits B only at its endpoint. Let us now consider the homotopy

$$\widehat{H} \colon (f^{-1}(B) \cup P^0) \times I \to X \ ,$$

$$\widehat{H}(x,t) = \begin{cases} f(x) & \text{if } x \in f^{-1}(B) \\ \gamma_x(t) & \text{if } x \in P^0 \setminus f^{-1}(B) \end{cases}$$

($\widehat{H}$ is indeed continuous since both $f^{-1}(B)$ and $P^0 \setminus f^{-1}(B)$ are closed in P). Since $f^{-1}(B) \cup P^0$ is a subcomplex of P, thanks to the Homotopy Extension Property for CW-pairs (see [Hat02, Proposition 0.16]) we can extend $\widehat{H}$ to a homotopy $H\colon P \to X$. Moreover, since the homotopy $\widehat{H}$ is relative to $f^{-1}(B)$, we may assume that H is also relative to $f^{-1}(B)$. In particular, H also defines a homotopy of pairs $H\colon (P, f^{-1}(B)) \times I \to (X, B)$. We then set $g = H(\cdot, 1)$, and we are left to show that every singular simplex appearing in $z' = g_*(z_P)$ is admissible and has all its vertices in B.

The fact that every singular simplex appearing in z' has all its vertices in B readily follows from the construction of g. Moreover, in order to prove that each singular simplex appearing in z' is admissible it is sufficient to show that $g^{-1}(B)$ is a subcomplex of P. However, from the definition of $\widehat{H}$ and from the explicit description of its extension H given in [Hat02, Proposition 0.16] we deduce that $g^{-1}(B) = f^{-1}(B) \cup P^0$. Since $f^{-1}(B)$ is a subcomplex of P, this concludes the proof. □

2.5. **Bounding mapping degrees.** Let us now turn to the proof of Theorem 2. Let $f\colon (M, \partial M) \to (N, \partial N)$ be a map of pairs between n-dimensional manifolds with boundary. We denote by (X_M, B_M) (resp. (X_N, B_N)) the marked space associated to M (resp. to N).

By Lemma 2.6, the map f is homotopic (as a map of pair) to a map $g\colon (M, \partial M) \to (N, \partial N)$ such that $g^{-1}(\partial N) = \partial M$. This condition ensures that the map $\overline{g}\colon (X_M, B_M) \to (X_N, B_N)$ induced by g is admissible. By looking at the commutative diagram

$$\begin{array}{ccc} H_n(M, \partial M) & \xrightarrow{H_n(g)} & H_n(N, \partial N) \\ \downarrow & & \downarrow \\ H_n(X_M, B_M) & \xrightarrow{H_n(\overline{g})} & H_n(X_N, B_N) \\ \uparrow & & \uparrow \\ H_n^{\mathcal{M}}(X_M, B_M) & \xrightarrow{H_n^{\mathcal{M}}(\overline{g})} & H_n^{\mathcal{M}}(X_N, B_N) \end{array}$$

we easily deduce that

$$H_n^{\mathcal{M}}(\overline{g})([M, \partial M]^{\mathcal{M}}) = \deg(g) \cdot [N, \partial N]^{\mathcal{M}} = \deg(f) \cdot [N, \partial N]^{\mathcal{M}} \ .$$

Moreover, the operator $H_n^{\mathcal{M}}(\overline{g})$ is norm non-increasing, since it is induced by a map on marked chains which sends every single admissible simplex to a single admissible simplex. Thus

$$\begin{aligned} \|M\|_{\mathcal{I}} &= \|[M, \partial M]^{\mathcal{M}}\|_1 \geq \|H_n^{\mathcal{M}}(\overline{g})([M, \partial M]^{\mathcal{M}})\|_1 \\ &= \|\deg(f) \cdot [N, \partial N]^{\mathcal{M}}\|_1 = |\deg(f)| \cdot \|N\|_{\mathcal{I}} \ . \end{aligned}$$

This concludes the proof of Theorem 2.

3. The universal covering of a marked space

When computing the (co)homology of a space, it is often useful to work with (co)invariant (co)chains on coverings. In order to implement this strategy in the context of marked (co)homology, we first need to define coverings of marked spaces. For the sake of simplicity (and since this will be sufficient for our purposes) we only define the universal covering of a marked space, even though our construction may be easily adapted to define a more general notion of covering between marked spaces.

Let (X, B) be a marked space, and assume for simplicity that $Y = X \setminus B$ admits a universal covering $q\colon \widetilde{Y} \to Y$. Then we define the universal covering $(\widetilde{X}, \widetilde{B})$ of (X, B) as follows. Let $\{F_b\}_{b\in B}$ be a collection of disjoint quasiconical closed neighbourhoods of the points of B. We pick a set $\widetilde{B}$ endowed with a fixed bijection with the set of connected components of $q^{-1}(\bigcup_{b\in B}(F_b \setminus \{b\}))$ and we define $\widetilde{X} = \widetilde{Y} \cup \widetilde{B}$. Observe that the covering projection $q\colon \widetilde{Y} \to Y$ extends to a map $\pi\colon \widetilde{X} \to X$ sending to $b \in B$ all the points of $\widetilde{B}$ corresponding to a component of $q^{-1}(F_b)$.

In order to turn the pair $(\widetilde{X}, \widetilde{B})$ into a marked space, we endow $\widetilde{X}$ with the unique topology such that the following conditions hold: the subset $\widetilde{Y} = \widetilde{X} \setminus \widetilde{B}$ is open in $\widetilde{X}$ and inherits from $\widetilde{X}$ the topology of $\widetilde{Y}$ as total space of the universal covering of Y; if $\widetilde{b} \in \widetilde{B}$ corresponds to a component $\widetilde{F}$ of $q^{-1}(F_b \setminus \{b\})$, then a basis of neighbourhoods of $\widetilde{b}$ in $\widetilde{X}$ is given by the collection $\{\widetilde{F} \cap \pi^{-1}(U)\}$, as U varies in a basis of neighbourhoods of b in X.

Lemma 3.1. *The pair $(\widetilde{X}, \widetilde{B})$ is indeed a marked space, and the map $\pi\colon (\widetilde{X}, \widetilde{B}) \to (X, B)$ is admissible.*

Proof. Let us check that every $\widetilde{b} \in \widetilde{B}$ admits a quasiconical neighbourhood in $\widetilde{X}$. Let $b = \pi(\widetilde{b})$ and let F_b be a quasiconical neighbourhood of b in X. Let also $H\colon F_b \times [0,1] \to F_b$ be a contracting homotopy as in Definition 1.1, and denote by

$$H'\colon (F_b \setminus \{b\}) \times [0,1) \to F_b \setminus \{b\}$$

the restriction of H' to $(F_b \setminus \{b\}) \times [0,1)$ (which is well defined thanks to the properties of H). Let also $\widetilde{F} \subseteq \widetilde{X} \setminus \widetilde{B}$ be the connected component of $q^{-1}(F_b \setminus \{b\})$ corresponding to $\widetilde{b}$. Covering theory ensures that the homotopy H' lifts to a homotopy

$$\widetilde{H}'\colon \widetilde{F} \times [0,1) \to \widetilde{F}$$

such that $\widetilde{H}'(x, 0) = x$ for every $x \in \widetilde{F}$. We now set $\widetilde{F}_{\widetilde{b}} = \widetilde{F} \cup \{\widetilde{b}\}$, and we extend $\widetilde{H}'$ to a homotopy

$$\widetilde{H}\colon \widetilde{F}_{\widetilde{b}} \times [0,1] \to \widetilde{F}_{\widetilde{b}}$$

by setting $\widetilde{H}(x,t) = \widetilde{b}$ whenever $x = \widetilde{b}$ or $t = 1$. It is now easy to check that the map $\widetilde{H}$ is continuous and satisfies the requirements of Definition 1.1, hence $\widetilde{F}_{\widetilde{b}}$ is a closed quasiconical neuighbourhood of $\widetilde{b}$ in $\widetilde{X}$. As $\widetilde{b}$ varies in $\widetilde{B}$, the construction just described provides disjoint sets, and this concludes the proof that $(\widetilde{X}, \widetilde{B})$ is a marked space.

The fact that $\pi^{-1}(B) = \widetilde{B}$ is obvious, while the continuity of π readily follows from the definitions. Thus π is admissible, whence the conclusion. □

In general, the map $\pi\colon \widetilde{X} \to X$ is no longer a covering (in fact, it is a covering if and only if the image of $\pi_1(F_b \setminus \{b\})$ into $\pi_1(X \setminus B)$ is trivial for every $b \in B$).

Remark 3.2. Let M be a compact manifold with boundary with associated marked space (X, B). Let $f\colon \widetilde{M} \to M$ be the universal covering of M, and let $p_{\widetilde{M}}\colon (\widetilde{M}, \partial\widetilde{M}) \to (X_{\widetilde{M}}, B_{\widetilde{M}})$ be the natural projection on the associated marked space. It is very natural to ask whether $(X_{\widetilde{M}}, B_{\widetilde{M}})$ may be identified with the universal covering $(\widetilde{X}, \widetilde{B})$ of (X, B).

First observe that the identification

$$\widetilde{M} \setminus \partial\widetilde{M} = \widetilde{M \setminus \partial M} \cong \widetilde{X \setminus B}$$

extends to a projection $p'_{\widetilde{M}}\colon \widetilde{M} \to \widetilde{X}$ which is continuous and sends every connected component of $\partial\widetilde{M}$ to a single point in $\widetilde{B}$. As a consequence, there exists a (unique) map $\alpha\colon X_{\widetilde{M}} \to \widetilde{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\quad p_{\widetilde{M}} \quad} & X_{\widetilde{M}} \\ & {\scriptstyle p'_{\widetilde{M}}} \searrow \quad \swarrow {\scriptstyle \alpha} & \\ & \widetilde{X}\ . & \end{array}$$

Since $X_{\widetilde{M}}$ is endowed with the quotient topology, the map α is continuous. Moreover, it is easy to check that it is bijective. However, we now show that α is *not* a homeomorphism if $\partial\widetilde{M}$ contains a non-compact connected component $\partial_0\widetilde{M}$.

Let $\widetilde{b} = p_{\widetilde{M}}(\partial_0\widetilde{M}) \in B_{\widetilde{M}}$ be the point corresponding to $\partial_0\widetilde{M}$. We claim that $\widetilde{b}$ does not admit a countable basis of neighbourhoods in $X_{\widetilde{M}}$. In fact, let $\{U_n\}_{n\in\mathbb{N}}$ be a countable collection of neighbourhoods of $\widetilde{b}$ in $X_{\widetilde{M}}$. Let d be a distance inducing the topology of $\widetilde{M}$ (every topological manifold is metrizable), and choose a diverging sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\partial_0\widetilde{M}$. For every $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ such that the d-ball $B(x_n, \varepsilon_n)$ is contained in $p^{-1}_{\widetilde{M}}(U_n)$. Using that the sequence $\{x_n\}_{n\in\mathbb{N}}$ diverges, it is not difficult to construct an open neighbourhood V of $\partial_0\widetilde{M}$ in $\widetilde{M}$ such that V does not contain $B(x_n, \varepsilon_n)$ for every $n \in \mathbb{N}$. The projection $p_{\widetilde{M}}(V)$ of V is now an

open neighbourhood of $\widetilde{b}$ in $X_{\widetilde{M}}$ which does not contain any of the U_n, $n \in \mathbb{N}$. This shows that $\widetilde{b}$ does not admit a countable basis of neighbourhoods in $X_{\widetilde{M}}$.

In order to show that α is not a homeomorphism it is now sufficient to observe that $\alpha(\widetilde{b})$ has a countable basis of neighbourhoods in $\widetilde{X}$. In fact, let $b = \pi(\alpha(\widetilde{b}))$ be the projection of $\alpha(\widetilde{b})$ under the covering of marked spaces $\pi\colon \widetilde{X} \to X$ introduced above. It follows from the very definition of the topology of $\widetilde{X}$ that a basis of neighbourhoods at $\alpha(\widetilde{b})$ is given by the set $\{\pi^{-1}(U)\}$ as U varies in a basis of neighbourhoods of b in X. Now it is easy to check that X is second countable (for example, it is metrizable and compact), hence b has a countable basis of neighbourhoods in X. This implies in turn that $\alpha(\widetilde{b})$ has a countable basis of neighbourhoods in $\widetilde{X}$, thus concluding the proof that α is not a homeomorphism.

This fact may seem a bit annoying at first sight, but our choice for the definition of the topology of $\widetilde{X}$ allows us to lift admissible simplices from X to its covering (see Lemma 3.3 and Remark 3.4).

3.1. **Lifting admissible simplices.** Let $Y = X \setminus B$ be as before, and let Γ denote the automorphism group of the universal covering $q\colon \widetilde{Y} \to Y$. It is immediate to check that every element of Γ extends to an admissible map of $(\widetilde{X}, \widetilde{B})$ into itself. As it is customary for ordinary (co)homology, we would like to compute the marked (co)homology of X by looking at (co)invariant (co)chains on the universal covering. To this aim we need the following important:

Lemma 3.3. *Let $\pi\colon \widetilde{X} \to X$ be the universal covering constructed above, and let $\sigma\colon \Delta^i \to X$ be an admissible singular simplex. Then, there exists an admissible singular simplex $\widetilde{\sigma}\colon \Delta^i \to \widetilde{X}$ such that $\pi \circ \widetilde{\sigma} = \sigma$. Moreover, if $\widetilde{\sigma}'\colon \Delta^i \to \widetilde{X}$ is any admissible singular simplex such that $\pi \circ \widetilde{\sigma}' = \pi \circ \widetilde{\sigma} = \sigma$, then there exists an element $\gamma \in \Gamma$ such that $\widetilde{\sigma}' = \gamma \circ \widetilde{\sigma}$.*

Proof. If $\sigma(\Delta^i) \subseteq B$, then by discreteness of B we have $\sigma(\Delta^i) = \{b\}$ for some $b \in B$. We then choose an element $\widetilde{b}$ in $\pi^{-1}(b)$ and define $\widetilde{\sigma}(x) = \widetilde{b}$ for every $x \in \Delta^i$. The conclusion follows from the fact that Γ acts transitively on the fiber of b.

We may thus set $K = \sigma^{-1}(B)$ and suppose that $A = \Delta^i \setminus K$ is non-empty. Since σ is admissible, the set K is a subcomplex of Δ^i, and A is a convex subset of Δ^i. In particular, A is simply connected. Since the restriction of π to $\widetilde{X} \setminus \widetilde{B}$ is a classical covering, this implies that there exists a continuous lift $\beta\colon A \to \widetilde{X}$ of the restriction $\sigma|_A$. Let us fix as usual a family F_b, $b \in B$, of disjoint closed quasiconical neighbourhoods of the points of B, and let $\widetilde{F_{\widetilde{b}}}$, $\widetilde{b} \in \widetilde{B}$, be the family of disjoint closed quasiconical neighbourhoods of points of $\widetilde{B}$ obtained by lifting the F_b (see the proof of Lemma 3.1).

Let K_0 be a connected component of K. Since B is discrete, there exists $b \in B$ such that $\sigma(K_0) = \{b\}$. Since σ is admissible and non-constant, the

subset A contains $\Delta^i \setminus \partial\Delta^i$. In particular, every point $x \in K_0$ is an accumulation point for A. By continuity of σ, there exists an open neighbourhood U of K_0 in Δ^i such that $U \cap (K \setminus K_0) = \emptyset$ and $\sigma(U) \subseteq F_b$. Moreover, we can choose U so that $U \setminus K_0 = U \cap A$ is path connected. We now have

$$\beta(U \setminus K_0) \subseteq \pi^{-1}(\sigma(U \setminus K_0)) \subseteq \pi^{-1}(F_b) = \bigsqcup_{\widetilde{b} \in \pi^{-1}(b)} \widetilde{F}_{\widetilde{b}} \,.$$

Since $U \setminus K_0$ is connected, this shows that there exists a unique $\widetilde{b} \in \pi^{-1}(B)$ such that $\beta(U \setminus K_0) \subseteq \widetilde{F}_{\widetilde{b}}$, and we set $\widetilde{\sigma}(x) = \widetilde{b}$ for every $x \in K_0$. Using the definition of the topology of $\widetilde{X}$ it is not difficult to show that $\widetilde{\sigma}\colon \Delta^i \to \widetilde{X}$ is continuous. By construction, it is also admissible, thus providing the desired lift of σ.

Suppose now that $\widetilde{\sigma}'\colon \Delta^i \to \widetilde{X}$ is an arbitrary lift of σ. The maps $\widetilde{\sigma}'|_A$ and $\widetilde{\sigma}|_A$ both lift $\sigma|_A$, so it readily follows from classical covering theory that $\widetilde{\sigma}'|_A = \gamma \circ \widetilde{\sigma}|_A$ for some $\gamma \in \Gamma$. Observe now for every $b \in B$, $x \in X \setminus \{b\}$ there exist disjoint neighbourhoods of b and x in X. Therefore, since A is dense in Δ^i the singular simplices $\widetilde{\sigma}'$ and $\gamma \circ \widetilde{\sigma}$ coincide on the whole of Δ^i, and this concludes the proof. □

Remark 3.4. Let M be a compact manifold with universal covering $\widetilde{M}$, and denote by X_M, $X_{\widetilde{M}}$ the marked spaces associated to M and $\widetilde{M}$, respectively. As observed in Remark 3.2, there exists a natural bijection between $\widetilde{X}$ and $X_{\widetilde{M}}$, so it makes sense to ask whether Lemma 3.3 would hold with $\widetilde{X}$ replaced by $X_{\widetilde{M}}$. The answer is negative in general.

For example, let $M = S^1 \times [0,1]$, so that $\widetilde{M} = \mathbb{R} \times [0,1]$, and let $\sigma\colon \Delta^1 \to \widetilde{M}$ be defined by $\sigma(t) = (1/t, t)$ if $t \neq 0$ and $\sigma(0) = (0,0)$. Let also $p_{\widetilde{M}}\colon \widetilde{M} \to X_{\widetilde{M}}$ be the natural projection, and $q\colon X_{\widetilde{M}} \to X_M$ the projection obtained by precomposing the map $\widetilde{X} \to X_M$ with the bijection $X_{\widetilde{M}} \cong \widetilde{X}$. Then it is not difficult to prove that the simplex $q \circ p_{\widetilde{M}} \circ \sigma\colon \Delta^1 \to X$ is continuous, hence admissible. On the contrary, the map $p_{\widetilde{M}} \circ \sigma\colon \Delta^1 \to X_{\widetilde{M}}$ is *not* continuous: in fact, if $U = \{(t,y) \in \mathbb{R} \times [0,1],\, t > 1 \text{ and } y < 1/t\} \subseteq \widetilde{M}$, then the set $p_{\widetilde{M}}(U)$ is open in $X_{\widetilde{M}}$ and intersects the image of $p_{\widetilde{M}} \circ \sigma$ only at the point $p_{\widetilde{M}}(\mathbb{R} \times \{0\})$, against the continuity of $p_{\widetilde{M}} \circ \sigma$. (On the other hand, the simplex $p_{\widetilde{M}} \circ \sigma\colon \Delta^1 \to X_{\widetilde{M}}$ would be continuous if we endowed the set $X_{\widetilde{M}}$ with the topology inherited from the bijection $X_{\widetilde{M}} \cong \widetilde{X}$.) Using this fact, one can easily show that the admissible simplex $q \circ p_{\widetilde{M}} \circ \sigma$ does not lift to an admissible simplex with values in $X_{\widetilde{M}}$.

4. Manifolds with amenable boundary

Let M be a manifold with boundary with associated marked space (X, B). As usual, we assume that M is compact, connected and oriented. Denote by $\widetilde{M}$ the universal covering of M, and let us fix an identification of the fundamental group of M with the group Γ of automorphisms of the covering

$\pi\colon \widetilde{M} \to M$. Let (X, B) be the marked space associated to M, and denote by $(\widetilde{X}, \widetilde{B})$ the universal covering of (X, B) as a marked space.

4.1. **Proof of Theorem 4.** Let us now suppose that the fundamental group of every component of ∂M is amenable. We would like to prove that $\|M\|_{\mathcal{I}} = \|M\|$. We already know that $\|M\|_{\mathcal{I}} \leq \|M\|$, so we need to show the converse inequality $\|M\| \leq \|M\|_{\mathcal{I}}$. The duality principle for ordinary singular (co)homology (see e.g. [Löh08] or [Fri17]) implies that there exists a bounded cohomology class $\psi \in H^n_b(M, \partial M)$ such that $\|\psi\|_\infty \leq 1$ and $\langle \psi, [M, \partial M]\rangle = \|M\|$.

Let $\varepsilon > 0$ be given. Since the fundamental group of each component of ∂M is amenable, by [Fri17, Corollary 5.18] there exists a representative $f \in C^n_b(M, \partial M)$ of ψ such that $\|f\|_\infty \leq \|\psi\|_\infty + \varepsilon$ which is *special* in the following sense.

- Let $\sigma, \sigma'\colon \Delta^n \to M$ be singular simplices which lift to maps $\widetilde{\sigma}, \widetilde{\sigma}'\colon \Delta^n \to \widetilde{M}$ such that, for every $i = 0, \dots, n$, at least one of the following conditions holds: either $\widetilde{\sigma}(e_i) = \widetilde{\sigma}'(e_i)$, or $\widetilde{\sigma}(e_i) \neq \widetilde{\sigma}'(e_i)$ but $\widetilde{\sigma}(e_i)$ and $\widetilde{\sigma}'(e_i)$ belong to the same connected component of $\partial\widetilde{M}$, where $e_0, \dots, e_n$ are the vertices of the standard n-simplex. Then $f(\sigma) = f(\sigma')$.

We are now ready to construct a cocycle $f^{\mathcal{M}} \in C^n_{\mathcal{M}}(X, B)$ out of the cocycle f. Let us denote by $p'_{\widetilde{M}}\colon \widetilde{M} \to \widetilde{X}$ the projection described in Remark 3.2.

Let $\sigma\colon \Delta^n \to X$ be an admissible simplex, and let $\widetilde{\sigma}\colon \Delta^n \to \widetilde{X}$ be a lift of σ, as described in Lemma 3.3. It may be the case that $\widetilde{\sigma}$ is not the projection on $\widetilde{X}$ of any singular simplex with values in $\widetilde{M}$. Nevertheless, we can arbitrarily choose a singular simplex $\widehat{\sigma}\colon \Delta^n \to \widetilde{M}$ such that $\widetilde{\sigma}$ and $p'_{\widetilde{M}} \circ \widehat{\sigma}$ coincide on the vertices of Δ^n. We then set

$$f^{\mathcal{M}}(\sigma) = f(\pi \circ \widehat{\sigma}) \ ,$$

where $\pi\colon \widetilde{M} \to M$ is the universal covering projection.

The fact that f is special implies that $f^{\mathcal{M}}$ is well defined. Moreover, it is readily seen that $f^{\mathcal{M}}$ is a cocycle, and

$$\|f^{\mathcal{M}}\|_\infty \leq \|f\|_\infty \leq \|\psi\|_\infty + \varepsilon \leq 1 + \varepsilon \ .$$

Let $\psi^M \in H^n_{\mathcal{M}}(X, B)$ be the class represented by $f^{\mathcal{M}}$. We are now going to evaluate ψ^M on the ideal fundamental class of M. So let $z \in C_n(M, \partial M)$ be a fundamental cycle for M. The proof of Theorem 2.7 shows that we can modify z into an admissible fundamental cycle without increasing its ℓ^1-norm. Therefore, we may assume that the chain z is admissible. As a consequence, the chain $z^{\mathcal{M}} = p_*(z) \in C^{\mathcal{M}}_n(X, B)$ is also admissible. By definition, it follows that $z^{\mathcal{M}}$ is an ideal fundamental cycle for M. Moreover, it readily follows from the definition of $f^{\mathcal{M}}$ that $f^{\mathcal{M}}(z^{\mathcal{M}}) = f(z)$. We then

have

$$\langle \psi^{\mathcal{M}}, [M, \partial M]^{\mathcal{M}} \rangle = f^{\mathcal{M}}(z^{\mathcal{M}}) = f(z) = \langle \psi, [M, \partial M] \rangle = \|M\| \ .$$

Moreover,

$$\|\psi^{\mathcal{M}}\|_\infty \leq \|f^{\mathcal{M}}\|_\infty \leq 1 + \varepsilon \ .$$

Therefore, by applying Proposition 2.4 with $\alpha = [M, \partial M]^{\mathcal{M}}$ and $\varphi = \psi^{\mathcal{M}}/(1+\varepsilon)$ we get

$$\|M\|_{\mathcal{I}} = \|[M, \partial M]^{\mathcal{M}}\|_1 \geq \frac{\langle \psi^M, [M, \partial M]^{\mathcal{M}} \rangle}{1+\varepsilon} = \frac{\|M\|}{1+\varepsilon} \ .$$

Since ε is arbitrary, we thus have

$$\|M\|_{\mathcal{I}} \geq \|M\| \ .$$

This concludes the proof of Theorem 4.

5. Hyperbolic manifolds with geodesic boundary

This section is devoted to the proof of Theorem 9, which computes the ideal simplicial volume of an infinite family of hyperbolic 3-manifolds with geodesic boundary. Mimicking the well-known computation of the classical simplicial volume for hyperbolic manifolds due to Gromov and Thurston, we will first establish the lower bound on the ideal simplicial volume in terms of the Riemannian volume described in Theorem 7. To this aim, rather than defining a proper straightening of simplices as in the classical case, we will directly exploit the duality principle described in Proposition 2.4. In the 3-dimensional case, building on a recent result on the volumes of peculiar classes of (partially) truncated tetrahedra in hyperbolic space, we will then show that the resulting lower bound is sharp for an infinite family of 3-manifolds.

5.1. **A geometric realization of the associated marked space.** We denote by $\mathcal{H}^n$ the hyperboloid model of hyperbolic space $\mathbb{H}^n$, i.e. we set

$$\mathcal{H}^n = \{x \in \mathbb{R}^{n+1} \,|\, \langle x, x \rangle = -1,\, x_0 > 0\} \ ,$$

where

$$\langle (x_0, \dots, x_n), (y_0, \dots, y_n) \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$$

is the Minkowsky scalar product. We also denote by $\mathcal{S}^n$ the hyperboloid

$$\mathcal{S}^n = \{x \in \mathbb{R}^{n+1} \,|\, \langle x, x \rangle = 1\} \ .$$

For every element $x \in \mathcal{S}^n$ we define the dual hyperplane $H(x)$ (resp. dual half-space $H^+(x)$) of $\mathcal{H}^n$ by setting

$$H^+(x) = \{y \in \mathcal{H}^n \,|\, \langle y, x \rangle \leq 0\} \,,$$
$$H(x) = \partial H^+(x) = \{y \in \mathcal{H}^n \,|\, \langle y, x \rangle = 0\} \ .$$

Let now M be a compact hyperbolic manifold with non-empty geodesic boundary. The universal covering $\widetilde{M}$ of M is a convex subset of $\mathbb{H}^n$ bounded

by infinitely many disjoint hyperplanes, and the fundamental group Γ of M acts by isometries on $\widetilde{M}$. In fact, every isometry of $\widetilde{M}$ is the restriction of a unique element in $SO(n,1)$, so Γ acts also on $\mathcal{S}^n$. By taking the dual vectors of the boundary component of $\widetilde{M}$, one can construct a Γ-invariant countable set $\widehat{D} \subseteq \mathcal{S}^n$ such that for every $\hat{q} \in \widehat{D}$ the dual hyperplane $H(q) \subseteq \mathcal{H}^n$ is a connected component of $\partial\widetilde{M}$, and

$$\widetilde{M} = \bigcap_{\hat{q}\in\widehat{D}} H^+(\hat{q}) \ .$$

Moreover, up to acting via an isometry of $\mathcal{H}^n$, we may suppose that $(1,0,\dots,0)$ belongs to $\mathrm{int}(\widetilde{M}) = \widetilde{M} \setminus \partial\widetilde{M}$, and this easily implies that the set $\widehat{D}$ is contained in the half space $\{x_0 > 0\}$. This allows us to fruitfully exploit the projective model of hyperbolic space: indeed, let $U \cong \mathbb{R}^n$ be the affine chart defined by $U = \{[x] \in \mathbb{P}^n(\mathbb{R}) \,|\, x_0 \neq 0\}$, consider the projection $\pi\colon \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{P}^n(\mathbb{R})$ and let $P \subseteq U$ be the projective model of hyperbolic space, i.e. set $P = \pi(\mathcal{H}^n)$. Also set $D = \pi(\widehat{D}) \subseteq U$. With a slight abuse, we denote simply by $\widetilde{M}$ the set $\pi(\widetilde{M}) \subseteq P$ and, for every $\hat{q} \in \widehat{D}$, we denote by $H^+(\hat{q})$ also the projection $\pi(H^+(\hat{q})) \cap U \subseteq U$.

Let now (X,B) be the marked space associated to M, and let $(\widetilde{X},\widetilde{B})$ be the universal covering of the marked space (X,B). Let us also set $Y = \mathrm{int}(\widetilde{M}) \cup D$. We will now fix the the following identification between $(\widetilde{X},\widetilde{B})$ and (Y,D): recall from Remark 3.2 that there exists a natural bijection between $(\widetilde{X},\widetilde{B})$ and $(X_{\widetilde{M}}, B_{\widetilde{M}})$; this induces in turn an identification between $\widetilde{X}\setminus\widetilde{B}$ and $X_{\widetilde{M}} \setminus B_{\widetilde{M}} \cong \mathrm{int}(\widetilde{M})$; moreover, $\widetilde{B} \cong B_{\widetilde{M}}$ admits a natural bijection with the set of connected components of $\widetilde{M}$, hence with D. We will endow the pair (Y,D) with the structure of marked space inherited from this identification. Observe that the identification $(\widetilde{X},\widetilde{B}) \cong (Y,D)$ is equivariant with respect to the action of Γ on (Y,D).

5.2. **(Partially) truncated simplices.** Just as ideal simplices are the fundamental building blocks for cusped hyperbolic manifolds, truncated simplices may be exploited to construct hyperbolic manifolds with geodesic boundary. Just as the name suggests, truncated simplices are obtained by truncating Euclidean simplices in the chart $U \subseteq \mathbb{P}^n(\mathbb{R})$ along the dual hyperplanes of their hyperideal vertices. Here is a precise definition:

Definition 5.1. Let $v_0,\dots,v_n$ be points of $(U \setminus \partial P) \subseteq \mathbb{P}^n(\mathbb{R})$, let $\Delta \subseteq U$ be the convex hull of the v_i, and let $I \subseteq \{0,\dots,n\}$ be the set of indices i such that $v_i \notin \overline{P}$. Also suppose that the following conditions hold:

(1) For every $i,j = 0,\dots,n$, $i \neq j$, the straight segment joining v_i with v_j intersects P;
(2) For every $i \in I$, let $\hat{v}_i \in \mathcal{S}^n$ be the lift of v_i such that $x_0(v_i) > 0$. Then $v_j \in H^+(\hat{v}_i)$ for every $j \notin I$.

Then the set

$$\Delta^* = \Delta \cap \left(\bigcap_{i \in I} H^+(\hat{v}_i) \right)$$

is a *partially truncated simplex* (see Figure 2).

The v_i are the *vertices* of Δ^*. A vertex v_i is *finite* if $v_i \in P$, and *ultraideal* if $v_i \in U \setminus \overline{P}$. The simplex Δ^* is *degenerate* if its vertices are all contained in a hyperplane of U.

The intersection of Δ^* with any face of Δ is an *internal* face of Δ^*. We say that an internal edge of Δ^* is *fully hyperideal* if both the vertices of the corresponding edge of Δ are hyperideal, and that Δ^* is fully truncated if every vertex of Δ is hyperideal. For every $i \in I$, the set $\Delta^* \cap H^+(\hat{v}_i)$ is the *truncation simplex* of Δ^*. When $n = 3$, truncation simplices are usually called truncation triangles. The dihedral angle between a truncation simplex and any internal face adjacent to it is equal to $\pi/2$. (Partially) truncated tetrahedra are compact; in particular, they have finite volume.

Remark 5.2. In our definition of (partially) truncated tetrahedra we did not allow ideal vertices, nor (partially) ideal truncation simplices (which occur when the straight segment joining two hyperideal vertices is tangent to P at a point in ∂P). Truncated tetrahedra with ideal vertices arise in the decomposition of hyperbolic n-manifolds with geodesic boundary and rank-$(n-1)$ cusps, while truncated tetrahedra with ideal truncation simplices arise in the decomposition of hyperbolic manifolds with non-compact geodesic boundary. This choice allows us to avoid some technicalities, and the study of compact truncated simplices is sufficient for our applications.

For any geodesic segment e in P we denote by $L(e)$ the hyperbolic length of e.

Definition 5.3. Let $\ell > 0$. Then we set for any $n \in \mathbb{N}$

$$V_\ell^n = \sup\{\mathrm{vol}(\Delta^*) \,|\, \Delta^* \text{ fully truncated } n-\text{simplex}\,, \\ L(e) \geq \ell \text{ for every internal edge of } \Delta^*\}\ .$$

5.3. **The marked volume form.** Let M be a compact hyperbolic manifold with geodesic boundary. Recall from the introduction that $\ell(M)$ denotes the smallest return length of M, i.e. the length of the shortest path with both endpoints on ∂M which intersects ∂M orthogonally at each of its endpoints (equivalently, it is the smallest distance between distinct boundary components of the universal covering of M). Let (X, B) be the marked space associated to M, and let $(Y, D) \cong (\widetilde{X}, \widetilde{B})$ be the marked space described in Subsection 5.1. We are going to define a marked volume form on M by assigning to every admissible simplex σ the signed volume of a partially truncated simplex having the same vertices as a lift of σ.

To this aim, let us first define the function

$$\mathrm{algvol}\colon Y^{n+1} \to \mathbb{R}$$

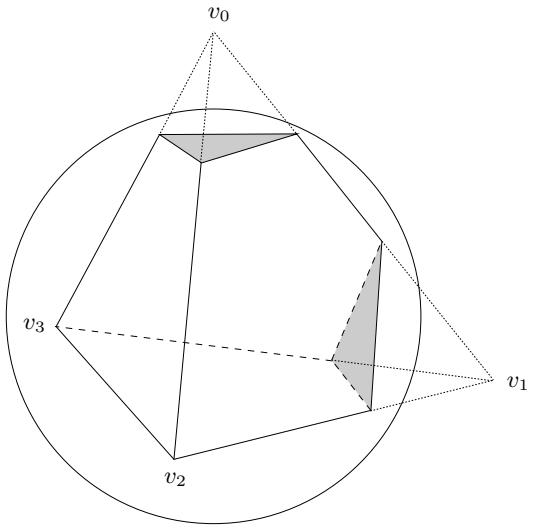

Figure 2. A partially truncated tetrahedron. The vertices v_0 and v_1 are hyperideal, while v_2 and v_3 are finite.

as follows. Take an element $(y_0, \dots, y_n) \in Y^{n+1}$. If the y_i are all contained in a hyperplane of P (e.g. if $y_i = y_j$ for some $i \neq j$), then we simply set $\mathrm{algvol}(y_0, \dots, y_n) = 0$. Otherwise, it is easy to check that $\{y_0, \dots, y_n\}$ is the set of vertices of a non-degenerate (partially) truncated simplex $\Delta^* \subseteq \widetilde{M}$, and we set

$$\mathrm{algvol}(y_0, \dots, y_n) = \varepsilon(y_0, \dots, y_n) \cdot \mathrm{vol}(\Delta^*) \ ,$$

where $\varepsilon(y_0, \dots, y_n) = 1$ (resp. $\varepsilon(y_0, \dots, y_n) = -1$) if the barycentric parametrization

$$\Delta^n \to U \ , \qquad (t_0, \dots, t_n) \mapsto t_0 y_0 + \dots t_n y_n$$

of the Euclidean simplex with vertices $v_0, \dots, v_n$ is orientation-preserving (resp. orientation-reversing). Here we are endowing U with the unique orientation which induces the positive orientation on $\widetilde{M} \subseteq P \subseteq U$.

Observe now that there is an obvious isometric isomorphism between $C^n_{\mathcal{M}}(X, B)$ and $C^n_{\mathcal{M}}(Y, D)^\Gamma$. We can then define the cochain $\omega \in C^n_{\mathcal{M}}(Y, D)^\Gamma$ by setting

$$\omega(\widetilde{\sigma}) = \mathrm{algvol}(v_0, \dots, v_n) \ ,$$

where $v_i = \widetilde{\sigma}(e_i)$ and e_i is the i-th vertex of the standard simplex Δ^n. It is esay to check that ω is a cocycle, hence we may denote by $[\omega] \in H^n_{\mathcal{M}}(X, B)$ the coclass represented by ω.

Proposition 5.4. *We have*

$$\langle [\omega], [M, \partial M]^{\mathcal{M}} \rangle = \mathrm{vol}(M) \ .$$

Proof. It is sufficient to exhibit an ideal fundamental cycle $z_M \in C_n^{\mathcal{M}}(X,B)$ for which $\omega(z_M) = \mathrm{vol}(M)$.

Kojima's canonical decomposition for hyperbolic manifolds with geodesic boundary [Koj90, Koj92] shows that M can be decomposed into fully truncated polyhedra. Unfortunately, it is not clear whether these polyhedra may be coherently subdivided to give a triangulation of M by non-degenerate truncated simplices. However, a decomposition of M into possibly degenerate truncated simplices may be obtained by subdividing Kojima's decomposition and, if needed, by inserting a finite number of degenerate truncated simplices between the faces of the original polyhedra. One can finally obtain the desired fundamental cycle z_M by alternating the barycentric parametrizations of these simplices just as we did in Subsection 2.3. It is then obvious that $\omega(z_M) = \mathrm{vol}(M)$, and this concludes the proof.

□

5.4. **Proof of Theorem 7.** We are now ready to conclude the proof of the lower bound on $\|M\|_{\mathcal{I}}$ in terms of the Riemannian volume of M. By Proposition 2.9, for every $\epsilon > 0$ we may find an ideal fundamental cycle

$$z_M = \sum_{i=1}^{k} a_i \sigma_i \ \in \ C_n^{\mathcal{M}}(X,B)$$

such that all the vertices of every σ_i lie in B, and $\|z_M\|_1 \leq \|M\|_{\mathcal{I}} + \varepsilon$. As a consequence, since the length of any internal edge of any fully truncated simplex with vertices in $Y = \mathrm{int}(\widetilde{M}) \cup D$ is not smaller than $\ell(M)$, for every $i = 0, \dots, k$ we have $|\omega(\sigma_i)| \leq V^n_{\ell(M)}$. Therefore,

$$\begin{aligned} \mathrm{vol}(M) = \omega(z_M) &\leq \sum_{i=0}^{k} |a_i| \cdot |\omega(\sigma_i)| \leq V^n_{\ell(M)} \sum_{i=0}^{k} |a_i| = V^n_{\ell(M)} \|z_M\|_1 \\ &\leq V^n_{\ell(M)} \left(\|M\|_{\mathcal{I}} + \varepsilon \right) \ . \end{aligned}$$

Since ε is arbitrary, this concludes the proof of Theorem 7.

5.5. **The class $\mathcal{M}_g$.** As in the introduction, let $\mathcal{M}_g$, $g \geq 2$, be the class of 3-manifolds M with boundary that admit an ideal triangulation by g tetrahedra and have Euler characteristic $\chi(M) = 1 - g$ (so $\chi(\partial M) = 2 - 2g$). We also denote by $\overline{\mathcal{M}}_g$ the set of hyperbolic 3-manifolds M with connected geodesic boundary such that $\chi(\partial M) = 2 - 2g$. It turns out that elements of $\mathcal{M}_g$ admit a geometric decomposition into *regular* truncated tetrahedra.

A fully truncated tetrahedron Δ is regular if any permutation of its vertices is realized by an isometry of the truncated tetrahdron. Equivalently, Δ is regular if and only if the dihedral angles along its edges are all equal to each other, and this happens if and only if the hyperbolic lengths of its internal edges are all equal to each other. Up to isometry, regular truncated tetrahedra are parametrized by their edge length (which may vary in $(0, +\infty)$, or by their dihedral angles (which may vary in $(0, \pi/3)$). It

is well known (see for example [Ush06]) that, if $\theta(\ell)$ denotes the dihedral angles along the internal edges of a regular truncated tetrahedron with edge lengths equal to ℓ, then the map $\ell \to \theta(\ell)$ is strictly increasing, and has limits

$$\lim_{\ell \to 0^+} \theta(\ell) = 0$$

(when the truncated tetrahedron is tending to a regular ideal hyperbolic octhedron), and

$$\lim_{\ell \to +\infty} \theta(\ell) = \pi/3$$

(when the truncated tetrahedron is tending to a regular ideal hyperbolic tetrahedron).

For every $g \geq 2$, we denote by ℓ_g the length of the internal edges of the regular truncated tetrahedron with dihedral angles equal to $\pi/(3g)$. Moreover, for every $\ell > 0$ we denote by Δ_ℓ the (isometry class of the) regular truncated tetrahedron with edge length equal to ℓ. It is proved e.g. in [KM91] that

$$\mathrm{vol}(\Delta_{\ell_g}) = v_8 - 3 \int_0^{\frac{\pi}{3g}} \mathrm{arccosh}\left(\frac{\cos t}{2\cos t - 1}\right) \mathrm{d}t \ ,$$

where v_8 is the volume of the regular ideal octahedron.

The following result is a restatement of the main theorem of [FM], and shows that, at least for every $\ell \leq \ell_2$, the regular tetrahedron Δ_ℓ has the largest volume among all fully truncated tetrahedra whose edge lengths are not smaller that ℓ:

Theorem 5.5. *Let $\ell \leq \ell_2$. Then*

$$V_\ell^3 = \mathrm{vol}(\Delta_\ell) \ .$$

The following result lists some known properties of manifolds belonging to $\mathcal{M}_g$. The last point implies that $\mathcal{M}_g$ coincides with the set of the elements of $\overline{\mathcal{M}}_g$ of smallest volume.

Proposition 5.6 ([FMP03, KM91])**.** *Let $g \geq 2$. Then:*

1. *the set $\mathcal{M}_g$ is nonempty;*
2. *every element of $\mathcal{M}_g$ admits a hyperbolic structure with geodesic boundary (which is unique up to isometry by Mostow Rigidity Theorem);*
3. *the boundary of every element of $\mathcal{M}_g$ is connected, so $\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$;*
4. *if $M \in \mathcal{M}_g$, then M decomposes into the union of g copies of Δ_{ℓ_g}, so in particular $\mathrm{vol}(M) = g\,\mathrm{vol}(\Delta_{\ell_g})$;*
5. *if $M \in \overline{\mathcal{M}}_g$, then $\mathrm{vol}(M) \geq g\,\mathrm{vol}(\Delta_{\ell_g})$;*
6. *if $M \in \mathcal{M}_g$, then the shortest return length $\ell(M)$ of M is equal to ℓ_g;*
7. *if $M \in \overline{\mathcal{M}}_g$, then $\ell(M) \geq \ell_g$.*

Proof. Items (2) and (3) and (4) are proved in [FMP03], items (1) and (5) in [KM91, Miy94]. Item (7) is proved in [Miy94, Lemma 5.3], and (6) follows

from (7) together with the fact that the internal egdes of the truncated tetrahedra in the decomposition described in (4) define return paths of length ℓ_g. □

5.6. **Proof of Theorem 9 and Corollary 10.** We are now ready to prove Theorem 9. Let $M \in \overline{\mathcal{M}}_g$. By Proposition 5.6 we have $\mathrm{vol}(M) = g \cdot \mathrm{vol}(\Delta_{\ell_g})$ and $\ell(M) = \ell_g$. Moreover, Theorem 5.5 ensures that $V^3_{\ell_g} = \mathrm{vol}(\Delta_{\ell_g})$, so plugging these equalities into the lower bound given by Theorem 7 we obtain

$$\|M\|_{\mathcal{I}} \geq \frac{\mathrm{vol}(M)}{V^3_{\ell(M)}} = \frac{g \cdot \mathrm{vol}(\Delta_{\ell_g})}{\mathrm{vol}(\Delta_{\ell_g})} = g \ .$$

On the other hand, the manifold M admits an ideal triangulation with g tetrahedra, so

$$\|M\|_{\mathcal{I}} \leq c(M) \leq g \ .$$

We thus get

$$\|M\|_{\mathcal{I}} = c(M) = g \ ,$$

which proves the first statement of Theorem 9.

If $M \in \overline{\mathcal{M}}_g$, Proposition 5.6 implies that $\mathrm{vol}(M) \geq g \cdot \mathrm{vol}(\Delta_{\ell_g})$ and $\ell(M) \geq \ell_g$, so that $V^3_{\ell(M)} \leq V^3_{\ell_g} = \mathrm{vol}(\Delta_g)$. Therefore, we have

$$\|M\|_{\mathcal{I}} \geq \frac{\mathrm{vol}(M)}{V^3_{\ell(M)}} \geq \frac{g \cdot \mathrm{vol}(\Delta_{\ell_g})}{\mathrm{vol}(\Delta_{\ell_g})} = g \ .$$

This concludes the proof of Theorem 9.

In order to prove Corollary 10, it is sufficient to observe that, if M is an oriented compact hyperbolic 3-manifold with geodesic boundary having an ideal simplicial volume not greater than 2, then Corollary 8 ensures that

$$2 \geq \|M\|_{\mathcal{I}} \geq \frac{\mathrm{vol}(M)}{v_8} \geq \mathrm{vol}(\partial M) \ ,$$

where the last inequality is due to Miyamoto [Miy94, Theorem 4.2]. In particular, from Gauss-Bonnet Theorem we deduce that ∂M is a connected surface of genus 2, so $\|M\|_{\mathcal{I}} \geq 2$ by Theorem 9. We have thus proved that the elements of $\mathcal{M}_2$ are exactly the compact hyperbolic 3-manifolds with geodesic boundary having the smallest possible ideal simplicial volume.

5.7. **An application to mapping degrees.** Take elements $M \in \mathcal{M}_g$ and $M' \in \mathcal{M}_{g'}$, where $g \geq g'$. As stated in Corollary 11, Theorems 2 and 9 imply that any map of pairs

$$f \colon (M, \partial M) \to (M', \partial M')$$

satisfies the inequality

$$\deg(f) \leq \frac{g}{g'} \ . \tag{1}$$

The following proposition implies that there are cases where this bound is sharp:

Proposition 5.7. *Let $g' \geq 2$ and let $g = k \cdot g'$, $k \in \mathbb{N} \setminus \{0\}$, be a multiple of g'. For every $M' \in \mathcal{M}_{g'}$ there exist a manifold $M \in \mathcal{M}_g$ and a map of pairs $f\colon (M, \partial M) \to (M', \partial M')$ such that*

$$\deg(f) = \frac{g}{g'} \ .$$

Proof. As proved e.g. in [FMP03], manifolds in $\mathcal{M}_{g'}$ can be characterized as those orientable manifolds that admit an ideal triangulation with g' tetrahedra and only one edge, meaning that all the edges of the truncated tetrahedra of the triangulation are identified to each other in the manifold.

Let us denote by e' the unique edge of the triangulation of M'. Then by removing from M' a small open neighbourhood of e' one gets a genus–$(g'+1)$ handlebody. In particular, a small loop encircling e' has infinite order in $\pi_1(M' \setminus e')$, so for every $k \geq 1$ we can consider the cyclic ramified covering $f\colon M \to M'$ having order k and ramification locus equal to e'. By construction, the ideal triangulation of M' lifts to an ideal triangulation of M having $g = k \cdot g'$ tetrahedra and only one edge $e = f^{-1}(e')$. Thus M belongs to $\mathcal{M}_g$. Since the map f has topological degree equal to k, this concludes the proof. □

One could bound the degrees of maps between manifolds with boundary also by looking at the ordinary simplicial volume both of the boundaries and of the doubles of the manifolds involved. Indeed, if $f\colon (M, \partial M) \to (M', \partial M')$ is a map of pairs between oriented manifolds with boundary of the same dimension, then f restricts to a map $g\colon \partial M \to \partial M'$ and extends to a map $F\colon DM \to DM'$ between the double DM of M and the double DM' of M'. Moreover,

$$\deg(F) = \deg(g) = \deg(f) \ .$$

Therefore, by exploiting the usual bounds of mapping degrees in terms of the ordinary simplicial volume, one gets

$$\deg(f) = \deg(F) \leq \frac{\|DM\|}{\|DM'\|} \ , \tag{2}$$

$$\deg(f) = \deg(g) \leq \frac{\|\partial M\|}{\|\partial M'\|} \ . \tag{3}$$

Let us show that, at least when $M \in \mathcal{M}_g$ and $M' \in \mathcal{M}_{g'}$, these bounds are less effective that the bound (1) obtained by exploiting the ideal simplicial volume. Indeed, in this case the inequality (2) ensures that, for every map of pairs $f\colon (M, \partial M) \to (M', \partial M')$, one has

$$\deg(f) \leq \frac{\|DM\|}{\|DM'\|} = \frac{\mathrm{vol}(DM)}{\mathrm{vol}(DM')} = \frac{\mathrm{vol}(M)}{\mathrm{vol}(M')} = \frac{g\,\mathrm{vol}(\Delta_{\ell_g})}{g'\,\mathrm{vol}(\Delta_{\ell_{g'}})} \ ,$$

and the right-hand side of this inequality is strictly bigger than g/g', since $g > g'$ implies $\ell_g < \ell_{g'}$ and $\mathrm{vol}(\Delta_{\ell_g}) > \mathrm{vol}(\Delta_{\ell_{g'}})$. For example, if $g' = 2$

and $g = 2k$ is very big, then $\mathrm{vol}(\Delta_{\ell_{g'}}) \approx 3.226$, while $\mathrm{vol}(\Delta_{\ell_g}) \approx v_8 = 3.664$, so the integral part of $(g\,\mathrm{vol}(\Delta_{\ell_g}))/(g'\,\mathrm{vol}(\Delta_{\ell_{g'}})) \approx 1.135(g/g')$ is strictly bigger than the (sharp) bound g/g'.

On the other hand, in this case the inequality (3) gives

$$\deg(f) \leq \frac{\|\partial M\|}{\|\partial M'\|} = \frac{|\chi(\partial M)|}{|\chi(\partial M')|} = \frac{g-1}{g'-1}\ ,$$

and again the right hand side of this inequality is strictly bigger than g/g'. For example, if $g' = 2$ this bound gives $\deg(f) \leq g - 1$, which is a much less restrictive condition than the inequality $\deg(f) \leq g/2$ provided by the study of ideal simplicial volume.

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