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Numerical Hints for Insulation Problems

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Abstract

In this work we analyze a problem of thermal insulation from the numerical point of view via finite element method. Physically, we are considering a domain of given temperature, thermally insulated by surrounding it with a constant amount of thermal insulator. From the mathematical point of view, this problem is composed by an elliptic partial differential equation with Robin–Dirichlet boundary conditions. Our question is related to the best (or worst) shape for the external domain, in terms of heat dispersion (of course, under prescribed geometrical constraints).

Keywords: Elliptic equations, Robin boundary conditions, Finite Element Method, Thermal insulation, dispersion function

2010 MSC: 65Nxx, 80M10

1. Introduction

In the last decades, shape optimization is becoming a really popular research field in applied mathematics and is relevant in several applications, like e.g. optimal thermal insulation [1]. Generally speaking, a shape optimization problem can be described by a minimization problem of the form

$$\min\{G(E) : E \in \mathcal{D}\}, \tag{1}$$

where $\mathcal{D}$ is a class of admissible domains and $G(E)$ a functional to be minimized over $\mathcal{D}$. In applications, like e.g. structural engineering, the functional $G(E)$ can be expressed in the following form

$$G(E) = \int_\Gamma g(x, u_E)\, dx, \tag{2}$$

with Γ a given bounded open subset of $\mathcal{R}^n (n \geq 2)$, u_E the solution of a partial differential equation (PDE) in E. Volume constraints or penalization terms are reasonable conditions to be considered [2].

Thermal insulation represents one of the major challenges for energy efficiency. Problems related to insulation are well-known and widely studied as optimization problems since the eighties [3, 4, 5]. Neverthless, mathematics involved is still very tricky especially when one looks at shape optimization issues [6, 7], and sometimes the answers are not so intuitive [8]. As far we know and as stated also in [9], in general there are few results in shape optimization where the existence of an optimal domain can be proved without imposing extra restrictive conditions, and the most of them hold considering Dirichlet boundary conditions. In [9], the authors identified a class of admissible domains containing the Lipschitz ones on which the minimization of the energy functional associated to a problem like (4) can be performed. Other related theoretical results can be found in [5, 10].

In this work we will consider two domains: an internal fixed domain and an external domain whose geometry varies. The aim is to find external domains which minimize the heat dispersion. After formulating the problem mathematically, deepening the case known analytically which is the two concentric case (Sect. 2), in Sect. 3 we move to the numerical resolution and results distinguishing three cases: fixed an internal domain as a unit ball $B_1(0)$, we start comparing different external domains which share the same area (Sect. 3.1), then a comparison under a maximum area constraint is addressed (Sect. 3.2). Finally, in Sect. 3.3 we consider different configurations for the internal domain in order to see if the circle is still the best external domain in terms of heat dispersion.

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2. Formulation of the problem

Let us consider two domains, Ω and D, inside one another, with Ω being the innermost one. From the thermal insulation point of view, the compact connected set Ω represents a conductor of constant temperature which is thermally insulated by surrounding it with a layer of thermal insulator, denoted by the open set $D \setminus \Omega$, with $\Omega \subset \bar{D}$. For a fixed internal domain, Ω, and a limited fixed amount of insulator, we are interested in seeking out suitable shapes of the insulator layer D which provide nice insulating performances, i.e., which guarantee “small values”for the heat dispersion functional defined as

$$F_\beta(D, \Omega) := \beta \int_{\partial D} u \, dx. \tag{3}$$

The function $u \in H'(D \cup \Omega)$ which appears in (3) represents the temperature of the conducting body and satisfies the Euler–Lagrange equation with Robin–Dirichlet boundary conditions

$$\begin{cases} \Delta u = 0, & \text{in } D \setminus \Omega, \\ \dfrac{\partial u}{\partial n} + \beta u = 0, & \text{on } \partial D, \\ u = 1, & \text{on } \partial \Omega, \end{cases} \tag{4}$$

where n is the exterior normal vector and $\beta > 0$ is a fixed parameter depending on the physical properties of the insulating layer. Our aim, in other words, is to numerically solve the problem

$$\min_D F_\beta(D, \Omega), \tag{5}$$

where u satisfies (4), Ω and β are fixed and D satisfies some geometrical constraints.

Remark 1. *It is always possible to find the best configuration with unlimited resources of material. In common situations, we have a limited amount of insulation material and the major interest is to find the best configuration which minimizes the heat dispersion functional, mathematically defined as in* (3)*.*

2.1. The two concentric circles case

Considering Ω and D as concentric circles of radius r and R, respectively, with $r < R$, it is possible to compute the analytical solution of (3-4). In fact, starting from the set of solutions to the Laplace's equation in the case of a circular crown, that is

$$A \log \sqrt{x^2 + y^2} + C = u(x, y), \tag{6}$$

where A and C are two constants, the functional (3) is the following:

$$\begin{aligned} \beta \int_{\partial D} u &= \beta \int_{\partial D} A \log \sqrt{x^2 + y^2} + C & (7) \\ &= \beta \int_0^{2\pi} (A \log \left(\sqrt{R^2 \cos^2(t) + R^2 \sin^2(t)} \right) + C) R dt & (8) \\ &= \beta 2\pi R (A \log(R) + C). & (9) \end{aligned}$$

In order to find the constants A and C, using the Robin–Dirichlet boundary conditions of the problem (4), we have to solve the following system of two equations:

$$\begin{cases} A \log r + C = 1, \\ \beta(A \log R + C) + A \frac{1}{R} = 0. \end{cases} \tag{10}$$

Solving the system (10) we obtain

$$A = -\frac{1}{\log(\frac{R}{r}) + \frac{1}{\beta R}}, \qquad C = 1 - A \log r = 1 + \frac{1}{\log(\frac{R}{r}) + \frac{1}{\beta R}} \log r. \tag{11}$$

Note that for $r = 1$, the constant C is always equal to one, for all the admissible values of R and β. For a fixed β and a fixed $r \in (0, R)$, the dispersion in (9), using A and C as in (11), only depends on R, is an increasing function for $R < 1/\beta$, and decreasing for $R > 1/\beta$. Since must be $R > r$, for $\beta > 1/r$ the dispersion is a decreasing function (the insulation increases adding insulator), whereas for $0 < \beta < 1/r$ the dispersion increases for $R < 1/\beta$ and decreases for $R > 1/\beta$. About the increasing phase, it might seem surprising that adding insulator increases the heat dispersion; however this is a well-known phenomenon, that from the physical point of view can be explained by the competing effects of the conduction resistance and the convection resistance (see [1], Sect. 3.3.1-3.3.2).

Remark 2. *In case $D \equiv \Omega$, no insulation is considered and the functional $F_\beta(D,\Omega)$ is simply equal to the perimeter P of the domain Ω multiplied by the parameter β, i.e. it is set to $F_\beta(\Omega,\Omega) = \beta P(\Omega)$.*

The behavior of the heat dispersion functional as the radius R increases is clearly depicted in Fig. 1. The

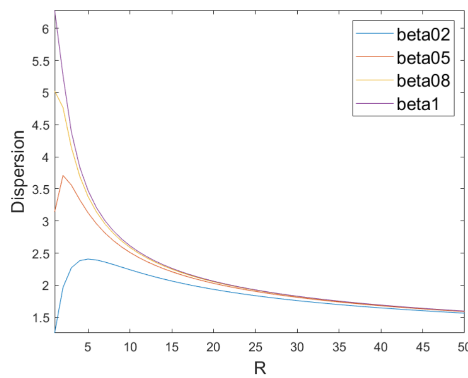

Figure 1: Plot of $(R, F_\beta(B_R(0), B_r(0)))$ for different values of β $(0.2, 0.5, 0.8, 1)$.

dependence of the dispersion function on the parameter β vanishes looking at its asymptotic behavior, as visible also in Fig. 1. In fact:

$$\begin{aligned} & \lim_{R\to+\infty} \beta 2\pi R(A\log(R) + C) \qquad (12)\\ = \; & \beta 2\pi\Big(\lim_{R\to+\infty} \frac{R}{\log(\frac{R}{r}) + \frac{1}{\beta R}}(-\log R + \log r) + R\Big) \\ = \; & \beta 2\pi\Big(\lim_{R\to+\infty} \frac{-R(\log(\frac{R}{r})) + R\log(\frac{R}{r}) + \frac{1}{\beta}}{\log(\frac{R}{r}) + \frac{1}{\beta R}}\Big) \\ = \; & 2\pi\Big(\lim_{R\to+\infty} \frac{1}{\log(\frac{R}{r}) + \frac{1}{\beta R}}\Big) = 0. \end{aligned}$$

Remark 3. *Considering a geometry for the domain D which is different from the circle of radius R, the qualitative behavior of the dispersion function is similar to the one which holds for the circle. See for example the plots in Fig. 2 for a comparison between a regular octagon and a circle as external domain.*

3. Numerical resolution and results

In order to solve problem (3-4) numerically, we compute the function u solving the weak formulation of the problem in (4), that is

$$\int_D \nabla u \cdot \nabla \phi + \int_{\partial D} \beta u \phi = 0, \qquad (13)$$

where ϕ is a suitable test function, using the Finite Element method implemented in MATLAB. From now on and until otherwise stated, we fix the internal domain Ω as a unit circle, i.e. $\Omega := B_1(0)$.

3.1. Comparison between domains with the same area

As a first step, we consider different convex geometries for the external domain, like circles, polygons, which share the same area. This comparison is aimed at understanding and evaluating which configuration for D must be preferred in order to surround the internal unit circle with a constant amount of thermal insulator. Because of the relation between exposed surface and heat dispersion, one should expect for the circle to be the best choice, since it minimizes the perimeter among convex domains for a given area. This guess is confirmed by most of our numerical experiments. Nevertheless, for a few cases we found irregular domains for some areas which produce a smaller dispersion $F_\beta(D,\Omega)$ with respect to the circle. As an example, Fig. 3 refers to an irregular octagon which disperses less than the circle with same area for the case of $\beta = 0.2$. However, we noted that such a behavior might be misleading: looking at Fig. 1 we noticed that for $\beta = 0.2$, $R \simeq 5$ corresponds to the maximum possible heat dispersion for the case of concentric circles. Thus the example of Fig. 3 refers to a case in which the prescribed quantity of insulator looks like an unwise choice if we want to have little heat dispersion, since in this case using no

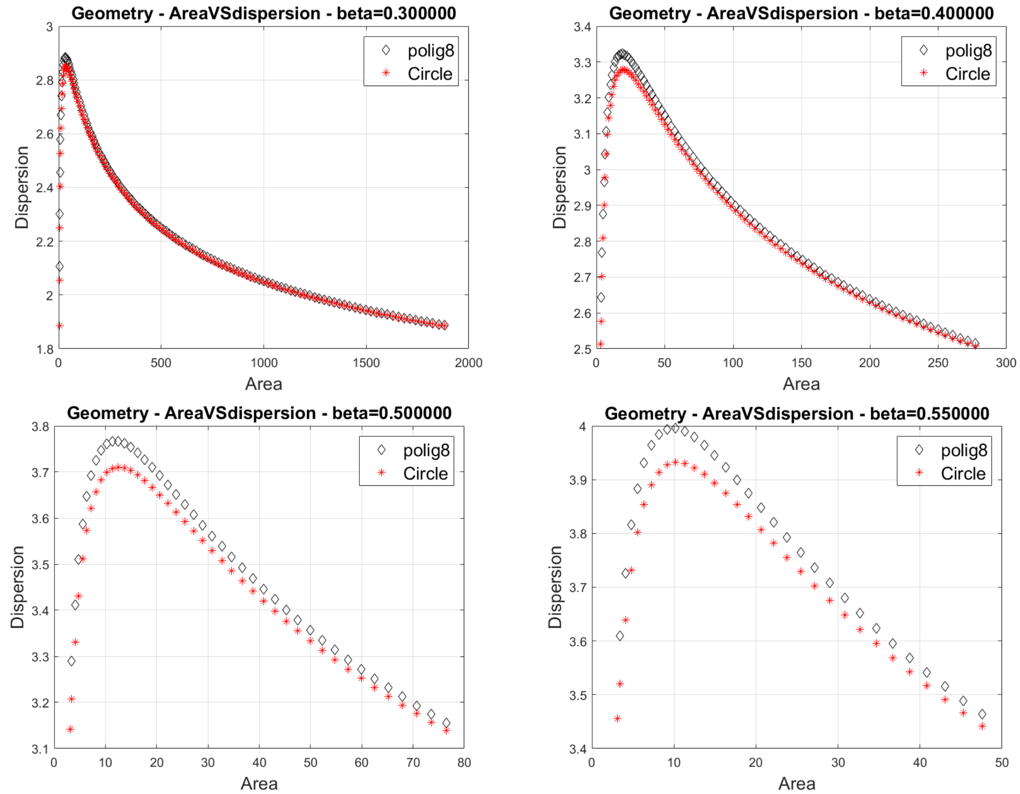

Figure 2: Plots of $(Area, F_\beta(D,\Omega))$ with D as a circle (red $*$) or D as a regular octagon (black $\diamond$), for different values of β, $\beta = 0.3, 0.4, 0.5, 0.55$, respectively. In all cases, Ω is the unit circle $B_1(0)$.

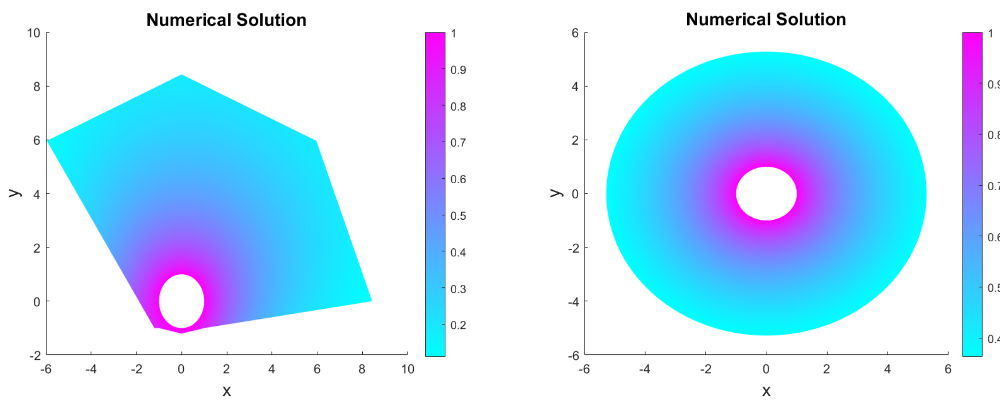

Figure 3: Numerical solutions in the case $\beta = 0.2$, area $A = 87.49$. On the left: Dispersion $F_{0.2}(D,\Omega) = 2.32$. On the right: Dispersion for the two circles $F_{0.2}(D,\Omega) = 2.41$.

insulator at all would be a much better solution. We also noted that the internal circle is located in a corner rather than in a symmetric central setting, and this could be interpreted as a possible way to get rid of the effect of the insulator. As a consequent remark of this example, other geometries better than the disc can be found. In fact, for example, any shape which consists on a very thin layer of insulator around Ω connected with a thin tube to another disc chosen such that the overall area constraint is satisfied will produce a lower value than the case when D is a disc. This is a consequence of the continuity of the functional with respect to the variation of the geometry. We observed a similar behavior for cases in which $\beta < 1$, and this suggested us to adopt less restrictive constraints on the area in order to get a more realistic picture.

The minimum heat dispersion problem with fixed area has been addressed differently in [11], where the area constraint is implicitly included in the definition of the functional (3) to be minimized by adding a penalization term, and the problem (4) is directly solved using a method based on fundamental solutions.

3.2. Comparison between domains under a maximum area constraint

Motivated by the considerations at the end of Subsect. 3.1, we decided to look at the problem of minimizing the heat dispersion under a prescribed maximum area $Amax$ on the insulator. Hence, we solved numerically the problem (5) under the constraints

$$\text{Area}(D) \leq Amax, \ D \in \Delta, \tag{14}$$

varying the class Δ of polygons. We considered both classes of regular and irregular polygons, comparing the results with those obtained considering a circle as external domain. The numerical minimization has been performed using the `patternsearch` MATLAB routine for global minimization. We noted that with `patternsearch` (so as in general for global optimization routines) cannot be ruled out the possibility to converge to a simple local minimum. As $Amax$, values ranging from 10 to 120 have been considered. We run the minimization code using five different starting points for each $Amax$ considered. By expressing the vertices of the external polygon in polar coordinates (ρ_i, θ_i), for $i = 1, \dots, nb$, where nb indicates the number of edges of the polygon, we imposed the following geometrical constraints on the outer polygon:

1. The distances ρ_i between each vertex of the external polygon and the center $(0, 0)$ of the unit circle must be greater than 1 (in the code, we require $\rho_i \geq 1.01, \forall i = 1, \dots, nb$)
2. Each segment r_i joining two consecutive vertices of the polygon must not intersect the inner circle Ω (i.e., $r_i \cap \Omega = \emptyset, \forall i = 1, \dots, nb$).
   Since in our code the constraints are all in the form $c(x) \leq 0$, in order to avoid troubles with the boundary conditions of the PDE, we avoid the intersection of the segments r_i with Ω by considering a circle of radius $r_{ext} = 1 + \epsilon$, where $\epsilon = 0.05$
3. The sum of the angles θ_i must be equal to 2π, i.e. $\sum_{i=1}^{nb} \theta_i = 2\pi$
4. In order to avoid intersections among sides when building polygons, we impose the angle between the vectors $p_0 p_i$ and $p_0 p_{i+1}$ to be positive, where $p_0 = (0, 0), p_i = (x_i, y_i), p_{i+1} = (x_{i+1}, y_{i+1})$.

By analogy with the circle, for $\beta > 1$ we expect the heat dispersion function to be decreasing with the area. As a consequence, at the solution of the minimization problem, we expect for the constraint on the area to be saturated. Our computational results on octagons are in agreement with that, and among the convex polygons with a fixed number of sides, the regular one is the best in terms of heat dispersion. However, the circle always guarantees a lower heat dispersion than any regular polygon.

One of our goals has been to check if the irregular polygons found in Subsect. 3.1 are "genuine"cases, that means if there are cases in which a polygonal shape for D guarantees better insulation than a circle concentric with Ω. Actually, for $\beta < 1$ the minimization process can be a bit more tricky, since different starting points can lead to different solutions. To make this point clear, let us consider a situation like the one depicted in the bottom-left picture of Fig. 2 (red $*$), and $Amax = 80$ in (14). Let A_0 be the area at the starting point, if, e.g., $A_0 = 10$ the minimization process is likely to end up at a (local) solution in which the area is at its minimum possible value (about 3.65). On the other hand, if $A_0 = 20$ (or greater), a (global) solution in which the area of the polygon is equal to $Amax$ will be probably reached. In Table 1, the results for external circle, dodecagon, and octagon, with fixed $\beta = 0.5$ are summarized. Two different values for the final area A mean that for different starting points different solutions, i.e. heat dispersion values, have been found (which correspond to the minimum and the maximum area for the feasible polygons). Comparing the cases $Amax = 120$ with $Amax = 10$, the global solution moves from the solution with maximum area to the solution with minimum area.

As for $\beta > 1$ among the convex polygons with a fixed number of sides, the regular one is always the one that minimizes the heat dispersion, with a value that is always slightly higher than the one with the circle. In any case, the dispersion reduces with the increase of the number of sides.

3.3. Results considering different internal domains

Finally, we consider internal domains different from the unit circle in order to understand if the outer circle remains the domain which minimizes the heat dispersion. The domain Ω is fixed as a regular octagon of area $\pi = A_{B_1(0)}$. The results obtained by using a circle, a regular octagon or a regular dodecagon as external domain are reported in Fig. 4, also compared with the case of internal and external circles (for which we know the analytical solution), and with the case external octagon and internal square, which results the worst one. Once again, it is confirmed that the circular shape for D guarantees the lowest heat dispersion, and the best overall configuration appears to be the two concentric circles one.

Table 1: Circle (C), dodecagon (P12), and octagon (P8) at fixed $\beta = 0.5$, $r_{ext} = 1.05$, by varying $Amax$.

Amax	Final A	DispersionC	Dispersion $P12$	Dispersion $P8$
120	3.6534	3.2554	3.2911	3.3369
120	120	2.9292	2.9343	2.9409
90	90	3.0632	3.0695	3.0776
90	3.6534	3.2554	3.2911	3.3370
80	3.6534	3.2554	3.2911	3.3370
80	80	3.1182	3.1252	3.1340
60	3.6534	3.2554	3.2911	3.3370
60	60	3.2514	3.2601	3.2710
40	40	3.4286	3.4404	3.4551
40	3.6534	3.2554	3.2911	3.3369
35	35	3.4819	3.4950	3.5111
35	3.6534	3.2554	3.2911	3.3369
30	30	3.5389	3.5535	3.5715
30	3.6534	3.2554	3.2911	3.3369
25	3.6534	3.2554	3.2911	3.3369
20	3.6534	3.2554	3.2911	3.3369
20	20	3.6569	3.6759	3.6996
15	3.6534	3.2554	3.2911	3.3369
10	3.6534	3.2554	3.2911	3.3369

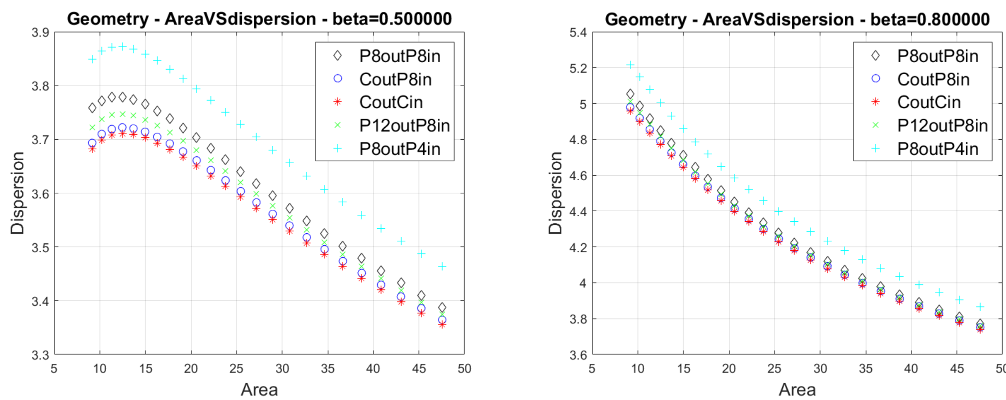

Figure 4: Plots of $(Area, F_\beta(D,\Omega))$ with D and Ω as regular octagons (black $\diamond$), D as circle and Ω as regular octagon (blue o), D and Ω as circles (red $*$), D as regular dodecagon and Ω as regular octagon (green x), D as regular octagon and Ω as square (cyan $+$), for $\beta = 0.5$ and $\beta = 0.8$.

4. Conclusions and future perspectives

In this work we have carried out an extensive computational analysis of a mathematical model of shape optimization related to thermal insulation problems. Our results seem to suggest that the most effective thermal insulation for a conductor of constant temperature, regardless of its shape, is obtained by surrounding it with insulating material disposed according to a circular geometry. Counterexamples to that, described in Sect. 3.1, appear to be very peculiar and of very limited interest in terms of real life applications. Our goal has been to give some insight into general problems of the optimal insulation of conductors, from both the applications and the mathematical modeling viewpoints. As an example, we gave numerical evidences relating to the Open Problem 3 in [6], i.e. in dimension 2, for $\beta > 0$, and for $0 < m < Amax$, there exists the minimum of

$$\{F_\beta(D,\Omega) : \Omega \subseteq D, |\Omega| = m, |D| \leq Amax\}, \tag{15}$$

which is attained when the two domains, Ω and D, are two concentric circles. On this line, we plan in a future work to deepen our analysis by considering a more complex mathematical model to better describe practical needs deriving from the engineering world or other application areas.

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