New extended thermodynamics balance equations for an electron gas confined in a metallic body

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New extended thermodynamics balance equations for an electron gas confined in a metallic body*

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Abstract

Sommerfeld's theory of metal electrons suggests that the electrons in a metal can be described as a gas of free fermion particles confined inside the metallic body. In this paper a new extended thermodynamics model is proposed starting from this idea. The model includes both the stress tensor and a quadratic expansion of the distribution function in the neighborhhod of an equilibrium state. Such contributions were neglected in the original model proposed by Müller in 1976; so this new set of balance laws represents an improvement aimed at the description of non-equilibrium phenomena. The application to a simple case is briefly analysed.

1 Introduction

Sommerfeld [1, 2] suggested a new way to treat physical problems that involve metal electrons. In particular, he thought the electrons as a gas of fermion free particles that from time to time collide with a lattice metallic ion. In order to construct a macroscopic model to describe thermomagnetic and galvanomagnetic effects related to metal electrons, it is natural to refer to rational extended thermodynamics (RET). In fact, this theory [3, 4] was developed to describe non-equilibrium phenomena in rarefied gases. Its new idea with respect to classical thermodynamics (CT) is to consider as field (independent) variables not only the usual ones (mass density, momentum, energy) but also non-equilibrium quantities such as the stress tensor, the heat flux and others. The corresponding field equations turn out to be balance laws supplemented by local and istantaneous constitutive equations that satisfy universal physical principles, like the entropy principle and the principle of relativity. [3, 4]. In order to fix such constitutive relations different methods were developed. The RET macroscopic

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approach identifies suitable constitutive relations through the validity requirement of the universal physical principles; a complete overview of this technique can be found in [3], while a simple application of this approach in the case of metal electrons is presented in [5]. At a microscopic level the set of balance laws can be obtained referring to the Grad method [6] or to the Maximum Entropy Principle (MEP) [3, 4]. As first step the infinite hierarchy of moments derived from the Boltzmann equation is truncated at some order. To close the truncated system, that is to say to express all the quantities as functions of the field variables, the form of the suitable "truncated" distribution function is identified through MEP and in this way all the constitutive relations are fixed. Usually the "truncated" distribution function is linearized in the neighborhood of an equilibrium state and this gives rise to a theory linearized with respect to the non-equilibrium variables. For an exhaustive description and comparison of such procedures see, for example, [3, 4, 7] and the references therein. The hyperbolic PDE systems of RET, initially proposed by Müller, Ruggeri and others researchers for monatomic gases [3], were capable to describe well non-stationary phenomena, overcoming the paradox of infinite velocity typical of parabolic PDEs. Recently, RET has been generalized to rarefied polyatomic gases both in the classical [4] and in the relativistic [8] framework and also to quantum systems [9], providing relevant results and good agreement with experimental data.

For metal electrons a RET model with 8 moments (mass density, momentum, energy and heat flux) was deduced by Müller and Ruggeri in [3], starting from a previous paper by Müller [10]. The model was also studied and generalized in [5]. It is well known that a good description of phenomena sufficiently far from equilibrium could require more moments and a nonlinear expansion of the systems with respect to non-equilibrium variables [3, 4, 7]. For this reason, in the present paper we introduce a RET model for metal electrons with 13 moments and take into account also quadratic non-equilibrium terms. The RET procedure is briefly summarized in section 2 with a particular attention to degenerate gases. Some preliminary calculations and notation are presented in section 3, while in section 4 and 5 the "linear" and the "quadratic" 13-moment model is constructed. An application of the quadratic model is presented in section 6 and, finally, section 7 is devoted to the conclusions.

2 From the kinetic theory to the closure of the truncated moment hierarchy

Following Sommerfel's ideas, the electrons are supposed to be point-particles moving inside the metallic body, while the ions form a periodic lattice and are modeled as rigid spheres (with a non-negligible radius) at rest, equipped with a mass much greater than the electron mass. Under such hypothesis, the collision between an electron and an ion is described as elastic, while the collisions between two electrons are neglected. The starting point for the construction of a model of free-electrons in a metal is obviously the Boltzmann equation[3, 10]:

$$\frac{\partial f}{\partial t} + c_k \frac{\partial f}{\partial x_k} + \dot{c}_k \frac{\partial f}{\partial c_k} = S, \tag{1}$$

where, as usual, here and in what follows repeated indexes imply their sum (in (1), for example, $\sum_{k=1}^{3}$ is omitted), t is the time, $\mathbf{x} = (x_1, x_2, x_3)$ the space variables, $\mathbf{c} = (c_1, c_2, c_3)$ the particle velocity vector, $f = f(\mathbf{x}, \mathbf{c}, t)$ is the phase density, so that

$$f(\mathbf{x}, \mathbf{c}, t) d\mathbf{x} d\mathbf{c} \tag{2}$$

represents the number of electron in an infinitesimal element $d\mathbf{x}d\mathbf{c}$ of the phase space. Moreover, S is the collision term and, concerning $\dot{c}_k$, it holds

$$\begin{aligned} &\dot{c}_k = \left[\psi_k + \sum_{l=1}^{3} \Gamma_{kl} c_l\right] \qquad \text{for} \quad k = 1, 2, 3, \\ &\text{with } \psi_k = -\frac{q_e}{m}\mathcal{E}_k + i_k^0 \quad \text{and} \quad \Gamma_{kl} = -\frac{q_e}{m}\epsilon_{kln} B_n + 2W_{kl} \end{aligned} \tag{3}$$

if $-q_e$ denotes the electron charge, m the electron mass, ϵ_{kln} the Levi-Civita tensor, $\mathcal{E}_k$ and B_k (for $k = 1, 2, 3$) are respectively the electromotive intensity and the magnetic flux density that for the metal electrons give rise to the specific external electromagnetic force, i_k^0 (for $k = 1, 2, 3$) denotes the k-components of velocity-independent part of the specific inertial force, while W_{kl} is related to the Coriolis force [3]. We stress that Γ_{in} represents an antisymmetric matrix.

Following [1, 10] and considering elastic electron-ion collisions, the velocity of an electron before the collision, $\mathbf{c}$, and the corresponding velocity after the collision, $\mathbf{c}'$, have to satisfy the following condition

$$c'_k = c_k - 2e_k(\mathbf{c} \cdot \mathbf{e}) \quad k = 1, 2, 3, \tag{4}$$

if $\cdot$ denotes the scalar product and $\mathbf{e}$ represents the unit vector that goes from the center of the ion sphere to the impact point of the electron [10]. In this way, it holds [1, 2, 10]

$$S = S(f) = \frac{1}{\pi \ell} \int_0^{2\pi} \int_0^{\pi/2} c(f' - f) \cos\theta \sin\theta \; d\theta d\epsilon, \tag{5}$$

if $c = |\mathbf{c}|$, ℓ is the mean free path of an electron between two collisions, $f' = f(\mathbf{c}', \mathbf{x}, t)$, θ is the angle between $\mathbf{e}$ and $\mathbf{c}$ (so that $(\mathbf{c} \cdot \mathbf{e}) = c \cos\theta$) and ϵ is the angle spanned by the plane containing $\mathbf{c}$ and $\mathbf{e}$ with a fixed plane containing $\mathbf{c}$ [10].

An infinite hierarchy of moment equations can be derived from the Boltzmann equation (1), introducing the following definitions

$$F = m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f dc_1 dc_2 dc_3$$

$$F_{j_1 j_2 \dots j_n} = m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_{j_1} c_{j_2} \dots c_{j_n} f dc_1 dc_2 dc_3 \quad n = 1, 2, 3 \dots, \tag{6}$$

$$Q_{j_1 j_2 \dots j_n} = m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_{j_1} c_{j_2} \dots c_{j_n} S(f) dc_1 dc_2 dc_3 \quad n = 1, 2, 3 \dots,$$

if $j_s = 1, 2, 3\ \forall s \in \mathbb{N} \backslash \{0\}$, and one refers to the usual notation (see, for example, [3, 7]). In this way the hierarchy reads

$$\begin{aligned}
&\partial_t F + \partial_{x_k} F_k = 0, \\
&\partial_t F_i + \partial_{x_k} F_{ki} - \psi_k F - \Gamma_{in} F_n = Q_i, \\
&\partial_t F_{ij} + \partial_{x_k} F_{kij} - \psi_i F_j - \psi_j F_i - \Gamma_{in} F_{jn} - \Gamma_{jn} F_{in} = Q_{ij}, \\
&\partial_t F_{ijl} + \partial_{x_k} F_{kjil} - \psi_i F_{jl} - \psi_j F_{il} - \psi_l F_{ij} - \Gamma_{in} F_{jln} - \Gamma_{jn} F_{iln} - \Gamma_{ln} F_{ijn} = Q_{ijl}, \\
&\dots,
\end{aligned} \tag{7}$$

with $\partial_t \cdot = \partial \cdot / \partial t$ and $\partial_{x_k} \cdot = \partial \cdot / \partial x_k$.

The infinite set of balance laws obtained in (7) is usually truncated at some truncation order N. In the next section, for example, we will consider the case $N = 3$. Usually the truncation practice leads to a closure problem, since the last fluxes and the production terms are not a priori expressed as a function of the density variables (which play the role of the independent field variables). Different methods were introduced in the past to close the system, as already recalled in the introduction.

In the present case, taking into account that we deal with fermion particles, the phase density that maximizes the entropy (MEP method) is [3]

$$f_N = \frac{y}{\exp(\chi_N / k_B) + 1} \qquad \text{with} \quad \chi_N = m\, \mathbf{\Lambda} \cdot \boldsymbol{\varphi}(\mathbf{c}) \tag{8}$$

where k_B denotes the Boltzmann constant, $\mathbf{\Lambda}$ denotes the main field vector [3] and

$$\begin{aligned}
&\boldsymbol{\varphi}(\mathbf{c}) = (1, c_{j_1}, c_{j_1} c_{j_2}, c_{j_1} c_{j_2} c_{j_3}, \cdots, c_{j_1} c_{j_2} \dots c_{j_N}) \\
&\mathbf{\Lambda} = (\Lambda, \Lambda_{j_1}, \Lambda_{j_1 j_2}, \Lambda_{j_1 j_2 j_3} \dots \Lambda_{j_1 j_2 \dots j_N}).
\end{aligned} \tag{9}$$

At equilibrium all the main field components vanish except $\Lambda^E = -g/T$ and $\Lambda^E_{ij} = 1/(6T)\delta_{ij}$ and the distribution function reduces to [3]

$$f_E = \frac{y}{\exp(\chi_E / k_B) + 1} \qquad \text{with} \quad \chi_E = -\frac{mg}{T} + \frac{mc^2}{2T} \tag{10}$$

with T temperature and g specific free-enthalpy. In order to simplify the calculations, usually, in Extended Thermodynamics [3, 4], f is approximated in the neighborhood of an equilibrium state thanks to a Taylor expansion of a certain prefixed order o ($o \geq 1$)

$$f_N^{(o)} = f_E + \frac{df_E}{d\chi_E}(\chi_N - \chi_E) + \frac{1}{2}\frac{d^2 f_E}{d\chi_E^2}(\chi_N - \chi_E)^2 + \cdots + \frac{1}{o!}\frac{d^o f_E}{d\chi_E^o}(\chi_N - \chi_E)^o, \quad (11)$$

that corresponds to the Taylor expansion of the main field $\mathbf{\Lambda}^{(o)}$, of the moments $\mathbf{F}^{(o)} = (F^{(o)}, F_{j_1}^{(o)} \cdots, F_{J_1 j_2 \dots j_N}^{(o)})$ and of the production terms $\mathbf{Q}^{(o)} = (0, Q_{j_1}^{(o)}, \dots Q_{j_1 j_2 \dots j_N}^{(o)})$ with

$$\begin{aligned} &F^{(o)} = m \int_{\mathbb{R}^3} f^{(o)} d\mathbf{c}, \qquad F^{(o)}{}_{j_1 j_2 \dots j_s} = m \int_{\mathbb{R}^3} c_{j_1} c_{j_2} \dots c_{j_s} f^{(o)} d\mathbf{c}, \\ &Q^{(o)}{}_{j_1 j_2 \dots j_s} = m \int_{\mathbb{R}^3} c_{j_1} c_{j_2} \dots c_{j_s} S(f^{(o)}) d\mathbf{c} \quad \text{with } s = 1, 2, \dots N, \end{aligned} \quad (12)$$

so that, one can write

$$\begin{aligned} \mathbf{\Lambda}^{(o)} &= \mathbf{\Lambda}^E + \Delta\mathbf{\Lambda}^{(1)} + \Delta\mathbf{\Lambda}^{(2)} + \cdots \Delta\mathbf{\Lambda}^{(o)}, \\ \mathbf{F}^{(o)} &= \mathbf{F}^E + \Delta\mathbf{F}^{(1)} + \Delta\mathbf{F}^{(2)} + \cdots + \Delta\mathbf{F}^{(o)}, \\ \mathbf{Q}^{(o)} &= \mathbf{Q}^E + \Delta\mathbf{Q}^{(1)} + \Delta\mathbf{Q}^{(2)} + \cdots + \Delta\mathbf{Q}^{(o)}, \end{aligned} \quad (13)$$

if $\Delta\mathbf{\Lambda}^{(k)} = \mathbf{\Lambda}^{(k)} - \mathbf{\Lambda}^{(k-1)}$, $\Delta\mathbf{F}^{(k)} = \mathbf{F}^{(k)} - \mathbf{F}^{(k-1)}$ and $\Delta\mathbf{Q}^{(k)} = \mathbf{Q}^{(k)} - \mathbf{Q}^{(k-1)}$, for $k = 1, 2 \dots o$ and $\mathbf{\Lambda}^{(0)} = \mathbf{\Lambda}^{(E)}$, $\mathbf{F}^{(0)} = \mathbf{F}^{(E)}$, while $\mathbf{Q}^{(0)} = \mathbf{Q}^E = \mathbf{0}$.

Referring to $f_N^{(o)}$ all the densities, fluxes and production terms of the truncated moment system can be determined [3, 4, 11, 7] as functions of the field variables. We are studying here a gas of charged particles in the presence of an external electromagnetic field. Thus, it is not reasonable to impose the Galilean invariance of the moment equations and from now on we will refer to a reference frame integral with the metallic body. This fact constitutes a relevant difference with respect to the common procedure used for non-degenerate gases [12]. Moreover, we underline that, here, the momentum density is related to the electric current, that is to say it represents a non-equilibrium quantity and this is a further difference with respect to the usual procedure for the construction of the field system. We recall that in the case of metal electrons the set of balance laws was already determined when the expansion order o is 1 and the truncation order $N = 3$, under the restrictive assumption that the stress tensor vanish [3]. Some further expansions of f with respect to the mean free path parameter were considered by Müller in [10], but no general expression of the balance laws were determined.

In the following we will construct the balance equation system both for a linear and a quadratic expansion of f without neglecting the stress tensor contribution. Before that, we recall briefly some notations and determine the general expression for the production terms.

3 Furhter notation, preliminary definitions and calculations

In what follows we will denote by α and ϑ the following quantities [3, 10]:

$$\alpha = -\frac{mg}{k_B T}, \qquad \vartheta = \frac{2k_B T}{m}, \tag{14}$$

so that the equilibrium distribution function can be rewritten as

$$f_E = \frac{y}{\exp(\alpha + \frac{c^2}{\vartheta}) + 1} \tag{15}$$

When a fermion gas is taken into account the following integral function is usually introduced (see for example [3, 10, 13]):

$$i_k(\alpha) = \int_0^\infty \frac{x^k}{\exp(\alpha + x^2) + 1} dx \qquad \text{if } k \in \mathbb{N}. \tag{16}$$

Since no analytic expression can be derived for $i_k(\alpha)$, several techniques have been developed to calculate it numerically, for an exhaustive review see [13]. Moreover, the function presents some peculiar properties, and in particular we recall, among the others, the following recurrence relation

$$\frac{di_k(\alpha)}{d\alpha} = -\frac{k-1}{2}\; i_{k-2}(\alpha) \quad \text{if } k \geq 2. \tag{17}$$

For the sake of brevity, we introduce also the following functions that depends on $i_k(\alpha)$ $(n, k \in \mathbb{N})$:

$$I_k(\alpha) = (k+1) i_k(\alpha), \qquad H_{n,k}(\alpha) = I_n(\alpha) I_{k+2}(\alpha) - I_{n+2}(\alpha) I_k(\alpha), \tag{18}$$

the main properties of $H_{n,k}(\alpha)$ are summarized and proved in Appendix A.

The production terms of the moment equations are determined starting from (5) and (6) [10, 3].

In particular, besides the relations already available in [10]

$$\begin{aligned} &\frac{1}{\pi\ell}\int_0^{2\pi}\int_0^{\pi/2} (c'_k - c_k)\cos\theta\sin\theta\; d\theta d\epsilon = -\frac{1}{\ell}c_k, \\ &\frac{1}{\pi\ell}\int_0^{2\pi}\int_0^{\pi/2} (c'_k c'_j - c_k c_j)\cos\theta\sin\theta\; d\theta d\epsilon = -\frac{1}{\ell}(c_k c_j - \frac{1}{3}c^2\delta_{kj}), \end{aligned} \tag{19}$$

we have also determined:

$$\begin{aligned} &\frac{1}{\pi\ell}\int_0^{2\pi}\int_0^{\pi/2} (c'_k c'_j c'_i - c_k c_j c_i)\cos\theta\sin\theta\; d\theta d\epsilon = -\frac{1}{\ell}c_k c_j c_i, \\ &\frac{1}{\pi\ell}\int_0^{2\pi}\int_0^{\pi/2} (c'_k c'_j c'_i c'_l - c_k c_j c_i c_l)\cos\theta\sin\theta\; d\theta d\epsilon = \\ &= -\frac{1}{\ell}(c_k c_j c_i c_l - \frac{1}{15}c^4\left(\delta_{kj}\delta_{il} + \delta_{ki}\delta_{jl} + \delta_{kl}\delta_{ji}\right)) . \end{aligned} \tag{20}$$

In what follows we will focus on the case $N = 3$ for a certain expansion order o and we will consider only the first 13 moment variables which are: mass density $\rho = F^{(o)}$, the components of the momentum density $J_i = F_i^{(o)}$ (we recall that $S_i = -q_e J_i/m$ is the i-component of the electric current), the momentum flux $P_{ij} = F_{ij}^{(o)}$, and the componets of heat flux $2q_i = \sum_{l=1}^{3} F_{ill}^{(o)}$ $(i, j = 1, 2, 3)$. The corresponding equation system reads

$$\begin{aligned}
&\partial_t F^{(o)} + \partial_{x_k} F_k^{(o)} = 0,\\
&\partial_t F_i^{(o)} + \partial_{x_k} F_{ki}^{(o)} - \psi_k F^{(o)} - \Gamma_{in} F_n^{(o)} = Q_i^{(o)},\\
&\partial_t F_{ij}^{(o)} + \partial_{x_k} F_{kij}^{(o)} - \psi_i F_j^{(o)} - \psi_j F_i^{(o)} - \Gamma_{in} F_{jn}^{(o)} - \Gamma_{jn} F_{in}^{(o)} = Q_{ij}^{(o)},\\
&\partial_t F_{ill}^{(o)} + \partial_{x_k} F_{kill}^{(o)} - \psi_i F_{ll}^{(o)} - 2\psi_l F_{il}^{(o)} - \Gamma_{in} F_{nll}^{(o)} = Q_{ill}^{(o)}.
\end{aligned} \tag{21}$$

4 The balance laws in the case of linear expansion of f_N

In this Section we will consider the linear expansion of f_N that is to say $o = 1$, so that the moments are expressed as

$$\begin{aligned}
&\rho = F^{(1)} = m \int_{\mathbb{R}^3} f_3^{(1)} d\mathbf{c},\\
&J_i = F_i^{(1)} = m \int_{\mathbb{R}^3} c_i f_3^{(1)} d\mathbf{c} \qquad i = 1, 2, 3,\\
&P_{ij} = F_{ij}^{(1)} = \sigma_{ij} + \frac{2e}{3}\delta_{ij} = m \int_{\mathbb{R}^3} c_i c_j f_3^{(1)} d\mathbf{c} \qquad i, j = 1, 2, 3,\\
&2q_i = F_{ill}^{(1)} = m \int_{\mathbb{R}^3} c_i c^2 f_3^{(1)} d\mathbf{c},
\end{aligned} \tag{22}$$

if $e = F_{ll}^E/2$ denotes the energy density, while $\sigma_{ij} = F_{<ij>}^{(1)}$ the deviatoric part of P_{ij}.

The equilibrium variables ρ and e turn out to be

$$\rho = \frac{4}{3}\pi m y \vartheta^{3/2} I_2(\alpha), \qquad e = \frac{2}{5}\pi m y \vartheta^{5/2} I_4(\alpha). \tag{23}$$

From the previous relations one can derive the explicit expression of the non-equilibrium main field components.

$$\begin{aligned}
&\Delta\Lambda^{(1)} = 0, \qquad \Delta\Lambda_k^{(1)} = \frac{3k_B(\vartheta I_6(\alpha) J_k - 2I_4(\alpha) q_k)}{2ym^2\pi\vartheta^{7/2} H_{4,2}(\alpha)}\\
&\Delta\Lambda_{ik}^{(1)} = -\frac{15 k_B \sigma_{ik}}{4ym^2\pi\vartheta^{7/2} I_4(\alpha)}, \qquad \Delta\Lambda_{kll}^{(1)} = -\frac{3k_B(\vartheta I_4(\alpha) J_k - 2I_2(\alpha) q_k)}{2ym^2\pi\vartheta^{9/2} H_{4,2}(\alpha)},
\end{aligned} \tag{24}$$

and consequently, if $a = a(\vartheta, \alpha) = 4\pi m y \vartheta^{7/2} I_6(\alpha)/21$, the last flux components read

$$\Delta F^{(1)}_{kij} = \frac{2}{5}(q_k\delta_{ij} + q_i\delta_{jk} + q_j\delta_{ik}), \qquad \Delta F^{(1)}_{kill} = \frac{I_6(\alpha)}{I_4(\alpha)}\vartheta\sigma_{ki} + a\delta_{ki}. \tag{25}$$

Furthermore, the production terms can be determined thanks to relations (5), (19) and (20); so, finally, the balance laws read

$$\begin{aligned}
&\partial_t \rho + \partial_{x_k} J_k = 0,\\
&\partial_t J_i + \partial_{x_k}\left(\sigma_{ki} + \frac{2}{3}e\delta_{ki}\right) - \psi_i\rho - \Gamma_{in}J_n = \frac{1}{\ell\sqrt{\vartheta}}\left[\vartheta J_i \Pi_1 + 2q_i\Pi_2\right],\\
&\partial_t\left(\sigma_{ij} + \frac{2}{3}e\delta_{ij}\right) + \partial_{x_k}\left(\frac{2}{5}q_k\delta_{ij} + \frac{2}{5}q_i\delta_{kj} + \frac{2}{5}q_j\delta_{ik}\right) - \psi_i J_j - \psi_j J_i -\\
&- \Gamma_{in}\sigma_{jn} - \Gamma_{jn}\sigma_{in} = -\frac{\sqrt{\vartheta}}{\ell}\frac{I_5(\alpha)}{I_4(\alpha)}\sigma_{ij},\\
&\partial_t(2q_i) + \partial_{x_k}\left(\sigma_{ki}\vartheta\frac{I_6(\alpha)}{I_4(\alpha)} + a\delta_{ki}\right) - 2\psi_n\left(\sigma_{in} + \frac{5}{3}e\delta_{in}\right) -\\
&- 2\Gamma_{in}q_n = \frac{\sqrt{\vartheta}}{\ell}\left[\vartheta J_i\Pi_3 + 2q_i\Pi_4\right],
\end{aligned} \tag{26}$$

where

$$\Pi_1 = \frac{H_{3,4}(\alpha)}{H_{4,2}(\alpha)}, \qquad \Pi_2 = \frac{H_{2,3}(\alpha)}{H_{4,2}(\alpha)}, \qquad \Pi_3 = \frac{H_{5,4}(\alpha)}{H_{4,2}(\alpha)}, \qquad \Pi_4 = \frac{H_{2,5}(\alpha)}{H_{4,2}(\alpha)}. \tag{27}$$

5 The balance laws in the case of quadratic expansion of f_N

The second order terms of the main field variables can be determined with the procedure described in [11, 7], obtaining:

$$\begin{aligned}
&\Delta\Lambda^{(2)} = \beta_1 J_n J_n + \beta_2 J_n q_n + \beta_3 q_n q_n + \beta_4 \sigma_{ns}\sigma_{ns},\\
&\Delta\Lambda^{(2)}_i = \beta_5\sigma_{in}J_n + \beta_6\sigma_{in}q_n,\\
&\Delta\Lambda^{(2)}_{ij} = \beta_7 J_i J_j + \beta_8 J_n J_n\delta_{ij} + \beta_9(J_i q_j + J_j q_i) + \beta_{10}J_n q_n\delta_{ij} + \beta_{11}q_i q_j +\\
&+ \beta_{12}q_n q_n\delta_{ij} + \beta_{13}\sigma_{in}\sigma_{jn} + \beta_{14}\sigma_{sn}\sigma_{sn}\delta_{ij},\\
&\Delta\Lambda^{(2)}_{ill} = \beta_{15}\sigma_{in}J_n + \beta_{16}\sigma_{in}q_n,
\end{aligned} \tag{28}$$

where

$$\beta_1 = \frac{3k_B(3I_6^2(\alpha)H_{2,0}(\alpha) - 7H_{4,2}^2(\alpha) - 9I_2(\alpha)I_6(\alpha)H_{4,2}(\alpha))}{16\pi^2 m^3 y^2 \vartheta^4 H_{2,0}(\alpha)H_{4,2}^2(\alpha)},$$
$$\beta_2 = \frac{9k_B I_4(\alpha)(3I_2(\alpha)H_{4,2}(\alpha) - I_6(\alpha)H_{2,0}(\alpha))}{4\pi^2 m^3 y^2 \vartheta^5 H_{2,0}(\alpha)H_{4,2}^2(\alpha)},$$
$$\beta_3 = \frac{k_B(9I_4^2(\alpha)H_{2,0}(\alpha) - 27I_2^2(\alpha)H_{4,2}(\alpha))}{4\pi^2 m^3 y^2 \vartheta^6 H_{4,2}^2(\alpha)H_{2,0}(\alpha)}, \qquad \beta_4 = \frac{15k_B I_2(\alpha)}{16\pi^2 m^3 y^2 \vartheta^5 I_4(\alpha)H_{2,0}(\alpha)},$$
$$\beta_5 = \frac{-45k_B I_6(\alpha)}{8\pi^2 m^3 y^2 \vartheta^5 I_4(\alpha)H_{4,2}(\alpha)}, \qquad \beta_6 = \frac{63k_B}{4\pi^2 m^3 y^2 \vartheta^6 H_{4,2}(\alpha)}, \tag{29}$$

$$\beta_7 = \frac{\beta_5}{2}, \quad \beta_8 = \frac{k_B(3I_2(\alpha)(7H_{4,2}^2(\alpha) - 3I_6^2(\alpha)H_{2,0}(\alpha) + 9I_2(\alpha)I_6(\alpha)H_{4,2}(\alpha))}{16\pi^2 m^3 y^2 I_4(\alpha)H_{2,0}(\alpha)H_{4,2}^2(\alpha)},$$
$$\beta_9 = \frac{63k_B}{8\pi^2 m^3 y^2 \vartheta^6 H_{4,2}(\alpha)} \quad \beta_{10} = \frac{9k_B I_2(\alpha)(I_6(\alpha)H_{2,0}(\alpha) - 3I_2(\alpha)H_{4,2}(\alpha))}{4\pi^2 m^3 y^2 \vartheta^6 H_{2,0}(\alpha)H_{4,2}^2(\alpha)},$$
$$\beta_{11} = -\frac{81k_B I_2(\alpha)}{4\pi^2 m^3 y^2 \vartheta^7 I_4(\alpha)H_{4,2}(\alpha)}, \quad \beta_{12} = \frac{9k_B I_2(\alpha)(3I_2^2(\alpha)H_{4,2}(\alpha) - I_4^2(\alpha)H_{2,0}(\alpha)}{4\pi^2 m^3 y^2 \vartheta^7 I_4(\alpha)H_{2,0}(\alpha)H_{4,2}^2(\alpha)},$$
$$\beta_{13} = \frac{225k_B}{16\pi^2 m^3 y^2 \vartheta^6 I_4^2(\alpha)}, \quad \beta_{14} = -\frac{15k_B(5H_{2,0}(\alpha) + 2I_0(\alpha)I_4(\alpha))}{32\pi^2 m^3 y^2 \vartheta^6 I_4^2(\alpha)H_{2,0}(\alpha)},$$
$$\beta_{15} = \frac{63k_B}{8\pi^2 m^3 y^2 \vartheta^6 H_{4,2}(\alpha)}, \quad \beta_{16} = \beta_{11}. \tag{30}$$

When the expansion order o is varied, the density vector remains the same; so, in the present case, $F^{(2)} = F^{(1)}$, $F_i^{(2)} = F_i^{(1)}$, $F_{ij}^{(2)} = F_{ij}^{(1)}$ and $F_{ill}^{(2)} = F_{ill}^{(1)}$ while the last components of the flux vector change as follows:

$$\begin{aligned}
\Delta F_{kij}^{(2)} =& b_1[(J_i\sigma_{jk} + J_j\sigma_{ik} + J_k\sigma_{ij}) - \frac{2}{5}J_n(\sigma_{kn}\delta_{ij} + \sigma_{in}\delta_{jk} + \sigma_{jn}\delta_{ik})]+ \\
&+ b_2[(q_i\sigma_{jk} + q_j\sigma_{ik} + q_k\sigma_{ij}) - \frac{2}{5}q_n(\sigma_{kn}\delta_{ij} + \sigma_{in}\delta_{jk} + \sigma_{jn}\delta_{ik})], \\
\Delta F_{kill}^{(2)} =& b_3 J_i J_k + b_4 J^2\delta_{ik} + b_5(J_k q_i + J_i q_k) + b_6 J_n q_n \delta_{ik}+ \\
&+ b_7 q_k q_i + b_8 q^2 \delta_{ik} + b_9 \sigma_{kn}\sigma_{in} + b_{10}\sigma_{sn}\sigma_{sn}\delta_{ik},
\end{aligned} \tag{31}$$

where the previous coeffiecients b_1, b_2, .., b_{10} depend on α and ϑ and read

$$
\begin{aligned}
b_1 &= \frac{3I_6(\alpha)}{14\pi m y \vartheta^{3/2} H_{4,2}(\alpha)}, \quad b_2 = \frac{3(9H_{4,2}(\alpha) - 2I_4^2(\alpha))}{14\pi m y \vartheta^{(5/2)} I_4(\alpha) H_{4,2}(\alpha)},\\
b_3 &= \frac{3(5I_6^2(\alpha)H_{4,2}(\alpha) - 11I_4^2(\alpha)H_{6,4}(\alpha))}{20\pi m y \vartheta^{1/2} I_4(\alpha) H_{4,2}^2(\alpha)},\\
b_4 &= \frac{35H_{4,2}^3(\alpha) - 33H_{2,0}(\alpha)H_{6,4}(\alpha)I_4^2(\alpha) + 45H_{4,2}^2(\alpha)I_2(\alpha)I_6(\alpha)}{40\pi m y \vartheta^{1/2} I_4(\alpha) H_{2,0}(\alpha) H_{4,2}^2(\alpha)},\\
b_5 &= -\frac{3(5I_6(\alpha)H_{4,2}(\alpha) - 11I_2(\alpha)H_{6,4}(\alpha)}{10\pi m y \vartheta^{3/2} H_{4,2}^2(\alpha)},\\
b_6 &= \frac{3I_2(\alpha)(11H_{2,0}(\alpha)H_{6,4}(\alpha) - 15H_{4,2}^2(\alpha)}{10\pi m y \vartheta^{3/2} H_{2,0}(\alpha) H_{4,2}^2(\alpha)},\\
b_7 &= \frac{3(7H_{4,2}^2(\alpha) - 11I_2^2(\alpha)H_{6,4}(\alpha) + 5I_2(\alpha)I_6(\alpha)H_{4,2}(\alpha))}{10\pi m y \vartheta^{5/2} I_4(\alpha) H_{4,2}^2(\alpha)},\\
b_8 &= \frac{3(-15I_2^2(\alpha)H_{4,2}^2(\alpha) + 11I_2^2(\alpha)H_{6,4}(\alpha)H_{2,0}(\alpha) - 2H_{2,0}(\alpha)H_{4,2}^2(\alpha))}{10\pi m y \vartheta^{5/2} I_4(\alpha) H_{2,0}(\alpha) H_{4,2}^2(\alpha)},\\
b_9 &= \frac{15I_6(\alpha)}{14\pi m y \vartheta^{3/2} I_4^2(\alpha)}, \quad b_{10} = \frac{5(3I_6(\alpha)H_{2,0}(\alpha) - 7I_2(\alpha)H_{4,2}(\alpha))}{56\pi m y \vartheta^{3/2} I_4^2(\alpha) H_{2,0}(\alpha)}.
\end{aligned}
\tag{32}
$$

Finally, also the quadratic terms of the production can be determined

$$
\begin{aligned}
\Delta Q^{(2)} =& 0,\\
\Delta Q_i^{(2)} =& \frac{3H_{3,4}(\alpha)\vartheta J_n \sigma_{in} + 6H_{3,2}(\alpha) q_n \sigma_{in}}{4\pi m y \ell \vartheta^3 I_4(\alpha) H_{4,2}(\alpha)},\\
\Delta Q_{ij}^{(2)} =& c_1(3J_iJ_j - J^2\delta_{ij}) + c_2(3q_iq_j - q^2\delta_{ij}) +\\
&+ c_3(3J_iq_j + 3J_jq_i - 2J_nq_n\delta_{ij}) + c_4(3\sigma_{in}\sigma_{jn} - \sigma_{sn}\sigma_{sn}\delta_{ij}),\\
\Delta Q_{ill}^{(2)} =& \frac{9H_{5,4}(\alpha)\vartheta J_n \sigma_{in} + 6H_{2,5}(\alpha) q_n \sigma_{in}}{4\pi m y \ell \vartheta^2 I_4(\alpha) H_{4,2}(\alpha)},
\end{aligned}
\tag{33}
$$

where $i, j = 1, 2, 3$ and the coefficients c_1, c_2, c_3, c_4 are

$$
\begin{aligned}
c_1 &= -\frac{5I_6^2(\alpha)H_{3,2}(\alpha) + I_6(\alpha)I_4(\alpha)H_{3,4}(\alpha) + 10I_4^2(\alpha)H_{4,5}(\alpha)}{20\pi m y \ell \vartheta I_4(\alpha) H_{4,2}^2(\alpha)},\\
c_2 &= -\frac{6I_4^2(\alpha)H_{3,2} + I_2(\alpha)I_4(\alpha)H_{2,5}(\alpha) + 9I_2^2(\alpha)H_{4,5}(\alpha)}{5\pi m y \ell \vartheta^3 I_4(\alpha) H_{4,2}^2(\alpha)},\\
c_3 &= -\frac{I_4(\alpha)H_{4,3}(\alpha) + 5I_6(\alpha)H_{2,3}(\alpha) + 10I_2(\alpha)H_{5,4}(\alpha)}{10\pi m y \ell \vartheta^2 H_{4,2}^2(\alpha)},\\
c_4 &= -\frac{5I_5(\alpha)}{28\pi m y \ell \vartheta^2 I_4^2(\alpha)}.
\end{aligned}
\tag{34}
$$

Referring to the previous relations, system (21) can be easily written explicitly.

6 A simple one-dimensional stationary case

To make a simple comparison between the present model and that proposed in [3], we investigate the case of one-dimensional phenomena, that take place, for example, in a metallic wire. Let x_1 be the one-dimensional space variable and assume that $\sigma_{ij} = 0 \; \forall i, j = 1, 2, 3$. Neglecting the differential equation for σ_{ij} in (21), and assuming that $q_2 = q_3 = J_2 = J_3 = 0$, the "quadratic" model derived in the previous sections reduces to

$$\begin{aligned}
&\partial_t \rho + \partial_{x_1} J_1 = 0, \\
&\partial_t J_i + \partial_{x_1} \left(\frac{2}{3} e \right) - \psi_1 \rho = \frac{1}{\ell\sqrt{\vartheta}} \left[\vartheta J_1 \Pi_1 + 2 q_1 \Pi_2 \right], \\
&\partial_t e + \partial_{x_1} q_1 - \psi_1 J_1 = 0, \\
&\partial_t (2q_1) + \partial_{x_1} \left(a + \underline{(b_3 + b_4) J_1^2 + (2b_5 + b_6) J_1 q_1 + (b_7 + b_8) q_1^2} \right) - \frac{10}{3} \psi_1 e = \\
&= \frac{\sqrt{\vartheta}}{\ell} \left[\vartheta J_1 \Pi_3 + 2 q_1 \Pi_4 \right].
\end{aligned} \tag{35}$$

The underlined terms are quadratic in the non-equilibrium variables, if they are neglected the equation system reduces to the one of [3].

Let us assume that the electric current vanish ($J_1 = 0$), there are no external forces and only stationary phenomena are examined, after some calculations the previous equations imply

$$\begin{aligned}
&q_1 = const \\
&\frac{d\alpha}{dx_1} = \frac{3\Pi_2}{e_\alpha \sqrt{\vartheta}\ell} q_1 - \frac{e_\vartheta}{e_\alpha} \frac{d\vartheta}{dx_1} \\
&\frac{d\vartheta}{dx_1} = \frac{2\vartheta \Pi_4 e_\alpha - (a_\alpha + \underline{(b_7 + b_8)_\alpha q_1^2}) 3\Pi_2}{\sqrt{\vartheta}\ell [e_a (a_\vartheta + \underline{(b_7 + b_8)_\vartheta q_1^2}) - e_\vartheta (a_\alpha + \underline{(b_7 + b_8)_\alpha q_1^2})]} q_1,
\end{aligned} \tag{36}$$

where for a generic variable u and a generic function z $z_u = \partial z / \partial u$ and $(\cdot)_u = \partial \cdot / \partial u$. Equation $(36)_3$ can be seen as a generalization of the Fourier law, written in an implicit form. In fact, if the underlined terms (which are the quadratic ones) are neglected, $(36)_3$ reduces to the the Fourier law already determined in [10]. In a similar way also the generalized Ohm's law can be determined.

7 Conclusions and final remarks

In this paper we introduced a RET model for metal electrons and we studied it in a simple one-dimensional stationary case. The equation system is a generalization of a previous one by Müller and Ruggeri [3] and takes into account both quadratic contributions and the deviatoric part of the momentum density. The goal of this work was the description of phenomena far from equilibrium that involve metal electrons. Therefore, the present paper constitutes the starting point for further investigations on the mathematical properties and on the range of applicability of the model, as well as for applications to realistic physical problems.

8 Appendix A

Property 1

Given $n, k \in \mathbb{N}$ and assuming that the fermion gas is not completely degenerate, for $H_{n,k}(\alpha)$ it holds

$$H_{n,k}(\alpha) \begin{cases} > 0 & \text{if } n < k \\ = 0 & \text{if } n = k \\ < 0 & \text{if } n > k \end{cases} \tag{37}$$

Proof of Property 1 Recalling the definition of $H_{n,k} = I_n(\alpha)I_{k+2}(\alpha) - I_{n+2}(\alpha)I_k(\alpha)$ it is immediate to show that $H_{n,n} = 0 \; \forall n \in \mathbb{N}$ and also that $H_{n,k} = -H_{k,n}$. Thus the proof reduces to the first case, i.e. $n < k$; in this framework we assume that $k = n + h$ with $h \in \mathbb{N}\backslash\{0\}$. From (17), we can deduce that $I_{j-2} = -2\frac{di_j(\alpha)}{d\alpha} \; \forall j \geq 2$ with $j \in \mathbb{N}$. So the function $H_{n,k}(\alpha)$ becomes

$$H_{n,k}(\alpha) = 4\left(\frac{di_{n+2}(\alpha)}{d\alpha}\frac{di_{k+4}(\alpha)}{d\alpha} - \frac{di_{n+4}(\alpha)}{d\alpha}\frac{di_{k+2}(\alpha)}{d\alpha}\right) \tag{38}$$

If we consider (16) and define $g(x,\alpha) = 1/(\exp(\alpha + x^2) + 1)$, (38) becomes

$$H_{n,k}(\alpha) = 4\int_0^{+\infty}\int_0^{+\infty} \frac{\partial g(x,\alpha)}{\partial\alpha}\frac{\partial g(y,\alpha)}{\partial\alpha}\left(x^{n+2}y^{n+h+4} - x^{n+4}y^{n+h+2}\right) dxdy. \tag{39}$$

Permuting the integration variables x and y in (39), it holds

$$\begin{aligned} H_{n,k}(\alpha) =& 2\int_0^{+\infty}\int_0^{+\infty} \frac{\partial g(x,\alpha)}{\partial\alpha}\frac{\partial g(y,\alpha)}{\partial\alpha}x^{n+2}y^{n+2}\left(y^{h+2} - x^2y^h + \right. \\ &\left. + x^{h+2} - y^2x^h\right) dxdy = \\ =& 2\int_0^{+\infty}\int_0^{+\infty} \frac{\partial g(x,\alpha)}{\partial\alpha}\frac{\partial g(y,\alpha)}{\partial\alpha}x^{n+2}y^{n+2}\left(y^2 - x^2\right)\left(y^h - x^h\right) dxdy. \end{aligned} \tag{40}$$

The integrand in (40) is a non-negative function $\forall h \in \mathbb{N}\backslash\{0\}$, moreover its support has clearly a non-zero mesaure in $\mathbb{R}^2$, therefore the double integral is strictly positive.

By contrast, when a gas is completely degenerate, i.e. when $\alpha \to -\infty$ the integrand function vanishes and the property is no more valid. $\square$

The property 1 and its proof are a generalization of theorem 1 in [5].

9 Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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