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SOME EXAMPLES OF NON-SMOOTHABLE GORENSTEIN FANO TORIC THREEFOLDS

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Abstract. We present a combinatorial criterion on reflexive polytopes of dimension 3 which gives a local-to-global obstruction for the smoothability of the corresponding Fano toric threefolds. As a result, we show an example of a singular Gorenstein Fano toric threefold which has compound Du Val, hence smoothable, singularities but is not smoothable.

1. Introduction

In this note we consider a specific feature of the deformation theory of Fano toric threefolds with Gorenstein singularities. Such varieties are in one-to-one correspondence with the 4319 reflexive polytopes of dimension 3, which were classified by Kreuzer and Skarke [6].

Fix such a polytope P and denote by X_P the corresponding Fano toric variety, i.e. the toric variety associated to the spanning fan of P. The singularities of X_P are detected by the shape of the facets of P. Here we will ignore the problem of understanding which singularities are smoothable. Instead, we will present a local-to-global obstruction to the smoothability of X_P. In other words, we will show examples where there exists an open non-affine subscheme $Y \hookrightarrow X_P$ such that Y is singular, Y has smoothable singularities, and Y is not smoothable (and consequently X_P is not smoothable). These examples are constructed by means of the following combinatorial criterion — the relevant definitions are given in §3.

Theorem 1.1. *Let P be a reflexive polytope of dimension 3 and let X_P be the Fano toric threefold associated to the spanning fan of P. If, for some integer $n \geq 1$, the polytope P has "two adjacent almost-flat A_n-triangles" as facets, then X_P is not smoothable.*

A particular polytope, which satisfies the hypothesis of Theorem 1.1, allows us to prove the following result.

Theorem 1.2. *There exists a singular Fano toric threefold X such that the singular locus of X is isomorphic to $\mathbb{P}^1$, X has only cA_1-singularities, and every infinitesimal deformation of X is trivial. In particular, X is not smoothable.*

This refutes a conjecture made by Prokhorov [10, Conjecture 1.9], according to which all Fano threefolds with only compound Du Val singularities are smoothable. This conjecture was motivated by Namikawa's result [8] on the smoothability of Fano threefolds with Gorenstein terminal singularities.

Idea of the proof of Theorem 1.1. Fix an integer $n \geq 1$. An A_n-triangle (see Definition 3.1) corresponds, via toric geometry, to the cA_n threefold singularity $\operatorname{Spec} \mathbb{C}[x, y, z, w]/(xy - z^{n+1})$.

If a reflexive polytope P of dimension 3 has two adjacent A_n-triangles as facets, then there is an open non-affine toric subscheme Y of X_P such that the singular locus of Y is isomorphic to $\mathbb{P}^1$ and the singularities are transverse A_n. Here A_n denotes the affine toric surface $\operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^{n+1})$. More precisely, Y is an A_n-bundle over $\mathbb{P}^1$ (see Definition 2.1), i.e. there exists a map $\pi \colon Y \to \mathbb{P}^1$ such that, Zariski locally on the target, it is the trivial projection with fibre A_n. The map π may be globally non-trivial, depending on the relative position of the two adjacent A_n-triangles. It is possible to express the sheaf $\pi_* \mathcal{E}xt^1_Y(\Omega^1_Y, \mathcal{O}_Y)$, which is a vector bundle on $\mathbb{P}^1$ of rank n, in terms of the combinatorics of the two triangles. In particular, we get to know when this sheaf is the direct sum of negative line bundles on $\mathbb{P}^1$. This gives a combinatorial condition for $\mathcal{E}xt^1_Y(\Omega^1_Y, \mathcal{O}_Y)$ not to have global sections; the condition is expressed by insisting that the two triangles almost lie on the same plane, i.e. they are "almost-flat" (see Definition 3.2). If this happens, then every infinitesimal deformation of Y is locally trivial and, thus, X_P is not smoothable.

Relation to Mirror Symmetry for Fano varieties. In the context of Mirror Symmetry for Fano varieties [1,3], Akhtar–Coates–Galkin–Kasprzyk [2] introduced the notion of "mutation". Starting from some combinatorial datum, a mutation transforms a Fano polytope (i.e. the lattice polytope associated to a Fano toric variety) into another Fano polytope. Varying the combinatorial datum gives different mutations of the same Fano polytope.

In the setting of Theorem 1.1, if a 3-dimensional reflexive polytope P has two adjacent A_n-triangle facets ($n \geq 1$), then these are almost-flat if and only if the polytope P does not admit a special kind of mutation, which we will not specify here. Therefore, Theorem 1.1 says that, in some cases, a Gorenstein Fano toric threefold is not smoothable if the corresponding polytope does not admit a special kind of mutation. This agrees with Ilten's observation [5] that mutations of Fano polytopes induce deformations of the corresponding Fano toric varieties.

Higher dimensions. The methods of this paper could be easily adapted to study obstructions to deformations of toric A_n-bundles on smooth toric varieties of any dimension. This would give a local-to-global obstruction to the smoothability of toric varieties of dimension $d \geq 4$ which contain, as an open toric subscheme, a toric A_n-bundle over a smooth toric variety of dimension $d - 2$.

Notation and conventions. We work over $\mathbb{C}$, but everything will hold over a field of characteristic zero or over a perfect field of large characteristic. If N is a lattice, its dual is denoted by $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and the symbol $\langle \cdot, \cdot \rangle$ denotes the duality pairing between M and N.

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2. A_n-bundles and their deformations

For any integer $n \geq 1$, let A_n denote the toric surface singularity associated to the cone spanned by $(0,1)$ and $(n+1,1)$ inside the lattice $\mathbb{Z}^2$, i.e. the affine hypersurface

$$A_n = \operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^{n+1}).$$

The conormal sequence of the closed embedding $A_n \hookrightarrow \mathbb{A}^3$ produces a free resolution of $\Omega^1_{A_n}$:

$$0 \longrightarrow I/I^2 = \mathcal{O}_{A_n} \xrightarrow{\begin{pmatrix} y \\ x \\ -(n+1)z^n \end{pmatrix}} \Omega^1_{\mathbb{A}^3}\big|_{A_n} = \mathcal{O}_{A_n}^{\oplus 3} \longrightarrow \Omega^1_{A_n} \longrightarrow 0 \tag{1}$$

where I is the ideal of A_n in $\mathbb{A}^3$. This allows us to compute

$$\operatorname{Ext}^1_{A_n}(\Omega^1_{A_n}, \mathcal{O}_{A_n}) = \operatorname{coker}\left(\mathcal{O}_{A_n}^{\oplus 3} \xrightarrow{(y,x,-(n+1)z^n)} \mathcal{O}_{A_n}\right) = \mathcal{O}_{A_n}/(y,x,z^n) = \mathcal{O}_{D_n}$$

where $D_n \simeq \operatorname{Spec} \mathbb{C}[z]/(z^n)$ is the closed subscheme of A_n defined by the ideal generated by y, x and z^n. Notice that D_n is the singular locus of A_n equipped with the schematic structure given by the second Fitting ideal of $\Omega^1_{A_n}$.

We want to define the notion of an A_n-bundle and globalise this computation of the Ext group. Informally, an A_n-bundle is a morphism $Y \to S$ which, Zariski-locally, is the projection $A_n \times S \to S$. More precisely we have to insist that an A_n-bundle is a closed subscheme in a split vector bundle over S of rank 3.

Definition 2.1. An *A_n-bundle* over a $\mathbb{C}$-scheme S is a morphism of schemes $\pi_Y \colon Y \to S$ such that there exist three line bundles $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z \in \operatorname{Pic}(S)$, a closed embedding of S-schemes

$$\iota \colon Y \hookrightarrow E = \operatorname{Spec}_S \mathcal{S}ym^{\bullet}_{\mathcal{O}_S}(\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z)^{\vee}$$

of Y into the total space of $\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z$, and an affine open cover $\{S_i\}_i$ of S satisfying the following condition: for each i, there are trivializations $\mathcal{L}_x|_{S_i} \simeq \mathcal{O}_{S_i}$, $\mathcal{L}_y|_{S_i} \simeq \mathcal{O}_{S_i}$, $\mathcal{L}_z|_{S_i} \simeq \mathcal{O}_{S_i}$ and a commutative diagram of S_i-schemes

$$\begin{array}{ccc} \pi_Y^{-1}(S_i) & \xrightarrow{\simeq} & \operatorname{Spec} \mathcal{O}_{S_i}(S_i)[x_i,y_i,z_i]/(x_iy_i - z_i^{n+1}) \\ {\scriptstyle \iota_{S_i}}\big\downarrow & & \big\downarrow \\ \pi_E^{-1}(S_i) & \xrightarrow{\simeq} & \operatorname{Spec} \mathcal{O}_{S_i}(S_i)[x_i,y_i,z_i] = \mathbb{A}^3_{S_i} \end{array}$$

where π_E denotes the projection $E \to S$, the coordinates $x_i \in \Gamma(S_i, \mathcal{L}_x^{\vee})$, $y_i \in \Gamma(S_i, \mathcal{L}_y^{\vee})$ and $z_i \in \Gamma(S_i, \mathcal{L}_z^{\vee})$ are the local sections corresponding to the trivializations above, the horizontal arrows are isomorphisms, the left vertical arrow is the restriction of the closed embedding $\iota \colon Y \hookrightarrow E$, and the right vertical arrow is the base change of the standard embedding $A_n \hookrightarrow \mathbb{A}^3$ to S_i.

Remark 2.2. A posteriori one can see that $\mathcal{L}_x \otimes \mathcal{L}_y \simeq \mathcal{L}_z^{\otimes(n+1)}$. This follows from the following easy fact in commutative algebra: let A be a ring and $f \in A$ be an invertible element; if the ideal of $A[x,y,z]$ generated by $xy - z^{n+1}$ coincides with the ideal generated by $xy - fz^{n+1}$, then $f = 1$.

Lemma 2.3. *Let S be a scheme with a line bundle $\mathcal{L} \in \operatorname{Pic}(S)$. Let D be the k-th order thickening of the zero section of the total space of $\mathcal{L}$, i.e. the closed subscheme of $\operatorname{Spec}_S \mathcal{S}ym^\bullet_{\mathcal{O}_S} \mathcal{L}^\vee$ locally defined by the equation $x^{k+1} = 0$ where x is a nowhere vanishing local section of $\mathcal{L}^\vee$. Let $\pi \colon D \to S$ be the projection. Then*

$$\pi_* \mathcal{O}_D = \bigoplus_{i=0}^{k} (\mathcal{L}^\vee)^{\otimes i}.$$

Proof. Let $\{S_i\}_i$ be an affine open cover of S which trivializes $\mathcal{L}$. Let $x_i \in \Gamma(S_i, \mathcal{L}^\vee)$ be a local coordinate. Then we have the isomorphism of S_i-schemes

$$\pi^{-1}(S_i) \simeq \operatorname{Spec} \mathcal{O}_S(S_i)[x_i]/(x_i^{k+1}).$$

Therefore $\pi_* \mathcal{O}_D|_{S_i}$ is the free $\mathcal{O}_{S_i}$-module with basis $\{1, x_i, \dots, x_i^k\}$, which is a local frame of $\mathcal{O}_S \oplus \mathcal{L}^\vee \oplus \cdots \oplus (\mathcal{L}^\vee)^{\otimes k}$.

Another way to see this is to notice that $D = \operatorname{Spec}_S(\mathcal{S}ym^\bullet_{\mathcal{O}_S} \mathcal{L}^\vee)/\mathcal{I}$, and consequently $\pi_* \mathcal{O}_D = (\mathcal{S}ym^\bullet_{\mathcal{O}_S} \mathcal{L}^\vee)/\mathcal{I}$, where $\mathcal{I} \subseteq \mathcal{S}ym^\bullet_{\mathcal{O}_S} \mathcal{L}^\vee$ is the ideal made up of elements of degree greater than k. □

Proposition 2.4. *Let S be a $\mathbb{C}$-scheme and $\pi_Y \colon Y \to S$ be an A_n-bundle, with $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z \in \operatorname{Pic}(S)$ as in Definition 2.1. Then there is an isomorphism of $\mathcal{O}_S$-modules*

$$(\pi_Y)_* \left(\mathcal{E}xt^1_Y(\Omega^1_{Y/S}, \mathcal{O}_Y) \right) \simeq \bigoplus_{2 \le j \le n+1} \mathcal{L}_z^{\otimes j}.$$

Proof. Assume we are in the setting of Definition 2.1, with projections $\pi_Y \colon Y \to S$ and $\pi_E \colon E \to S$, closed embedding $\iota \colon Y \hookrightarrow E$, and a trivialising affine open cover $\{S_i\}_i$ of S with local sections x_i, y_i, z_i.

We consider the conormal sequence of $Y \overset{\iota}{\hookrightarrow} E \overset{\pi_E}{\to} S$:

$$\mathcal{I}_{Y/E}/\mathcal{I}^2_{Y/E} \longrightarrow \Omega^1_{E/S}|_Y \longrightarrow \Omega^1_{Y/S} \longrightarrow 0, \tag{2}$$

where $\mathcal{I}_{Y/E}$ is the ideal sheaf of the closed embedding $\iota \colon Y \hookrightarrow E$. We restrict this sequence to S_i and we get the conormal sequence of $Y_i = \pi_Y^{-1}(S_i) \overset{\iota_{S_i}}{\hookrightarrow} E_i = \pi_E^{-1}(S_i) \to S_i$:

$$\mathcal{I}_{Y_i/E_i}/\mathcal{I}^2_{Y_i/E_i} \longrightarrow \Omega^1_{E_i/S_i}|_{Y_i} \longrightarrow \Omega^1_{Y_i/S_i} \longrightarrow 0; \tag{3}$$

this is the base change to S_i of (1), the conormal sequence of $A_n \hookrightarrow \mathbb{A}^3 \to \operatorname{Spec} \mathbb{C}$. As $S_i \to \operatorname{Spec} \mathbb{C}$ is flat, we have that (3) is left exact for all i. As $\{S_i\}_i$ is an open cover of S, we have that also (2) is left exact.

Since $\pi_E \colon E \to S$ is the vector bundle whose sheaf of sections is $\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z$, we have that $\Omega^1_{E/S} = \pi_E^*(\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z)^\vee$. Therefore $\Omega^1_{E/S}|_Y = \pi_Y^*(\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z)^\vee$.

One can check that $\mathcal{I}_{Y/E}/\mathcal{I}^2_{Y/E} \simeq \pi_Y^*(\mathcal{L}_x \otimes \mathcal{L}_y)^\vee$. On the intersection $S_{ij} = S_i \cap S_j$ we have the equalities $x_i = g^x_{ij} x_j$, $y_i = g^y_{ij} y_j$, and $z_i = g^z_{ij} z_j$, where $g^x_{ij}, g^y_{ij}, g^z_{ij} \in \Gamma(S_{ij}, \mathcal{O}_S^*)$ are invertible functions such that $g^x_{ij} g^y_{ij} = (g^z_{ij})^{n+1}$ (by Remark 2.2). Then the restriction of the map

$$\pi_Y^*(\mathcal{L}_x \otimes \mathcal{L}_y)^\vee = \mathcal{I}_{Y/E}/\mathcal{I}^2_{Y/E} \longrightarrow \Omega^1_{E/S}|_Y = \pi_Y^*(\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z)^\vee$$

in (2) to $Y_{ij} = \pi_Y^{-1}(S_{ij})$ produces the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{O}_{Y_{ij}} & \xrightarrow{\begin{pmatrix} y_i \\ x_i \\ -(n+1)z_i^n \end{pmatrix}} & \mathcal{O}_{Y_{ij}}^{\oplus 3} \\
{\scriptstyle g_{ij}^x g_{ij}^y}\downarrow & & \downarrow{\scriptstyle \mathrm{diag}(g_{ij}^x, g_{ij}^y, g_{ij}^z)} \\
\mathcal{O}_{Y_{ij}} & \xrightarrow[\begin{pmatrix} y_j \\ x_j \\ -(n+1)z_j^n \end{pmatrix}]{} & \mathcal{O}_{Y_{ij}}^{\oplus 3}
\end{array}$$

Therefore the sequence (2) becomes

$$0 \longrightarrow \pi_Y^*(\mathcal{L}_x \otimes \mathcal{L}_y)^\vee \longrightarrow \pi_Y^*(\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z)^\vee \longrightarrow \Omega^1_{Y/S} \longrightarrow 0,$$

which gives a locally free resolution of $\Omega^1_{Y/S}$. Hence

$$\begin{aligned}
\mathcal{E}xt^1_Y(\Omega^1_{Y/S}, \mathcal{O}_Y) &= \mathrm{coker}\left(\pi_Y^*(\mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z) \longrightarrow \pi_Y^*(\mathcal{L}_x \otimes \mathcal{L}_y)\right) \\
&= \pi_Y^*(\mathcal{L}_x \otimes \mathcal{L}_y) \otimes_{\mathcal{O}_Y} \mathcal{O}_D \\
&= \pi_Y^*(\mathcal{L}_z)^{\otimes(n+1)} \otimes_{\mathcal{O}_Y} \mathcal{O}_D
\end{aligned}$$

where $D \hookrightarrow Y$ is the closed subscheme locally defined by $x_i = y_i = z_i^n = 0$. Denote by $\pi_D \colon D \to S$ the projection. It is clear that D is the $(n-1)$-th order thickening of the zero section in the total space $\mathcal{L}_z$ over S. By Lemma 2.3 we have

$$(\pi_D)_* \mathcal{O}_D = \bigoplus_{i=0}^{n-1} (\mathcal{L}_z^\vee)^{\otimes i}.$$

Thus

$$\begin{aligned}
(\pi_Y)_* \mathcal{E}xt^1_Y(\Omega^1_{Y/S}, \mathcal{O}_Y) &= (\pi_Y)_*(\pi_Y^* \mathcal{L}_z^{\otimes(n+1)} \otimes_{\mathcal{O}_Y} \mathcal{O}_D) \\
&= (\pi_D)_*(\pi_D^* \mathcal{L}_z^{\otimes(n+1)}) \\
&= (\pi_D)_* \mathcal{O}_D \otimes_{\mathcal{O}_S} \mathcal{L}_z^{\otimes(n+1)} \\
&= \bigoplus_{i=0}^{n-1} (\mathcal{L}_z^\vee)^{\otimes i} \otimes_{\mathcal{O}_S} \mathcal{L}_z^{\otimes(n+1)} \\
&= \bigoplus_{2 \le j \le n+1} \mathcal{L}_z^{\otimes j}.
\end{aligned}$$

This concludes the proof of Proposition 2.4. □

The following lemma is well known in deformation theory.

Lemma 2.5. *Let Y be a reduced $\mathbb{C}$-scheme. Assume that $Y \to \mathrm{Spec}\,\mathbb{C}$ is a local complete intersection morphism and that* $\mathrm{H}^0(Y, \mathcal{E}xt^1_Y(\Omega^1_Y, \mathcal{O}_Y)) = 0$.

Then all infinitesimal deformations of Y are locally trivial. In particular, if Y is not smooth, then Y is not smoothable.

Proof. Let (Art) be the category of local artinian $\mathbb{C}$-algebras with residue field $\mathbb{C}$. Let Def_Y be the functor of infinitesimal deformations of Y, i.e. the covariant functor from (Art) to the category of sets which maps each $A \in$ (Art) to the set $Def_Y(A)$ of isomorphism classes of deformations of Y over Spec A and acts on arrows by base change. For every $A \in (\mathrm{Art})$, let $Def'_Y(A)$ be the subset of $Def_Y(A)$ made up of the locally trivial deformations. This gives a subfunctor $\phi\colon Def'_Y \hookrightarrow Def_Y$. We refer the reader to [11, §2.4] or to [7] for details.

We want to show that the natural transformation ϕ is an isomorphism. It is enough to show that the injective function $\phi_A\colon Def'_Y(A) \hookrightarrow Def_Y(A)$ is surjective for every $A \in (\mathrm{Art})$. This is implied by the smoothness of ϕ (see [7, Definition 3.9]). This is what we will prove below.

Let $\mathcal{T}_Y = \mathcal{H}om_Y(\Omega^1_Y, \mathcal{O}_Y)$ be the sheaf of derivations on Y. By [11, Theorem 2.4.1] the tangent space of Def'_Y is $\mathrm{H}^1(Y, \mathcal{T}_Y)$ and the tangent space of Def_Y is $\mathrm{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y)$. By [11, Proposition 2.4.6], $\mathrm{H}^2(Y, \mathcal{T}_Y)$ is an obstruction space for Def'_Y. By [11, Proposition 2.4.8] or [13, Theorem 4.4], $\mathrm{Ext}^2_Y(\Omega^1_Y, \mathcal{O}_Y)$ is an obstruction space for Def_Y.

The local-to-global spectral sequence for Ext gives the following five term exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1(Y, \mathcal{T}_Y) \longrightarrow \mathrm{Ext}^1_Y(\Omega^1_Y, \mathcal{O}_Y) \longrightarrow \mathrm{H}^0(Y, \mathcal{E}xt^1_Y(\Omega^1_Y, \mathcal{O}_Y)) \longrightarrow \\ \longrightarrow \mathrm{H}^2(Y, \mathcal{T}_Y) \longrightarrow \mathrm{Ext}^2_Y(\Omega^1_Y, \mathcal{O}_Y). \end{aligned}$$

With the identifications above, the vanishing of $\mathrm{H}^0(Y, \mathcal{E}xt^1_Y(\Omega_Y, \mathcal{O}_Y))$ implies that ϕ induces an isomorphism on tangent spaces and an injection on obstruction spaces. By [7, Remark 4.12] we get that ϕ is smooth. □

Corollary 2.6. *Let S be a smooth $\mathbb{C}$-scheme and $\pi_Y\colon Y \to S$ be an A_n-bundle, with $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z \in \mathrm{Pic}(S)$ as in Definition 2.1. Then we have:*

(i) *the sheaf $\mathcal{E}xt^1_Y(\Omega^1_Y, \mathcal{O}_Y)$ is isomorphic to $\mathcal{E}xt^1_Y(\Omega^1_{Y/S}, \mathcal{O}_Y)$;*

(ii) *if $\mathrm{H}^0(S, \mathcal{L}_z^{\otimes j}) = 0$ for all $2 \le j \le n+1$, then all infinitesimal deformations of Y are locally trivial and Y is not smoothable.*

Proof. As $Y \to S$ is a Zariski-locally trivial fibration, the sequence of Kähler differentials of $Y \to S \to \operatorname{Spec}\mathbb{C}$ is left exact and locally split:

$$0 \longrightarrow \pi_Y^*\Omega^1_S \longrightarrow \Omega^1_Y \longrightarrow \Omega^1_{Y/S} \longrightarrow 0.$$

This implies that the dual sequence

$$0 \longrightarrow \mathcal{H}om_Y(\Omega^1_{Y/S}, \mathcal{O}_Y) \longrightarrow \mathcal{H}om_Y(\Omega^1_Y, \mathcal{O}_Y) \longrightarrow \mathcal{H}om_Y(\pi_Y^*\Omega^1_S, \mathcal{O}_Y) \longrightarrow 0$$

is exact. From the long exact sequence of Ext sheaves we get the following exact sequence of $\mathcal{O}_Y$-modules:

$$0 \longrightarrow \mathcal{E}xt^1_Y(\Omega^1_{Y/S}, \mathcal{O}_Y) \longrightarrow \mathcal{E}xt^1_Y(\Omega^1_Y, \mathcal{O}_Y) \longrightarrow \mathcal{E}xt^1_Y(\pi_Y^*\Omega^1_S, \mathcal{O}_Y).$$

But the last sheaf is zero because S is smooth over $\mathbb{C}$. This proves (i).

By Proposition 2.4 we deduce that

$$\mathrm{H}^0(Y, \mathcal{E}xt^1_Y(\Omega^1_Y, \mathcal{O}_Y)) = \bigoplus_{2 \le j \le n+1} \mathrm{H}^0(S, \mathcal{L}_z^{\otimes j}) = 0.$$

From Lemma 2.5 we deduce (ii). □

3. Toric A_n-bundles over $\mathbb{P}^1$

Definition 3.1. Fix an integer $n \ge 1$ and a 3-dimensional lattice N. An *A_n-triangle* in N is a lattice triangle $T \subseteq N_\mathbb{R}$ such that:

(1) there are no lattice points in the relative interior of T;

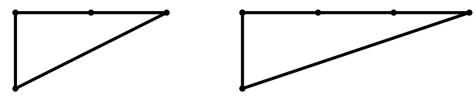

FIGURE 1. An A_1-triangle and an A_2-triangle

(2) the edges of T have lattice lengths 1, 1, and $n+1$;
(3) T is contained in a plane which has height 1 with respect to the origin, i.e. there exists a linear form $w \in M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ such that T is contained in the affine plane $H_{w,1} := \{v \in N_{\mathbb{R}} \mid \langle w, v \rangle = 1\}$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between M and N.

If T is an A_n-triangle in the 3-dimensional lattice N, consider the cone $\sigma \subseteq N_{\mathbb{R}}$ spanned by the vertices of T. Then the affine toric variety associated to the cone σ, namely $\operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$, is isomorphic to $\operatorname{Spec} \mathbb{C}[x, y, z, w]/(xy - z^{n+1})$; every point with $x = y = z = 0$ is a cA_n singularity.

Definition 3.2. Fix an integer $n \geq 1$ and a 3-dimensional lattice N. *Two adjacent A_n-triangles* in N are two A_n-triangles T_0 and T_1 in N such that:

(4) $T_0 \cap T_1$ is the edge of length $n+1$ for both T_0 and T_1;
(5) T_0 and T_1 lie in the two different half-spaces of $N_{\mathbb{R}}$ defined by the plane $\operatorname{span}_{\mathbb{R}}(T_0 \cap T_1)$.

We say that T_0 and T_1 are *almost-flat* if $\langle w_1, \rho_0 \rangle = 0$, where ρ_0 is the vertex of the triangle T_0 not in the segment $T_0 \cap T_1$ and $w_1 \in M$ is the linear form such that T_1 is contained in the plane $H_{w_1,1}$.

Notice that the condition of almost-flatness is symmetric between T_0 and T_1 because $\langle w_1, \rho_0 \rangle = \langle w_0, \rho_1 \rangle$.

Remark 3.3. Let P be a reflexive polytope in the lattice N of rank 3 and let T_0 and T_1 be two adjacent A_n-triangles which are facets of P. The convexity of P implies $\langle w_1, \rho_0 \rangle \leq 0$.

Consider the dual polytope

$$P^* = \{u \in M_{\mathbb{R}} \mid \forall v \in P,\ \langle u, v \rangle \geq -1\}.$$

The dual face of T_0 (resp. T_1) is the vertex $-w_0$ (resp. $-w_1$) of P^*. The dual face of the edge $T_0 \cap T_1$ is the edge $\operatorname{conv}\{-w_0, -w_1\}$ of P^*. The segment $\operatorname{conv}\{-w_0, -w_1\}$ has lattice length equal to $1 - \langle w_1, \rho_0 \rangle$.

Setup 3.4. Let T_0 and T_1 be two adjacent A_n-triangles in a 3-dimensional lattice N. We denote by ρ_u and ρ_v the vertices of the segment $T_0 \cap T_1$. Let ρ_0 (resp. ρ_1) be the vertex of T_0 (resp. T_1) which does not lie on $T_0 \cap T_1$ (see Figure 2). Let Y be the toric variety associated to the fan in N generated by $\operatorname{cone}\{\rho_0, \rho_u, \rho_v\}$ and $\operatorname{cone}\{\rho_1, \rho_u, \rho_v\}$. The projection $N \to N/(N \cap (\mathbb{R}\rho_u + \mathbb{R}\rho_v)) \simeq \mathbb{Z}$ induces a toric morphism $\pi \colon Y \to \mathbb{P}^1$.

Proposition 3.5. *Let T_0 and T_1 be two adjacent A_n-triangles in a 3-dimensional lattice N. Then the toric morphism $\pi \colon Y \to \mathbb{P}^1$, constructed in Setup 3.4, is an A_n-bundle and there exists an isomorphism*

$$\pi_* \mathcal{E}xt^1_Y(\Omega^1_Y, \mathcal{O}_Y) \simeq \bigoplus_{2 \leq j \leq n+1} \mathcal{O}_{\mathbb{P}^1}\left(-j\left(\langle w_1, \rho_0 \rangle + 1\right)\right). \tag{4}$$

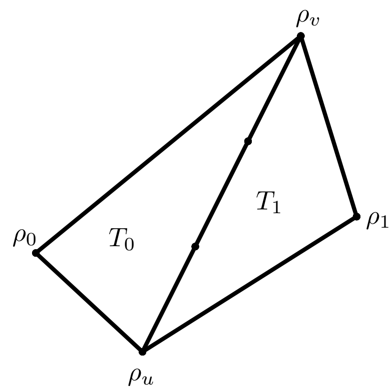

FIGURE 2. Two adjacent A_2-triangles

Moreover, if $\langle w_1, \rho_0 \rangle \geq 0$ *then all infinitesimal deformations of* Y *are locally trivial and* Y *is not smoothable.*

Before proving this proposition we prove the following lemma.

Lemma 3.6. *After a* $\mathrm{GL}_3(\mathbb{Z})$*-transformation, in Setup 3.4 we may assume that* $N = \mathbb{Z}^3$ *and*

$$\rho_0 = \begin{pmatrix} a \\ b \\ -1 \end{pmatrix}, \ \rho_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \rho_u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \rho_v = \begin{pmatrix} -n \\ n+1 \\ 0 \end{pmatrix},$$

for some $a, b \in \mathbb{Z}$.

Proof. Let $\hat{\rho} \in N$ be the lattice point on the segment between ρ_u and ρ_v which is the closest one to ρ_u. The triangle with vertices $\rho_u, \rho_1, \hat{\rho}$ is an empty triangle at height 1, so $\{\rho_u, \rho_{x_1}, \hat{\rho}\}$ is a basis of N. Without loss of generality we may assume that $\rho_u = (1,0,0)$, $\hat{\rho} = (0,1,0)$ and $\rho_1 = (0,0,1)$. Since on the edge between ρ_u and ρ_v there are $n+2$ lattice points, we have $\rho_v = \rho_u + (n+1)(\hat{\rho} - \rho_u) = (-n, n+1, 0)$.

Assume $\rho_0 = (a,b,c)$ for some $a,b,c \in \mathbb{Z}$. Since $\rho_u, \hat{\rho}, \rho_0$ are the vertices of an empty triangle at height 1, they constitute a basis of N. Therefore $c = \det(\rho_u | \hat{\rho} | \rho_0) = \pm 1$.

Since ρ_0 and ρ_1 have to be in the two different half-spaces in which the plane $\mathbb{R}\rho_u + \mathbb{R}\rho_v = (0,0,1)^\perp$ divides $N_\mathbb{R}$, we have $c < 0$, so $c = -1$. □

Proof of Proposition 3.5. By Lemma 3.6, the ray map $\mathbb{Z}^4 \to N = \mathbb{Z}^3$ of Y is given by the matrix

$$\begin{pmatrix} a & 0 & 1 & -n \\ b & 0 & 0 & n+1 \\ -1 & 1 & 0 & 0 \end{pmatrix}.$$

One can see that the ideal of $\mathbb{Z}$ generated by the 2×2 minors is $\mathbb{Z}$ itself and the ideal generated by the 3×3 minors is $r\mathbb{Z}$, where $r = \gcd(n+1, b) > 0$. Let $p, q \in \mathbb{Z}$ be such that $b = rp$ and $n+1 = rq$. The kernel of the ray map is generated by the primitive vector $(q, q, -np - aq, -p)$. By Bézout let $s, t \in \mathbb{Z}$ be such that $sp + tq = 1$. The cokernel of the transpose of the ray map is the homomorphism $\mathbb{Z}^4 \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z}$

given by the matrix

$$\begin{pmatrix} q & q & -qa-pn & -p \\ \bar{s} & \bar{s} & -\bar{s}\bar{a}+\bar{t}\bar{n} & \bar{t} \end{pmatrix},$$

where $\bar{\cdot}$ denotes the reduction modulo r. By [4, Theorem 4.1.3], the divisor class group of Y is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z}$.

Let the group

$$G = \left\{ \left(\lambda^q \varepsilon^s, \lambda^q \varepsilon^s, \lambda^{-qa-pn} \varepsilon^{-sa+tn}, \lambda^{-p} \varepsilon^t\right) \in \mathbb{G}_{\mathrm{m}}^4 \,\middle|\, \lambda \in \mathbb{G}_{\mathrm{m}}, \varepsilon \in \boldsymbol{\mu}_r \right\}$$

act linearly on the affine space $\mathbb{A}^4 = \operatorname{Spec} \mathbb{C}[x_0, x_1, u, v]$. By [4, §5.1], Y is the geometric quotient of $\mathbb{A}^4 \smallsetminus \mathrm{V}(x_0, x_1) = \operatorname{Spec} \mathbb{C}[x_0^{\pm}, x_1, u, v] \cup \operatorname{Spec} \mathbb{C}[x_0, x_1^{\pm}, u, v]$ with respect to this action. The variables x_0, x_1, u, v can be identified with the Cox coordinates of Y associated to the rays $\rho_0, \rho_1, \rho_u, \rho_v$, respectively. The toric morphism $\pi\colon Y \to \mathbb{P}^1$ is defined by

$$[x_0 : x_1 : u : v] \mapsto [x_0 : x_1],$$

where $[x_0 : x_1 : u : v]$ denotes the point of Y corresponding to the G-orbit of the point $(x_0, x_1, u, v) \in \mathbb{A}^4$.

We consider the following integers

$$\begin{aligned} d_x &= b - (n+1)(a+b), \\ d_y &= -b, \\ d_z &= -a - b. \end{aligned}$$

We consider the line bundles $\mathcal{L}_x = \mathcal{O}_{\mathbb{P}^1}(d_x)$, $\mathcal{L}_y = \mathcal{O}_{\mathbb{P}^1}(d_y)$, $\mathcal{L}_z = \mathcal{O}_{\mathbb{P}^1}(d_z)$ and the sheaf $\mathcal{E} = \mathcal{L}_x \oplus \mathcal{L}_y \oplus \mathcal{L}_z$ on $\mathbb{P}^1$. Let $\pi_E\colon E \to \mathbb{P}^1$ be the total space of $\mathcal{E}$ over $\mathbb{P}^1$. Then E is the geometric quotient of $\operatorname{Spec} \mathbb{C}[x_0, x_1, x, y, z] \smallsetminus \mathrm{V}(x_0, x_1)$ with respect to the linear action of $\mathbb{G}_{\mathrm{m}}$ with weights $(1, 1, d_x, d_y, d_z)$. The variables x_0, x_1, x, y, z can be identified with the Cox coordinates of the toric variety E. We denote by $[x_0 : x_1 : x : y : z]$ the point of E corresponding to the $\mathbb{G}_{\mathrm{m}}$-orbit of $(x_0, x_1, x, y, z) \in \mathbb{A}^5$.

It is easy to check that the map $\iota\colon Y \to E$ given by

$$[x_0 : x_1 : u : v] \mapsto [x_0 : x_1 : u^{n+1} : v^{n+1} : uv]$$

is a closed embedding, locally defined by $xy - z^{n+1} = 0$. So $\pi\colon Y \to \mathbb{P}^1$ is an A_n-bundle and we are in the situation of Definition 2.1.

The triangle T_1 is contained in the plane $H_{w_1,1}$, where $w_1 = (1,1,1)$. Therefore $\langle w_1, \rho_0 \rangle = a + b - 1 = -d_z - 1$. By Proposition 2.4 and Corollary 2.6 we have the isomorphism (4).

The inequality $\langle w_1, \rho_0 \rangle \geq 0$ implies that $\mathcal{L}_z$ is a negative line bundle on $\mathbb{P}^1$ and, by Corollary 2.6, that all infinitesimal deformations of Y are locally trivial. □

Proof of Theorem 1.1. It is an immediate consequence of Proposition 3.5. □

Remark 3.7. There are 273 reflexive polytopes of dimension 3 which satisfy the condition of Theorem 1.1: the complete list is given in [9, Remark 4.15]. Therefore, there are at least 273 non-smoothable Gorenstein Fano toric threefolds.

Proof of Theorem 1.2. In the lattice $N = \mathbb{Z}^3$ consider the reflexive polytope P that is the convex hull of the following vectors:

$$\rho_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \rho_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \ \rho_u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \rho_v = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}, \ \xi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let Σ be the spanning fan of P. The maximal cones of Σ are:

$$\begin{aligned} &\text{cone}\{\rho_0, \rho_u, \rho_v\}, \qquad &&\text{cone}\{\rho_1, \rho_u, \rho_v\}, \\ &\text{cone}\{\rho_0, \rho_u, \xi\}, \qquad &&\text{cone}\{\rho_1, \rho_u, \xi\}, \\ &\text{cone}\{\rho_0, \rho_v, \xi\}, \qquad &&\text{cone}\{\rho_1, \rho_v, \xi\}. \end{aligned}$$

The singular cones of Σ are the ones in the first row and cone $\{\rho_u, \rho_v\}$. The corresponding facets of P are two adjacent A_1-triangles. We have $w_1 = (-1, 1, 0)$ and $\langle w_1, \rho_0 \rangle = 0$, so the two A_1-triangles are almost flat.

Let X be the Fano toric threefold associated to the fan Σ. The singular locus of X is the curve C, which is the closure of the torus-orbit corresponding to cone $\{\rho_u, \rho_v\}$. The curve C is isomorphic to $\mathbb{P}^1$ and the singularities of X along C are transverse A_1.

By Proposition 3.5 the sheaf $\mathcal{E}xt^1_X(\Omega^1_X, \mathcal{O}_X)$ is the line bundle $\mathcal{O}_C(-2)$ on C. Therefore $\mathrm{H}^0(X, \mathcal{E}xt^1_X(\Omega^1_X, \mathcal{O}_X)) = 0$.

Let $j \colon U \hookrightarrow X$ be the inclusion of the smooth locus of X. Notice that the sheaf of derivations $\mathcal{T}_X = \mathcal{H}om_X(\Omega^1_X, \mathcal{O}_X)$ is isomorphic to $j_* \Omega^2_U \otimes \mathcal{O}_X(-K_X)$, because these two sheaves are both reflexive and coincide on the open subset U whose complement has codimension 2. As $-K_X$ is ample, by Bott–Steenbrink–Danilov vanishing [4, Theorem 9.3.1] we have $\mathrm{H}^1(X, \mathcal{T}_X) = 0$. This argument comes from the proof of [12, Theorem 5.1].

From the five term exact sequence for Ext, which is rewritten in the proof of Lemma 2.5, we deduce that $\mathrm{Ext}^1_X(\Omega^1_X, \mathcal{O}_X) = 0$. This implies that all infinitesimal deformations of X are trivial. In particular, X is not smoothable. □

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