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Bezout-like polynomial equations associated with dual univariate interpolating subdivision schemes

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Abstract The algebraic characterization of dual univariate interpolating subdivision schemes is investigated. Specifically, we provide a constructive approach for finding dual univariate interpolating subdivision schemes based on the solutions of certain associated polynomial equations. The proposed approach also makes it possible to identify conditions for the existence of the sought schemes.

Keywords Bezout equation; Univariate dual subdivision; Higher arity; Interpolation

Mathematics Subject Classification (2000) 65F05 · 68W30 · 65D05 · 65D17

1 Introduction

Subdivision schemes are useful tools for the fast generation of graphs of functions, smooth curves and surfaces by the application of iterative refinements to an initial set of discrete data. The major fields of application of subdivision schemes are Computer Graphics and Animation, Computer-Aided Geometric Design and Signal/Image Processing, but a further motivation for their study is also their close relation to multiresolution analysis and wavelets. The last connection was especially investigated in the case of interpolating subdivision schemes and it was pointed out that the interpolatory subdivision schemes of Dubuc-Deslauriers [11] are connected to orthonormal wavelets of Daubechies [6, 23]. Interpolating subdivision schemes were also deeply studied, because they are considered to be very efficient in representing smooth curves and surfaces passing through a given set of points. In fact, after five or six subdivision iterations only, they are capable of

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providing the refined set of points needed to represent on the screen the desired smooth limit shape interpolating the given data. The main properties of interpolating subdivision schemes were investigated over the past 20 years by several researchers (see, e.g., [12, 15, 18]) and many approaches were proposed to design their refinement rules. However, as far as we are aware, before the papers [25, 26], no one ever tried to construct interpolating subdivision schemes that do not satisfy the stepwise interpolation property and are thus not defined via refinement rules that at each stage of the iteration leave the previous set of points unchanged. Stepwise interpolating subdivision schemes - also known as primal interpolating subdivision schemes [11, 17] - are defined by finite subdivision masks of odd width that contain as a special submask the sequence $\boldsymbol{\delta} = \{\delta_{0,j},\ j \in \mathbb{Z}\}$. Differently, members of the most recently introduced class of non-stepwise interpolating subdivision schemes –also known as dual interpolating subdivision schemes– are characterized by finite masks with an even number of entries that do not necessarily contain as a special submask the $\boldsymbol{\delta}$ sequence. A first algorithm to construct dual interpolating quaternary schemes was proposed in [25] and successively extended to arbitrary arity greater than two in [26]. Precisely, in the latter it was shown that, under some suitable auxiliary assumptions, the coefficients of the subdivision mask of a dual interpolating scheme can be (possibly) determined by the solution of an associated rectangular linear system. This system can be clearly inconsistent for some choices of input data and/or size (length) of the mask. For a given input data set the approach taken in [26] consists of an exhaustive analysis of the associated linear systems of increasing sizes in order to identify possible consistent configurations.

In this paper we pursue a different method for constructing dual interpolating subdivision schemes based on the reduction of the matrix formulation into a functional setting to solving a certain Bezout-like polynomial equation. The method makes it possible to address the consistency issues by detecting suitable conditions on the input data which guarantee the existence of a dual interpolating scheme. Additionally, it yields a full characterization of the set of solutions which can be exploited to fulfil additional demands and properties of the solution mask. From the point of view of applications, such a computational approach allows the user to meet specific requests in terms of polynomial reproduction, support size and regularity. Even though a general result concerning convergence and/or smoothness of a dual interpolating subdivision scheme is not yet available, in all the considered examples the regularity analysis is done via joint spectral radius techniques (see [4, 20, 22]), rather than by means of the restricted spectral radius approach (see, e.g., [3]), and the best Hölder exponent for each scheme is computed up to the 15th decimal digit.

2 Background and notation

In this section we briefly recall some needed background on subdivision schemes of arbitrary arity $m \in \mathbb{N}$, $m \geqslant 2$.

Any linear, stationary subdivision scheme is identified by a *refinement mask* $\boldsymbol{a} := \{a_i \in \mathbb{R},\ i \in \mathbb{Z}\}$ that is usually assumed to have finite support, *i.e.* to satisfy $a_i = 0$ for $i \notin [-L, L]$ for suitable $L > 0$.

The *subdivision scheme* identified by the mask $\boldsymbol{a}$ consists of the subsequent application of the *subdivision operator*

$$S_{\boldsymbol{a}} \ : \ \ell(\mathbb{Z}) \to \ell(\mathbb{Z}) \ , \qquad (S_{\boldsymbol{a}}\, \boldsymbol{p})_i \ := \ \sum_{j\in\mathbb{Z}} a_{i-mj}\, p_j, \quad i \in \mathbb{Z} \ ,$$

which provides the linear rules determining the successive refinements of the initial sequence of discrete data $\boldsymbol{p} := (p_i \in \mathbb{R}, i \in \mathbb{Z}) \in \ell(\mathbb{Z})$. Introducing the notation $\boldsymbol{p}^{(0)} := \boldsymbol{p}$, we can thus describe the subdivision scheme as an iterative method that at the k-th step generates the refined scalar sequence

$$\boldsymbol{p}^{(k+1)} \ := \ S_{\boldsymbol{a}}\, \boldsymbol{p}^{(k)}, \qquad k \geqslant 0. \tag{1}$$

Attaching the data $p_i^{(k)}$ generated at the k-th step to the parameter values $t_i^{(k)}$ with

$$t_i^{(k)} \ < \ t_{i+1}^{(k)}, \qquad \text{and} \qquad t_{i+1}^{(k)} - t_i^{(k)} \ = \ m^{-k}, \qquad k \ \geqslant \ 0$$

(these are usually set as $t_i^{(k)} := m^{-k}i$) we see that the subdivision process generates denser and denser sequences of data so that a notion of convergence can be established by taking into account the piecewise linear function $P^{(k)}$ that interpolates the data, namely

$$P^{(k)}(t_i^{(k)}) \ = \ p_i^{(k)}, \qquad P^{(k)}|_{[t_i^{(k)},t_{i+1}^{(k)}]} \in \Pi_1, \qquad i \in \mathbb{Z}, \quad k \geqslant 0,$$

where Π_1 is the space of linear polynomials. If the sequence of the continuous functions $\{P^{(k)},\ k \geqslant 0\}$ converges uniformly, then we denote its limit by

$$f_{\boldsymbol{p}} \ := \ \lim_{k\to\infty} P^{(k)}$$

and say that $f_{\boldsymbol{p}}$ is the *limit function* of the subdivision scheme based on the rule (1) for the data $\boldsymbol{p}$ [2]. When $\mathbf{p} = \boldsymbol{\delta}$, $f_{\boldsymbol{\delta}}$ is called *basic limit function.*
The analysis of convergence of a subdivision scheme can be accomplished by studying the properties of the so-called *symbol* of the subdivision mask [14]. The symbol of a finitely supported sequence $\boldsymbol{a}$ is defined as the Laurent polynomial

$$a(z) \ := \ \sum_{i\in\mathbb{Z}} a_i\, z^i, \qquad z \in \mathbb{C}\backslash\{0\}.$$

Besides convergence and smoothness, many other properties of a subdivision scheme, like polynomial generation and reproduction, can be checked by investigating algebraic conditions on the subdivision symbol [7]. While the term *polynomial generation* refers to the capability of the subdivision scheme of providing polynomials as limit functions, with *polynomial reproduction* we mean the capability of a subdivision scheme of reproducing in the limit exactly the same polynomial from which the data are sampled. The property of polynomial reproduction is very important since strictly connected to the approximation order of the subdivision scheme and to its regularity [5, 16]. With respect to the capability of reproducing polynomials up to a certain degree, the standard parametrization (corresponding to the choice $t_i^{(k)} := m^{-k}i$, $i \in \mathbb{Z}$) is not always the optimal one. Indeed, the choice $t_i^{(k)} := m^{-k}(i + \sigma/(m-1))$ with $\sigma = a^{(1)}(1)/m$, turns out to be the recommended selection [8]. The subdivision schemes for which $\sigma \in \mathbb{Z}$ are termed *primal*, whereas the ones for which $\sigma \in (2\mathbb{Z}+1)/2$ are called *dual*. The target of this work are dual schemes. While dual approximating schemes were investigated extensively (see, e.g., [8, 13] and references therein), to the best of our knowledge dual interpolating schemes were only considered in the recent papers [25, 26]. However, as already acknowledged in [25], the open problem treated in these papers was suggested by Malcolm Sabin, who has the merit of being the first who foresaw the existence of dual m-ary schemes (with $m > 2$) that are capable of interpolating the initial data.

3 The proposed approach

The aim of this section is to investigate the algebraic characterization of univariate dual interpolating subdivision schemes of arity m. According to the results shown in [26], the construction of such schemes requires as input the desired degree of polynomial reproduction (denoted in the following by $d-1$, $d \in \mathbb{N}$) and some samples of the resulting basic limit function $f_{\boldsymbol{\delta}}$, i.e.,

$$f_{\boldsymbol{\delta}}\left(\frac{1}{2}+\ell\right) = \varphi\left(\frac{1}{2}+\ell\right), \qquad \forall \ell \in \mathbb{Z}, \tag{2}$$

for a given $\varphi\colon (2\mathbb{Z}+1)/2 \to \mathbb{R}$. A similar procedure was investigated in [9, 10], where the samples of the basic limit function at the integers were required: here instead the samples at the integers are fixed to be the $\boldsymbol{\delta}$ sequence and information about the samples at the half-integers are required.
More specifically, in [26] it is seen that taking Fourier transforms on both sides of the refinement equation for the basic limit function $f_{\boldsymbol{\delta}}$ allows one to describe the mask of dual interpolatory schemes in a matrix setting in terms of the solution of certain bi-infinite Toeplitz-like linear systems in banded form. In this paper we exploit the interplay between the functional and the matrix settings into more details. In particular, from the matrix setting we come back to the functional one by relying upon the connection of Toeplitz-like systems with corresponding Bezout-like polynomial equations. This connection yields a constructive approach to determine the associated symbols. Moreover, the proposed approach also makes it possible to identify conditions for the existence of the sought dual interpolatory schemes. In the following, to simplify the presentation, we distinguish between the odd and even arity cases.

3.1 The odd arity case

Now let us consider the solution of the linear system (35) in [26] for the case where m is an odd integer. The system is defined as follows:

$$M\boldsymbol{a} = \boldsymbol{c}, \qquad M = (\mu_{i,j})_{i,j\in\mathbb{Z}}, \qquad \boldsymbol{c} = (c_i)_{i\in\mathbb{Z}},$$

where

$$\mu_{i,j} = \begin{cases} \varphi\left(\frac{i+1}{2} - j\right), & \text{if } i \in 2m\mathbb{Z}, \\ 1, & \text{if } i \in m(2\mathbb{Z}+1),\ j = \frac{i+1}{2}, \\ 0, & \text{otherwise}, \end{cases}$$

$$c_i = \begin{cases} 1, & \text{if } i = 0, \\ \varphi\left(\frac{i}{2m}\right), & \text{if } i \in m(2\mathbb{Z}+1), \\ 0, & \text{otherwise}. \end{cases}$$

By suppressing the zero rows in both M and $\boldsymbol{c}$ we obtain the equivalent linear system

$$\widehat{M}\boldsymbol{a} = \widehat{\boldsymbol{c}}, \qquad \widehat{M} = (\widehat{\mu}_{i,j})_{i,j\in\mathbb{Z}}, \qquad \widehat{\boldsymbol{c}} = (\widehat{c}_i)_{i\in\mathbb{Z}}, \tag{3}$$

where

$$\widehat{\mu}_{i,j} = \begin{cases} \varphi\left(\frac{im+1}{2} - j\right), & \text{if } \mod(i,2) = 0, \\ 1, & \text{if } \mod(i,2) = 1,\ j = \frac{im+1}{2}, \end{cases}$$

$$\widehat{c}_i = \begin{cases} 1, & \text{if } i = 0, \\ \varphi\left(\frac{i}{2}\right), & \text{if } \mod(i,2) = 1, \\ 0, & \text{otherwise}. \end{cases}$$

The interplay between computations with polynomials and Toeplitz-like matrices can be exploited to recast the solution of the linear system (3) in terms of solving an associated Bezout-like polynomial equation. Indeed from the proof of Theorem 4.1 in [26] one deduces that the entries of the unknown vector $\boldsymbol{a}$ satisfy

$$\begin{cases} \displaystyle\sum_{\alpha\in m(2\mathbb{Z}+1)} \varphi\left(\frac{\alpha}{2m}\right) z^\alpha = \sum_{\alpha\in m(2\mathbb{Z}+1)} a_{\frac{\alpha+1}{2}} z^\alpha, \\ \displaystyle 1 = \sum_{\alpha\in 2m\mathbb{Z}} \sum_{\beta\in\mathbb{Z}} a_\beta\, \varphi\left(\frac{\alpha+1}{2} - \beta\right) z^\alpha, \end{cases} \tag{4}$$

which implies

$$\begin{cases} a_{mi+\frac{m+1}{2}} = \varphi\left(\frac{2i+1}{2}\right), \qquad i \in \mathbb{Z}, \\ \displaystyle 1 - \sum_{\alpha\in 2m\mathbb{Z}} \sum_{\beta\in m\mathbb{Z}+\frac{m+1}{2}} a_\beta \varphi\left(\frac{\alpha+1}{2} - \beta\right) z^\alpha = \\ \displaystyle \qquad \sum_{\alpha\in 2m\mathbb{Z}} \sum_{\substack{\beta\in\mathbb{Z} \\ \mod(m,\beta)\neq\frac{m+1}{2}}} a_\beta \varphi\left(\frac{\alpha+1}{2} - \beta\right) z^\alpha. \end{cases} \tag{5}$$

The system (5) can be rewritten into a more compact form by using the decomposition of $a(z) = \sum_{i\in\mathbb{Z}} a_i z^i$ that involves the sub-symbols of the scheme given by

$$a(z) = \sum_{i=0}^{m-1} a_i(z^m)z^i, \qquad a_\ell(z) = \sum_{i\in\mathbb{Z}} a_{mi+\ell} z^i, \qquad 0 \leqslant \ell \leqslant m-1. \tag{6}$$

Let us introduce the corresponding decomposition of the Laurent polynomial $\phi(z) = \sum_{\ell\in\mathbb{Z}} \varphi\left(\frac{1}{2}+\ell\right) z^\ell$ defined by

$$\phi(z) = \sum_{i=0}^{m-1} \phi_i(z^m)z^{-i}, \qquad \phi_\ell(z) = \sum_{i\in\mathbb{Z}} \varphi\left(\frac{2mi+1}{2} - \ell\right) z^i, \qquad 0 \leqslant \ell \leqslant m-1. \tag{7}$$

The first equation of (5) determines

$$a_{\frac{m+1}{2}}(z) \;=\; \phi(z). \tag{8}$$

Then the second equation can be read as follows

$$1 - a_{\frac{m+1}{2}}(z^m)\phi_{\frac{m+1}{2}}(z^m) \;=\; \sum_{\substack{i=0\\ i\neq\frac{m+1}{2}}}^{m-1} a_i(z^m)\phi_i(z^m)$$

or, equivalently,

$$1 - \phi(z)\phi_{\frac{m+1}{2}}(z) \;=\; \sum_{\substack{i=0\\ i\neq\frac{m+1}{2}}}^{m-1} a_i(z)\phi_i(z). \tag{9}$$

Our computational task is therefore reduced to computing a Laurent polynomial $a(z)$ defined as in (6) satisfying the Bezout-like polynomial equation (9). It is quite natural for convergence and reproducibility issues to impose some other constraints of the form

$$\begin{aligned} &a_i(1) = 1, \quad 0 \leqslant i \leqslant m-1, \\ &a(z) = \left(\frac{1+z+\ldots+z^{m-1}}{m}\right)^d b(z), \end{aligned} \tag{10}$$

for some $b(z) \in \mathbb{R}[z, z^{-1}]$ with $b(\xi_k) \neq 0$, $1 \leqslant k \leqslant m-1$, where $\xi_k = e^{2\pi \mathrm{i} k/m}$. Our proposed construction of such a polynomial $a(z)$ works under some additional assumptions on the input data $\{\varphi((2k+1)/2)\}_{k=-\kappa}^{\kappa-1}$ encoded in the function $\phi(z)$. More specifically:

ASSUMPTION 1 *: We suppose that*

$$1 \;-\; z\phi(z^2) \;=\; (z-1)^d\,\gamma(z), \tag{11}$$

for a certain $\gamma(z) \in \mathbb{R}[z, z^{-1}]$ the ring of Laurent polynomials in z, z^{-1} over $\mathbb{R}$.

ASSUMPTION 2 *: We suppose that*

$$g(z) \;:=\; gcd\left\{\phi_0(z), \ldots, \phi_{\frac{m-1}{2}}(z), \phi_{\frac{m+3}{2}}(z), \ldots, \phi_{m-1}(z)\right\},$$

divides $1 - \phi(z)\phi_{\frac{m+1}{2}}(z)$ and is such that $g(1) \neq 0$.

Assumption 1 is necessary in order to achieve polynomial reproduction of order d. Indeed, by definition of polynomial reproduction, we should have, for every polynomial p of degree $d-1$,

$$p(x) \;=\; \sum_{k\in\mathbb{Z}} p(k)\; f_{\boldsymbol{\delta}}(x-k), \qquad \forall x \in \mathbb{R}.$$

In particular, taking $x = 1/2+i$, $i \in \mathbb{Z}$, the (compactly supported) vector $[f_{\boldsymbol{\delta}}(1/2+\ell) = \varphi(1/2+\ell)]_{\ell\in\mathbb{Z}}$ defines column-wise a (bandlimited) Toeplitz matrix $\mathbf{T}$ such that, for every polynomial p of degree $d-1$,

$$\mathbf{T}\;[p\,(k)]_{k\in\mathbb{Z}} \;=\; \left[p\left(\frac{1}{2}+i\right)\right]_{i\in\mathbb{Z}}.$$

Thus, one can naively define a binary primal interpolating refinement mask as

$$\mathbf{r} \;=\; \{\, r_i \in \mathbb{R},\;\; i \in \mathbb{Z}\,\}, \qquad \text{with} \qquad r_i \;=\; \begin{cases} \varphi\left(\frac{1}{2}+\ell\right), & \text{if } i = 2\ell+1, \\ \delta_{0,\ell}, & \text{if } i = 2\ell, \end{cases}$$

which is not guaranteed to be associated with a convergent subdivision scheme, but it always satisfies

$$\sum_{i\in\mathbb{Z}} r_i\, z^i \;=\; (z+1)^d\, \tilde{\gamma}(z), \tag{12}$$

for some Laurent polynomial $\tilde{\gamma}(z)$. Now it is easy to check that, replacing z with $-z$ in (12) and using (7), one indeed obtains Assumption 1, i.e., $1 - z\phi(z^2) = (z-1)^d\gamma(z)$, with $\gamma(z) = (-1)^d\tilde{\gamma}(-z)$.

Remark 1 The previous observation is also the reason why a suitable way to construct the starting sequence $\{\varphi((2k+1)/2)\}_{k\in\mathbb{Z}}$ is using the mask of a binary primal interpolating scheme with the desired reproduction properties. In the following Example 1 and 2 ((26) and (42) respectively) we choose the mask of the binary 6-point Dubuc-Deslauriers interpolating scheme [11] since it forms the shortest symmetric sequence that guarantees polynomial reproduction of order 6.

As for Assumption 2, requiring $g(z)$ to divide $1 - \phi(z)\phi_{\frac{m+1}{2}}(z)$ is also necessary due to equation (9), while asking $g(1) \neq 0$ is only a sufficient condition as it will be clear in the following. When 1 is a root of $g(z)$, the construction we propose is still viable but a price has to be paid in terms of polynomial reproduction (see Remark 2).

Under Assumption 1 and Assumption 2 our composite approach for computing $a(z)$ proceeds by the following steps. The first step consists of determining the values $a_i^{(s)}(1)$, $0 \leqslant i \leqslant m-1$, $s = 0,\ldots,d-1$. From (10) one gets immediately $a_i^{(0)}(1) = a_i(1) = 1$, $0 \leqslant i \leqslant m-1$. Due to (8) and Assumption 1, we have that

$$1 - z\phi(z^2) \;=\; 1 - z a_{\frac{m+1}{2}}(z^2) \;=\; (z-1)^d\gamma(z),$$

from which we can compute the values of $a_{\frac{m+1}{2}}(z)$ and its derivatives at $z = 1$.

Theorem 1 *Under Assumption 1, it holds*

$$\begin{cases} a_{\frac{m+1}{2}}(1) = \phi(1) = 1, \\ a^{(k)}_{\frac{m+1}{2}}(1) = \phi^{(k)}(1) = (-1)^k \frac{(2k-1)!!}{2^k}, \qquad 1 \leqslant k \leqslant d-1. \end{cases}$$

Proof Substituting $z = \sqrt{w}$ in (11), we get

$$\phi(w) - w^{-1/2} = \frac{(1-\sqrt{w})^d(-1)^{d+1}\gamma(\sqrt{w})}{\sqrt{w}}.$$

The proof easily follows by differentiating this relation at $w = z = 1$. ▫

The remaining unknowns $a_i^{(s)}(1)$, $0 \leqslant i \leqslant m-1$, $i \neq (m+1)/2$, $s = 1, \ldots, d-1$, are computed by solving the linear system obtained by differentiation of (10). Specifically, by differentiating s times the expression of $a(z)$ in (6) with respect to the variable z we find that

$$a^{(s)}(z) = \sum_{i=0}^{m-1} \sum_{p=0}^{s} \frac{a_i^{(p)}(z^m)}{p!} \left(\sum_{j=max\{s-i,p\}}^{s} \binom{s}{j} A_{j,p}(z) \frac{i!}{(i-(s-j))!} z^{i-(s-j)} \right), \quad (13)$$

where $A_{j,p}(z)$ are polynomials defined by Hoppe's formula (see, e.g., [21]) for the differentiation of composite function according to

$$A_{j,p}(z) = \sum_{\ell=0}^{j} \binom{p}{\ell} (-f(z))^{p-\ell} \frac{d^j}{dz^j}(f(z))^{\ell}, \qquad f(z) = z^m.$$

If $\xi_k = e^{2\pi \mathrm{i} k/m}$, $1 \leqslant k \leqslant m-1$, are the m-th roots of unity, then from (10) it follows that $a^{(s)}(\xi_k) = 0$, $s = 0, \ldots, d-1$, $1 \leqslant k \leqslant m-1$. In the view of (13) this implies that the values $a_i^{(s)}(1)$, $0 \leqslant i \leqslant m-1$, $i \neq (m+1)/2$, $s = 1, \ldots, d-1$, can be computed recursively by solving

$$\sum_{i=0}^{m-1} \sum_{p=0}^{s} \frac{a_i^{(p)}(1)}{p!} \left(\sum_{j=max\{s-i,p\}}^{s} \binom{s}{j} A_{j,p}(\xi_k) \frac{i!}{(i-(s-j))!} \xi_k^{i-(s-j)} \right) = 0,$$

with $1 \leqslant k \leqslant m-1$. The system can be expressed in matrix form as

$$m^s \mathcal{M} \left[a_0^{(s)}(1), \ldots, a^{(s)}_{\frac{m-1}{2}}(1), a^{(s)}_{\frac{m+3}{2}}(1), \ldots, a^{(s)}_{m-1}(1) \right]^T = \boldsymbol{b}_s, \quad (14)$$

with

$$\mathcal{M} := \mathcal{D}\left(\xi_1^{(m-1)s}, \ldots, \xi_{m-1}^{(m-1)s}\right) \mathcal{V}(\xi_1, \ldots, \xi_{m-1}).$$

Here $\mathcal{D}(\boldsymbol{v})$, $\boldsymbol{v} = [v_1, \ldots, v_{m-1}]^T$, is the diagonal matrix with diagonal entries v_k, $1 \leqslant k \leqslant m-1$, $\mathcal{V}(\xi_1, \ldots, \xi_{m-1})$ is the Vandermonde matrix with nodes ξ_k, $1 \leqslant k \leqslant m-1$, and

$$(\boldsymbol{b}_s)_k = -\sum_{i=0}^{m-1} \sum_{p=0}^{s-1} \frac{a_i^{(p)}(1)}{p!} \left(\sum_{j=max\{s-i,p\}}^{s} \binom{s}{j} A_{j,p}(\xi_k) \frac{i!}{(i-(s-j))!} \xi_k^{i-(s-j)} \right)$$

$$- \frac{a^{(s)}_{\frac{m+1}{2}}(1)}{s!} A_{s,s}(\xi_k) \xi_k^{\frac{m+1}{2}}, \qquad 1 \leqslant k \leqslant m-1.$$

Since ξ_k, $1 \leqslant k \leqslant m-1$, are distinct and non-zero, the coefficient matrix is nonsingular and $a_i^{(s)}(1)$, $0 \leqslant i \leqslant m-1$, $i \neq (m+1)/2$, are uniquely determined.

Once the quantities $a_i^{(s)}(1)$, $0 \leqslant i \leqslant m-1$, $s = 0, \ldots, d-1$, are calculated, then the sub-symbols $a_i(z)$, $0 \leqslant i \leqslant m-1$, $i \neq (m+1)/2$, can be represented as follows

$$a_i(z) \;=\; 1 + \sum_{j=1}^{d-1} \frac{a_i^{(j)}(1)}{j!}(z-1)^j + (z-1)^d \widehat{a}_i(z) \;=\; \breve{a}_i(z) + (z-1)^d \widehat{a}_i(z), \qquad (15)$$

for suitable $\widehat{a}_i(z) \in \mathbb{R}[z, z^{-1}]$. This representation is exploited in the second step to find a solution of (9). Combining (9) with (15) we obtain

$$1 \;-\; \phi(z)\phi_{\frac{m+1}{2}}(z) \;-\; \sum_{\substack{i=0\\ i\neq\frac{m+1}{2}}}^{m-1} \breve{a}_i(z)\phi_i(z) \;=\; (z-1)^d \sum_{\substack{i=0\\ i\neq\frac{m+1}{2}}}^{m-1} \widehat{a}_i(z)\phi_i(z). \qquad (16)$$

Thus, setting

$$\theta(z) \;:=\; 1 \;-\; \phi(z)\phi_{\frac{m+1}{2}}(z) \;-\; \sum_{\substack{i=0\\ i\neq\frac{m+1}{2}}}^{m-1} \breve{a}_i(z)\phi_i(z), \qquad (17)$$

the condition

$$\theta^{(s)}(1) \;=\; 0, \qquad 0 \leqslant s \leqslant d-1,$$

is needed, but it is always guaranteed by the following result.

Theorem 2 *The function $\theta(z)$ in (17) satisfies $\theta^{(s)}(1) = 0$ for $s = 0, \ldots, d-1$.*

Proof Let us introduce the truncated representation $\breve{a}(z)$ of the symbol $a(z)$, that is,

$$\breve{a}(z) \;=\; \sum_{\substack{i=0\\ i\neq\frac{m+1}{2}}}^{m-1} \breve{a}_i(z^m)z^i \;+\; \phi(z^m)z^{\frac{m+1}{2}},$$

and consider the auxiliary function $q(z) = z^{-\frac{m+1}{2}}\breve{a}(z^2)z\phi(z^2)$. From (11) it follows that $q(z) = z^{-\frac{m+1}{2}}\breve{a}(z^2) - z^{-\frac{m+1}{2}}\breve{a}(z^2)(-1)^d(1-z)^d\gamma(z)$. By construction $\breve{a}(z)$ satisfies relations (10). By using the representation of $\breve{a}(z)$ provided by (10) this gives

$$q(z) \;=\; z^{-\frac{m+1}{2}}\breve{a}(z^2) + \frac{(1-z^m)^d(1+z^m)^d}{(1+z)^d}\widehat{\rho}(z)$$

with $\widehat{\rho}(z) \in \mathbb{R}[z, z^{-1}]$. Observe that

$$z^{-\frac{m+1}{2}}\breve{a}(z^2) \;=\; z^{\frac{m+1}{2}}\phi(z^{2m}) + \sum_{\substack{i=0\\ i\neq\frac{m+1}{2}}}^{m-1} \breve{a}_i(z^{2m})z^{2i-\frac{m+1}{2}},$$

and, hence,

$$q(z) \;=\; z^{\frac{m+1}{2}}\phi(z^{2m}) + \sum_{\substack{i=0\\ i\neq\frac{m+1}{2}}}^{m-1} \breve{a}_i(z^{2m})z^{2i-\frac{m+1}{2}} + \frac{(1-z^m)^d(1+z^m)^d}{(1+z)^d}\widehat{\rho}(z). \qquad (18)$$

Moreover it can be easily seen that the two sets $[0, m-1] \cap \mathbb{N}$ and $\{n \in \mathbb{N}\colon n = 2i - (m+1)/2 \pmod m, 0 \leqslant i \leqslant m-1\}$ coincide. Besides this, by direct multiplication of $a(z^2)$ and $\phi(z^2)$, we can write

$$q(z) \;=\; z^{\frac{1-m}{2}} \left(\sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \breve{a}_i(z^{2m}) \phi_i(z^{2m}) + \phi(z^{2m}) \phi_{\frac{m+1}{2}}(z^{2m}) \right) + \\ + \; z^{\frac{1-m}{2}} \sum_{\substack{0 \leqslant i,j \leqslant m-1 \\ i \neq j}} z^{2(i-j)} \eta_{i,j}(z^{2m}), \tag{19}$$

for suitable Laurent polynomials $\eta_{i,j}(z) \in \mathbb{R}[z, z^{-1}]$. Since $(1-m)/2 \equiv (m+1)/2 \pmod m$ the class of integers congruent to $(1-m)/2$ modulo m is $\{n \in \mathbb{Z}\colon n = (1-m)/2 + \ell m, \ell \in \mathbb{Z}\}$. It follows that $n = (1-m)/2 + 2(i-j)$, $i \neq j, 0 \leqslant i, j \leqslant m-1$, is such that $n \not\equiv (1-m)/2 \pmod m$. Hence, by comparison of classes mod m in (18) and (19), we obtain that

$$z^m \phi(z^{2m}) \;=\; \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \breve{a}_i(z^{2m}) \phi_i(z^{2m}) \;+\; \phi(z^{2m}) \phi_{\frac{m+1}{2}}(z^{2m}) \;+\; (1-z^m)^d \widetilde{\rho}(z),$$

for some $\widetilde{\rho}(z) \in \mathbb{R}[z, z^{-1}]$. From (11) this implies that

$$\sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \breve{a}_i(z^2) \phi_i(z^2) + \phi(z^2) \phi_{\frac{m+1}{2}}(z^2) \;=\; 1 + (1-z)^d \rho(z), \qquad \rho(z) \in \mathbb{R}[z, z^{-1}],$$

which concludes the proof. □

Theorem 2, along with Assumption 2, guarantees the existence of $\widehat{\theta}(z) \in \mathbb{R}[z, z^{-1}]$ such that

$$\theta(z) \;=\; (z-1)^d \, g(z) \, \widehat{\theta}(z). \tag{20}$$

Thus, due to (16), the polynomial corrections $\widehat{a}_i(z)$, $0 \leqslant i \leqslant m-1$, $i \neq (m+1)/2$, must satisfy the Bezout equation

$$\widehat{\theta}(z) \;=\; \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \widehat{a}_i(z) \frac{\phi_i(z)}{g(z)}. \tag{21}$$

Under Assumption 2 the Laurent polynomials $\phi_i(z)/g(z)$, $i \neq (m+1)/2$, are relatively prime and thus equation (21) is solvable. In particular, following [19] every solution of (21) can be written as

$$\widehat{a}_i(z) \;=\; \widetilde{a}_i(z) \;+\; \sum_{\substack{j=i+1 \\ j \neq \frac{m+1}{2}}}^{m-1} H_{i,j}(z) \frac{\phi_j(z)}{g(z)} \;-\; \sum_{\substack{j=0 \\ j \neq \frac{m+1}{2}}}^{i-1} H_{j,i}(z) \frac{\phi_j(z)}{g(z)}, \tag{22}$$

where $\{\widetilde{a}_i(z),\ i \neq (m+1)/2\}$ is a particular solution of (21) and $H_{i,j}(z)$ is any element of $\mathbb{R}[z, z^{-1}]$. Upper bounds for the minimal length of the coefficient vectors associated to the solution of (21) are known a priori [19]. Using these bounds the computation of a particular solution $\widetilde{a}_i(z), 0 \leqslant i \leqslant m-1, i \neq \frac{m+1}{2}$, reduces to solving a square linear system.

Remark 2 If $g(z) = (z-1)^q \widehat{g}(z)$ for some $q \in \mathbb{N}$, $q < d$, and $\widehat{g}(z) \in \mathbb{R}[z, z^{-1}]$, with $\widehat{g}(1) \neq 0$, then the result of Theorem 2 is unchanged but, differently from (20), we can only factorize $\theta(z)$ as

$$\theta(z) \;=\; (z-1)^{d-q}\, g(z)\, \widehat{\theta}(z).$$

Thus, in this case, one should consider

$$a_i(z) \;=\; 1 + \sum_{j=1}^{d-q-1} \frac{a_i^{(j)}(1)}{j!}(z-1)^j + (z-1)^{d-q}\widehat{a}_i(z) \;=\; \breve{a}_i(z) + (z-1)^{d-q}\widehat{a}_i(z), \tag{23}$$

instead of (15), and the illustrated procedure will lead to a symbol $a(z)$ of the form

$$a(z) \;=\; \left(\frac{1+z+\ldots+z^{m-1}}{m}\right)^{d-q} b(z) \tag{24}$$

rather than (10). This means that the scheme associated to $a(z)$ would reproduce only polynomials up to degree $d-q-1$.

Remark 3 Combining (6), (15) and (22), we get

$$\begin{aligned} a(z) \;=\; & \sum_{i=0}^{m-1} \breve{a}_i(z^m) z^i \;+\; (z^m-1)^d \sum_{i=0}^{m-1} \widetilde{a}_i(z^m) z^i \;+ \\ & +\; (z^m-1)^d \sum_{i=0}^{m-1} \left(\sum_{\substack{j=i+1 \\ j \neq \frac{m+1}{2}}}^{m-1} H_{i,j}(z^m) \frac{\phi_j(z^m)}{g(z^m)} - \sum_{\substack{j=0 \\ j \neq \frac{m+1}{2}}}^{i-1} H_{j,i}(z^m) \frac{\phi_j(z^m)}{g(z^m)} \right) z^i, \end{aligned}$$

where on the right-hand-side the unique unknowns are the coefficients of the Laurent polynomials $H_{i,j}(z)$, $i, j = 0, \ldots, m-1$. Knowing the first and the last non-zero coefficients of

$$\sum_{i=0}^{m-1} \breve{a}_i(z^m) z^i \;+\; (z^m-1)^d \sum_{i=0}^{m-1} \widetilde{a}_i(z^m) z^i, \tag{25}$$

it is possible to establish the indices of the first and the last non-zero coefficients of each $H_{i,j}(z)$, so that the range of the powers in (25) and in

$$(z^m-1)^d \sum_{i=0}^{m-1} \left(\sum_{\substack{j=i+1 \\ j \neq \frac{m+1}{2}}}^{m-1} H_{i,j}(z^m) \frac{\phi_j(z^m)}{g(z^m)} - \sum_{\substack{j=0 \\ j \neq \frac{m+1}{2}}}^{i-1} H_{j,i}(z^m) \frac{\phi_j(z^m)}{g(z^m)} \right) z^i,$$

are the same. After that one can start imposing the first (or the last) coefficient of $a(z)$ to be 0, which is a linear condition with respect to the coefficients of all $H_{i,j}(z)$. It is possible then to add linear constraints in the same unknowns in order to annihilate additional coefficients of $a(z)$ as long as the new added linear condition is compatible with the previous ones. Since (22) encodes all possible solutions of (21), when there are no more compatible conditions to be added, the mask with minimal support has been reached. A naive implementation of this strategy has been used in our experiments to produce the interpolatory mask of minimal support. A more general algorithmic description of this strategy should incorporate some preprocessing algebraic computations such as the reduction of the polynomials in reduced form as described in [1].

Remark 4 Of great importance for applications is the case of symbols $a(z)$ that satisfy the symmetry condition $a(z) = za(z^{-1})$. The existence of such a symmetric symbol can be proven under the auxiliary assumption that $\varphi(1/2+\ell) = \varphi(-1/2-\ell)$, $\ell \in \mathbb{N} \cup \{0\}$. Under this assumption, we obtain that the coefficients of $a(z)$ satisfy (4) if and only if the coefficients of $za(z^{-1})$ also satisfy (4). By linearity this implies that the coefficients of $(a(z)+za(z^{-1}))/2$ satisfy (4) too, with $(a(z)+za(z^{-1}))/2$ fulfilling the symmetry condition.

The presented procedure for the odd arity case can be summarized as in Algorithm 1, at the end of which Remark 3 and Remark 4 can be exploited to reduce the support of the resulting mask and/or to obtain a symmetric mask. The following example is used to illustrate our composite approach for the odd arity case. Here we construct the dual ternary interpolating scheme, reproducing quintic polynomials, sharing with the primal binary Dubuc-Deslauriers 6-point scheme the same samples at the half integers, and having symmetric mask with shortest support.

Example 1 We choose $m = 3$, $d = 6$ and (see Remark 1)

$$\varphi\left(\frac{1}{2}+\ell\right) = \begin{cases} \frac{3}{256}, & \text{if } \ell \in \{-3,2\}, \\ -\frac{25}{256}, & \text{if } \ell \in \{-2,1\}, \\ \frac{75}{128}, & \text{if } \ell \in \{-1,0\}, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

These values are taken from the mask of the primal binary 6-point interpolating scheme which reproduces quintic polynomials and it has a basic limit function supported in $[-5,5]$ with best Hölder exponent 2.830074998557687. Its primal ternary counterpart (see, e.g., [24]) reproduces quintic polynomials as well, but it has a basic limit function supported in $[-4,4]$ with best Hölder exponent 2.319856140753624. According to (7), we have

$$\begin{aligned}\phi(z) &= \frac{3}{256\,z^3} - \frac{25}{256\,z^2} + \frac{75}{128\,z} + \frac{75}{128} - \frac{25\,z}{256} + \frac{3\,z^2}{256} \\ &= \phi_0(z^3) + \phi_1(z^3)z^{-1} + \phi_2(z^3)z^{-2},\end{aligned}$$

with

$$\phi_0(z) = \frac{3}{256\,z} + \frac{75}{128}, \qquad \phi_1(z) = \frac{75}{128} + \frac{3\,z}{256}, \qquad \phi_2(z) = -\frac{25}{256} - \frac{25\,z}{256}.$$

In particular, we observe that

$$1 - z\phi(z^2) = -(z-1)^6\,\frac{3z^4+18z^3+38z^2+18z+3}{256z^5} \tag{27}$$

and

$$g(z) = gcd\{\phi_0(z), \phi_1(z)\} = 1.$$

Thus, Assumption 1 and Assumption 2 are satisfied. After solving the linear system (14), we have from (15)

$$a_0(z) = \breve{a}_0(z) + (z-1)^6\hat{a}_0(z), \qquad a_1(z) = \breve{a}_1(z) + (z-1)^6\hat{a}_1(z)$$

Algorithm 1 [odd arity case]

Input: $m \in 2\mathbb{N}+1$ and a compactly supported sequence $\left\{\varphi\left(\frac{2k+1}{2}\right) \in \mathbb{R}\right\}_{k\in\mathbb{Z}}$ such that the Laurent polynomials

$$\phi_\ell(z) = \sum_{i\in\mathbb{Z}} \varphi\left(\tfrac{2mi+1}{2} - \ell\right) z^i, \qquad \ell \in \{0,\dots,m-1\},$$

$$g(z) = gcd\left\{\phi_0(z),\dots,\phi_{\frac{m-1}{2}}(z), \phi_{\frac{m+3}{2}}(z),\dots,\phi_{m-1}(z)\right\},$$

$$\phi(z) = \sum_{i=0}^{m-1} \phi_i(z^m) z^{-i},$$

satisfy

(a) $1 - z\phi(z^2) = (z-1)^d\,\gamma(z)$ for some $d \in \mathbb{N}$, $\gamma \in \mathbb{R}[z,z^{-1}]$;

(b) $g(z)$ divides $1 - \phi(z)\phi_{\frac{m+1}{2}}(z)$;

(c) $g(z) = (z-1)^q \widehat{g}(z)$ for some $0 \leqslant q < d$, and $\widehat{g}(z) \in \mathbb{R}[z,z^{-1}]$ with $\widehat{g}(1) \neq 0$.

Procedure:

(i) set $a_{\frac{m+1}{2}}(z) = \phi(z)$;

(ii) for $s \in \{1,\dots,d-q-1\}$, solve linear system (14) for $\{a_i^{(s)}(1)\}_{i\in\{0,\dots,m-1\}\setminus\{\frac{m+1}{2}\}}$;

(iii) for $i \in \{0,\dots,m-1\}\setminus\{\frac{m+1}{2}\}$, define

$$\breve{a}_i(z) = 1 + \sum_{s=1}^{d-q-1} \frac{a_i^{(s)}(1)}{s!}(z-1)^s;$$

(iv) compute

$$\widehat{\theta}(z) = \frac{(z-1)^{q-d}}{g(z)} \left(1 - \phi(z)\phi_{\frac{m+1}{2}}(z) - \sum_{\substack{i=0\\ i\neq\frac{m+1}{2}}}^{m-1} \breve{a}_i(z)\phi_i(z) \right);$$

(v) follow the strategy in [19] to compute Laurent polynomials $\{\widehat{a}_i(z)\}_{i\in\{0,\dots,m-1\}\setminus\{\frac{m+1}{2}\}}$ such that

$$\widehat{\theta}(z) = \sum_{\substack{i=0\\ i\neq\frac{m+1}{2}}}^{m-1} \widehat{a}_i(z)\frac{\phi_i(z)}{g(z)};$$

Output: the symbol

$$a(z) = \sum_{i=0}^{m-1} a_i(z^m) z^i \quad \text{with} \quad a_i(z) = \breve{a}_i(z) + \widehat{a}_i(z)(z-1)^{d-q},\ i \neq \frac{m+1}{2},$$

of an m-ary dual interpolating subdivision scheme reproducing polynomials up to degree $d-q-1$ and having basic limit function with the given samples at $\mathbb{Z}/2$.

with

$$\breve{a}_0(z) \;=\; 1+\frac{(z-1)}{6}-\frac{5\,(z-1)^2}{72}+\frac{55\,(z-1)^3}{1296}-\frac{935\,(z-1)^4}{31104}+\frac{4301\,(z-1)^5}{186624},$$
$$\breve{a}_1(z) \;=\; 1-\frac{(z-1)}{6}+\frac{7\,(z-1)^2}{72}-\frac{91\,(z-1)^3}{1296}+\frac{1729\,(z-1)^4}{31104}-\frac{8645\,(z-1)^5}{186624},$$

and

$$a_2(z) \;=\; \phi(z).$$

To search for compatible $\widehat{a}_0(z)$ and $\widehat{a}_1(z)$, we first compute

$$\widehat{\theta}(z) \;=\; \frac{8645\,z^3+215471\,z^2-24300\,z+18225}{15925248\,z^3}$$

in such a way that (20) holds, i.e.,

$$(z-1)^6\widehat{\theta}(z) \;=\; 1 \;-\; a_2(z)\,\phi_2(z) \;-\; \sum_{i=0}^{1}\breve{a}_i(z)\,\phi_i(z).$$

Then we look for particular solutions $\widetilde{a}_0(z)$ and $\widetilde{a}_1(z)$ such that

$$\widehat{\theta}(z) \;=\; \widetilde{a}_0(z)\,\phi_0(z) \;+\; \widetilde{a}_1(z)\,\phi_1(z).$$

A possible choice is

$$\widetilde{a}_0(z) \;=\; -\frac{9903400\,z-45544275}{466373376\,z^2},$$
$$\widetilde{a}_1(z) \;=\; \frac{21603855\,z-46560721}{466373376\,z^2}.$$

To obtain a shorter mask, according to Remark 3, we search for a suitable $H_{0,1}(z)$ so that replacing

$$\widehat{a}_0(z) \;=\; \widetilde{a}_0(z) \;+\; H_{0,1}(z)\,\phi_1(z),$$
$$\widehat{a}_1(z) \;=\; \widetilde{a}_1(z) \;-\; H_{0,1}(z)\,\phi_0(z),$$

in the previous expressions of $a_0(z)$ and $a_1(z)$, leads to a symbol

$$a(z) \;=\; a_0(z^3) \;+\; a_1(z^3)\,z \;+\; a_2(z^3)\,z^2$$

with a shorter associated mask. The choice of $H_{0,1}(z)$ that leads to the shortest mask is

$$H_{0,1}(z) \;=\; -\,\frac{844799}{5465313\,z^2},$$

and, after symmetrization (see Remark 4), the resulting symmetric mask $\boldsymbol{a}$ is such that $a_i = 0$ for $i \notin [-14, 15]$, with the first half of its entries being

$$\begin{aligned}&\left\{\frac{16567}{466373376},\,0,\,-\frac{414175}{233186688},\,\frac{224821}{66624768},\,\frac{3}{256},\,\frac{589847}{33312384},\right.\\ &\left.-\frac{83995}{2776032},\,-\frac{25}{256},\,-\frac{2042857}{22208256},\,\frac{1290971}{8328096},\,\frac{75}{128},\,\frac{63152905}{66624768}\right\}.\end{aligned} \tag{28}$$

The basic limit function $f_{\boldsymbol{\delta}}$ related to the mask in (28) is shown in Figure 1, and two examples of interpolating curves can be found in Figure 2. We have that $\mathrm{supp}(f_{\boldsymbol{\delta}}) = [-23/4, 23/4]$ and $f_{\boldsymbol{\delta}} \in \mathcal{C}^{\omega}(\mathbb{R})$ with the best Hölder exponent ω being 3.006664260760692. By construction the corresponding subdivision scheme reproduces polynomials of degree 5.

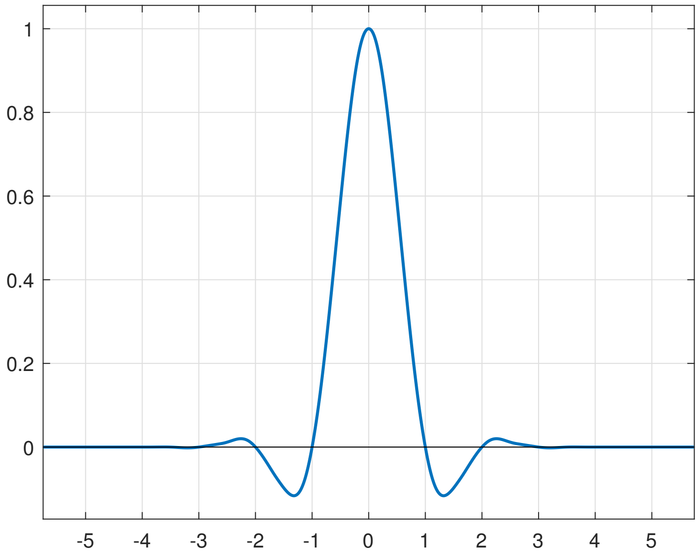

Fig. 1: The graph of the basic limit function $f_{\boldsymbol{\delta}}$ related to the mask in (28).

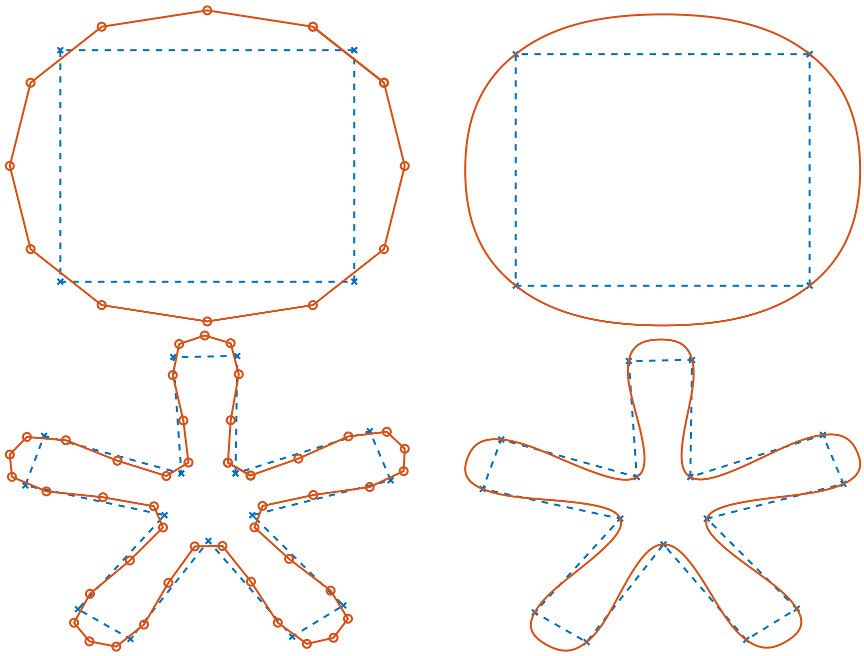

Fig. 2: Two examples of interpolating curves given by the subdivision scheme associated to the mask in (28). On the left, the first level of subdivision starting with the dotted control polygons; on the right, the corresponding interpolating limit curves.

3.2 The even arity case

Let us now consider the solution of the linear system (35) in [26] for the case where m is an even integer. The system is defined as follows:

$$M\boldsymbol{a} \;=\; \boldsymbol{c}, \qquad M \;=\; (\mu_{i,j})_{i,j\in\mathbb{Z}}, \qquad \boldsymbol{c} \;=\; (c_i)_{i\in\mathbb{Z}} \tag{29}$$

where

$$\mu_{i,j} \;=\; \begin{cases} \varphi\left(\dfrac{i+1}{2}-j\right), & \text{if } i \in m\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases} \qquad c_i \;=\; \begin{cases} 1, & \text{if } i = 0, \\ \varphi\left(\dfrac{i}{2m}\right), & \text{if } i \in m(2\mathbb{Z}+1), \\ 0, & \text{otherwise}. \end{cases}$$

By suppressing the zero rows in both M and $\boldsymbol{c}$ we obtain the equivalent linear system

$$\widehat{M}\boldsymbol{a} \;=\; \hat{\boldsymbol{c}}, \qquad \widehat{M} \;=\; (\widehat{\mu}_{i,j})_{i,j\in\mathbb{Z}}, \qquad \hat{\boldsymbol{c}} \;=\; (\widehat{c}_i)_{i\in\mathbb{Z}} \tag{30}$$

where

$$\widehat{\mu}_{i,j} \;=\; \varphi\left(\frac{im+1}{2}-j\right), \quad i,j\in\mathbb{Z}, \qquad \widehat{c}_i \;=\; \begin{cases} 1, & \text{if } i = 0, \\ \varphi\left(\dfrac{i}{2}\right), & \text{if } \mod(i,2) = 1, \\ 0, & \text{otherwise}. \end{cases}$$

According to [26], (29) and (30) can be expressed in functional form as

$$\sum_{\alpha\in m\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} a_\beta \varphi\left(\frac{\alpha+1}{2}-\beta\right) z^\alpha \;=\; 1 + \sum_{\alpha\in m(2\mathbb{Z}+1)} \varphi\left(\frac{\alpha}{2m}\right) z^\alpha$$

which can be rewritten as

$$\begin{aligned} \sum_{\ell\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} a_\beta \varphi\left(\frac{m\ell+1}{2}-\beta\right) z^\ell &= 1 + \sum_{\ell\in\mathbb{Z}} \varphi\left(\frac{2\ell+1}{2}\right) z^{2\ell+1} \\ &= 1 + \sum_{\ell\in\mathbb{Z}} \varphi\left(\ell+\frac{1}{2}\right) z^{2\ell+1}. \end{aligned} \tag{31}$$

Supposing that, as in the odd arity case, Assumption 1 holds for

$$\phi(z) \;=\; \sum_{\ell\in\mathbb{Z}} \varphi\left(\frac{1}{2}+\ell\right) z^\ell,$$

the right-hand side of (31) satisfies

$$\begin{aligned} 1 + \sum_{\ell\in\mathbb{Z}} \varphi\left(\ell+\frac{1}{2}\right) z^{2\ell+1} &= 1 + z\phi(z^2) \\ &= (z+1)^d(-1)^d\gamma(-z) \\ &= (z+1)^d\widetilde{\gamma}(z), \qquad \widetilde{\gamma}(z)\in\mathbb{R}[z,z^{-1}]. \end{aligned}$$

Concerning the representation of the left-hand side of (31), let us introduce the modified subsymbols defined by

$$\widehat{\phi}_\ell(z) \;=\; \sum_{i\in\mathbb{Z}} \varphi\left(\frac{mi+1}{2}-\ell\right) z^i, \qquad 0 \leqslant \ell \leqslant m-1. \tag{32}$$

Notice that if $\phi_\ell(z)$, $0 \leqslant \ell \leqslant m/2-1$, denote the subsymbols of the mask of arity $m/2$, then we have

$$\widehat{\phi}_\ell(z) \;=\; \phi_\ell(z), \qquad \widehat{\phi}_{\ell+m/2}(z) = z\widehat{\phi}_\ell(z), \qquad 0 \leqslant \ell \leqslant m/2-1. \tag{33}$$

In particular this implies that

$$\widehat{\phi}_{\ell+m/2}(-1) \;=\; -\widehat{\phi}_\ell(-1), \qquad \widehat{\phi}_{\ell+m/2}(1) \;=\; \widehat{\phi}_\ell(1), \qquad 0 \leqslant \ell \leqslant m/2-1.$$

Moreover, from $1+z\phi(z^2) = (z+1)^d\widetilde{\gamma}(z)$ and $1-z\phi(z^2) = (z-1)^d\gamma(z)$ one deduces that

$$(z+1)^d\widetilde{\gamma}(z) \;=\; 2-(z-1)^d\gamma(z). \tag{34}$$

Then for the left-hand side of (31) it holds

$$\sum_{\ell\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} a_\beta \varphi\left(\frac{m\ell+1}{2}-\beta\right) z^\ell \;=\; a_0(z^2)\widehat{\phi}_0(z)+\ldots+a_{m-1}(z^2)\widehat{\phi}_{m-1}(z).$$

Hence, it follows that relation (31) can be reformulated as the Bezout-like polynomial equation

$$a_0(z^2)\widehat{\phi}_0(z)+\ldots+a_{m-1}(z^2)\widehat{\phi}_{m-1}(z) \;=\; (z+1)^d\widetilde{\gamma}(z). \tag{35}$$

From (33) it follows that equation (35) can be equivalently rewritten as

$$\left(a_0(z^2)+za_{m/2}(z^2)\right)\phi_0(z)+\ldots+\left(a_{m/2-1}(z^2)+za_{m-1}(z^2)\right)\phi_{m/2-1}(z) = (z+1)^d\widetilde{\gamma}(z). \tag{36}$$

To proceed we consider the following assumption that plays the same role as Assumption 2 in the odd arity case.

ASSUMPTION 3 *: We suppose that*

$$g(z) \;:=\; gcd\left\{\phi_0(z),\ldots,\phi_{m/2-1}(z)\right\},$$

divides $(z+1)^d\widetilde{\gamma}(z)$ *and is such that* $g(\pm 1) \neq 0$.

Requiring $g(z)$ to divide $(z+1)^d\widetilde{\gamma}(z)$ and satisfy $g(1) \neq 0$ is clearly a necessary condition because of (36) and Assumption 1. Condition $g(-1) \neq 0$ however is only sufficient to construct $a(z)$ as in (10) and, when it is not satisfied, a price has to be paid in terms of polynomial reproduction (see Remark 5).

Under Assumption 3 the solution to equation (35) can be found similarly to the odd arity case. Specifically, at the first step the unknowns $a_i^{(s)}(1)$, $0 \leqslant i \leqslant m-1$, $s = 1,\ldots,d-1$, are computed by solving a Vandermonde linear system. The system is formed as follows. The first $m-1$ equations are obtained by differentiation of (10) complemented with relation (35). The last equation is found by imposing the property (34) on the left hand-side of (35). If $\xi_k = e^{2\pi \mathrm{i} k/m}$, $1 \leqslant k \leqslant m$, denote the m-th roots of unity, then the system is of the form

$$m^s\,\mathcal{D}\left(\xi_1^{(m-1)s}, \ldots, \xi_{m-1}^{(m-1)s}, (2/m)^s\right) \mathcal{V}(\xi_1, \ldots, \xi_{m-1}, \boldsymbol{\phi}) \left[a_0^{(s)}(1), \ldots, a_{m-1}^{(s)}(1)\right]^T \;=\; \boldsymbol{b}_s, \quad (37)$$

where $\mathcal{D}(\boldsymbol{v})$ is the diagonal matrix with diagonal entries v_k and $\mathcal{V}(\xi_1, \ldots, \xi_{m-1}, \boldsymbol{\phi})$ is a Vandermonde-like matrix with nodes ξ_k of the form

$$\mathcal{V}(\xi_1, \ldots, \xi_{m-1}, \boldsymbol{\phi}) = \begin{bmatrix} \xi_1^0 & \cdots\cdots & \xi_1^{m-1} \\ \vdots & \vdots\ \vdots\ \vdots & \vdots \\ \xi_{m-1}^0 & \cdots\cdots & \xi_{m-1}^{m-1} \\ (\boldsymbol{\phi})_1 & \cdots\cdots & (\boldsymbol{\phi})_m \end{bmatrix} \quad \text{with} \quad (\boldsymbol{\phi})_j = \widehat{\phi}_{j-1}(1),\ 1 \leqslant j \leqslant m.$$

The solvability of the systems (37) follows from the next lemma.

Lemma 1 *For any* $\boldsymbol{v} = [v_1, \ldots, v_{m/2}] \in \mathbb{R}^{m/2}$ *and* $\boldsymbol{w} = [\boldsymbol{v}, \boldsymbol{v}] \in \mathbb{R}^m$, *it holds*

$$\det \mathcal{V}(\xi_1, \ldots, \xi_{m-1}, \boldsymbol{w}) = \frac{2}{m} \prod_{1 \leqslant i,j \leqslant m} (\xi_i - \xi_j) \sum_{i=1}^{m/2} v_i.$$

Proof By Laplace's rule we find that $\det \mathcal{V}(\xi_1, \ldots, \xi_{m-1}, \boldsymbol{w})$ is linear in $v_1, \ldots, v_{m/2}$. If $\sum_{i=1}^{m/2} v_i = 0$, then $\det \mathcal{V}(\xi_1, \ldots, \xi_{m-1}, \boldsymbol{w}) = 0$ since the all-ones vector $\mathbf{1}$ belongs to the kernel of the matrix. This implies that $\det \mathcal{V}(\xi_1, \ldots, \xi_{m-1}, \boldsymbol{w}) = \gamma \sum_{i=1}^{m/2} v_i$ for a suitable γ. The value of γ can be determined by setting $\boldsymbol{w} = \mathbf{1}$ which amounts to consider the customary Vandermonde matrix. ▫

As a consequence of Assumption 1 with $z = 1$, we have that

$$\sum_{i=1}^{m/2} \widehat{\phi}_{i-1}(1) \;=\; \sum_{i=1}^{m/2} \phi_{i-1}(1) \;=\; 1.$$

Therefore, by Lemma 1, the coefficient matrix in (37) is non-singular and $a_i^{(s)}(1)$, $0 \leqslant i \leqslant m-1$, are uniquely determined. Once these quantities are computed, the sub-symbols can be represented as follows

$$\begin{aligned} a_i(z) \;&=\; \breve{a}_i(z) + (z-1)^d \widehat{a}_i(z), \qquad 0 \leqslant i \leqslant m-1, \\ \breve{a}_i(z) \;&=\; 1 + \sum_{j=1}^{d-1} \frac{a_i^{(j)}(1)}{j!} (z-1)^j, \end{aligned} \quad (38)$$

for suitable $\widehat{a}_i(z) \in \mathbb{R}[z, z^{-1}]$. This representation is exploited in the second step to find a solution of (35). If we set

$$\theta(z) \;:=\; (z+1)^d \widetilde{\gamma}(z) - \sum_{j=0}^{m-1} \breve{a}_j(z^2) \widehat{\phi}_j(z), \quad (39)$$

by using similar arguments as in the proof of Theorem 2, together with Assumption 3, it is shown that there exists $\widehat{\theta}(z) \in \mathbb{R}[z, z^{-1}]$ such that

$$\theta(z) \;=\; (z^2-1)^d\, g(z)\, \widehat{\theta}(z). \quad (40)$$

In this way equation (35) can be simplified as follows

$$\widehat{a}_0(z^2)\frac{\widehat{\phi}_0(z)}{g(z)} \;+\; \ldots \;+\; \widehat{a}_{m-1}(z^2)\frac{\widehat{\phi}_{m-1}(z)}{g(z)} \;=\; \widehat{\theta}(z),$$

which yields to its reduced analogue

$$\left(\widehat{a}_0(z^2)+z\widehat{a}_{m/2}(z^2)\right)\frac{\phi_0(z)}{g(z)}+\ldots+\left(\widehat{a}_{m/2-1}(z^2)+z\widehat{a}_{m-1}(z^2)\right)\frac{\phi_{m/2-1}(z)}{g(z)} \;=\; \widehat{\theta}(z).$$

By setting $\widetilde{a}_i(z)=\widehat{a}_i(z^2)+z\widehat{a}_{i+m/2}(z^2)$, $0\leqslant i\leqslant m/2-1$, thanks to Assumption 3 we deduce that the equation

$$\widetilde{a}_0(z)\frac{\phi_0(z)}{g(z)} \;+\; \ldots \;+\; \widetilde{a}_{m/2-1}(z)\frac{\phi_{m/2-1}(z)}{g(z)} \;=\; \widehat{\theta}(z) \tag{41}$$

is solvable and every solution can be written as

$$\overline{a}_i(z) \;=\; \widetilde{a}_i(z)+\sum_{j=i+1}^{m/2-1}H_{i,j}(z)\frac{\phi_j(z)}{g(z)}-\sum_{j=0}^{i-1}H_{j,i}(z)\frac{\phi_j(z)}{g(z)},$$

where $\widetilde{a}_i(z)$ is a particular solution of (41) and $H_{i,j}(z)$ is any element of $\mathbb{R}[z,z^{-1}]$.

Remark 5 If $g(z)=(z+1)^q\widehat{g}(z)$ for some $q\in\mathbb{N}$, $q<d$, and $\widehat{g}(z)\in\mathbb{R}[z,z^{-1}]$, with $\widehat{g}(-1)\neq 0$, then one can only factorize $\theta(z)$ as

$$\theta(z) \;=\; (z^2-1)^{d-q}\,(z-1)^q\,g(z)\,\widehat{\theta}(z),$$

rather than (40). In this case, similarly to what was done in Remark 2, one should consider (23) instead of (38) and the illustrated procedure will lead to a symbol $a(z)$ of the form (24) instead of (10). This means that the subdivision scheme associated to $a(z)$ would reproduce only polynomials up to degree $d-q-1$.

Remark 6 For $m=2$ equation (35) becomes

$$(a_0(z^2)+za_1(z^2))\phi(z) \;=\; 1+z\phi(z^2)$$

which implies that the first and the last non-zero elements of $a(z)$ must be equal to 1. It follows that the associated subdivision scheme cannot be convergent [26].

The presented procedure for the even arity case can be summarized as in Algorithm 2, after which similar arguments as in Remark 3 and Remark 4 can be exploited, to reduce the support of the resulting mask and/or to obtain a symmetric one. Next, we conclude with the illustration of our composite approach, in the even arity case, by means of a computational example where we construct the dual quaternary interpolating scheme, reproducing quintic polynomials, sharing with the primal binary Dubuc-Deslauriers 6-point scheme the same samples at the half integers, and having symmetric mask with shortest support.

Algorithm 2 [even arity case]

Input: $m \in 2\mathbb{N}\backslash\{2\}$ and a compactly supported sequence $\left\{\varphi\left(\frac{2k+1}{2}\right) \in \mathbb{R}\right\}_{k\in\mathbb{Z}}$ such that the Laurent polynomials

$$\phi_\ell(z) = \sum_{i\in\mathbb{Z}} \varphi\left(\tfrac{mi+1}{2} - \ell\right) z^i, \qquad \ell \in \{0,\dots,m/2-1\},$$

$$g(z) = gcd\left\{\phi_0(z),\dots,\phi_{m/2-1}(z)\right\},$$

$$\phi(z) = \sum_{i=0}^{m/2-1} \phi_i(z^m) z^{-i},$$

satisfy

(a) $1 - z\phi(z^2) = (z-1)^d\,\gamma(z)$ for some $d \in \mathbb{N}$, $\gamma \in \mathbb{R}[z,z^{-1}]$;

(b) $g(z)$ divides $1 + z\phi(z^2)$;

(c) $g(z) = (z+1)^q \widehat{g}(z)$ for some $0 \leqslant q < d$, and $\widehat{g}(z) \in \mathbb{R}[z,z^{-1}]$ with $\widehat{g}(-1) \neq 0$.

Procedure:

(i) for $s \in \{1,\dots,d-q-1\}$, solve linear system (37) for $\{a_i^{(s)}(1)\}_{i=0,\dots,m-1}$;

(ii) for $i \in \{0,\dots,m-1\}$, define

$$\breve{a}_i(z) = 1 + \sum_{s=1}^{d-q-1} \frac{a_i^{(s)}(1)}{s!}(z-1)^s;$$

(iii) compute

$$\widehat{\theta}(z) = \frac{(z^2-1)^{q-d}}{(z-1)^q\, g(z)} \left(1 + z\phi(z^2) - \sum_{i=0}^{m/2-1} \left(\breve{a}_i(z^2) + z\breve{a}_{m/2+i}(z^2)\right)\phi_i(z) \right);$$

(iv) follow the strategy in [19] to compute Laurent polynomials $\{\widetilde{a}_i(z)\}_{i=0,\dots,m/2-1}$ such that

$$\widehat{\theta}(z) = \sum_{i=0}^{m/2-1} \widetilde{a}_i(z)\frac{\phi_i(z)}{g(z)};$$

(v) for $i \in \{0,\dots,m/2-1\}$, compute the Laurent polynomials $\widehat{a}_i(z)$ and $\widehat{a}_{m/2+i}(z)$ uniquely defined by the relation

$$\widetilde{a}_i(z) = \widehat{a}_i(z^2) + z\widehat{a}_{m/2+i}(z^2).$$

Output: the symbol

$$a(z) = \sum_{i=0}^{m-1} a_i(z^m)z^i \quad \text{with} \quad a_i(z) = \breve{a}_i(z) + \widehat{a}_i(z)(z-1)^{d-q},$$

of an m-ary dual interpolating subdivision scheme reproducing polynomials up to degree $d-q-1$ and having basic limit function with the given samples at $\mathbb{Z}/2$.

Example 2 We choose $m = 4$, $d = 6$ and (see Remark 1)

$$\varphi\left(\frac{1}{2}+\ell\right) = \begin{cases} \frac{3}{256}, & \text{if } \ell \in \{-3, 2\}, \\ -\frac{25}{256}, & \text{if } \ell \in \{-2, 1\}, \\ \frac{75}{128}, & \text{if } \ell \in \{-1, 0\}, \\ 0, & \text{otherwise.} \end{cases} \tag{42}$$

These values are again taken from the mask of the primal binary 6-point interpolating scheme which reproduces quintic polynomials and it has a basic limit function supported in $[-5, 5]$ with best Hölder exponent 2.830074998557687. Its primal quaternary counterpart (see, e.g., [24]) reproduces quintic polynomials as well, but it has a basic limit function supported in $[-11/3, 11/3]$ with best Hölder exponent 2.099550050039848. In view of (32) and (33), we have

$$\widehat{\phi}_0(z) = \phi_0(z) = -\frac{25}{256\,z} + \frac{75}{128} + \frac{3\,z}{256},$$

$$\widehat{\phi}_1(z) = \phi_1(z) = \frac{3}{256\,z} + \frac{75}{128} - \frac{25\,z}{256},$$

$$\widehat{\phi}_2(z) = z\phi_0(z) = -\frac{25}{256} + \frac{75\,z}{128} + \frac{3\,z^2}{256},$$

$$\widehat{\phi}_3(z) = z\phi_1(z) = \frac{3}{256} + \frac{75\,z}{128} - \frac{25\,z^2}{256}.$$

Assumption 1 is satisfied since $\phi(z)$ is the same as in Example 1 (27), while Assumption 3 holds because

$$\phi_1(z) = \phi_0(1/z) \quad \Longrightarrow \quad gdc\{\phi_0(z), \phi_1(z)\} = 1.$$

After solving the linear system (37), from (38) we obtain

$$a_i(z) = \breve{a}_i(z) + (z-1)^d \widehat{a}_i(z), \qquad 0 \leqslant i \leqslant 3,$$

with

$$\breve{a}_0(z) = 1 + \frac{(z-1)}{8} - \frac{7\,(z-1)^2}{128} + \frac{35\,(z-1)^3}{1024} - \frac{805\,(z-1)^4}{32768} + \frac{4991\,(z-1)^5}{262144},$$

$$\breve{a}_1(z) = 1 - \frac{(z-1)}{8} + \frac{9\,(z-1)^2}{128} - \frac{51\,(z-1)^3}{1024} + \frac{1275\,(z-1)^4}{32768} - \frac{8415\,(z-1)^5}{262144},$$

$$\breve{a}_2(z) = 1 - \frac{3\,(z-1)}{8} + \frac{33\,(z-1)^2}{128} - \frac{209\,(z-1)^3}{1024} + \frac{5643\,(z-1)^4}{32768} - \frac{39501\,(z-1)^5}{262144},$$

$$\breve{a}_3(z) = 1 - \frac{5\,(z-1)}{8} + \frac{65\,(z-1)^2}{128} - \frac{455\,(z-1)^3}{1024} + \frac{13195\,(z-1)^4}{32768} - \frac{97643\,(z-1)^5}{262144}.$$

To search for compatible $\widehat{a}_0(z)$, $\widehat{a}_1(z)$, $\widehat{a}_2(z)$ and $\widehat{a}_3(z)$, we first compute

$$\widehat{\theta}(z) = \frac{3}{256\,z^5} - \frac{7}{256\,z^3} + \frac{5086563}{16777216\,z} - \frac{580643}{16777216}$$

such that, according to (39) and (40),

$$(z^2-1)^6\widehat{\theta}(z) \;=\; (z+1)^6\,\widetilde{\gamma}(z) \;-\; \sum_{i=0}^{3} \breve{a}_i(z^2)\,\widehat{\phi}_i(z),$$

with

$$\widetilde{\gamma}(z) \;=\; \frac{3}{256\,z^5} - \frac{9}{128\,z^4} + \frac{19}{128\,z^3} - \frac{9}{128\,z^2} + \frac{3}{256\,z},$$

due to (34). Then we search for $\widetilde{a}_0(z)$ and $\widetilde{a}_1(z)$ that solve the reduced Bezout equation in (41), i.e.,

$$\widehat{\theta}(z) \;=\; \widetilde{a}_0(z)\,\phi_0(z) \;+\; \widetilde{a}_1(z)\,\phi_1(z). \tag{43}$$

A possible choice is

$$\begin{aligned}\widetilde{a}_0(z) \;&=\; \frac{2126507351527}{157810688\,z} - \frac{176620228675}{78905344},\\ \widetilde{a}_1(z) \;&=\; \frac{1}{z^4} - \frac{50}{z^3} + \frac{2506}{z^2} - \frac{2118539063675}{157810688\,z} - \frac{21194427441}{78905344}.\end{aligned}$$

Once we have a solution of (43), we search for

$$\begin{aligned}\overline{a}_0(z) \;&=\; \widetilde{a}_0(z) + H_{0,1}(z)\,\phi_1(z),\\ \overline{a}_1(z) \;&=\; \widetilde{a}_1(z) - H_{0,1}(z)\,\phi_0(z),\end{aligned}$$

so that $\{\widehat{a}_k(z)\}_{k=0,\ldots,3}$ fulfilling

$$\overline{a}_i(z) \;=\; \widehat{a}_i(z^2) + z\,\widehat{a}_{i+2}(z^2), \qquad i \in \{0,1\},$$

lead to a symbol $a(z)$ satisfying $a(z) = za(z^{-1})$. For example, the choice

$$\begin{aligned}H_{0,1}(z) = &-\frac{7064809147}{308224\,z} + \frac{281633113}{616448\,z^2} - \frac{2817667}{308224\,z^3} + \frac{119853}{616448\,z^4}\\ &+\frac{7302199}{596413440\,z^5} - \frac{3127}{1232896\,z^6} + \frac{947}{1331280\,z^7}\end{aligned}$$

leads to

$$\begin{aligned}\overline{a}_0(z) = &\frac{39501}{262144\,z} - \frac{4991}{262144\,z^2} - \frac{5643}{262144\,z^3} + \frac{24415849}{4362338304\,z^4} + \frac{394938757}{40715157504\,z^5}\\ &-\frac{61600783}{43623383040\,z^6} + \frac{15760091}{40715157504\,z^7} + \frac{947}{113602560\,z^8}\\ \overline{a}_1(z) = &\frac{97643}{262144\,z} + \frac{8415}{262144\,z^2} - \frac{7917}{262144\,z^3} - \frac{49446367}{7270563840\,z^4} + \frac{482174039}{40715157504\,z^5}\\ &+\frac{116624327}{43623383040\,z^6} - \frac{27054815}{40715157504\,z^7} + \frac{4735}{68161536\,z^8},\end{aligned}$$

and so

$$\begin{aligned}\widehat{a}_0(z) \;&=\; -\frac{4991}{262144\,z} + \frac{24415849}{4362338304\,z^2} - \frac{61600783}{43623383040\,z^3} + \frac{947}{113602560\,z^4},\\ \widehat{a}_1(z) \;&=\; \frac{8415}{262144\,z} - \frac{49446367}{7270563840\,z^2} + \frac{116624327}{43623383040\,z^3} + \frac{4735}{68161536\,z^4},\end{aligned}$$

$$\widehat{a}_2(z) = \frac{39501}{262144\,z} - \frac{5643}{262144\,z^2} + \frac{394938757}{40715157504\,z^3} + \frac{15760091}{40715157504\,z^4},$$

$$\widehat{a}_3(z) = \frac{97643}{262144\,z} - \frac{7917}{262144\,z^2} + \frac{482174039}{40715157504\,z^3} - \frac{27054815}{40715157504\,z^4}.$$

Replacing the previous expressions in the above equations of $a_0(z)$, $a_1(z)$, $a_2(z)$ and $a_3(z)$ and using

$$a(z) = a_0(z^4) + a_1(z^4)\,z + a_2(z^4)\,z^2 + a_3(z^4)\,z^3,$$

the first half of the resulting symmetric mask $\boldsymbol{a}$ is

$$\left\{ \frac{947}{113602560}, \frac{4735}{68161536}, \frac{15760091}{40715157504}, -\frac{27054815}{40715157504}, -\frac{63782671}{43623383040}, \frac{98441927}{43623383040}, \frac{42911173}{5816451072}, \frac{92071847}{5816451072}, \frac{154804477}{10905845760}, -\frac{79247347}{3635281920}, -\frac{143318065}{1938817024}, -\frac{215643011}{1938817024}, -\frac{71706399}{969408512}, \frac{4869166267}{43623383040}, \frac{2428957997}{5816451072}, \frac{4331006815}{5816451072}, \frac{528433771}{545292288} \right\}. \tag{44}$$

The basic limit function $f_{\boldsymbol{\delta}}$ related to this mask is shown in Figure 3, and two examples of interpolating curves can be found in Figure 4. We have that $\mathrm{supp}(f_{\boldsymbol{\delta}}) = [-11/2, 11/2]$ and, via joint spectral radius techniques, one can prove that $f_{\boldsymbol{\delta}} \in \mathcal{C}^{\omega}(\mathbb{R})$ with the best Hölder exponent ω being 3.050871089158321. By construction the corresponding subdivision scheme reproduces polynomials of degree 5.

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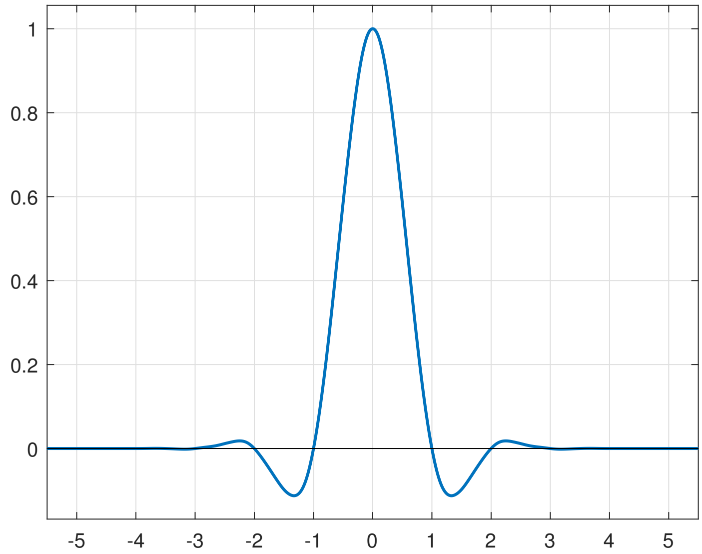

Fig. 3: The graph of the basic limit function f_δ related to the mask in (44).

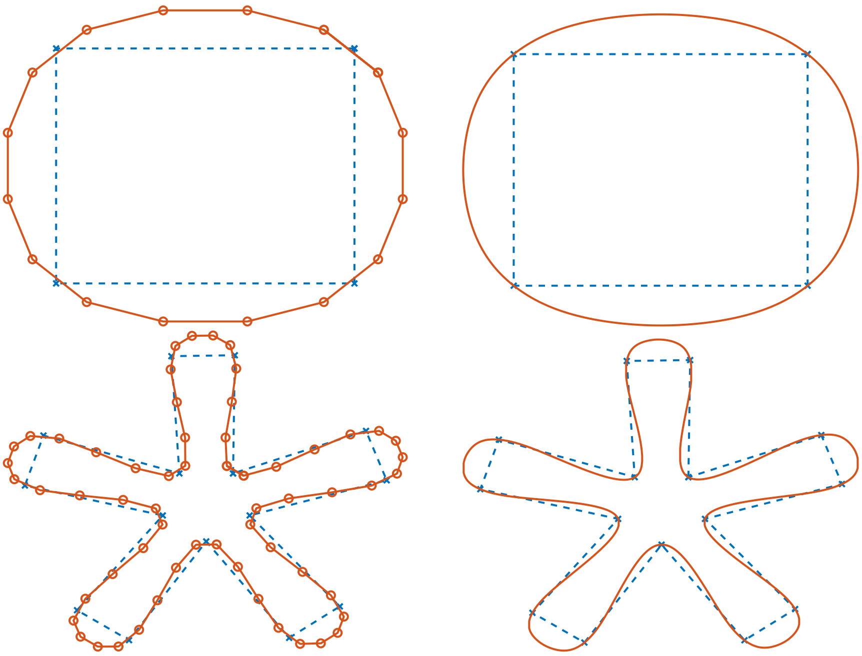

Fig. 4: Two examples of interpolating curves given by the subdivision scheme associated to the mask in (44). On the left, the first level of subdivision starting with the dotted control polygons; on the right, the corresponding interpolating limit curves.

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