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On the solvability of a class of second order degenerate operators

Serena Federico and Alberto Parmeggiani

Abstract In this paper we will be concerned with the problem of solvability of second order degenerate operators that are not of principal type. We will describe some recent results we have obtained about local solvability in the Sobolev spaces of a class of degenerate operators which is an elaboration of the class considered by Colombini-Cordaro-Pernazza (in turn, an elaboration of the adjoint of the Kannai operator).
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Dedicated to Luigi Rodino

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1 Introduction

In this paper we will survey some results concerning the solvability (in L^2-based Sobolev spaces) of an interesting class of degenerate operators, whose symbol may be complex valued with a real part which may change sign. Such a class is interesting and natural, for it is built upon the operator

$$P = D_{x_2} x_1 D_{x_2} + i D_{x_1}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad D = -i\partial, \tag{1.1}$$

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which is the (formal) adjoint of the important example of Kannai [18]

$$K = P^* = D_{x_2} x_1 D_{x_2} - iD_{x_1},$$

of an operator which is hypoelliptic but not locally solvable at $x_1 = 0$. The class of operators of the kind (1.1) was then extended by the adjoints of the class considered by Beals and Fefferman in [2] (in their study of hypoellipticity of degenerate operators) and next by Colombini, Cordaro and Pernazza in [4] (in their study of the solvability of operators of the form $X(x,D)^* f X(x,D) + iY(x,D) + a_0$, where iX, iY are *real* vector-fields, f is real analytic and a_0 is smooth). The class we consider contains operators of the kind (1.1) but also operators whose formal adjoint is not hypoelliptic (see [29]) so that the solvability is *not* a consequence of the hypoellipticity of the adjoint.

The study of solvability of linear degenerate PDEs (even after Dencker's resolution of the Nirenberg-Treves conjecture on condition (Ψ), see [16, 17]) is still largely open and unsettled, especially for operators not of principal type. Many are the examples, coming from several complex variables or linearization of nonlinear operators involved in physical and geometrical problems, of degenerate operators that are interesting to study. One may look at [20] for some history and basic problems on local solvability and at [29] for some history, survey, bibliography and considerations related to the solvability of degenerate operators along with some results (see also [28]) related to the solvability of operators with multiple transversal symplectic characteristics.

It is important to keep in mind that the hypoellipticity of an operator P implies the local solvability of P^* (or ${}^tP$), thus the issue of local solvability is very much related to that of hypoellipticity and hence also to that of propagation of singularities (see, e.g., [7, 14, 16, 17, 21, 28, 29]). However, Kannai's example shows that there are operators that are locally solvable but *not* hypoelliptic, and that the operation of taking adjoints may preserve local solvability but may also destroy hypoellipticity.

Interesting solvability results for degenerate operators of the form $P_1P_2 + Q$ (where P_1, P_2, Q are first order operators) with double characteristics are given in a paper by Helffer [13] (in which he actually studies the problem of the hypoellipticity with a loss of 1 derivative) and by Treves [33] (in which he studies the solvability of an operator of the form $X_1(x,D)X_2(x,D) + iY(x,D) + a_0$, where iX_1, iX_2, iY are *real* vector fields, proving that under certain conditions one has solvability with a loss of one derivative), and for operators of the form sums of squares $\sum_{j=1}^N X_j^* X_j$ by Kohn [19], in which the vector fields involved are *complex* (see also Treves [34] for the study of the solvability of vector fields with critical points). Furthermore, it is important to mention the recent work due to Dencker [5, 6] concerned with necessary conditions for the solvability of degenerate operators whose principal symbol may be *complex* but with a non-radial involutive double-characteristic set (that is, the characteristic points where the principal symbol and its differential vanish is a non-radial manifold which is involutive), based on the behavior of *limit bicharacteristics* and the so-called *sub*-Ψ condition that were introduced by Mendoza and Uhlmann in [23] (see also [22]), and the work of Müller (see [24, 25]) for operators whose principal symbol is *complex* with double characteristics, where in [25] the necessary conditions for solvability are described in terms of "dissipative pairs" (a condition related to the Hessian of the *principal symbol only*, at double characteristic points; hence the condition does not "see" the information carried by the other invariants one has at a double point) and in [24] he shows the sufficiency of such condition in the important instance of left-invariant second order operators on the Heisenberg group (thus having a particular algebraic structure on the lower-order terms). For sums of squares of left-invariant vector fields on a Lie group, one has extensive work by Müller, Ricci and Peloso (see for instance [26, 27, 31]), for operators with double involutive characteristics one has an interesting result by Popivanov [32], and for the semi-global solvability of operators with transversal multiple symplectic characteristics one has the results by Parenti-Parmeggiani (see [28] and also [29]).

We next introduce the class of operators we shall be considering here. The class is subdivided in three types, that will be described in the subsequent sections. The first kind of operators, which is a direct generalization of the class considered by Colombini, Cordaro and Pernazza, was introduced in [11] and studied also in [12]. Notice that an interesting and meaningful variation of it, with non-smooth coefficients and invariant under affine transformations, was studied by Federico in [8]. The other two kinds have been introduced in [12] and in [10].

Let $\Omega \subset \mathbb{R}^n$ be open and let $N \geq 1$ be an integer. We consider the following operators

$$(MT) \qquad P_1 = \sum_{j=1}^{N} X_j^* f X_j + X_{N+1} + iX_0 + a_0,$$

$$(ST) \qquad P_2 = \sum_{j=1}^{N} X_j^* f_j X_j + X_{N+1} + a_0,$$

$$(MST) \qquad P_3 = \sum_{j=1}^{N} X_j^* f_j X_j + X_{N+1} + iX_0 + a_0,$$

where (MT) stands for "mixed type", (ST) for "Schrödinger type" and (MST) for "mixed Schrödinger type", respectively. The above operators are constructed from a given *system* $(X_0, X_1, \dots, X_{N+1})$ *of first order homogeneous partial differential operators* $X_j(x, D)$ (that we shall also call, somewhat improperly, "vector fields"; as a matter of fact, the iX_j are indeed vector fields). The symbols of X_{N+1} and X_0 will be *always supposed to be real*, whereas those of $X_1, \dots, X_N$ will be supposed to be *real* in the (MT) and (MST) cases, respectively, and *complex* in the (ST) case. We shall denote by $X_j(x, \xi) = \langle \alpha_j(x), \xi \rangle$ the symbols of the X_j, where $\alpha_j \in C^\infty(\Omega; \mathbb{R}^n)$, $0 \leq j \leq N+1$, in the (MT) and (MST) cases, and $\alpha_{N+1} \in C^\infty(\Omega; \mathbb{R}^n)$ and $\alpha_j \in C^\infty(\Omega; \mathbb{C}^n)$, $1 \leq j \leq N$, in the (ST) case. The functions $f, f_1, \dots, f_N \in C^\infty(\Omega; \mathbb{R})$ are assumed to be smooth, and may vanish and/or change sign somewhere on Ω, where in particular for f we assume that $f^{-1}(0) \neq \emptyset$ and $df\big|_{f^{-1}(0)} \neq 0$. Finally $a_0 \in C^\infty(\Omega;\mathbb{C})$. Notice that a main difference among the operators P_1, P_2 and P_3 is in the symbol of the first order part (*the subprincipal symbol* $\mathrm{sub}(P)(x,\xi)$, that at points (x,ξ) such that $X_j(x,\xi) = 0$ for all $1 \leq j \leq N$ is given by $X_{N+1}(x,\xi) + iX_0(x,\xi)$): in the (MT) case $\mathrm{sub}(P_1)$ is *complex* so that P_1 is a sort of parabolic-Schrödinger operator, in the (ST) case $\mathrm{sub}(P_2)$ is *real* so that P_2 is a sort of degenerate Schrödinger operator, and in the (MST) case $\mathrm{sub}(P_3)$ is again complex but with a degeneracy in the principal part which may depend on the various functions f_j and may be thought of as a blend of the previous two types.

Our interest in these classes, that are invariantly defined, comes from the interplay between the degeneracy due to the vanishing, and the (assumed or possible) change of sign, of the various f and f_j involved, and the *characteristic set* Σ *of the system of vector fields* $(X_0, X_1, \ldots, X_N)$, defined as

$$\Sigma = \bigcap_{j=0}^{N} \Sigma_j \subset T^*\Omega \setminus 0, \quad \Sigma_j = \{(x,\xi) \in T^*\Omega \setminus 0;\ X_j(x,\xi) = 0\},\ 0 \leq j \leq N \quad (1.2)$$

(recall that $T^*\Omega\setminus 0$ denotes the cotangent bundle of Ω with the zero-section removed). Note that in our setting Σ *does not* depend on the characteristics of X_{N+1}. The reason why this is the case will be made clear in the sequel.

In the first, second and third section, respectively, we will state the hypotheses on the class of operators P_1, P_2 and P_3, respectively, and state the related solvability results, explaining the main solvability estimates that give, in some cases, "better" solvability results (if compared to L^2 to L^2 local solvabiity, see Definition 2 below). For each class of operators we shall also give a number of examples.

We remark once more that our classes of operators contain operators whose formal adjoint is *not* hypoelliptic (see [29]; see also [1, 35] for the study of hypoellipticity of degenerate operators whose coefficients may change sign) so that our solvability results are *not* a consequence of the hypoellipticity of the adjoints.

We close the introduction by recalling the definition of *local solvability* and by giving the definition of H^s *to* $H^{s'}$ *local solvability* we will be interested in, where $H^s = H^s(\mathbb{R}^n)$ is the L^2-based Sobolev space of order $s \in \mathbb{R}$, whose norm will be denoted by $\|\cdot\|_s$.

Definition 1 (**Local solvability**) Let P be an mth-order partial differential operator with smooth coefficients on an open set $\Omega \subset \mathbb{R}^n$. We say that *$P$ is locally solvable at* $x_0 \in \Omega$ if there exists a neighborhood $V \subset \Omega$ of x_0 such that for all $v \in C^\infty(\Omega)$ there is $u \in \mathscr{D}'(\Omega)$ satisfying $Pu = v$ in V.

Definition 2 (**H^s to $H^{s'}$ local solvability**) Let P be an mth-order partial differential operator with smooth coefficients on an open set $\Omega \subset \mathbb{R}^n$. Given $s, s' \in \mathbb{R}$ and $x_0 \in \Omega$ we say that *P is H^s to $H^{s'}$ locally solvable near* x_0 if there is a compact $K \subset \Omega$ with $x_0 \in \mathring{K}$ (the interior of K) such that for all $v \in H^s_{\mathrm{loc}}(\Omega)$ there exists $u \in H^{s'}_{\mathrm{loc}}(\Omega)$ with $Pu = v$ in $\mathring{K}$. We will call the number $s - s'$ *the gain of smoothness* (near x_0) of the solution. We will say that P *is* H^s *to* $H^{s'}$ *locally solvable near* $V \subset \Omega$ if P is H^s to $H^{s'}$ locally solvable near x_0 for all $x_0 \in V$. When one has H^s to $H^{s'}$ local solvability for all $s \in \mathbb{R}$ where $s' = s + m - r$, then one calls r *the loss of derivatives*.

Recall that to obtain the H^s to $H^{s'}$ local solvability in the interior $\mathring{K}$ of a compact K, by the Hahn-Banach Theorem one needs to establish the a priori estimate

$$\exists C > 0 \text{ such that } \quad \|\varphi\|_{-s} \leq C\|P^*\varphi\|_{-s'}, \quad \forall \varphi \in C_0^\infty(K).$$

Throughout the paper $\{\cdot,\cdot\}$ denotes the Poisson bracket and $\pi\colon T^*\Omega\setminus 0 \longrightarrow \Omega$ the canonical projection. By $(\cdot,\cdot)$ we will denote the L^2-scalar product. Finally, given $A, B \geq 0$ we will write $A \lesssim B$ (or $B \gtrsim A$) if there is $C > 0$ such that $A \leq CB$.

2 The mixed-type case

In this section we introduce the following set of hypotheses on the operator

$$P_1 = \sum_{j=1}^{N} X_j^* f X_j + X_{N+1} + iX_0 + a_0$$

of the kind (MT), where $f \in C^\infty(\Omega;\mathbb{R})$, with $f^{-1}(0) \neq \emptyset$ and $df\big|_{f^{-1}(0)} \neq 0$. We will write $\mathsf{d}_{X_j} = -i\mathsf{div}(\alpha_j)$ for the "divergence" of X_j. Notice that in this case the principal symbol of P_1 is *real and changing sign* across $f^{-1}(0)$.

Hypotheses (HM1) to (HM5):

(HM1) $iX_0 f\big|_{f^{-1}(0)} > 0$;

(HM2) For all compact $K \subset \Omega$ there exists $C > 0$ such that for all $j = 1, \dots, N+1$

$$\{X_j, X_0\}(x,\xi)^2 \leq C \sum_{k=0}^{N} X_k(x,\xi)^2, \quad \forall (x,\xi) \in K \times \mathbb{R}^n; \tag{2.3}$$

(HM3) For all compact $K \subset \Omega$ there exists $C > 0$ such that

$$|(\mathsf{Im}\,\mathsf{d}_{X_0}(x))X_{N+1}(x,\xi)| \leq C\Big(\sum_{k=0}^{N} X_k(x,\xi)^2\Big)^{1/2}, \quad \forall (x,\xi) \in K \times \mathbb{R}^n; \tag{2.4}$$

(HM4) For $\rho \in \Sigma$ (see (1.2)) let

$$H_{X_j}(\rho) = \sum_{k=1}^{n} \Big(\frac{\partial X_j}{\partial \xi_k}(\rho)\frac{\partial}{\partial x_k} - \frac{\partial X_j}{\partial x_k}(\rho)\frac{\partial}{\partial \xi_k}\Big)$$

be the Hamilton vector-field of X_j at ρ, let $V(\rho) := \mathsf{Span}\{H_{X_0}(\rho), \dots, H_{X_N}(\rho)\}$, let $J(\rho) \subset \{0, \dots, N\}$ be a set of indices for which $\{H_{X_j}(\rho)\}_{j \in J(\rho)}$ is a basis of $V(\rho)$, and let $M(\rho) = [\{X_j, X_{j'}\}(\rho)]_{j,j' \in J(\rho)}$ be the $r \times r$ matrix of the Poisson brackets of the corresponding symbols $X_j, X_{j'}$, where $r = \mathsf{card}(J(\rho))$. *We say that hypothesis (HM4) is fulfilled at* $x_0 \in f^{-1}(0)$ *if* $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ *and*

$$\mathsf{rank}\, M(\rho) \geq 2, \quad \forall \rho \in \pi^{-1}(x_0) \cap \Sigma; \tag{2.5}$$

(HM5) Let

$$\mathscr{L}_k(x) = \mathsf{Span}_{\mathbb{R}}\{iX_0, \dots, iX_N \text{ and their commutators up to length } k \text{ at } x\}$$

(recall that iX_j has length 1 and $[iX_j, iX_{j'}]$ has length 2, and so on). *We say that hypothesis (HM5) is fulfilled at* $x_0 \in f^{-1}(0)$ *if* $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ *and there exists* $k \geq 1$ *such that*

$$\dim \mathscr{L}_k(x_0) = n. \tag{2.6}$$

Remark 1 Note that if condition (2.5) holds at x_0 then there is a neighborhood V_{x_0} of x_0 such that the condition holds for all $\rho \in \pi^{-1}(V_{x_0}) \cap \Sigma$. Since the subprincipal symbol of $\sum_{j=0}^N X_j^* X_j$ is zero (here the symbols X_j are real) one has (see [11]) that condition (2.5) amounts to Melin's strong Tr+ condition (see [15])

$$\mathrm{sub}\Big(\sum_{j=0}^N X_j^* X_j\Big)(\rho) + \mathsf{Tr}^+ F_{\sum_{j=0}^N X_j^* X_j}(\rho) > 0, \quad \forall \rho \in \pi^{-1}(V_{x_0}) \cap \Sigma,$$

so that for all compact $K \subset V_{x_0}$ one has the sharp Melin inequality [15]: *There are constants* $c_K, C_K > 0$ *such that*

$$(\sum_{j=0}^N X_j^* X_j\, u, u) = \sum_{j=0}^N \|X_j u\|_0^2 \geq c_K \|u\|_{1/2}^2 - C_K \|u\|_0^2, \quad \forall u \in C_0^\infty(K). \tag{2.7}$$

Remark 2 Condition (2.6) yields the Rothschild-Stein sharp subelliptic estimate in a sufficiently small neighborhood V_{x_0} of x_0, that is, *for any given compact* $K \subset V_{x_0}$ *there exist* $c_K, C_K > 0$ *such that*

$$(\sum_{j=0}^N X_j^* X_j\, u, u) = \sum_{j=0}^N \|X_j u\|_0^2 \geq c_K \|u\|_{1/k}^2 - C_K \|u\|_0^2, \quad \forall u \in C_0^\infty(K). \tag{2.8}$$

One may prove that when hypothesis (HM4) is fulfilled then also hypothesis (HM5) is fulfilled with $k = 2$ (see Federico [9]). However, it is still interesting to distinguish the two cases, because of the fact that the subprincipal symbol and the positive trace are symplectic invariants of an operator with double characteristics.
Notice also that having $\dim \mathscr{L}_1(x_0) = n$ is equivalent to saying that the system $(X_0, X_1, \dots, X_N)$ is elliptic near x_0, that is, $\pi^{-1}(V_{x_0}) \cap \Sigma = \emptyset$ for some neighborhood V_{x_0} of x_0, so that inequality (2.8) becomes the well-known Gårding inequality.

For the class of operators (MT) we have the following solvability result (see [12]) near $S := f^{-1}(0)$ (which is the region of interest for us).

Theorem 1 *Supposing hypotheses (HM1), (HM2) and (HM3), one has:*

(i) For all $x_0 \in S$ the operator P_1 is L^2 to L^2 locally solvable at x_0;
(ii) If $x_0 \in S$ is such that $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ and (HM4) holds at x_0, then P_1 is $H^{-1/2}$ to L^2 locally solvable at x_0;
(iii) If $x_0 \in S$ is such that $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ and (HM5) holds at x_0 for some $k \geq 2$, then P_1 is $H^{-1/k}$ to L^2 locally solvable at x_0;
(iv) If $x_0 \in S$ is such that $\pi^{-1}(x_0) \cap \Sigma = \emptyset$, then P_1 is H^{-1} to L^2 locally solvable at x_0.

Remark 3 Notice that the operator given in (1.1) falls in case (iv) of the theorem.

Of course, we won't be giving the proof of the theorem (which can be found in [12]). Instead, to explain the role of the assumptions we next recall the main estimate needed to prove the theorem, that is: *For all sufficiently small compact $K \subset \Omega$ with $x_0 \in \mathring{K}$ there are constants $c_K, C_K > 0$ such that*

$$2\,\mathrm{Re}(P_1^* u, -iX_0 u) \geq c_K \sum_{j=0}^{N} \|X_j u\|_0^2 + \frac{3}{2}\|X_0 u\|_0^2 - C_K \|u\|_0^2, \quad \forall u \in C_0^\infty(K). \tag{2.9}$$

A fundamental step to obtain the main estimate (2.9) is the use the Fefferman-Phong inequality for the operator

$$\widehat{P}_{\varepsilon,\gamma} := \sum_{j=0}^{N}\Big(X_j^* X_j - \frac{\varepsilon}{\gamma}[X_j, X_0]^*[X_j, X_0]\Big) + \frac{1}{\gamma}Y,$$

where

$$Y = -\mathrm{Re}\Big((\mathrm{Im}\,\mathrm{d}_{X_0})X_{N+1}\Big),$$

$\varepsilon = \|f\|_{L^\infty(K)} \to 0$ when $K \searrow \{x_0\}$ and γ is an auxiliary parameter to be picked. Hence, hypotheses (HM1), (HM2) and (HM3) and the fact that $x_0 \in S$ allow one to choose K sufficiently small containing x_0 (in its interior) so as to have, by then picking γ, the Fefferman-Phong estimate, that is, the existence of $C_K > 0$ such that

$$(\widehat{P}_{\varepsilon,\gamma} u, u) \geq -C_K \|u\|_0^2, \quad \forall u \in C_0^\infty(K).$$

Such control allows one to bound from below $\mathrm{Re}(P_1^* u, -iX_0 u)$, $u \in C_0^\infty(K)$, by the right-hand side of (2.9), provided K is sufficiently small about x_0.

Once the main estimate is obtained (which, remark, holds under the assumptions (HM1) to (HM3)), one gets, according to hypotheses (HM4), or (HM5), or $\pi^{-1}(x_0) \cap \Sigma = \emptyset$, by virtue of the Melin, or the Rothschild-Stein, or the Gårding estimates, respectively, the control from below

$$\|P_1^* u\|_0^2 \geq c_0 \|X_0 u\|_0^2 + c_1 \|u\|_s^2 - C_2 \|u\|_0^2, \quad \forall u \in C_0^\infty(K),$$

where $c_0, c_1, C_2 > 0$ (depending on K) and where

- $s = 0$ when only (HM1), (HM2) and (HM3) hold;
- $s = 1/2$ when, in addition to the first three hypotheses, (HM4) holds;
- $s = 1/k$ when, in addition to the first three hypotheses, (HM5) holds for some $k \geq 2$;
- $s = 1$ when, in addition to the first three hypotheses, $\pi^{-1}(x_0) \cap \Sigma = \emptyset$.

One finally gets rid of the L^2-error terms by using the Poincaré inequality for the term $\|X_0 u\|_0^2$ and by possibly shrinking further the compact K (keeping x_0 in its interior). At last, the solvability estimate

$$\|P_1^* u\|_0^2 \geq C_K(\|u\|_s^2 + \|u\|_0^2), \quad \forall u \in C_0^\infty(K),$$

is obtained and an application of the Hahn-Banach Theorem gives the result.

We next give a few examples of operators P_1 in the class (MT) for which we can conclude local solvability near quite degenerate points.

2.1 Example.

Let $x = (x_1, x_2)$ be coordinates in $\mathbb{R}^2$, let $g(x_2) = 1 + x_2^2$, $f(x) = x_1 - (x_2 + x_2^3/3)$. Let

$$A(x_2) = \begin{bmatrix} g(x_2) & 1 \\ 1 & 1/g(x_2) \end{bmatrix}.$$

We have $\dim \mathsf{Ker} A(x_2) = 1$ for all x_2. Consider

$$P_1 = \sum_{j_1, j_2 = 1}^{2} D_{j_1}\Big(f(x) a_{j_1 j_2}(x_2) D_{j_2}\Big) + X_3 + iX_0 + a_0,$$

where

$$X_3(x,\xi) = \mu_1(x) X(x,\xi) + \mu_2(x) X_0(x,\xi),$$

with

$$X(x,\xi) = g(x_2)\xi_1 + \xi_2, \quad X_0(x,\xi) = \alpha\xi_1 + \frac{1}{g(x_2)}\xi_2,$$

where $\alpha > 1$ is a constant and μ_1, μ_2 are smooth real-valued functions. Then, putting

$$X_1(x,\xi) = \sqrt{g(x_2)}\frac{X(x,\xi)}{\sqrt{1 + g(x_2)^2}}, \quad X_2(x,\xi) = \frac{1}{\sqrt{g(x_2)}}\frac{X(x,\xi)}{\sqrt{1 + g(x_2)^2}},$$

gives that P_1 may be written in the form

$$P_1 = \sum_{j=1}^{2} X_j^* f X_j + X_3 + iX_0 + a_0,$$

where conditions (HM1), (HM2) and (HM3) hold, but (HM4) and (HM5) (including the case $k = 1$) do not, since $X_1(x,\xi), X_2(x,\xi)$ and $\{X, X_0\}(x,\xi)$ are always proportional to $X(x,\xi)$. Therefore P_1 is L^2 to L^2 locally solvable near $f^{-1}(0)$.

2.2 Example.

The next example deals with a situation in which one has better (than L^2 to L^2) local solvability. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and let $k \geq 0$ be an integer. Take $f(x) = x_2$, and the following system of vector fields

$$X_1 = D_{x_1}, \quad X_2 = x_1^k D_{x_3}, \quad X_3 = \beta(x) D_{x_1}, \quad X_0 = D_{x_2},$$

where $\beta \in C^\infty(\mathbb{R}^3; \mathbb{R})$. Let

$$P_1 = \sum_{j=1}^{2} X_j^* f X_j + X_3 + iX_0 + a_0.$$

Then it is readily seen that $\mathsf{d}_{X_0} \equiv 0$, that $\{X_j, X_0\} = 0$ for $j = 1, 2$ and that $|\{X_0, X_3\}(x,\xi)|^2 \lesssim X_1(x,\xi)^2$ for all (x,ξ) (locally for x in compact sets), so that hypotheses (HM1), (HM2) and (HM3) are fulfilled. We therefore have that P is $H^{-1/(k+1)}$ to L^2 locally solvable near $x_2 = 0$ with $k+1$ given by

- $k + 1 = 1$, that is in the case $\Sigma = \emptyset$ (whence $\pi^{-1}(x_0) \cap \Sigma = \emptyset$ for all $x_0 \in f^{-1}(0)$);
- $k + 1 = 2$, that is in the case in which (HM4) is fulfilled;
- $k + 1 \geq 2$, that is in the case in which (HM5) is fulfilled.

2.3 Example.

In this example, we show that condition (HM4) might not always be satisfied at $f^{-1}(0)$ so that the gain of derivatives may vary depending on the position of $\pi^{-1}(x_0)$, $x_0 \in f^{-1}(0)$, with respect to Σ. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and let $\Omega \subset \mathbb{R}^3$ be an open set such that $\Omega \cap \{x;\ x_1 = -1\} \neq \emptyset$. Let $\Omega_\pm := \{x \in \Omega;\ x_1 \gtrless -1\}$ and $f(x) = x_2 + x_2^3/3 - x_1 x_3$. Introduce the following system of vector fields

$$X_1(x,\xi) = \xi_1 - x_3\xi_3, \quad X_2(x,\xi) = (1 + x_1)\xi_3, \quad X_0(x,\xi) = \xi_2 - x_1\xi_3,$$

$$X_3(x,\xi) = \sum_{j=0}^{2} \Big(\beta_j(x) X_j(x,\xi) + \gamma(x)\{X_0, X_j\}(x,\xi)\Big),$$

where $\beta_j, \gamma \in C^\infty(\Omega; \mathbb{R})$. Let

$$P_1 = \sum_{j=1}^{2} X_j^* f X_j + X_3 + iX_0 + a_0.$$

Since $\mathsf{d}_{X_0} = 0$ and

$$\{X_1, X_0\} = -X_2, \quad \{X_1, X_2\} = (2 + x_1)\xi_3, \quad \{X_2, X_0\} = 0,$$

one has that hypotheses (HM1), (HM2) and (HM3) are satisfied. Therefore P_1 is always L^2 to L^2 locally solvable near $f^{-1}(0)$. However, since

$$\Sigma = \{(x,\xi); \xi_1 = x_3\xi_3,\ (1 + x_1)\xi_3 = 0,\ \xi_2 = x_1\xi_3,\ \xi \neq 0\},$$

so that $\xi_3 \neq 0$ and therefore also $x_1 = -1$ when $(x,\xi) \in \Sigma$, it follows that

$$\Sigma = \{(x,\xi);\ x_1 = -1,\ \xi_2 + \xi_3 = 0,\ \xi_1 = x_3\xi_3,\ \xi_3 \neq 0\}.$$

At any given $\rho = (x,\xi) \in \Sigma$ we have

$$H_{X_0}(\rho) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ \xi_3 \\ 0 \\ 0 \end{bmatrix}, \quad H_{X_1}(\rho) = \begin{bmatrix} 1 \\ 0 \\ -x_3 \\ 0 \\ 0 \\ \xi_3 \end{bmatrix}, \quad H_{X_2}(\rho) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\xi_3 \\ 0 \\ 0 \end{bmatrix},$$

whence $\dim V(\rho) = 3$ (here $J(\rho) = \{0,1,2\}$),

$$M(\rho) := \left[\{X_j, X_{j'}\}(\rho)\right]_{0\leq j,j'\leq 2} = \begin{bmatrix} 0 & X_2(x,\xi) & 0 \\ -X_2(x,\xi) & 0 & (2 + x_1)\xi_3 \\ 0 & -(2 + x_1)\xi_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi_3 \\ 0 & -\xi_3 & 0 \end{bmatrix},$$

has rank 2 for all $\rho \in \Sigma$ and condition (HM4) is fulfilled at $x_0 = \pi(\rho)$ (when $\pi(\rho) \in f^{-1}(0)$). Equivalently, condition (HM5) with $k = 2$ is fulfilled, for one has at $x_0 = (-1, x_2^0, x_3^0) = \pi(\rho)$, $\rho \in \Sigma$,

$$\begin{aligned} \mathscr{L}_2(x_0) &= \mathsf{Span}_{\mathbb{R}}\{iX_0(x_0,D),\ iX_1(x_0,D),\ [iX_1, iX_2](x_0,D)\} \\ &= \mathsf{Span}_{\mathbb{R}}\left\{\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} - x_3^0\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_3}\right\} = \mathbb{R}^3. \end{aligned}$$

In general

$$\pi^{-1}(\Omega_\pm) \cap \Sigma = \emptyset,$$

and when $x_0 = (-1, x_2^0, x_3^0)$ then

$$\pi^{-1}(x_0) \cap \Sigma \neq \emptyset,$$

so that we have the following different cases:

- when $x_0 \in f^{-1}(0) \cap \Omega_\pm$ then P_1 is H^{-1} to L^2 locally solvable near x_0 (in fact, in this case we have $\pi^{-1}(x_0) \cap \Sigma = \emptyset$);
- when $x_0 = (-1, x_2^0, x_3^0) \in f^{-1}(0) \cap \Omega$ then $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ and (HM4) holds (equivalently, (HM5) with $k = 2$ holds), whence P_1 is $H^{-1/2}$ to L^2 locally solvable near x_0.

2.4 Example: A mildly complex case.

We conclude the section by briefly discussing an example involving a complex operator of mixed type. In [11] we considered

$$P = \sum_{j=1}^{N} Z_j^* f Z_j + iZ_0 + a_0,$$

where $Z_1, \ldots, Z_N$ have a *complex* symbol and Z_0 has a *real* symbol. For it, under suitable assumptions, similar in nature to (HM1) through (HM5), we could give a $H^{-1/k}$ to L^2 local solvability result near $f^{-1}(0)$. We consider it a *mildly* complex case, because the subprincipal parts of $\sum_{j=1}^{N} Z_j^* Z_j$ and of $\sum_{j=1}^{N} [Z_j, Z_0]^* [Z_j, Z_0]$ are in addition assumed (see [11]) to be controlled by $\sum_{k=0}^{N} |Z_k|^2$ (and hence to *vanish* on $\bigcap_{j=0}^{N} Z_j^{-1}(0)$).

For instance, an example of operator that can be considered is given by

$$P = Z_1^* f Z_1 + Z_2^* f Z_2 + iZ_0 + a_0,$$

where, for $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$,

$$Z_1(x,\xi) = \xi_1 + i x_2^k \xi_3, \quad Z_2(x,\xi) = e^{ig(x_1,x_2)} \xi_2, \quad Z_0(x,\xi) = \xi_4,$$

with $g \in C^\infty(\mathbb{R}^2_{x_1,x_2}; \mathbb{R})$ is such that $\partial g / \partial x_2 \neq 0$ and $f(x) = x_4 + \tilde{f}(x_1, x_2, x_3)$. Then P is $H^{-1/(k+1)}$ to L^2 locally solvable near $f^{-1}(0)$.

3 The Schrödinger-type case

We next turn our attention to the Schrödinger-type case (ST). Recall that in this case

$$P_2 = \sum_{j=1}^{N} X_j^* f_j X_j + X_{N+1} + a_0,$$

where $f_1, \ldots, f_N \in C^\infty(\Omega; \mathbb{R})$ (notice that $X_0 \equiv 0$). Notice that also in this case the principal symbol of P_2 is *real and may change sign*. Here we make the following assumptions.

Hypotheses (HS1) and (HS2):

(HS1) The operators $X_1,\dots,X_N$ have complex coefficients;

(HS2) For all $x_0 \in \Omega$ there exists a connected neighborhood $V_{x_0} \subset \Omega$ of x_0 and $g \in C^\infty(V_{x_0};\mathbb{R})$ such that

(i) $X_j g = 0$ on V_{x_0}, for all $1 \leq j \leq N$;

(ii) $X_{N+1} g \neq 0$ on V_{x_0}.

Remark 4 Note that, since we are interested in a degenerate setting where the vanishing of the functions f_j has an interplay with the degeneracies of the operators $X_1,\dots,X_N$ appearing in the second order part of P_2, we do not assume any nondegeneracy conditions on the X_j for $1 \leq j \leq N$ (in the sense that iX_j may have a critical point at some x, i.e. $\alpha_j(x) = 0$). At the same time, since we prove solvability by a priori estimates, we need to put a nondegeneracy condition somewhere on P_2, which is indeed placed on the first order part X_{N+1} (condition (HS2-(ii)).
We also remark that, since the sets $f_j^{-1}(0)$ may be disjoint, the local solvability of P_2 is studied at each point of Ω and not near any particular zero-set $f_j^{-1}(0)$.

In this case we have the following result (see [12]).

Theorem 2 *In the above hypotheses the operator P_2 is L^2 to L^2 locally solvable at any given $x_0 \in \Omega$.*

Note that now the way the functions f_j may degenerate in Ω is no longer important, neither does the characteristic set of the system $(X_0 = 0, X_1,\dots,X_N)$ play any role. The point now is to estimate, given $x_0 \in V_{x_0} \subset \Omega$ and a compact $K \subset V_{x_0}$ containing x_0, the quantity

$$\mathsf{Im}(e^{\lambda g} P^* u, e^{\lambda g} u)$$

for $\lambda \geq 1$ large and $u \in C_0^\infty(K)$. By the hypotheses, this is indeed possible. It is worth mentioning that the estimate follows from a general framework. In fact, let $B\colon C_0^\infty(V_{x_0}) \longrightarrow C_0^\infty(V_{x_0})$ be a 0th-order properly supported ψdo, such that $B^* = B + R$, where R is a smoothing operator. Then

$$\begin{aligned}\mathsf{Im}(P^*\varphi, B\varphi) = \sum_{j=1}^{N} \mathsf{Im}(X_j\varphi, f_j[X_j,B]\varphi) + \frac{1}{2}\sum_{j=1}^{N}\mathsf{Im}(X_j\varphi,[f_j,B]X_j\varphi)\\ +\mathsf{Im}(\varphi,[X_{N+1},B]\varphi) + O(\|\varphi\|_0^2),\ \forall\varphi \in C_0^\infty(V_{x_0}),\end{aligned} \tag{3.10}$$

where in $O(\|\varphi\|_0^2)$ we gathered the contributions of $[R, X_j]\varphi$, $[R, X_{N+1}]\varphi$ and of $[B, \mathsf{d}_{X_{N+1}}]\varphi$. The first two terms on the right-hand side of (3.10) are delicate, in that we cannot control terms like $\|X_j\varphi\|_0$ and are able to control only the third term, and this is where we make use of assumption (HS2)-(ii), a reasonable choice of B being $e^{\lambda g}$. Then one obtains the estimate ($c_0 > 0$)

$$\mathsf{Im}(e^{\lambda g}P^*u, e^{\lambda g}u) \geq \Big(\lambda c_0 - \frac{\|\mathsf{d}_{X_{N+1}}\|_{L^\infty(K)}}{2} - \|a_0\|_{L^\infty(K)}\Big)\|e^{\lambda g}u\|_0^2, \quad \forall u \in C_0^\infty(K),$$

whence by picking $\lambda > 0$ sufficiently large and by using the Cauchy-Schwarz inequality, one obtains the L^2 solvability estimate $\|P^*u\|_0 \gtrsim \|u\|_0$, for all $u \in C_0^\infty(K)$.

We next wish to give a few examples of operators in the class (ST) of the P_2.

3.1 Example.

In $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_y^m$ we may consider the operators

$$P_2 = -\Delta_x \pm \Delta_y + D_t, \quad \text{or} \quad P_2 = f_1(t)\Delta_x + f_2(t)\Delta_y + D_t,$$

where f_1, f_2 are smooth, real-valued (and not identically zero). In all cases, P_2 is L^2 to L^2 locally solvable.

3.2 Example.

This example is related to the so-called Mizohata structures. Let $\Omega_0 \subset \mathbb{R}_x^n \times \mathbb{R}_y$ be open. Then $\Omega := \mathbb{R}_t \times \Omega_0 \subset \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_y$ is open. Take $Q = Q(x)$ to be a real-valued quadratic form and let

$$X_j = D_{x_j} - i\frac{\partial Q}{\partial x_j}(x)D_y, \quad 1 \le j \le n.$$

Let $Y = Y(x, y, D_x, D_y)$ be a first order homogeneous operator with real symbol and let $X_{N+1} = D_t + Y$. One immediately has that $g = g(t) = t$ satisfies hypotheses (HS1) and (HS2) whence

$$P_2 = \sum_{j=1}^{n} X_j^* f_j X_j + X_{N+1} + a_0$$

is L^2 to L^2 locally solvable near any given point of Ω, regardless the choice of the (non indentically zero) functions $f_j \in C^\infty(\Omega; \mathbb{R})$ (and, of course, of $a_0 \in C^\infty(\Omega; \mathbb{C})$).

3.3 Example.

Another example is the following. Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and let $\Omega \subset \mathbb{R}^4$ be open. Let

$$X_1 = D_{x_1} - i\frac{x_2}{2}D_{x_3}, \quad X_2 = D_{x_2} + i\frac{x_1}{2}D_{x_3}, \quad X_3 = D_{x_4} + \alpha(x)D_{x_3},$$

where $\alpha \in C^\infty(\Omega; \mathbb{R})$. We then choose $g = g(x) = x_4$ and have that, whatever the (non indentically zero) functions $f_1, f_2 \in C^\infty(\Omega; \mathbb{R})$ (and of course $a_0 \in C^\infty(\Omega; \mathbb{C})$)

the operator

$$P_2 = X_1^* f_1 X_1 + X_2^* f_2 X_2 + X_3 + a_0$$

is L^2 to L^2 locally solvable near any given point of Ω.

Remark 5 It may be interesting to think of the operators P_2 appearing in Examples 3.2 and 3.3 as evolution operators in the direction iX_{N+1} associated with the involutive/hypoanalytic structure (see [3]) on the leaves $g^{-1}(c)$, spanned by the system $(iX_1, \dots, iX_N)$ tangential to $g^{-1}(c)$ (c near some regular value of g).

4 The mixed-Schrödinger-type case

We finally consider the case (MST), that is, recall, an operator of the kind

$$P_3 = \sum_{j=1}^{N} X_j f_j X_j + X_{N+1} + iX_0 + a_0,$$

where the novelty with respect to the class (MT), where we had only one f and had the presence of X_0, and with respect to the class (ST), where we had many smooth real-valued f_j but no X_0, lies in the fact that we may now allow the presence of many f_j and a non-zero X_0. Notice that also in this case the principal symbol of P_3 is *real and may change sign*. The hypotheses to deal with such a case are the following.

Hypotheses (HMS1), (HMS2) and (HMS3):

(HMS1) $X_0 \neq 0$ throughout Ω (i.e. iX_0 has no critical points), and $iX_0 f_j \geq 0$ on Ω, $1 \leq j \leq N$;
(HMS2) $\{X_0, X_j\}(x, \xi) = 0$ for all $(x, \xi) \in T^*\Omega$ and all $j = 1, \dots, N$;
(HMS3) For all $x_0 \in \Omega$ there exists a compact $K \subset \Omega$ with non-empty interior containing x_0, and a positive constant C_K such that for all $(x, \xi) \in T^*K$

$$|\{X_0, X_{N+1}\}(x, \xi)|^2 \leq C_K \Big(\sum_{j=1}^{N} (iX_0 f_j(x)) X_j(x, \xi)^2 + X_0(x, \xi)^2 \Big).$$

Remark 6 In this case, as well as in the (ST) case above, the presence of several possibly vanishing functions f_j (which may have nonintersecting zeros) in Ω motivates the study of the local solvability of P_3 at any point of Ω. This explains why our conditions are given on Ω and, especially, why the nondegeneracy condition (HMS1) is required on Ω (and not on any particular $f_j^{-1}(0)$). In this regard, note that in the present (MST) case we assume $iX_0 f_j \geq 0$ on Ω for all $1 \leq j \leq N$ and *not* $iX_0 f|_{f^{-1}\{0\}} > 0$ as in the (MT) case.

In this case we have the following result (see [10]).

Theorem 3 *Suppose hypotheses (HMS1) to (HMS3) hold. Then P_3 is L^2 to L^2 locally solvable near any given $x_0 \in \Omega$.*

The main point, as before, is to estimate from below the quantity

$$\mathrm{Re}\big(P^* u, -iX_0 u\big), \quad u \in C_0^\infty(K).$$

One obtains the main estimate, with $c_K, C_K > 0$,

$$2\mathrm{Re}(P^* u, -iX_0 u) \geq (P_0 u, u) + c_K \|X_0 u\|_0^2 - C_K \|u\|_0^2, \quad u \in C_0^\infty(K),$$

where

$$P_0 = \sum_{j=1}^{N} [iX_0, X_j^* f_j X_j] + X_0^2 - \epsilon [X_0, X_{N+1}]^* [X_0, X_{N+1}] \tag{4.11}$$

($\epsilon > 0$ is small depending on a compact K_0 containing K), which then satisfies the Fefferman-Phong inequality (by virtue of hypotheses (HMS1) to (HMS3)), so that one gets the inequality ($c, C > 0$)

$$\|P^* u\|_0^2 \geq c\|X_0 u\|_0^2 - C\|u\|_0^2, \quad u \in C_0^\infty(K).$$

Then using, as in the (MT) case, the Poincaré inequality for the term $\|X_0 u\|_0^2$ makes it possible to get rid of the term $-C\|u\|_0^2$ (it is here that one exploits hypothesis (HMS1) and has possibly to shrink the compact set K keeping x_0 in its interior, process that does not change the above constants $c, C > 0$), whence the L^2 to L^2 local solvability estimate

$$\|P^* u\|_0^2 \gtrsim \|u\|_0^2, \quad \forall u \in C_0^\infty(K).$$

Notice that requiring stronger assumptions on the vector fields yields stronger properties of P_0, whence the possibility of exploiting the Gårding inequality, or the Melin inequality, or the Rothschild-Stein inequality to obtain improved solvability results (i.e. with a better gain of derivatives).

4.1 Example.

Consider in $\mathbb{R}^n$, $n \geq 3$, the operator

$$P_3 = x_1(D_1^2 - D_2^2) + i(D_1 + D_2) + X_3(x, D) = D_1 x_1 D_1 - D_2 x_1 D_2 + iD_2 + X_3(x, D),$$

with

$$X_3(x, D) = g_1(x) D_1 + g_2(x) D_2 + \sum_{j=3}^{n} g_j(x) D_j,$$

where the g_j, $1 \leq j \leq n$, are smooth real-valued, and where g_1 and g_j, $3 \leq j \leq n$, are *independent of* x_2. In this case

$$X_0 = D_2, \quad X_1 = D_1, \quad X_2 = D_2 \quad f_1(x) = f_2(x) = x_1.$$

Since we then have

$$\{X_0, X_3\}(x,\xi) = -i\frac{\partial g_2}{\partial x_2}(x)X_0(x,\xi), \quad \{X_0, X_1\} = \{X_0, X_2\} = 0,$$

for such an operator conditions (HMS1) to (HMS3) are satisfied and P is L^2 to L^2 locally solvable in $\mathbb{R}^n$.

4.2 Example.

Consider next the operator in $\mathbb{R}^n_x \times \mathbb{R}_t$

$$P_3 = \sum_{j=1}^{M} D_j x_j^{m_1} D_j \pm \sum_{k=M+1}^{n} D_k x_k^{m_2} D_k + ig(t)D_t + \sum_{h=1}^{n} g_h(x)D_h,$$

where $1 \leq M \leq n-1$, $m_1, m_2 \geq 1$, and g and the g_h, $1 \leq h \leq n$, are smooth real-valued functions, with g nowhere vanishing. Once more, since in this case (here $N = n$)

$$X_0 = g(t)D_t, \quad X_{N+1} = \sum_{h=1}^{n} g_h(x)D_h,$$

conditions (HMS1) to (HMS3) are satisfied and P is L^2 to L^2 locally solvable in $\mathbb{R}^n \times \mathbb{R}$.

4.3 Example.

In this final example one sees that there are cases in which an operator of the kind (MST) can be locally solvable with a better gain of derivatives. Consider in fact in $\mathbb{R}^2$ the operator

$$P_3 = D_1x_1D_1 - D_2x_2D_2 + i(D_1 - D_2) + x_2D_1 = x_1D_1^2 - x_2D_2^2 + x_2D_1,$$

in which case

$$X_1 = D_1, \quad X_2 = D_2, \quad X_0 = D_1 - D_2, \quad X_3 = x_2D_1, \quad f_1(x) = x_1, \quad f_2(x) = -x_2.$$

In this case the associated operator P_0 (see (4.11)) is

$$P_0 = D_1^2 + D_2^2 + (D_1 - D_2)^2 - \epsilon D_1^2,$$

whence for $0 < \epsilon < 1$ the operator P_0 is elliptic and one can use the Gårding inequality in place of the Fefferman-Phong inequality, thus obtaining an H^{-1} to L^2 local solvability result near any given point of $\mathbb{R}^2$.

5 Concluding remarks

There is a number of problems raised by the study of this class of degenerate operators. Among them, we wish to mention the following two.

Problems:

1. For the operators considered here one should study whether given any $s \in \mathbb{R}$ one has H^s to H^{s+2-r} local solvability, where r is the loss of derivatives. Of course, this problem is very difficult because the operator is very degenerate and even microlocalization gives lower order terms that may be too big to control. It might very well be the case that the Sobolev regularity s cannot range freely in $\mathbb{R}$ but that there might be thresholds due to the kernel of P^*.

2. One should study semi-global solvability (see [16]) for these operators, and for that one needs to understand the propagation of singularities. One should expect that things abruptly change, depending on the different classes (mixed-type, Schrödinger-type, mixed-Schrödinger-type, respectively), depending on the behavior of the bicharacteristics of $\sum_{j=0}^{N} X_j^* X_j$ when hitting the sets $f^{-1}(0)$, $f_1^{-1}(0), \ldots, f_N^{-1}(0)$. In addition, as we saw in Example 2.3 of Section 2, the gain of regularity may wildly change depending on the position of $\pi^{-1}(S)$ with respect to Σ (recall that $S = f^{-1}(0)$), whence an approach based on the propagation of Sobolev microlocal regularity along the bicharacteristics (of P or $\sum_{j=0}^{N} X_j^* X_j$) might be the appropriate one. A first important step in this direction was taken by Parenti and Rodino in [30], where they studied the hypoellipticity and microlocal hypoellipticity of a class of anisotropic operator in terms of the Lascar anisotropic wave-front set.

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