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Published Version:
Ferrari F. (2021). Some Extension Results for Nonlocal Operators and Applications. Cham : Springer Science and Business Media Deutschland GmbH [10.1007/978-3-030-69236-0_9].

Availability:
This version is available at: https://hdl.handle.net/11585/843775 since: 2022-01-02

Published:
DOI: http://doi.org/10.1007/978-3-030-69236-0_9

Some extension results for nonlocal operators and applications

Fausto Ferrari

Abstract In this paper, we deal with some recent and old results, concerning fractional operators, obtained via the extension technique. This approach is particularly fruitful for exploiting some of those well known properties, true for the local operators obtained via the extension approach, for deducing some parallel results about the underlaying nonlocal operators.
Keywords: fractional Laplacian, Marchaud derivative, Weyl derivative, extension techniques, Carnot groups, Harnack inequality.

1 Some nonlocal operators

We start this note by reviewing some recent and old results concerning fractional operators, putting in evidence the extension technique used in [12] for the fractional Laplace operator and successively applied to other operators, see e.g. [52, 7, 10]. This approach results particularly fruitful for exploiting some, perhaps, well known properties of the local operators obtained from the nonlocal ones, for which we would like to prove some results. The typical easier example is given in [12] where, for proving the Harnack inequality for the fractional Laplace operator, the authors reduce themselves to apply the Harnack inequality associated with a degenerate local operator in divergence form and for which there exist in literature many results, see [17]. We point out that Harnack inequality for the fractional Laplace operator was already known in literature for $s-$harmonic functions being $s \in (0, 1)$. Namely, there exists a positive constant C such that for every sufficiently smooth function $u : \mathbb{R}^n \to \mathbb{R}$, $u \geq 0$ in $\mathbb{R}^n$, such that $(-\Delta)^s u = 0$ in $\Omega \subset \mathbb{R}^n$, then for every ball $B_{2R} \subset \Omega$ of radius $2R$ we have:

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$$\sup_{B_R} u \le C \inf_{B_R} u.$$

The classical proof can be found, for instance, in [48], [40] using potential theory techniques and, via probabilistic approach, in [6]. In the Harnack inequality, the different hypotheses that have to be done between the local and the nonlocal case are explicit. In fact, without extra assumptions on the positivity of u in all of $\mathbb{R}^n$, instead of the required positivity of u only on the set Ω as well as in the local setting, the result is false, see e.g. [38].

This approach can be adapted to different operators. In particular the case of Weyl-Marchaud derivative has been faced in [10], while the case of degenerate operators in [23]. Recently, some further generalizations and applications to Kolmogorov operators type have been obtained in [31], see also [29, 30] and [54] for a recent review of these results.

We continue this section introducing some basic nonlocal operators, while in Section 2 we deal with the extension technique. More precisely: In Subsection 2.1 we deal with the extension approach adapted to the case of the Marchaud derivative, in Subsection 2.2 we introduce the main tools for working in a non-commutative framework and successively, in Subsection 2.3, we describe the extension approach technique in Carnot groups. The applications are collected in Section 3. We conclude with Section 4 discussing the construction of the solutions of the extended problem associated with a periodic function, via the Fourier series approach.

The following notation concerns the difference of fractional order $\alpha \in \mathbb{R}$ for a function f. Let

$$(\Delta_h^\alpha f)(x) := \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x - kh), \tag{1}$$

where

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha+n-1)}.$$

The Grünwald-Letnikov derivative, [33, 41], is:

$$\mathbb{D}^\alpha f(x) = \lim_{h\to 0^+} \frac{(\Delta_h^\alpha f)(x)}{h^\alpha},$$

whenever the pointwise limit exists.

In addition, see Theorem 20.4, [51], for $L^p(\mathbb{R})$ functions, we know that Grüwald-Letnikov-derivative exists if and only if Marchaud-derivative exists in the following sense:

$$\lim_{h\to 0^+, L^p(\mathbb{R})} \frac{(\Delta_h^\alpha f)(x)}{h^\alpha} = \lim_{\epsilon\to 0^+, L^p(\mathbb{R})} \frac{\alpha}{\Gamma(1-\alpha)} \int_\epsilon^\infty \frac{f(x) - f(x-\tau)}{\tau^{1+\alpha}} d\tau,$$

being $\alpha \in (0, 1)$ and

$$\mathbf{D}_+^\alpha f(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x-\tau)}{\tau^{1+\alpha}} d\tau$$

the Marchaud-derivative, see [45] and [50, 10, 22]. For further information about these relationships, we recall the classical handbooks [50, 39, 46] as well. We have to say that, for function defined in $[a,t]$, sometime the Caputo derivative

$$ {}_aD_t^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^\alpha} ds \tag{2} $$

is used, see e.g. [14], instead of the Riemann-Lioville one, defined as:

$$ {}_a\mathcal{D}_+^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f'(s)}{(t-s)^\alpha} ds, $$

see [51].

The Riemann-Liouville derivative is closely related to Marchaud derivative. In fact, the interested reader may check this fact one more time in [51] or e.g. in [22], but, in any case, also Caputo derivative is strictly linked to Marchaud derivative. In fact, integrating by parts (2), we obtain:

$$ \Gamma(1-\alpha)_a D_t^\alpha f(t) = \frac{f(t)-f(a)}{(t-a)^\alpha} + \alpha \int_a^t \frac{f(t)-f(s)}{(t-s)^{1+\alpha}} ds. $$

Thus for functions f defined in $(-\infty, t]$, letting $a = -\infty$, we obtain, after a change of variable:

$$ {}_{-\infty}D_t^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{f(t)-f(s)}{(t-s)^{1+\alpha}} ds = \mathbf{D}_+^\alpha f(t) $$

that is exactly the Marchaud derivative.

Keeping in mind the subject contained in Section 4, as well as the Weyl definition of derivative, see [55], we wish to remark how the Weyl derivative can be justified for pointing out its relationship with Marchaud derivative.

Let us consider a periodic function, let us say for simplicity a 2π-periodic function having zero average. We can associate to this function its own Fourier series, see [22]:

$$ \sum_{k=-\infty}^{+\infty} c_k e^{ikx}, $$

where of course $\{c_k\}_{k\in\mathbb{Z}}$ is the sequence of the Fourier coefficients. Thus, formally, the derivative may be written as:

$$ \sum_{k=-\infty}^{+\infty} c_k (ik) e^{ikx}. $$

Then, defining a new function for each fixed $\alpha < 1$:

$$ \sum_{k=-\infty}^{+\infty} \frac{c_k}{(ik)^\alpha} e^{ikx}, $$

(morally the integral), we formally obtain, by taking its derivative:

$$D\left(\sum_{k=-\infty}^{+\infty}\frac{c_k}{(ik)^{\alpha}}e^{ikx}\right)=\sum_{k=-\infty}^{+\infty}\frac{c_k}{(ik)^{\alpha-1}}e^{ikx}. \tag{3}$$

Following this path, Weyl analogously introduces the parallel fractional integral, see [55]. Thus, it is natural to define the fractional derivative of f as:

$$\sum_{k=-\infty,}^{+\infty}c_k(ik)^{\alpha}e^{ikx}.$$

We recall that, given two periodic functions f, g, the new function

$$\frac{1}{2\pi}\int_0^{2\pi}g(t)f(x-t)dt$$

is represented by the Fourier series

$$\sum_{k=-\infty}^{\infty}g_k c_k e^{ikx},$$

where $\{g_k\}_{k\in\mathbb{Z}}$ and $\{c_k\}_{k\in\mathbb{Z}}$ are the respective Fourier coefficients. As a consequence, considering

$$\sum_{k=-\infty}^{+\infty}\frac{c_k}{(ik)^{\alpha}}e^{ikx}$$

as representing the Fourier series of an integral like the following one:

$$\frac{1}{2\pi}\int_0^{2\pi}g(t)f(x-t)dt,$$

we deduce that previous integral has to be written in the following form:

$$\frac{1}{2\pi}\int_0^{2\pi}f(x-t)\left(\sum_{k=-\infty,k\neq 0}^{+\infty}\frac{e^{ikt}}{(ik)^{\alpha}}\right)dt.$$

It can be proved that (see [51])

$$\sum_{k=-\infty,k\neq 0}^{+\infty}\frac{e^{ikt}}{(ik)^{\alpha}}=2\sum_{k=1}^{\infty}\frac{\cos(kt-\alpha\frac{\pi}{2})}{k^{\alpha}}.$$

Then, denoting the kernel

$$\psi_+^{\alpha}(t):=\sum_{k=-\infty,k\neq 0}^{+\infty}\frac{e^{ikt}}{(ik)^{\alpha}},$$

Weyl obtains the fractional integral

$$I_+^{(\alpha)} f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-t)\psi_+^{\alpha}(t)dt.$$

At this point, see [51], Weyl defines the fractional derivative as

$$\mathcal{D}_+^{(\alpha)}(x) := D\left(I_+^{(1-\alpha)} f\right)(x).$$

This definition corresponds to the Weyl-Riemann-Lioville version of this derivative, see one more time [51] for the details. Then taking formally the derivative, Weyl obtains the Weyl-Marchaud derivative that is:

$$\mathbf{D}_+^{(\alpha)} f(x) := \frac{1}{2\pi} \int_0^{2\pi} \left(f(x) - f(x-t)\right) \frac{d}{dt}\psi_+^{1-\alpha}(t)dt.$$

Of course, the case concerning $\mathbf{D}_-^{(\alpha)} f$ is analogous.

The Weyl derivative and the Marchaud derivative of 2π periodic functions in $L^p(0, 2\pi)$ coincide a.e., whenever they exist, see Lemma 19.4 in [50]. It is worth to point out that the numerical evaluation of these derivatives is particularly important, see e.g. [32] or [19], so that it is useful to understand their properties considering their different representations.

Concerning the applications of these nonlocal operators, it would be very difficult to list all the papers published on this subject, nevertheless we like to cite [3], where Riemann-Liouville-Marchaud-Weyl-Caputo derivative as well as the fractional Laplace operator appeared in the description of a porous medium flow problem, and [43] for a description of many other problems.

We have just quoted the fractional operator, so that we formally introduce it. Let $\alpha \in (0, 1)$, we recall that the fractional Laplace operator is defined, let us say for every $f \in \mathcal{S}(\mathbb{R}^n)$, the set of rapidly decreasing set functions on $\mathbb{R}^n$, as:

$$\begin{aligned}(-\Delta)^{\alpha} f(x) &= C_{\alpha,n} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2\alpha}} dy \\ &:= \lim_{r \to 0^+} C_{\alpha,n} \int_{\mathbb{R}^n \setminus B_r(x)} \frac{f(x) - f(y)}{|x-y|^{n+2\alpha}} dy,\end{aligned} \tag{4}$$

where $B_r(x)$ denotes the ball centered at x pf radius r and the constant $C_{\alpha,n}$ depends only on α, n and represents a normalizing constant determined by the following condition

$$\mathcal{F}^{-1}(|\xi|^{2\alpha}\mathcal{F}(f))(x) = (-\Delta)^{\alpha} f(x).$$

Here we denote, as usual, by $\mathcal{F}$ the Fourier transform. We remark indeed that $\mathcal{F}^{-1}(|\xi|^{2\alpha}\mathcal{F}(f))$ is often taken as the definition of the fractional Laplace operator itself $(-\Delta)^{\alpha} f$, see e.g. [51] and [20].

There exists a relationship between the Marchaud derivative and the fractional Laplace operator that is given for $\alpha \in (0, 1)$ by:

$$\int_{\partial B_1(0)} \mathbf{D}_h^\alpha f(x) d\mathcal{H}^{n-1}(h) = (-\Delta)^{\alpha/2} f(x),$$

where

$$\mathbf{D}_h^\alpha f(x) = c_\alpha \int_0^\infty \frac{f(x) - f(x - th)}{t^{1+\alpha}} dt,$$

$h \in \partial B_1(0)$, $h \in \mathbb{R}^n$ and c_α is the suitable normalizing constant, see Lemma 26.2 in [50] and also [22] for an equivalent statement. In addition, we remark that in [50] further relationships between Marchaud derivative and hypergeometric integrals are studied.

2 Extension approach to nonlocal operators

In this section we discuss few cases in which we adapt the extension approach to the Marchaud derivative and to Carnot groups, respectively introduced in Subsection 2.1 and Subsection 2.3. Moreover, for dealing with Carnot groups, we arranged the preparatory Subsection 2.2, where we introduce the main notations and definitions useful in that non-commutative field.

2.1 The Marchaud case

In [12], the authors observed that the fractional Laplace operator can be represented as well as:

$$(-\Delta)^\alpha f(x) = C \lim_{y \to 0} y^{1-2\alpha} \frac{\partial V(x, y)}{\partial y},$$

where $0 < \alpha < 1$ and V is the solution to the following problem:

$$\begin{cases} \operatorname{div}(y^{1-2\alpha} \nabla_{x,y} V) = 0, & \mathbb{R}^n \times \mathbb{R}^+, \\ V(x, 0) = f, & x \in \mathbb{R}^n, \end{cases}$$

and C a suitable constant. We will come back to this point in Section 2.2 discussing the fractional Laplace case.

This extension approach, in defining the Weyl-Marchaud derivative, has been faced in [10]. We describe below the rough idea in case $\alpha = \frac{1}{2}$ for obtaining $\mathbf{D}^{\frac{1}{2}}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a given function, sufficiently smooth. Let U be a solution to the problem

$$\begin{cases} \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, & (x, t) \in (0, \infty) \times \mathbb{R} \\ U(0, t) = \varphi(t), \ t \in \mathbb{R}. \end{cases} \tag{5}$$

We point out that (5) is not the usual Cauchy problem associated with the heat operator, but a heat conduction problem.

It is known that, without extra assumptions, we can not expect to have a unique solution of the problem (5), see e.g. [53]. Nevertheless, if we denote by $T_{1/2}$ the operator that associates to φ the partial derivative $\frac{\partial U}{\partial x}$, whenever U is sufficiently regular, we have that

$$T_{1/2}T_{1/2}\varphi = \frac{d\varphi}{dt}.$$

That is the operator $T_{1/2}$ acts like an half derivative, that is a fractional derivative of order $1/2$, indeed

$$\frac{\partial}{\partial x}\frac{\partial U}{\partial x}(x,t) = \frac{\partial U}{\partial t}(x,t) \underset{x\to 0}{\longrightarrow} \frac{d\varphi(t)}{dt}.$$

The solution of problem (5) under the reasonable assumptions that φ is bounded and Hölder continuous, is explicitly known (check e.g. [53]) to be

$$\begin{aligned} U(x,t) &= cx\int_{-\infty}^{t} e^{-\frac{x^2}{4(t-\tau)}}(t-\tau)^{-\frac{3}{2}}\varphi(\tau)\,d\tau \\ &= cx\int_0^\infty e^{-\frac{x^2}{4\tau}}\tau^{-\frac{3}{2}}\varphi(t-\tau)\,d\tau, \end{aligned} \tag{6}$$

where (6) is obtained performing a change of variable.

We get moreover that

$$\int_0^\infty xe^{-\frac{x^2}{4\tau}}\tau^{-\frac{3}{2}}\,d\tau = 2\int_0^\infty e^{-t}t^{-\frac{1}{2}}\,dt = 2\Gamma(\frac{1}{2}).$$

Hence,

$$\frac{U(x,t)-U(0,t)}{x} = c\int_0^\infty e^{-\frac{x^2}{4\tau}}\tau^{-\frac{3}{2}}\left(\varphi(t-\tau)-\varphi(t)\right)d\tau, \tag{7}$$

choosing c that takes into account the right normalization. This yields, by passing to the limit, that

$$-\lim_{x\to 0^+}\frac{U(x,t)-U(0,t)}{x} = c\int_0^\infty \frac{\varphi(t)-\varphi(t-\tau)}{\tau^{\frac{3}{2}}}d\tau, \tag{8}$$

that, possibly up to a multiplicative constant, is exactly $\mathbf{D}^{\frac{1}{2}}\varphi$.

The previous description enters as a particular case of the following results proved in [10], see also [7],

Theorem 1 *Let $s\in(0,1)$ and $\varphi\colon\mathbb{R}\to\mathbb{R}$ be a bounded, locally $C^{\bar\gamma}$ function for $s<\bar\gamma\le 1$. Let $U\colon[0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a solution to the problem*

$$\begin{cases} \frac{\partial U(x,t)}{\partial t} = \frac{1-2s}{x}\frac{\partial U(x,t)}{\partial x} + \frac{\partial^2 U(x,t)}{\partial x^2}, & (x,t)\in(0,\infty)\times\mathbb{R} \\ U(0,t)=\varphi(t), & t\in\mathbb{R} \\ \lim_{x\to\infty}U(x,t)=0, & t\in\mathbb{R}. \end{cases} \tag{9}$$

Then U defines the extension operator for ϕ, such that

$$\mathbf{D}^s\varphi(t) = -\lim_{x\to 0^+} c_s x^{-2s}(U(x,t)-\varphi(t)), \tag{10}$$

where

$$c_s = 4^s\Gamma(s).$$

Adapting the constant c_s given in Theorem 1 by fixing $c_s = \frac{4^s\Gamma(s)s}{\Gamma(1-s)}$, in

$$\mathbf{D}^s\varphi(t) = -\lim_{x\to 0^+} c_s x^{1-2s}\frac{\partial U}{\partial x}(x,t), \tag{11}$$

in analogy with formula (3.1) in [12] we straightforwardly obtain the definition

$$\mathbf{D}^s_{\pm} f(t) = \frac{s}{\Gamma(1-s)}\int_0^\infty \frac{f(t)-f(t\mp\tau)}{\tau^{1+s}}d\tau. \tag{12}$$

The advantage of this choice is that $\mathbf{D}^s_{\pm}\varphi \to \varphi$ as $s\to 0^+$ and $\mathbf{D}^s_{\pm}\varphi \to \varphi'$ as $s\to 1^-$. Eventually, for recent result about the regularity theory in this framework, see [2].

2.2 Carnot setting

In this subsection, preparatory to the next one, we introduce the basic language for dealing with Carnot groups. We recall that fractional operators may be defined also for degenerate PDEs associated with non-negative quadratic forms on non-commutative structures. In fact, a stratified Carnot group of step m $(\mathbb{G},\circ)$ is a set, in general endowed with a non-commutative law and a Lie algebra g with m stratifications. More precisely there exist $\{g_i\}_{1\le i\le m}$, $m\in\mathbb{N}$, $m\le N\in\mathbb{N}$, vector spaces such that:

$$g_1\bigoplus g_2\bigoplus\cdots\bigoplus g_m = g \equiv \mathbb{R}^N \equiv \mathbb{G},$$

$$[g_1,g_1]=g_2,\quad [g_1,g_2]=g_3,\quad \dots,[g_1,g_{m-1}]=g_m\neq\{0\}$$

and

$$[g_1,g_m]=0.$$

Moreover

$$x\in\mathbb{G}\equiv\mathbb{R}^N=\mathbb{R}^{k_1}\times\dots\mathbb{R}^{k_m},\quad \sum_{j=1}^m k_j = N$$

and $\sum_{j=1}^m jk_j = Q$ is called the homogeneous dimension. For every $\lambda>0$ is defined the anisotropic dilation:

$$\delta_\lambda(x) = (\lambda x^{(1)},\lambda^2x^{(2)},\dots,\lambda^m x^{(m)}),\quad \text{where}\quad x^{(j)}\in\mathbb{R}^{k_j},\quad j=1,\dots,m.$$

In addition if $Z_1, \ldots, Z_{k_1} \in g_1$ are left invariant vector fields such that $Z_j(0) = \frac{\partial}{\partial x_j}_{|x=0}$, $j = 1, \ldots, k_1$ then

$$\mathrm{rank}(\mathrm{Lie}\{Z_1, \ldots, Z_{k_1}\})(x) = N, \quad \text{(Hörmander condition)}$$

for every $x \in \mathbb{R}^N \equiv \mathbb{G}$. Let us consider the sublaplacian on the stratified Carnot group $\mathbb{G}$ given by

$$\mathcal{L}_{\mathbb{G}} \equiv -\Delta_{\mathbb{G}} = -\sum_{j=1}^{k_1} X_j^2, \tag{13}$$

where $\mathrm{span}\{X_1, \ldots, X_{k_1}\} = g_1$.

In particular there exists a $N \times k_1$ matrix σ such that $\sigma \cdot \sigma^T$ is a $N \times N$ matrix such that

$$\mathrm{div}(\sigma \cdot \sigma^T \nabla \cdot) = \Delta_{\mathbb{G}}.$$

Moreover

$$\sigma^T \nabla u = \sum_{j=1}^{k_1} X_j u X_j \equiv \nabla_{g_1} u,$$

is the so called horizontal gradient of u.

Hence

$$A = \sigma \cdot \sigma^T$$

The simplest example is the Heisenberg group, that is

$$\mathbb{G} = \mathbb{H}^1 \equiv \mathbb{R}^3, \quad (\mathbb{H}^1, \circ)$$

where for every $(x_1, y_1, t_1), (x_2, y_2, t_2) \in \mathbb{H}^1$

$$(x_1, y_1, t_1) \circ (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(x_2 y_1 - x_1 y_2)).$$

The opposite of $\xi := (x, y, t) \in \mathbb{H}^1$ is usually denoted by ξ^{-1} and $\xi^{-1} := (-x, -y, -t)$ and the dilation by $\lambda > 0$ is: $\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t)$.

Moreover,

$$g_1 = \mathrm{span}\{X, Y\}, \quad g_2 = \mathrm{span}\{T\}$$

where $X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}$, $Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$, and $T = -4\frac{\partial}{\partial t}$, and

$$[X, Y] = T,$$

so that $[g_1, g_1] = g_2$, $[g_1, g_2] = \{0\}$,

$$g_1 \bigoplus g_2 = \mathbb{R}^3.$$

Hence, Heisenberg group is a 2 step groups and g_1 is the first layer, namely the horizontal vector space. Moreover

$$\Delta_{\mathbb{H}^1} = X^2 + Y^2 = \frac{\partial^2}{\partial x^2} + 4y\frac{\partial^2}{\partial x \partial t} + \frac{\partial^2}{\partial y^2} - 4x\frac{\partial^2}{\partial y \partial t} + 4(x^2+y^2)\frac{\partial^2}{\partial t^2}$$

Thus

$$\Delta_{\mathbb{H}^1} = \mathrm{Tr}\left(\begin{bmatrix} 1, & 0, & 2y \\ 0, & 1, & -2x \\ 2y, & -2x, & 4(x^2+y^2) \end{bmatrix}\right) = \mathrm{div}\left(\begin{bmatrix} 1, & 0, & 2y \\ 0, & 1, & -2x \\ 2y, & -2x, & 4(x^2+y^2) \end{bmatrix}\nabla u\right)$$
$$= \mathrm{div}\left(\begin{bmatrix} 1, & 0 \\ 0, & 1 \\ 2y, & -2x \end{bmatrix}\begin{bmatrix} 1, \ 0, \ 2y \\ 0, \ 1, \ -2x \end{bmatrix}\nabla u\right) = X^2u + Y^2u.$$

In our framework

$$\mathcal{L} \equiv -\Delta_{\mathbb{G}} := -\sum_{j=1}^{k_1} X_j^2,$$

and considering the example given by the Heisenberg group, we have $k_1 = 2$, $X_1 \equiv X$ and $X_2 \equiv Y$, so that $\mathcal{L} \equiv -\Delta_{\mathbb{H}^1}$.

We point out that with this presentation we include, as a very particular case, the Laplace operator on $\mathbb{R}^n$ that is a commutative structure. The main difference with the general elliptic case, usually given by the Laplace operator, for instance considering the Kohn-Laplace operator on the Heisenberg group $\mathbb{H}^n$ is that $\Delta_{\mathbb{H}^n}$ represents a degenerate operator associated with a non-negative quadratic form having the smallest eigenvalue always zero. This structure is in some way associated with the quantum description of a system of moving particles with classical position and momentum coordinates, see [26], [47]. The fundamental contribution about the analytic properties of sublaplacians (13) can be found in [36]. Thus, the properties of fractional operators in this framework is particularly interesting.

Thus, following [25], Section 3, see also [57, 37, 5], it results:

Theorem 2 *The operator $\mathcal{L}$, see* (13)*, is a positive self-adjoint operator with domain $W^{2,2}_{\mathbb{G}}(\mathbb{G})$. Denote now by $\{E(\lambda)\}$ the spectral resolution of $\mathcal{L}$ in $L^2(\mathbb{G})$. If $\alpha > 0$ then*

$$\mathcal{L}^{\alpha/2} = \int_0^{+\infty} \lambda^{\alpha/2} dE(\lambda)$$

with domain

$$W^{\alpha,2}_{\mathbb{G}}(\mathbb{G}) := \{u \in L^2(\mathbb{G}) \ : \ \int_0^{+\infty} \lambda^\alpha d\langle E(\lambda)u, u\rangle < \infty\},$$

endowed with the graph norm.

Let us denote by $h = h(t, x)$ the fundamental solution of $\mathcal{L} + \partial/\partial t$. Recall that Q denotes the homogeneous dimension as well. Then:

Theorem 3 ([25], Proposition 3.3)

Suppose $Q \geq 3$ and $0 < \beta < Q$. Then the integral

$$R_\beta(x) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\frac{\beta}{2}-1} h(t,x)\,dt$$

converges absolutely for $x \neq 0$*. In addition,* R_β *is a kernel such that:*

i) R_2 *is the* $\mathcal{L}$ *fundamental solution;*
ii) if $\alpha \in (0,2)$ *and* $u \in \mathcal{D}(\mathbb{G})$*, then* $\mathcal{L}^{\alpha/2}u = \mathcal{L}u * R_{2-\alpha}$*.*
iii)the kernels R_α *admit the following convolution rule: if* $\alpha > 0$*,* $\beta > 0$ *and* $x \neq 0$*, then* $R_{\alpha+\beta}(x) = R_\alpha(x) * R_\beta(x)$*.*

In this case we cannot apply straightforwardly the Fourier transform, because the operator may have variable coefficients. We faced in [23] the problem, see also [12], by considering the following result.

Lemma 1 *If* $-\infty < \alpha < 1$*, the boundary value problem*

$$\begin{cases} -t^\alpha \phi'' + \phi = 0 \\ \phi(0) = 1 \\ \lim_{t\to+\infty} \phi(t) = 0 \end{cases} \tag{14}$$

has a solution $\phi \in \mathbf{C}^{2-\alpha}([0,\infty))$ *of the form*

$$\phi(t) = c_\alpha\, t^{1/2} K_{1/2k}(k^{-1}t^k),$$

where $c_\alpha := 2^{1-1/2k}\Gamma(1/2k)^{-1}k^{-1/2k} > 0$ *is a positive constant,* $k = \frac{2-\alpha}{2}$*, and* $K_{1/2k}$ *is the modified Bessel function of second kind (see [56]).*

In addition:

i) $0 < \phi < 1$. Moreover $\phi'(t)$ has a finite limit as $t \to 0$ and, recursively,

$$t^{\alpha+h-2}\phi^{(h)}(t) \text{ has a finite limit as } t \to 0$$

for $h = 2, 3, \ldots$;
ii) $\phi' \in L^2((0,\infty))$;
iii) $\phi(t) = c\sqrt{\frac{\pi k}{2}}\, t^{\alpha/2}\, e^{-t^k/k}\left(1 + O(\frac{1}{t})\right)$ as $t \to \infty$;
iv) $\phi^{(h)}(t) = c_h\, t^{\alpha(1-h)/2}\, e^{-t^k/k}\left(1 + o(1)\right)$ as $t \to \infty$ for $h = 1, 2, \ldots$.

The problem (14) in Lemma 1 takes the place of the problem that we obtain when we apply the Fourier transform to the Laplace operator.

We explain now why in Carnot groups we need to a new tool that takes the place of the Fourier transform and how the extension approach may be applied as it will be clear reading the subsequent explanation.

Following [12], for studying problem (14), we consider the Euclidean case for the Laplace operator Δ in $\mathbb{R}^n$. Recalling the argument that we have already described for the extension problem in the construction of the Marchaud derivative, we reduce, supposing to deal with the fractional operator associated with Δ, to consider the following PDE:

$$\Delta_x U + \frac{a}{y}\frac{\partial U}{\partial y} + \frac{\partial^2 U}{\partial y^2} = 0, \quad \text{in} \quad \mathbb{R}^n \times (0, +\infty). \tag{15}$$

Then applying the Fourier transform to (15) with respect to x, we obtain:

$$\mathcal{F}\Delta_x U + \mathcal{F}\frac{a}{y}\frac{\partial U}{\partial y} + \mathcal{F}\frac{\partial^2 U}{\partial y^2} = 0. \tag{16}$$

Thus it results:

$$-|\xi|^2 \mathcal{F}U(\xi, y) + \frac{a}{y}\frac{\partial \mathcal{F}U(\xi, y)}{\partial y} + \frac{\partial^2 \mathcal{F}U(\xi, y)}{\partial y^2} = 0.$$

If $U(x, 0) = u$, by considering the problem,

$$\begin{cases} v'' + \frac{a}{y}v' = |\xi|^2 v \\ v(0) = 1 \\ \lim_{y\to\infty} v(y) = 0, \end{cases} \tag{17}$$

then it results, (knowing the solution v to (17))) via a scaling argument:

$$\mathcal{F}U = \mathcal{F}uv(|\xi|y).$$

If $a = 0$ then $v = e^{-|\xi|y}$ and

$$U(x, y) = u * \mathcal{F}^{-1}(e^{-|\xi|y}).$$

In general:

$$\mathcal{F}^{-1}(v(|\xi|y)) \equiv P_a(x, y) = C_{n,a}\frac{y^{1-a}}{(|x|^2 + |y|^2)^{\frac{n+1-a}{2}}}.$$

and we obtain:

$$U(x, y) = u * P_a(\cdot, y).$$

Now: with a change of variable we transform problem (17) in problem (14).

Unfortunately, in our Carnot groups framework, we can not use the classical Fourier transform having a dependence on the coefficients that is much complicate with respect to the benchmark case represented by the Laplace operator in the Euclidean setting. Thus, we start from Lemma 1 for applying the approach described in [23] in Carnot groups.

2.3 The extension approach in Carnot groups

We have already pointed out in the previous Subsection 2.2 that we cannot apply the Fourier transform, so we use the solution ϕ of problem (1) to define a new operator via the spectral resolution of $\mathcal{L}$, see (13) for recalling the definition.

Hence, for every $u \in L^2(\mathbb{G})$ and $y > 0$, we set, see [23]:

$$v(\cdot, y) := \phi(\theta y^{1-a}\, \mathcal{L}^{(1-a)/2})u := \int_0^\infty \phi(\theta y^{1-a}\, \lambda^{(1-a)/2})\, dE(\lambda)u, \tag{18}$$

where $\theta := (1-a)^{a-1}$, ϕ solves (14), and $\{E(\lambda)\}$ is the spectral resolution of $\mathcal{L}$ in $L^2(\mathbb{G})$ and therefore $v \in L^2(\mathbb{G})$ for $y > 0$.

Moreover we proved in [23] the following result.

Theorem 4 (generalized subordination identity)

Let $h(t, \cdot)$ be the heat kernel associated with $\mathcal{L} + \frac{\partial}{\partial t}$. We denote by $P_{\mathbb{G}}(\cdot, y)$ the "Poisson kernel"

$$P_{\mathbb{G}}(\cdot, y) := C_a\, y^{1-a} \int_0^\infty t^{(a-3)/2} e^{-\frac{y^2}{4t}}\, h(t, \cdot)\, dt, \tag{19}$$

where

$$C_a = \frac{2^{a-1}}{\Gamma((1-a)/2)}.$$

Then

$$P_{\mathbb{G}}(\cdot, y) \geq 0,$$

and

$$v(\cdot, y) = u * P_{\mathbb{G}}(\cdot, y). \tag{20}$$

The last part is a consequence of some results contained in [37], [25], Theorem 3.1. In addition, we recall that the existence of the heat kernel h is proved in [25]. As a byproduct of Theorem 4, we obtain, see [23], that

$$\mathcal{L}^{\frac{1-a}{2}} u(x) = \tilde{C}_a \int_{\mathbb{G}} (u(\xi) - u(x)) \tilde{R}_{a-1}(\xi) d\xi. \tag{21}$$

In fact, whenever u is sufficiently smooth:

$$\begin{aligned} y^a \frac{v(x,y) - v(x,0)}{y} &= y^a \frac{u * P_{\mathbb{G}}(\cdot, y) - u(x)}{y} \\ &= \Big(C_a \int_0^\infty t^{(a-3)/2} e^{-\frac{y^2}{4t}} u * h(t, \cdot)\, dt \\ &\quad - C_a u(x) \int_{\mathbb{G}} \int_0^\infty t^{(a-3)/2} e^{-\frac{y^2}{4t}} h(t, \xi^{-1}x)\, dt d\xi \Big) \\ &= C_a \int_{\mathbb{G}} \int_0^\infty t^{(a-3)/2} e^{-\frac{y^2}{4t}} h(t, \xi^{-1}x)\, dt (u(\xi) - u(x)) d\xi. \end{aligned} \tag{22}$$

On the other hand

$$\lim_{y \to 0^+} C_a \int_0^\infty t^{(a-3)/2} e^{-\frac{y^2}{4t}} h(t, \xi^{-1}x)\, dt = \tilde{C}_a \tilde{R}_{a-1}. \tag{23}$$

Thus

$$\lim_{y\to 0^+} y^a \frac{v(x,y)-v(x,0)}{y} = C_a \int_{\mathbb{G}} (u(\xi)-u(x))\tilde{R}_{a-1}(\xi)d\xi = \tilde{C}_a \mathcal{L}^{\frac{1-a}{2}} u(x),$$

where

$$\widetilde{R}_\beta(x) = \frac{\frac{\beta}{2}}{\Gamma(\beta/2)} \int_0^\infty t^{\frac{\beta}{2}-1} h(t,x)\, dt. \tag{24}$$

It boils down:

$$\mathcal{L}^{\frac{1-a}{2}} u(x) = \lim_{y\to 0^+} y^a \frac{v(x,y)-v(x,0)}{y} = \tilde{C}_a \int_{\mathbb{G}} (u(\xi)-u(x))\tilde{R}_{a-1}(\xi)d\xi.$$

This construction is very general, nevertheless it is not always easy to write explicitly the kernel $\tilde{R}_{a-1}$ even in the simplest non-commutative case like in the Heisenberg group $\mathbb{H}^1$. In fact, the heat kernel in the Heisenberg group $\mathbb{H}^1$ is written via an integral. More precisely, if $(z,s) \in \mathbb{C}\times\mathbb{R} \equiv \mathbb{H}^1$, then the heat kernel h of $-\Delta_{\mathbb{H}^1} + \frac{\partial}{\partial t}$ in $]0,+\infty[\times\mathbb{H}^1$ is:

$$h(t,(z,s)) = (4\pi t)^{-2} \int_{\mathbb{R}} e^{-\frac{f(z,s,\kappa)}{t}} V(\kappa)d\kappa,$$

where

$$f(z,s,\kappa) = \frac{1}{2}(-i\kappa s + \frac{\kappa|z|^2}{2\tanh(2\kappa)})$$

and

$$V(\kappa) = \frac{2\kappa}{\sinh(2\kappa)}.$$

Thus, recalling (24), we obtain in the Heisenberg case $\mathbb{H}^1$ the following kernel:

$$\widetilde{R}_\beta(z,s) = \frac{\frac{\beta}{2}}{\Gamma(\beta/2)} \int_0^\infty t^{\frac{\beta}{2}-1} \left((4\pi t)^{-2} \int_{\mathbb{R}} e^{-\frac{f(z,s,\kappa)}{t}} V(\kappa)d\kappa \right) dt. \tag{25}$$

It is clear that the double integration in the representation of $\widetilde{R}_\beta$ produces some technical difficulties that do not appear in dealing with the usual heat kernel in $\mathbb{R}^3$.

In fact, if $(\mathbb{G},\circ) = (\mathbb{R}^3,+)$, then for every $\alpha > 0$ it results

$$\widetilde{R}_\alpha(x) = -\frac{\alpha}{2\Gamma(-\alpha/2)} \frac{1}{(4\pi)^{\frac{3}{2}}} \int_0^\infty t^{-\alpha/2-3/2-1} e^{-\frac{|x|^2}{4t}}\, dt,$$

being $\frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4t}}$ the heat kernel of $-\Delta + \frac{\partial}{\partial t}$ in $]0,+\infty[\times\mathbb{R}^3$. Thus, after a changing of variables, we obtain:

$$\begin{aligned} \widetilde{R}_\alpha(x) &= -\frac{\alpha}{2\Gamma(-\alpha/2)} \frac{4^{\frac{\alpha}{2}+\frac{3}{2}}}{(4\pi)^{\frac{3}{2}}} |x|^{-\alpha-3} \int_0^\infty y^{\frac{1+\alpha}{2}} e^{-y} dy \\ &= -\frac{\alpha}{2\Gamma(-\alpha/2)} \frac{4^{\frac{\alpha}{2}+\frac{3}{2}}}{(4\pi)^{\frac{3}{2}}} \Gamma\Big(\frac{\alpha+3}{2}\Big)|x|^{-\alpha-3}. \end{aligned} \tag{26}$$

Moreover, denoting $c := -\frac{\alpha}{2\Gamma(-\alpha/2)}\frac{4^{\frac{\alpha}{2}}}{\pi^{\frac{3}{2}}}\Gamma\Big(\frac{\alpha+3}{2}\Big)$, we get the usual kernel that appears in the representation of $(-\Delta)^{\frac{\alpha}{2}}$ that is:

$$\widetilde{R}_\alpha(x) = c\frac{1}{|x|^{3+\alpha}},$$

see (4) when $n = 3$.

As far as we are concern, we don't know if a simpler representation of the kernel (25) exists when we are in a non-commutative group.

We conclude this section pointing out that some tentatives to define the fractional operator, starting from the notion of the intrinsic translation, has been done in [21]. Moreover, it is known that in CR structures the extension can be done with a little different approach, see [28].

3 Applications

In this section we discuss some applications of the extension method to solutions of nonlocal operators. In particular in Subsection 3.1 we review the Harnack inequality for positive solutions of the equation $\mathbf{D}^\alpha u = 0$ in $I \subset \mathbb{R}$. In Subsection 3.2 we recall how to prove Harnack inequality for $\alpha-$harmonic function in Carnot groups, while in Subsection 3.3 we revisit an application of the extension approach to the notion of perimeter in Carnot groups.

3.1 Weyl-Marchaud derivative: the Harnack inequality

Concerning the PDE of the problem (9), in this particular case, the conductivity coefficient (i.e. the coefficient in front of the x derivative) and the specific heat (the coefficient of the t derivative) coincide. In [16], this type of equation has been studied in a more general framework. In fact a more general form of that equation is given as follows:

$$w(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(a(x)\frac{\partial U}{\partial x}). \tag{27}$$

We assume that:

$$\lambda^{-1}w(x) \le a(x) \le \lambda w(x)$$

and that the following integrability condition (known as a Muckehoupt, or A_2 weight condition) on the weight w holds as well:

$$\sup_J(\frac{1}{|J|}\int_J w(x)\,dx)\,(\frac{1}{|J|}\int_J \frac{1}{w(x)}\,dx) = c_0 < \infty, \tag{28}$$

for every interval $J \subseteq (-R, R)$. The constant c_0 is indicated as the A_2 constant of w. In our case, of course, we are left with the condition (28). In [16] the authors also proved the following Harnack inequality.

Theorem 5 (Chiarenza-Serapioni) *Let U be a positive solution in $(-R, R) \times (0, T)$ of the equation in (9) and assume that condition (28) holds, with constant c_0. Then there exists $\gamma = \gamma(c_0) > 0$ such that*

$$\sup_{(\frac{\rho}{2},\frac{\rho}{2})\times(t_0-\frac{3\rho^2}{4},t_0-\frac{\rho^2}{4})} U \leq \gamma \inf_{(\frac{\rho}{2},\frac{\rho}{2})\times(t_0+\frac{3\rho^2}{4},t_0+\rho^2)} U \tag{29}$$

holds for $t_0 \in (0, T)$ and any ρ such that $0 < \rho < R/2$ and $[t_0 - \rho^2, t_0 + \rho^2] \subset (0, T)$.

As a consequence, in [10], has been proved the following Harnack inequality for the Marchaud derivative:

Corollary 1 *Let $s \in (0, 1)$. There exists a positive constant γ such that, if $\mathbf{D}^s\phi = 0$ in $J \subseteq \mathbb{R}$ and $\phi \geq 0$ in $\mathbb{R}$, then*

$$\sup_{[t_0-\frac{3}{4}\delta,t_0-\frac{1}{4}\delta]} \phi \leq \gamma \inf_{[t_0+\frac{3}{4}\delta,t_0+\delta]} \phi \tag{30}$$

for every $t_0 \in \mathbb{R}$ and for every $\delta > 0$ such that $[t_0 - \delta, t_0 + \delta] \subset J$.

3.2 Fractional operators of sublaplacians in Carnot groups: the Harnack inequality

Having in hand the characterization of the fractional operator recalled in (21) and (22), we may reduce ourselves to work with local operator (as well as we have already remarked for the Marchaud derivative), see also [23] for this part. In fact, if Y is the following vector field $\frac{\partial}{\partial y}$ and $\hat{\mathbb{G}} := \mathbb{G} \times \mathbb{R}$, then $\hat{\mathbb{G}}$ is still a Carnot group and its Lie algebra $\hat{g}$ admits the stratification

$$\hat{g} = \hat{g}_1 \bigoplus g_2 \bigoplus \cdots \bigoplus g_m,$$

where $\hat{g}_1 = \text{span}\{Y, g_1\}$.

Then the following result holds.

Theorem 6 ([23])

Let $u \in W^{1-a,2}_{\mathbb{G}}(\mathbb{G})$ be given, $u \geq 0$, and assume $\mathcal{L}^{(1-a)/2}u = 0$ in an open set Ω. Denoting by $\hat{v}$ the function on $\hat{\mathbb{G}}$ obtained by continuing v by parity across $y = 0$. Then

i) $\hat{v} \geq 0$;
ii) $\hat{v} \in W^{1,2}_{\hat{\mathbb{G}},\text{loc}}(\hat{\Omega};\ y^a dx\, dy)$, *where* $\hat{\Omega} := \Omega \times (-1, 1)$;

iii) $\hat{v}$ *is a weak solution of the equation*

$$div_{\hat{\mathbb{G}}}(|y|^a \nabla_{\hat{\mathbb{G}}} \hat{v}) = 0 \quad in\ \hat{\Omega}. \tag{31}$$

We recall the following well known definition in the study of operators with weights.

Definition 1 (see [17]) A function $\omega \in L^1_{\text{loc}}(\mathbb{G})$ is said to be a A_2-weight with respect to the cc-metric of $\mathbb{G}$ if

$$\sup_{x\in\mathbb{G},\, r>0} |B_c(x,r))|^{-1} \int_{B_c(x,r)} \omega(y)\, dy \cdot |B_c(x,r))|^{-1} \int_{B_c(x,r)} \omega(y)^{-1}\, dy < \infty.$$

Remark. The function $\omega(x,y) = |y|^a$ is a A_2-weight with respect to the CC-metric of $\mathbb{G} \times \mathbb{R}$ if and only if $-1 < a < 1$.

The following result is well known in literature, see: [34, 35, 42].

Theorem 7 *Let* $\mathbb{G}$ *be a Carnot group, and let* $\Omega \subset \mathbb{G}$ *be an open set. Let now* $\omega \in L^1_{\text{loc}}(\mathbb{G})$ *be a* A_2*-weight with respect to the Carnot-Carathéodory metric* d_c *of* $\mathbb{G}$*. If* $u \in W^{1,2}_{\mathbb{G}}(\Omega, \omega dx)$ *is a weak solution to*

$$\text{div}_{\mathbb{G}}\,(\omega\, \nabla_{\mathbb{G}} u) = 0, \tag{32}$$

then u is locally Hölder continuous in Ω*. Moreover, if* $u \geq 0$*, then there exist* $C, b > 0$ *(independent of u) such that the following invariant Harnack inequality holds:*

$$\sup_{B_c(x,r)} u \leq C \inf_{B_c(x,r)} u \tag{33}$$

for any metric ball $B_c(x,r)$ *such that* $B_c(x,br) \subset \Omega$*.*

In addition, if Ω satisfies the following local condition: for any $x_0 \in \partial\Omega$ there exist $r_0 > 0$ and $\alpha > 0$ such that

$$|B_c(x_0,r) \cap \Omega^c| \geq \alpha |B_c(x_0,r)| \quad \text{for } r < r_0.$$

Then u is locally Hölder continuous in $\overline{\Omega}$. Thus, applying Theorem 7 we obtain the Harnack inequality. In fact we get the following theorem.

Theorem 8 ([23])

Let $-1 < a < 1$ *and let* $u \in W^{1-a,2}_{\mathbb{G}}(\mathbb{G})$ *be given,* $u \geq 0$ *on all of* $\mathbb{G}$*. Assume* $\mathcal{L}^{(1-a)/2}u = 0$ *in an open set* $\Omega \subset \mathbb{G}$*.*

Then there exist $C, b > 0$ *(independent of u) such that the following invariant Harnack inequality holds:*

$$\sup_{B_c(x,r)} u \leq C \inf_{B_c(x,r)} u$$

for any metric ball $B_c(x,r)$ *such that* $B_c(x,br) \subset \Omega$*.*

In fact, the proof of Theorem 8 is consequence of the following argument. Since the function (18), that may be written as (20) as well, after to be prolonged by parity and denoted by $\hat{v}$, is solution to the local problem (31) in an extended set obtained by parity. Then, recalling that $\hat{v}$ is positive, the Harnack inequality holds true for $\hat{v}$ applying the well known theory of the operators with weights, see Theorem 7. In this way, we straightforwardly obtain the desired inequality from (33), because $u(x) = \hat{v}(x, 0) = v(x, 0)$.

3.3 Carnot groups: a perimeter notion

Having in mind the previous results described in Section 2.2, we would like to show some applications of them to fractional perimeter in Carnot groups. This is a first tentative of extending a research theme already explored in the Euclidean case, see e.g. [11], to the non-commutative setting.

We start as usual recalling some remarks $\mathbb{R}^n$. Let $h_\alpha(t, z)$ be the fundamental solution of the fractional heat equation in $\mathbb{R}_+ \times \mathbb{R}^n$

$$u_t + (-\Delta)^\alpha u = 0. \tag{34}$$

Setting $\tilde{h}_\alpha(z) = h_\alpha(1, z)$, h_α satisfies

$$\int_{\mathbb{R}^n} h_\alpha(t, z)\, dz = 1 \qquad \forall\, t > 0, \qquad\qquad h_\alpha(t, z) = \frac{1}{t^{n/2\alpha}} \tilde{h}_\alpha(t^{-1/2\alpha} z) \tag{35}$$

and

$$\lim_{t \to 0} \frac{h_\alpha(t, x)}{t} = \frac{C_{n,\alpha}}{|x|^{n+2\alpha}}, \tag{36}$$

see Theorem 2.1 in [9], where the exact value of the constant is given. The fractional heat semigroup that gives the solution of (34) with initial datum f is given by

$$e^{-t(-\Delta)^\alpha} f(x) = \int_{\mathbb{R}^n} h_\alpha(t, y) f(x - y)\, dy, \qquad f \in L^1(\mathbb{R}^n),$$

and, since the kernel h_α has integral one, we have

$$e^{-t(-\Delta)^\alpha} f(x) - f(x) = \int_{\mathbb{R}^n} h_\alpha(t, y) \big(f(x - y) - f(x)\big)\, dy.$$

On the other hand:

$$
\begin{aligned}
&\|(-\Delta)^{\alpha/2} f\|^2_{L^2(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} f(-\Delta)^\alpha f\, dx = C(n,\alpha) \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2\alpha}}\, dy dx \\
&= \frac{C(n,\alpha)}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\, dy dx = \frac{C(n,\alpha)}{2} [f]^2_{W^{\alpha,2}}.
\end{aligned}
\tag{37}
$$

Thus we consider the following quantity

$$
Q_t^\alpha(f) = \int_{\mathbb{R}^n\times\mathbb{R}^n} h_\alpha(t,y)\Big(f(x-y)-f(x)\Big)^2\, dx\, dy.
$$

Using (36) we get

$$
\lim_{t\to 0} \frac{Q_t^\alpha(f)}{t} = C_{n,\alpha}[f]^2_{W^{\alpha,2}(\mathbb{R}^n)}.
$$

Hence we have that $f \in W^{\alpha,2}(\mathbb{R}^n)$ if and only if

$$
\lim_{t\to 0} \frac{Q_t^\alpha(f)}{t} < \infty,
$$

see [4] and [18].

The following properties have been proved in [27].

Theorem 9 *There exists a function h defined in $\hat{\mathbb{G}}$ such that:*

(i) $h \in C^\infty(\hat{\mathbb{G}} \setminus \{(0,0)\})$
(ii) $h(\lambda^2 t, \delta_\lambda(x)) = \lambda^{-Q} h(t,x)$ *for every* $t > 0$, $x \in \mathbb{G}$ *and* $\lambda > 0$;
(iii) $h(t,x) = 0$ *for every* $t < 0$ *and* $\int_{\mathbb{G}} h(t,x)dx = 1$ *for every* $t > 0$;
(iv) $h(t,x) = h(t,x^{-1})$ *for every* $t > 0$ *and* $x \in \mathbb{G}$;
(v) there exists $c > 0$ *such that for every* $x \in \mathbb{G}$ *and* $t > 0$

$$
c^{-1} t^{-Q/2} \exp\Big(-\frac{\|x\|^2)}{c^{-1}t}\Big) \le h(x,t) \le c t^{-Q/2} \exp\Big(-\frac{\|x\|^2}{ct}\Big).
\tag{38}
$$

As well as in the Euclidean case, we introduce the heat semigroup

$$
e^{-t\mathcal{L}} f(x) := \int_{\mathbb{G}} h(t, y^{-1}\circ x) f(y) dy, \quad f \in L^1(\mathbb{G}).
\tag{39}
$$

For every $\alpha > 0$ let

$$
\tilde{R}_\alpha(x) := -\frac{\alpha}{2\Gamma(-\alpha/2)} \int_0^\infty t^{-\frac{\alpha}{2}-1} h(t,x) dt,
\tag{40}
$$

where

$$
R_\alpha(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1} h(t,x) dt.
$$

Then, see [23], $\tilde{R}_\alpha$ and R_α are smooth functions in $\mathbb{G}\setminus\{0\}$ and $\mathcal{L}R_{2-\alpha} = \tilde{R}_{-\alpha}$. In addition, $\tilde{R}_\alpha$ is positive and homogeneous of degree $-\alpha - Q$.

Moreover, using (iv) and (v) of Theorem 9, we get

$$\tilde{R}_\alpha(x) = \tilde{R}_\alpha(x^{-1}), \tag{41}$$

and

$$c^{-1}\|x\|^{-\alpha-Q} \le \tilde{R}_\alpha(x) \le c\|x\|^{-\alpha-Q} \qquad \forall x \in \mathbb{G}. \tag{42}$$

Thus, defining

$$\|x\|_\alpha := \left(\tilde{R}_\alpha(x)\right)^{-\frac{1}{\alpha+Q}}, \tag{43}$$

we deduce that $\|x\|_\alpha$ is a homogeneous symmetric norm because from (42) follows that there exists a constant $c > 0$, depending only on α, such that for every $x \in \mathbb{G}$

$$c^{-1}\|x\| \le \|x\|_\alpha \le c\|x\|.$$

After a straightforward calculation we obtain the following result.

Lemma 2 *If $u \in \mathcal{S}(\mathbb{G})$ then*

$$\mathcal{L}^\alpha u(x) = -\frac{1}{2}\int_{\mathbb{G}} \frac{u(x \circ y) + u(x \circ y^{-1}) - 2u(x)}{\|y\|_\alpha^{Q+2\alpha}} dy. \tag{44}$$

Moreover for any $u \in \mathcal{S}(\mathbb{G})$:

$$\lim_{\alpha \to 1^-} (1-\alpha)\mathcal{L}^\alpha u(x) = \mathcal{L}u(x), \qquad \forall x \in \mathbb{G}.$$

For the proof see e.g. [24] and [23]. The notion of α-horizontal perimeter in Carnot groups can be introduce as follow, see [24].

Definition 2 For a Borel set $E \subset \mathbb{G}$ and $\alpha \in (0,1)$ the fractional $\alpha-$horizontal perimeter of E is

$$\mathrm{Per}_{\alpha,\mathbb{G}}(E) := \int_E \int_{E^c} \frac{1}{\|y^{-1} \circ x\|_\alpha^{Q+\alpha}} dx dy.$$

We say that $E \subset \mathbb{G}$ has finite fractional $\alpha-$horizontal perimeter if $\mathrm{Per}_{\alpha,\mathbb{G}}(E) < \infty$.

We recall here, for permitting a quick comparison, that the fractional perimeter in $\mathbb{R}^n$ of a Borel set $E \subset \mathbb{R}^n$, in $\mathbb{R}^n$, assuming that $\alpha \in (0,1)$ is defined as follows:

$$\mathrm{Per}_{\alpha,\mathbb{R}^n}(E) := \int_E \int_{E^c} \frac{1}{|x-y|^{n+\alpha}} dx dy.$$

The interested reader may easily compare this definition keeping in mind the role of the kernel $\frac{1}{|x-y|^{n+\alpha}}$ in defining the fractional Laplace operator in $\mathbb{R}^n$, see (4). For further details about the notion of fractional perimeter of a set E in a set Ω that is not necessarily all of $\mathbb{R}^n$ see [1] and [13].

Moreover, continuing our description in Carnot groups, we remark that

$$\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|\chi_E(x) - \chi_E(y)|}{\|y^{-1}\circ x\|_\alpha^{Q+\alpha}} dxdy = \int_{E\cup E^c}\int_{E\cup E^c} \frac{|\chi_E(x) - \chi_E(y)|}{\|y^{-1}\circ x\|_\alpha^{Q+\alpha}} dxdy$$
$$= \int_E \int_{E^c} \frac{2}{\|y^{-1}\circ x\|_\alpha^{Q+\alpha}} dxdy.$$

So that, the function

$$Q_t^\alpha(\chi_E) = \int_{\mathbb{G}\times\mathbb{G}} h_\alpha(t,y)|\chi_E(y^{-1}\circ x) - \chi_E(x)|dxdy$$

establishes a relationship between the fractional heat semigroup and the fractional perimeter. In fact, see the following result whose detailed proof is given in [24].

Theorem 10 *There are constants* $c_1(\alpha), c_2(\alpha) > 0$ *such that for every Borel set* E *there holds.*

$$c_1(\alpha)\mathrm{Per}_{\alpha,\mathbb{G}}(E) \le \liminf_{t\to 0} \frac{Q_t^{\alpha/2}(\chi_E)}{t} \le \limsup_{t\to 0} \frac{Q_t^{\alpha/2}(\chi_E)}{t} \le c_2(\alpha)\mathrm{Per}_{\alpha,\mathbb{G}}(E). \tag{45}$$

More precisely, the upper estimates follow from [15] where it is proved that:

$$c^{-1}\Big(t^{-Q/2\alpha} \wedge \frac{t}{\|z\|^{Q+2\alpha}}\Big) \le h_\alpha(t,z) \le c\Big(t^{-Q/2\alpha} \wedge \frac{t}{\|z\|^{Q+2\alpha}}\Big), \tag{46}$$

and the following lemma that has been proved in [49], see Theorems 9 and 14.

Lemma 3 *Let* $u \in W^{\alpha/2,2}(\mathbb{G})$*; then there exists* $c_\alpha > 0$ *such that for all* $z \in \mathbb{G}$*, denoting by* $\tau_z u(x) := u(z^{-1}x)$*, there holds*

$$\|\tau_z u - u\|^2_{L^2(\mathbb{G})} \le c_\alpha \|z\|^\alpha \int_{\mathbb{G}\times\mathbb{G}} \frac{|u(x) - u(w^{-1}\circ x)|^2}{\|w\|^{Q+\alpha}} dxdw.$$

We sketch the proof of Theorem 10 for helping the reader, recalling that the detail can be found in [24]:

$$Q_t^{\alpha/2}(\chi_E) = \int_{\mathbb{G}\times\mathbb{G}} h_{\alpha/2}(t,z)|\chi_E(z^{-1}\circ x) - \chi_E(x)|dxdz$$
$$\le c_2 t^{-Q/\alpha} \int_{B(t^{1/\alpha})} \|z\|^\alpha [\chi_E]^2_{W^{\alpha/2,2}} dz$$
$$+ c_2 t \int_{B^c(t^{1/\alpha})\times\mathbb{G}} \frac{|\chi_E(z^{-1}\circ x) - \chi_E(x)|}{\|z\|^{Q+\alpha}} dxdz \le 2tc_2|B(1)|\mathrm{Per}_{\alpha,\mathbb{G}}(E)$$
$$+ c_2 t \int_{B^c(t^{1/\alpha})\times\mathbb{G}} \frac{|\chi_E(z^{-1}\circ x) - \chi_E(x)|}{\|z\|^{Q+\alpha}} dxdz \le c_2(\alpha) t \mathrm{Per}_{\alpha,\mathbb{G}}(E).$$

Concerning the lower bound, we start from (46), keeping in mind that on the complement of the ball $B(t^{\frac{1}{\alpha}})$ we have the estimate

$$h_{\alpha/2}(t,y) \geq c_1 \frac{t}{\|y\|^{Q+\alpha}},$$

so that we deduce

$$Q_t^{\alpha/2}(\chi_E) = \int_{\mathbb{G}\times\mathbb{G}} h_{\alpha/2}(t,y)|\chi_E(y^{-1}\circ x) - \chi_E(x)|dxdy$$
$$\geq c_1(\alpha)t \int_{\mathbb{G}\setminus B(t^{\frac{1}{\alpha}})} \int_{\mathbb{G}} \frac{|\chi(y^{-1}\circ x) - \chi_E(x)|}{\|y\|_\alpha^{Q+\alpha}} dxdy.$$

It follows

$$c_1(\alpha) \int_{\mathbb{G}\times\mathbb{G}} \frac{|\chi(y^{-1}\circ x) - \chi_E(x)|}{\|y\|_\alpha^{Q+\alpha}} = \lim_{t\to 0} c_1 \int_{\mathbb{G}\setminus B(t^{\frac{1}{\alpha}})} \int_{\mathbb{G}} \frac{|\chi(y^{-1}\circ x) - \chi_E(x)|}{\|y\|_\alpha^{Q+\alpha}}$$
$$\leq \liminf_{t\to 0} \frac{Q_t^{\alpha/2}(\chi_E)}{t},$$

ending the proof. Of course Theorem 10 can be generalized to every function $u \in L^2(\mathbb{G})$, see [24] , Theorem 3.5. So that $u \in W^{\alpha,2}(\mathbb{G})$ if and only if

$$\limsup_{t\to 0^+} \frac{Q_t^\alpha(u)}{t} < +\infty.$$

4 Extension approach: the periodic function case via Fourier series tool

In this section we test the extension approach considering a periodic function. To do this we face the problem considering Fourier series. In particular, in the case $s = \frac{1}{2}$, we obtain as a byproduct a Poincaré inequality, see (63).

In fact let $L \in (0,\infty)$ be a fixed number and $s \in (0,1)$. Let $\varphi \in C_{2\pi}(\mathbb{R})$ be a 2π periodic, continuous given function. We want to solve the following problem

$$\begin{cases} \frac{\partial U}{\partial t}(x,t) = \frac{1-2s}{x}\frac{\partial U}{\partial x}(x,t) + \frac{\partial^2 U}{\partial x^2}(x,t), & (x,t) \in (0,L)\times(-\pi,\pi) \\ U(0,t) = \varphi(t), & t \in [-\pi,\pi] \\ U(x,-\pi) = U(x,\pi), & x \in [0,L], \end{cases} \tag{47}$$

compare with [10] and Section 2, Theorem 1.

We look for a solution of the previous problem in the following form:

$$U = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k(x)\cos(kt) + b_k(x)\sin(kt)),$$

where $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ are functions defined in $[0,L]$ that have to be determined, but assuming that

$$a_0(0) = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\tau)d\tau$$

and

$$a_k(0) = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\tau)\cos(k\tau)d\tau, \quad b_k(0) = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\tau)\sin(k\tau)d\tau.$$

We can suppose without, any restriction, that $a_0(0) = 0$ simply considering $\varphi - \frac{\varphi_0}{2}$, where $\varphi_0 := \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\tau)d\tau$. Inserting formally our formal solution U in the equation of the problem (47), we get the following sequence of ODE systems

$$a_0'' + a_0'\frac{1-2s}{x} = 0, \quad x \in (0, L) \tag{48}$$

and for every $k \in \mathbb{N}$, $k \geq 1$

$$\begin{cases} b_k'' + \frac{1-2s}{x}b_k' = -ka_k, & x \in (0, L) \\ a_k'' + \frac{1-2s}{x}a_k' = kb_k & x \in (0, L) \end{cases} \tag{49}$$

with the initial conditions $a_0(0) = 0$, and such that for every $k \in \mathbb{N}$, $k \geq 1$,

$$a_k(0) = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\tau)\cos(k\tau)d\tau, \quad b_k(0) = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\tau)\sin(k\tau)d\tau.$$

4.1 The case $s = \frac{1}{2}$

In case $s = \frac{1}{2}$, from (48) and (49) we get:

$$a_0'' = 0, \quad x \in (0, L)$$

and for every $k \in \mathbb{N}$, $k \geq 1$

$$\begin{cases} b_k'' = -ka_k, & x \in (0, L) \\ a_k'' = kb_k & x \in (0, L) \end{cases} \tag{50}$$

with the initial conditions $a_0(0) = 0$, and for every $k \in \mathbb{N}$, $k \geq 1$,

$$a_k(0) = \varphi_{a_k}(0), \quad b_k(0) = \varphi_{b_k}(0),$$

where

$$\varphi_{a_k}(0) = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\tau)\cos(k\tau)d\tau \quad b_k(0) = \varphi_{b_k}(0) = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\tau)\sin(k\tau)d\tau$$

In this special case we get, for every $k \geq 1$, $b_k^{(iv)} + k^2 b_k = 0$ and $b_k'' = -ka_k$. As a consequence the solution of $b_k^{(iv)} + k^2 b_k = 0$, $b_k(0) = \varphi_{b_k}(0)$ takes the following form:

$$
\begin{aligned}
b_k &= \varphi_{b_k}(0)\cosh(\frac{\sqrt{2k}}{2}x)\cos(\frac{\sqrt{2k}}{2}x) + c_2\cosh(\frac{\sqrt{2k}}{2}x)\sin(\frac{\sqrt{2k}}{2}x)\\
&+ c_3\sinh(\frac{\sqrt{2k}}{2}x)\cos(\frac{\sqrt{2k}}{2}x) + c_4\sinh(\frac{\sqrt{2k}}{2})\sin(\frac{\sqrt{2k}}{2}x)\\
&= \cosh(\frac{\sqrt{2k}}{2}x)\left(\varphi_{b_k}(0)\cos(\frac{\sqrt{2k}}{2}x) + c_2\sin(\frac{\sqrt{2k}}{2}x)\right)\\
&+ \sinh(\frac{\sqrt{2k}}{2}x)\left(c_3\cos(\frac{\sqrt{2k}}{2}x) + c_4\sin(\frac{\sqrt{2k}}{2}x)\right).
\end{aligned}
\tag{51}
$$

From $b_k'' = -ka_k$, $a_k(0) = \varphi_{a_k}(0)$ we get

$$
\begin{aligned}
b_k &= \cosh(\frac{\sqrt{2k}}{2}x)\left(\varphi_{b_k}(0)\cos(\frac{\sqrt{2k}}{2}x) + c_2\sin(\frac{\sqrt{2k}}{2}x)\right)\\
&+ \sinh(\frac{\sqrt{2k}}{2}x)\left(c_3\cos(\frac{\sqrt{2k}}{2}x) - \varphi_{a_k}(0)\sin(\frac{\sqrt{2k}}{2}x)\right)
\end{aligned}
\tag{52}
$$

and

$$
\begin{aligned}
a_k &= \cosh(\frac{\sqrt{2k}}{2}x)\left(\varphi_{a_k}(0)\cos(\frac{\sqrt{2k}}{2}x) + c_2\sin(\frac{\sqrt{2k}}{2}x)\right)\\
&+ \sinh(\frac{\sqrt{2k}}{2}x)\left(c_3\cos(\frac{\sqrt{2k}}{2}x) + \varphi_{b_k}(0)\sin(\frac{\sqrt{2k}}{2}x)\right).
\end{aligned}
\tag{53}
$$

While

$$a_0(x) = c_0x + \varphi_0,$$

for some $c_0 \in \mathbb{R}$.

Then

$$a_k'(0) = \frac{\sqrt{2k}}{2}(c_{2,k} + c_{3,k}), \quad b_k'(0) = \frac{\sqrt{2k}}{2}(c_{2,k} + c_{3,k})$$

and

$$
\begin{aligned}
\frac{\partial U}{\partial x}(0,t) &= \frac{c_0}{2} + \sum_{k=1}^{\infty} (a_k'(0)\cos(kt) + b_k'(0)\sin(kt)) \\
&= \frac{c_0}{2} + \frac{\sqrt{2}}{2}\sum_{k=1}^{\infty} \sqrt{k}(c_{2,k} + c_{3,k})\,(\cos(kt) + \sin(kt)) \\
&= \frac{c_0}{2} + \frac{\sqrt{2}}{2}\sum_{k=1}^{\infty} \sqrt{k}(c_{2,k} + c_{3,k})\left(\cos(kt) + \cos(kt - \frac{\pi}{2})\right) \\
&= \frac{c_0}{2} + \sqrt{2}\sum_{k=1}^{\infty} \sqrt{k}(c_{2,k} + c_{3,k})\cos(kt - \frac{\pi}{4})\cos\frac{\pi}{4} \\
&= \frac{c_0}{2} + \sum_{k=1}^{\infty} \sqrt{k}(c_{2,k} + c_{3,k})\cos(kt - \frac{\pi}{4}).
\end{aligned}
\tag{54}
$$

In this way we do not have the convergence of the Fourier solution, in general.

Thus from the fundamental system of solutions

$$
\begin{aligned}
&\{\exp(\frac{\sqrt{2k}}{2}x)\cos(\frac{\sqrt{2k}}{2}x), \exp(\frac{\sqrt{2k}}{2}x)\sin(\frac{\sqrt{2k}}{2}x), \\
&\exp(-\frac{\sqrt{2k}}{2}x)\cos(\frac{\sqrt{2k}}{2}x), \exp(-\frac{\sqrt{2k}}{2}x)\sin(\frac{\sqrt{2k}}{2}x)\}
\end{aligned}
\tag{55}
$$

we consider only the functions

$$
\{\exp(-\frac{\sqrt{2k}}{2}x)\cos(\frac{\sqrt{2k}}{2}x), \exp(-\frac{\sqrt{2k}}{2}x)\sin(\frac{\sqrt{2k}}{2}x)\}.
$$

Let us consider the following linear combination

$$
c_1\exp(-\frac{\sqrt{2k}}{2}x)\cos(\frac{\sqrt{2k}}{2}x) + c_2\exp(-\frac{\sqrt{2k}}{2}x)\sin(\frac{\sqrt{2k}}{2}x).
$$

We impose that $c_1 = \varphi_{b_k}$, so that

$$
b_k = \varphi_{b_k}\exp(-\frac{\sqrt{2k}}{2}x)\cos(\frac{\sqrt{2k}}{2}x) + c_2\exp(-\frac{\sqrt{2k}}{2}x)\sin(\frac{\sqrt{2k}}{2}x).
$$

Moreover

$$
b_k' = \frac{\sqrt{2k}}{2}\exp(-\frac{\sqrt{2k}}{2}x)\left((c_2 - \varphi_{b_k})\cos(\frac{\sqrt{2k}}{2}x) - (c_2 + \varphi_{b_k})\sin(\frac{\sqrt{2k}}{2}x)\right).
$$

and

$$
b_k'' = -k\exp(-\frac{\sqrt{2k}}{2}x)\left(c_2\cos(\frac{\sqrt{2k}}{2}x) - \varphi_{b_k}\sin(\frac{\sqrt{2k}}{2}x)\right).
$$

so that $b_k'' = -a_k$ if $c_2 = \varphi_{a_k}$. Hence

$$a_k = \exp(-\frac{\sqrt{2k}}{2}x)\left(\varphi_{a_k}\cos(\frac{\sqrt{2k}}{2}x) - \varphi_{b_k}\sin(\frac{\sqrt{2k}}{2}x)\right)$$

and

$$b_k = \exp(-\frac{\sqrt{2k}}{2}x)\left(\varphi_{b_k}\cos(\frac{\sqrt{2k}}{2}x) + \varphi_{a_k}\sin(\frac{\sqrt{2k}}{2}x)\right)$$

satisfies the system.

In particular

$$\begin{aligned}
a'_k &= -\frac{\sqrt{2k}}{2}\exp(-\frac{\sqrt{2k}}{2}x)\left(\varphi_{a_k}\cos(\frac{\sqrt{2k}}{2}x) - \varphi_{b_k}\sin(\frac{\sqrt{2k}}{2}x)\right) \\
&+ \exp(-\frac{\sqrt{2k}}{2}x)\left(-\frac{\sqrt{2k}}{2}\varphi_{a_k}\sin(\frac{\sqrt{2k}}{2}x) - \frac{\sqrt{2k}}{2}\varphi_{b_k}\cos(\frac{\sqrt{2k}}{2}x)\right) \\
&= -\frac{\sqrt{2k}}{2}\exp(-\frac{\sqrt{2k}}{2}x)\left((\varphi_{a_k}+\varphi_{b_k})\cos(\frac{\sqrt{2k}}{2}x) + (\varphi_{a_k}-\varphi_{b_k})\sin(\frac{\sqrt{2k}}{2}x)\right)
\end{aligned} \tag{56}$$

and

$$b'_k = \frac{\sqrt{2k}}{2}\exp(-\frac{\sqrt{2k}}{2}x)\left((\varphi_{a_k}-\varphi_{b_k})\cos(\frac{\sqrt{2k}}{2}x) - (\varphi_{a_k}+\varphi_{b_k})\sin(\frac{\sqrt{2k}}{2}x)\right).$$

As a consequence

$$a'_k(0) = -\frac{\sqrt{2k}}{2}(\varphi_{a_k}+\varphi_{b_k})$$

and

$$b'_k(0) = \frac{\sqrt{2k}}{2}(\varphi_{a_k}-\varphi_{b_k}).$$

$$\begin{aligned}
\frac{\partial U}{\partial x}(0,t) &= \frac{c_0}{2} + \sum_{k=1}^{\infty}(a'_k(0)\cos(kt) + b'_k(0)\sin(kt)) \\
&= \frac{c_0}{2} - \frac{\sqrt{2}}{2}\sum_{k=1}^{\infty}\sqrt{k}\left((\varphi_{a_k}+\varphi_{b_k}))\cos(kt) - (\varphi_{a_k}-\varphi_{b_k})\sin(kt)\right).
\end{aligned} \tag{57}$$

Moreover now $\psi_{a_k}(0) = -\sqrt{k}\frac{\sqrt{2}}{2}(\varphi_{a_k}+\varphi_{b_k})$ and $\psi_{b_k}(0) = \sqrt{k}\frac{\sqrt{2}}{2}(\varphi_{a_k}-\varphi_{b_k})$

$$
\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial U}{\partial x}(0,t) &= -\frac{\sqrt{2}}{2}\sum_{k=1}^{\infty}\sqrt{k}\,\left((\psi_{a_k}+\psi_{b_k})\cos(kt)-(\psi_{a_k}-\psi_{b_k})\sin(kt)\right)\\
&= -\frac{1}{2}\sum_{k=1}^{\infty}k\,\left((-(\varphi_{a_k}+\varphi_{b_k})+(\varphi_{a_k}-\varphi_{b_k}))\cos(kt)-(\psi_{a_k}-\psi_{b_k})\sin(kt)\right)\\
&= -\frac{1}{2}\sum_{k=1}^{\infty}k\,\left(-2\varphi_{b_k}\cos(kt)-(-(\varphi_{a_k}+\varphi_{b_k})-(\varphi_{a_k}-\varphi_{b_k}))\sin(kt)\right)\\
&= -\frac{1}{2}\sum_{k=1}^{\infty}k\,\left(-2\varphi_{b_k}\cos(kt)+2\varphi_{a_k}\sin(kt)\right)\\
&= \sum_{k=1}^{\infty}k\,\left(\varphi_{b_k}\cos(kt)-\varphi_{a_k}\sin(kt)\right) = \frac{\partial U}{\partial x}(0,t).
\end{aligned}
$$

From (57) it follows that, fixing $c_0 = 0$, we can define

$$
\frac{d^{\frac{1}{2}}\varphi}{d^{\frac{1}{2}}t}(t) = -\frac{\sqrt{2}}{2}\sum_{k=1}^{\infty}\sqrt{k}\,\left((\varphi_{a_k}+\varphi_{b_k}))\cos(kt)-(\varphi_{a_k}-\varphi_{b_k})\sin(kt)\right). \tag{58}
$$

as the representative of a class of functions $[\frac{d^{\frac{1}{2}}\varphi}{dt^{\frac{1}{2}}}]$ such that for every $\eta \in [\frac{d^{\frac{1}{2}}\varphi}{dt^{\frac{1}{2}}}]$ then

$$
\eta - \frac{d^{\frac{1}{2}}\varphi}{dt^{\frac{1}{2}}}
$$

is constant.

Moreover for every $c_0 \in \mathbb{R}$, the operator U_{c_0} acts on φ as it follows

$$
\frac{c_0 x + \varphi_0}{2} + \sum_{k=1}^{\infty}(a_k(x)\cos(kt)+b_k(x)\sin(kt)) \tag{59}
$$

that is

$$
U_{c_0}(\varphi) := U_{c_0} \equiv \frac{c_0 x + \varphi_0}{2} + \sum_{k=1}^{\infty}(a_k(x)\cos(kt)+b_k(x)\sin(kt))
$$

is a solution of the extension problem and since

$$
\frac{\partial U_{c_0}}{\partial x}(0,t) = \frac{c_0}{2} - \frac{\sqrt{2}}{2}\sum_{k=1}^{\infty}\sqrt{k}\,\left((\varphi_{a_k}+\varphi_{b_k}))\cos(kt)-(\varphi_{a_k}-\varphi_{b_k})\sin(kt)\right)
$$

we define

$$
T_{1/2,c_0}(\varphi) = \frac{c_0}{2} - \frac{\sqrt{2}}{2}\sum_{k=1}^{\infty}\sqrt{k}\,\left((\varphi_{a_k}+\varphi_{b_k}))\cos(kt)-(\varphi_{a_k}-\varphi_{b_k})\sin(kt)\right).
$$

In case $c_0 \neq 0$, then applying the operator $T_{1/2,c_1}$ to $T_{1/2,c_0}(\varphi)$, after a further extension where we generate a new operator U_{c_1} associated with the constant $c_1 \in \mathbb{R}$, we get

$$\begin{aligned} T_{1/2,c_1}T_{1/2,c_0}(\varphi) &= \frac{c_1}{2} - \frac{\sqrt{2}}{2}\sum_{k=1}^{\infty}\sqrt{k}\left((\varphi_{a_k}+\varphi_{b_k}))\cos(kt) - (\varphi_{a_k}-\varphi_{b_k})\sin(kt)\right) \\ &= \frac{c_1}{2} + \sum_{k=1}^{\infty} k\left(\varphi_{b_k}\cos(kt) - \varphi_{a_k}\sin(kt)\right) = \frac{c_1}{2} + \frac{d\varphi}{dt}(t), \end{aligned} \tag{60}$$

because

$$\begin{aligned} &-\frac{\sqrt{2}}{2}\sqrt{k}\left(-\sqrt{k}\frac{\sqrt{2}}{2}(\varphi_{a_k}+\varphi_{b_k}) + \sqrt{k}\frac{\sqrt{2}}{2}(\varphi_{a_k}-\varphi_{b_k})\right) \\ &= k\varphi_{b_k} \end{aligned} \tag{61}$$

and

$$\begin{aligned} &\frac{\sqrt{2}}{2}\sqrt{k}\left(-\sqrt{k}\frac{\sqrt{2}}{2}(\varphi_{a_k}+\varphi_{b_k}) - \sqrt{k}\frac{\sqrt{2}}{2}(\varphi_{a_k}-\varphi_{b_k})\right) \\ &= -k\varphi_{a_k} \end{aligned} \tag{62}$$

and

$$\phi'(t) = \sum_{k=1}^{\infty}(-ka_k(x)\sin(kt)+kb_k(x)\cos(kt)) = \sum_{k=1}^{\infty}(kb_k(x)\cos(kt)-ka_k(x)\sin(kt))$$

We remark that if

$$f = \frac{d^{\frac{1}{2}}\varphi}{dt^{\frac{1}{2}}}(t) = -\frac{\sqrt{2}}{2}\sum_{k=1}^{\infty}\sqrt{k}\left((\varphi_{a_k}+\varphi_{b_k})\cos(kt) - (\varphi_{a_k}-\varphi_{b_k})\sin(kt)\right),$$

then

$$||f||^2_{L^2} = \frac{\pi}{2}\sum_{k=1}^{\infty} k\left((\varphi_{a_k}+\varphi_{b_k})^2 + (\varphi_{a_k}-\varphi_{b_k})^2\right) = \pi\sum_{k=1}^{\infty} k\left(\varphi^2_{a_k}+\varphi^2_{b_k}\right).$$

Thus if $\varphi_0 = 0$, then

$$||\varphi||^2_{L^2(-\pi,\pi)} \leq \pi\sum_{k=1}^{\infty} k\left(\varphi^2_{a_k}+\varphi^2_{b_k}\right) = \pi||\frac{d^{\frac{1}{2}}\varphi}{dt^{\frac{1}{2}}}||^2_{L^2(-\pi,\pi)}. \tag{63}$$

That is we have proved the following Poincaré inequality:

$$||\varphi - \frac{\varphi_0}{2}||^2_{L^2(-\pi,\pi)} \le \pi || \frac{d^{\frac{1}{2}}\varphi}{dt^{\frac{1}{2}}}||^2_{L^2(-\pi,\pi)}. \tag{64}$$

4.2 General case with Bessel functions: a short remark

Let

$$a_0'' + a_0' \frac{1-2s}{x} = 0, \quad x \in (0, L)$$

and for every $k \in \mathbb{N}$, $k \ge 1$

$$\begin{cases} b_k'' + \frac{1-2s}{x} b_k' = -k a_k, \; x \in (0, L) \\ a_k'' + \frac{1-2s}{x} a_k' = k b_k \quad x \in (0, L) \end{cases} \tag{65}$$

It is convenient to consider the following differential equation in $\mathbb{C}$

$$\frac{d^2 w_k}{dx^2} + \frac{1-2s}{x}\frac{dw_k}{dx} + ikw = 0, \tag{66}$$

where $k \in \mathbb{N}$. Indeed $\Re w$ and $\Im w$ satisfy the following system

$$\begin{cases} \Im w_k'' + \frac{1-2s}{x} \Im w_k' = -k \Re w_k, \\ \Re w_k'' + \frac{1-2s}{x} \Re w_k' = k \Im w_k \end{cases} \tag{67}$$

Then, see [44] Section 3.5 p. 77, a solution of previous equation (66) is given by

$$w_k = x^s Z_s(\sqrt{\frac{k}{2}}(1+i)x),$$

or also

$$w_k = x^s Z_s(-\sqrt{\frac{k}{2}}(1+i)x),$$

where Z_ν is an arbitrary solution of Bessel's differential equation:

$$z^2 \frac{d^2 w_k}{dz^2} + z \frac{dw_k}{dz} + (z^2 - \nu^2) w = 0. \tag{68}$$

Then solving the problem we get

$$U_k(x,t) = \phi_{a_k} \frac{(L-x)^s Z_s(\sqrt{\frac{k}{2}}(1+i)(L-x))}{L^s Z_s(\sqrt{\frac{k}{2}}(1+i)L)} e^{-ikt}.$$

Using this functions we may expect to obtain inequalities for $\alpha \in (0,1)$ analogous to (64) obtained for $\alpha = \frac{1}{2}$.

Acknowledgements The author is supported by MURST, Italy, INDAM-GNAMPA project 2018: *Costanti critiche e problemi asintotici per equazioni completamente non lineari* and INDAM-GNAMPA project 2019: *Proprietà di regolarità delle soluzioni viscose con applicazioni a problemi di frontiera libera.*

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