On stochastic Langevin and Fokker-Planck equations: The two-dimensional case

Published Version:
Pascucci A., Pesce A. (2022). On stochastic Langevin and Fokker-Planck equations: The two-dimensional case. JOURNAL OF DIFFERENTIAL EQUATIONS, 310, 443-483 [10.1016/j.jde.2021.11.004].

Availability:
This version is available at: https://hdl.handle.net/11585/843491 since: 2021-12-30

Published:
DOI: http://doi.org/10.1016/j.jde.2021.11.004

Submitted to the Annals of Probability

ON STOCHASTIC LANGEVIN AND FOKKER-PLANCK EQUATIONS: THE TWO-DIMENSIONAL CASE

By Andrea Pascucci and Antonello Pesce

Università di Bologna

We prove existence, regularity in Hölder classes and estimates from above and below of the fundamental solution of the stochastic Langevin equation. This degenerate SPDE satisfies the weak Hörmander condition. We use a Wentzell's transform to reduce the SPDE to a PDE with random coefficients; then we apply a new method, based on the parametrix technique, to construct a fundamental solution. This approach avoids the use of the Duhamel's principle for the SPDE and the related measurability issues that appear in the stochastic framework. Our results are new even for the deterministic equation.

1. Introduction We consider the stochastic version of the Fokker-Planck equation

$$\partial_t u + \sum_{j=1}^{n} v_j \partial_{x_j} u = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \partial_{v_i v_j} u. \tag{1}$$

Here the variables $t \geqslant 0$, $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ respectively stand for time, position and velocity, and the unknown $u = u_t(x,v) \geqslant 0$ stands for the density of particles in phase space. The vector field $\mathbf{Y} := \partial_t + v \cdot \nabla_x$ on the left-hand side of (1) describes transport; the coefficients a_{ij} describe some kind of collision among particles and in general may depend on the solution u through some integral expressions. Linear Fokker-Planck equations (cf. Desvillettes and Villani (2001) and Risken (1989)), non-linear Boltzmann-Landau equations (cf. Lions (1994) and Cercignani (1988)) and non-linear equations for Lagrangian stochastic models commonly used in the simulation of turbulent flows (cf. Bossy, Jabir and Talay (2011)) can be written in the form (1). In mathematical finance, (1) describes path-dependent financial contracts such as Asian options (see, for instance, Pascucci (2011)).

In this note we study a kinetic model where the position and the velocity of a particle are stochastic processes (X_t, V_t) only partially observable through some observation process O_t. We consider the two-dimensional case, $n = 1$, which is already challenging enough, and propose an approach that hopefully can be extended to the multi-dimensional case. If $\mathcal{F}_t^O = \sigma(O_s, s \leqslant t)$ denotes the filtration of the observations then, under natural assumptions, the *conditional* density $p_t(x,v)$ of (X_t, V_t) given $\mathcal{F}_t^O$ solves a linear SPDE of the form

$$d_{\mathbf{Y}} u_t(x,v) = \frac{a_t(x,v)}{2} \partial_{vv} u_t(x,v) dt + \sigma_t(x,v) \partial_v u_t(x,v) dW_t, \qquad \mathbf{Y} = \partial_t + v\partial_x. \tag{2}$$

MSC 2010 subject classifications: Primary 60H15, 35R60; secondary 35K70

Keywords and phrases: stochastic partial differential equations, Langevin equation, Fokker-Planck equation, fundamental solution

In (2) W is a Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $(\mathcal{F}_t)_{t\geqslant 0}$ satisfying the usual conditions. The symbol $d_{\mathbf{Y}}$ indicates that the equation is solved in the Itô (or strong) sense: a solution to (2) is a continuous process $u_t = u_t(x, v)$ that is twice differentiable in v and such that

$$u_t\left(\gamma_t^B(x,v)\right) = u_0(x,v) + \frac{1}{2}\int_0^t \left(a_s\partial_{vv}u_s\right)(\gamma_s^B(x,v))ds + \int_0^t \left(\sigma_s\partial_v u_s\right)(\gamma_s^B(x,v))dW_s$$

where $t \mapsto \gamma_t^B(x,v)$ denotes the integral curve, starting from (x,v), of the advection vector field $v\partial_x$, that is

$$\gamma_t^B(x,v) = e^{tB}(x,v) = (x+tv, v), \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3}$$

Clearly, in case the observation process O is independent of X and V, the SPDE (2) boils down to the deterministic PDE (1) with $n = 1$.

The main goal of this paper is to show existence, regularity and Gaussian-type estimates of a stochastic fundamental solution of (2). As far as the authors are aware, such kind of results was never proved for SPDEs that satisfy the *weak* Hörmander condition, that is under the assumption that the drift has a key role in the noise propagation. We mention that hypoellipticity for SPDEs under the strong Hörmander condition was studied by Chaleyat-Maurel and Michel Chaleyat-Maurel and Michel (1984), Kunita Kunita (1982), Krylov Krylov (2015) and Jinniao Qiu (2018). Even in the deterministic case, our results are new in that they extend the recent results Delarue and Menozzi (2010), Menozzi (2018) for Kolmogorov equations with general drift.

Our method is based on a Wentzell's reduction of the SPDE to a PDE with random coefficients to which we apply the parametrix technique to construct a fundamental solution. This approach avoids the use of the Duhamel's principle for the SPDE and the related measurability issues that appear in the stochastic framework as discussed, for instance, in Sowers (1994). As in Pascucci and Pesce (2019), Wentzell's reduction of the SPDE is done globally: to control the behavior as $|x|, |v| \to \infty$ of the random coefficients of the resulting PDE, we impose some flattening condition at infinity on the coefficient $\sigma_t(x, v)$ in (2) (cf. Assumption 2.5). Compared to the uniformly parabolic case, two main new difficulties arise:

i) the Itô-Wentzell transform drastically affects the drift $\mathbf{Y}$: in particular, after the random change of coordinates, the new drift has no longer polynomials coefficients. Consequently, a careful analysis is needed to check the validity of the Hörmander condition in the new coordinates. This question is discussed in more detail in Section 1.1;

ii) in the deterministic case, the parametrix method has been applied to degenerate Fokker-Planck equations, including (2) with $\sigma \equiv 0$, by several authors, Polidoro (1994), Di Francesco and Pascucci (2005), Menozzi (2011), Kohatsu-Higa and Yûki (2018), using *intrinsic* Hölder spaces. Loosely speaking, the intrinsic Hölder regularity reflects the geometry of the PDE and is defined in terms of the translations and homogeneous norm associated to the Hörmander vector fields: this kind of regularity is natural for the study of the singular kernels that come into play in the parametrix iterative procedure. Now, under the weak Hörmander condition, the intrinsic regularity properties in space and time are closely intertwined and cannot be studied separately. However, assuming that the coefficients are merely predictable, we have no good control on the regularity in the time variable; for instance, even in the deterministic case, the coefficients are only measurable in t and consequently they cannot be Hölder continuous in (x, v) in the intrinsic sense. On the other hand, assuming that the coefficients are Hölder continuous in (x, v) in the classical Euclidean sense, the parametrix method still works as long as we use a suitable *time-dependent* parametrix and exploit the fact that the intrinsic translations coincide with the Euclidean ones for points (t, x, v) and (t, ξ, η) at the same time level. We comment on this question more thoroughly in Section 1.2.

The rest of the paper is organized as follows. In Sections 1.1 and 1.2 we go deeper into the issues mentioned above. In Section 2 we set the assumptions, introduce the functional setting and state the main result, Theorem 2.6. In Section 3 we prove some crucial estimate for stochastic flows of diffeomorphisms: these estimates, which can be of independent interest, extend some result of Kunita (1990). In Section 4 we formulate a version of the Itô-Wentzell formula and exploit the results of Section 3 to perform a stochastic change of variable in order to reduce the SPDE to a PDE with random coefficients. In Section 5 we build on the work by Delarue and Menozzi Delarue and Menozzi (2010) to develop a parametrix method for Kolmogorov PDEs with general drift (Theorem 5.5). Finally, in Section 6 we complete the proof of Theorem 2.6.

1.1. *Stochastic Langevin equation and the Hörmander condition* For illustrative purposes, we examine the case of constant coefficients and introduce the stochastic counterpart of the classical Langevin PDE.

Let B, W be independent real Brownian motions, $a > 0$ and $\sigma \in [0, \sqrt{a}]$. The Langevin model is defined in terms of the system of SDEs

$$\begin{cases} dX_t = V_t dt, \\ dV_t = \sqrt{a - \sigma^2} dB_t - \sigma dW_t. \end{cases} \tag{4}$$

We interpret W as the observation process: if $\sigma = 0$ the velocity V is unobservable, while for $\sigma = \sqrt{a}$ the velocity V is completely observable, being equal to W. To shorten notations, we denote by $z = (x, v)$ and $\zeta = (\xi, \eta)$ the points in $\mathbb{R}^2$. Setting $Z_t = (X_t, Y_t)$, equation (4) can be rewritten as

$$dZ_t = BZ_t dt + \mathbf{e}_2 d(\sqrt{a - \sigma^2} B_t - \sigma W_t), \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5}$$

where B is the matrix in (3).

In this section we show in two different ways that the SPDE

$$d_{\mathbf{Y}} u_t = \frac{a}{2} \partial_{vv} u_t dt + \sigma \partial_v u_t dW_t, \qquad \mathbf{Y} := \partial_t + v\partial_x, \tag{6}$$

is the *forward Kolmogorov* (or *Fokker-Planck*) equation of the SDE (4) conditioned to the Brownian observation given by $\mathcal{F}_t^W = \sigma(W_s,\, s \leqslant t)$. In the uniformly parabolic case, this is a well-known fact, proved under diverse assumptions by several authors (see, for instance, Zakai (1969), Krylov and Rozovskii (1977) and Pardoux (1979)).

In the first approach, we solve explicitly the linear SDE (5) and find the expression of the conditional transition density Γ of the solution Z: by Itô formula, we directly infer that Γ is the fundamental solution of the SPDE (6). The second approach, inspired by Krylov and Zatezalo (2000), is much more general because it does not require the explicit knowledge of Γ: we first prove the existence of the fundamental solution of the SPDE (6) and then show that it is the conditional transition density of the solution of (4).

The following result is an easy consequence of the Itô formula and isometry.

PROPOSITION 1.1. *The solution $Z = Z^\zeta$ of (5), with initial condition $\zeta = (\xi, \eta) \in \mathbb{R}^2$, is given by*

$$Z_t^\zeta = e^{tB}\left(\zeta + \int_0^t e^{-sB}\mathbf{e}_2\, d(\sqrt{a-\sigma^2}B_s - \sigma W_s)\right)$$

with $\mathbf{e}_2$ as in (5). Conditioned to $\mathcal{F}_t^W$, Z_t^ζ has normal distribution with mean and covariance matrix given by

$$m_t(\zeta) := E\left[Z_t^\zeta \mid \mathcal{F}_t^W\right] = e^{tB}\left(\zeta - \sigma\int_0^t e^{-sB}\mathbf{e}_2 dW_s\right) = \begin{pmatrix} \xi + t\eta - \sigma\int_0^t (t-s)dW_s \\ \eta - \sigma W_t \end{pmatrix}, \tag{7}$$

$$\mathcal{C}_t := cov\left(Z_t^\zeta \mid \mathcal{F}_t^W\right) = (a-\sigma^2)Q_t, \qquad Q_t := \int_0^t \left(e^{sB}\mathbf{e}_2\right)\left(e^{sB}\mathbf{e}_2\right)^* ds = \begin{pmatrix} \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{pmatrix}. \tag{8}$$

In particular, if $\sigma = \sqrt{a}$ then the distribution of Z_t^ζ conditioned to $\mathcal{F}_t^W$ is a Dirac delta centered at $m_t(\zeta)$; if $\sigma \in [0, \sqrt{a})$ and $t > 0$ then Z_t^ζ has density, conditioned to $\mathcal{F}_t^W$, given by

$$\Gamma(t, z; 0, \zeta) = \frac{1}{2\pi\sqrt{\det \mathcal{C}_t}} \exp\left(-\frac{1}{2}\langle \mathcal{C}_t^{-1}(z - m_t(\zeta)), (z - m_t(\zeta))\rangle\right), \qquad z \in \mathbb{R}^2. \tag{9}$$

More explicitly, we have $\Gamma(t, z; 0, \zeta) = \Gamma_0(t, z - m_t(\zeta))$ where

$$\Gamma_0(t, x, v) = \frac{\sqrt{3}}{\pi t^2(a-\sigma^2)} \exp\left(-\frac{2}{a-\sigma^2}\left(\frac{v^2}{t} - \frac{3vx}{t^2} + \frac{3x^2}{t^3}\right)\right), \qquad t > 0,\ (x, v) \in \mathbb{R}^2. \tag{10}$$

By the Itô formula, $\Gamma(t, z; 0, \zeta)$ is the stochastic fundamental solution of SPDE (6), with pole at $(0, \zeta)$.

As an alternative approach, we construct the fundamental solution of the SPDE (6) by performing some suitable change of variables. First we transform (6) into a PDE with random coefficients, satisfying the weak Hörmander condition; by a second change of variables, we remove the drift of the equation and transform it into a deterministic heat equation. Going back to the original variables, we find the stochastic fundamental solution of (6), which obviously coincides with Γ in (9). Eventually, we prove that $\Gamma(t,\cdot;0,\zeta)$ is a density of Z_t^ζ conditioned to $\mathcal{F}_t^W$. We split the proof in three steps.

[Step 1] We set

$$\hat{u}_t(x,v)=u_t(x,v-\sigma W_t). \tag{11}$$

By Itô formula, u solves (6) if and only if $\hat{u}$ solves the Langevin PDE

$$\partial_t\hat{u}+(v-\sigma W_t)\partial_x\hat{u}=\frac{a-\sigma^2}{2}\partial_{vv}\hat{u}. \tag{12}$$

By this change of coordinates we get rid of the stochastic part of the SPDE; however, this is done at the cost of introducing a random drift term. For the moment, this is not a big issue because σ is constant and, in particular, independent of v: for this reason, the weak Hörmander condition is preserved since the vector fields ∂_v, $\partial_t+(v-\sigma W_t)\partial_x$ and their Lie bracket

$$[\partial_v,\partial_t+(v-\sigma W_t)\partial_x]=\partial_x$$

span $\mathbb{R}^3$ at any point.

[Step 2] In order to remove the random drift, we perform a second change of variables:

$$g_t(x,v)=\hat{u}_t(\gamma_t(x,v)),\qquad \gamma_t(x,v):=\left(x+tv-\sigma\int_0^t W_s ds, v\right). \tag{13}$$

The spatial Jacobian of γ_t equals

$$\nabla\gamma_t(x,y)=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$$

so that γ_t is one-to-one and onto $\mathbb{R}^2$ for any t. Then, (12) is transformed into the deterministic heat equation with time-dependent coefficients

$$\partial_t g_t(x,v)=\frac{a-\sigma^2}{2}\left(t^2\partial_{xx}-2t\partial_{xv}+\partial_{vv}\right)g_t(x,v). \tag{14}$$

Equation (14) is not uniformly parabolic because the matrix of coefficients of the second order part

$$a_t:=(a-\sigma^2)\begin{pmatrix}t^2 & -t\\ -t & 1\end{pmatrix}$$

is singular. However, in case of partial observation, that is $\sigma\in[0,\sqrt{a})$, the diffusion matrix

$$A_t=\int_0^t a_s ds=(a-\sigma^2)\begin{pmatrix}\frac{t^3}{3} & -\frac{t^2}{2}\\ -\frac{t^2}{2} & t\end{pmatrix}$$

is positive definite for any $t>0$ and therefore (14) admits a Gaussian fundamental solution. For $\sigma=0$, this result was originally proved by Kolmogorov Kolmogorov (1934) (see also the introduction in Hörmander (1967)). Going back to the original variables we recover the explicit expression of Γ in (9).

Incidentally, we notice that (14) also reads

$$\partial_t g_t(x,v) = \frac{a-\sigma^2}{2}\bar{\mathbf{V}}_t^2 g_t(x,v), \qquad \bar{\mathbf{V}}_t := \partial_v - t\partial_x,$$

where the vector fields ∂_t and $\bar{\mathbf{V}}_t$ satisfy the weak Hörmander condition in $\mathbb{R}^3$ because $[\bar{\mathbf{V}}_t, \partial_t] = \partial_x$.

[Step 3] We show that Γ is the conditional transition density of Z: the proof is based on a combination of the arguments of Krylov and Zatezalo (2000) with the gradient estimates for Kolmogorov equations proved in Di Francesco and Pascucci (2007).

THEOREM 1.2. *Let Z^ζ denote the solution of the linear SDE (5) starting from $\zeta \in \mathbb{R}^2$ and let $\Gamma = \Gamma(t,\cdot;0,\zeta)$ in (9) be the fundamental solution of the Langevin SPDE (6) with $\sigma \in [0,\sqrt{a})$. For any bounded and measurable function φ on $\mathbb{R}^2$, we have*

$$E\left[\varphi(Z_t^\zeta) \mid \mathcal{F}_t^W\right] = \int_{\mathbb{R}^2} \varphi(z)\Gamma(t,z;0,\zeta)dz.$$

PROOF. It is not restrictive to assume that φ is a test function. Let

$$I_t(\zeta) := \int_{\mathbb{R}^2} \varphi(z)\Gamma(t,z;0,\zeta)dz, \qquad t>0,\ \zeta \in \mathbb{R}^2.$$

By (7)-(9), $I_t(\zeta)$ is $\mathcal{F}_t^W$-measurable: thus, to prove the thesis it suffices to show that, for any continuous and non-negative function $c = c_s(w)$ on $[0,t]\times\mathbb{R}$, we have

$$E\left[e^{-\int_0^t c_s(W_s)ds}\varphi(Z_t^\zeta)\right] = E\left[e^{-\int_0^t c_s(W_s)ds} I_t(\zeta)\right]. \tag{15}$$

Let

$$\mathcal{L}^{(\sigma)} = \frac{a}{2}\left(\partial_{vv} - 2\sigma\partial_{vw} + \partial_{ww}\right) + v\partial_x$$

be the infinitesimal generator of the three-dimensional process (X,V,W). For $\sigma \in [0,\sqrt{a})$, $\partial_t + \mathcal{L}^{(\sigma)}$ satisfies the weak Hörmander condition in $\mathbb{R}^4$ and has a Gaussian fundamental solution (see, for instance, formula (2.9) in Di Francesco and Pascucci (2007)). We denote by $f = f_s(x,v,w)$ the classical solution of the *backward* Cauchy problem

$$\begin{cases} \left(\partial_s + \mathcal{L}^{(\sigma)}\right) f_s(x,v,w) - c_s(w) f_s(x,v,w) = 0, & (s,x,v,w) \in [0,t)\times\mathbb{R}^3, \\ f_t(x,v,w) = \varphi(x,v), & (x,v,w) \in \mathbb{R}^3, \end{cases}$$

and set

$$M_s := e^{-\int_0^s c_\tau(W_\tau)d\tau} \int_{\mathbb{R}^2} f_s(z,W_s)\Gamma(s,z;0,\zeta)dz, \qquad s \in [0,t].$$

By definition, we have

$$E\left[M_t\right] = E\left[e^{-\int_0^t c_s(W_s)ds} I_t(\zeta)\right]$$

and, by the Feynman-Kac representation of f,

$$E\left[M_0\right] = f_0(\zeta, 0) = E\left[e^{-\int_0^t c_s(W_s)ds} \varphi(Z_t^\zeta)\right].$$

Hence (15) follows by proving that M is a martingale. By the Itô formula, we have

$$\begin{aligned} df_s(x, v, W_s) &= \left(\partial_s f_s + \frac{1}{2}\partial_{ww} f_s\right)(x, v, W_s)ds + \left(\partial_w f_s\right)(x, v, W_s)dW_s \\ &= \left(-\mathcal{L}^{(\sigma)} f_s + c_s f_s + \frac{1}{2}\partial_{ww} f_s\right)(x, v, W_s)ds + \left(\partial_w f_s\right)(x, v, W_s)dW_s. \end{aligned}$$

Moreover, since Γ solves the SPDE (6), setting $e_s := e^{-\int_0^s c_\tau(W_\tau)d\tau}$ for brevity, we get

$$\begin{aligned} dM_s = -c_s(W_s)M_s ds &+ e_s \int_{\mathbb{R}^2} \left(-\mathcal{L}^{(\sigma)} f_s + c_s f_s + \frac{1}{2}\partial_{ww} f_s\right)(x, v, W_s)\Gamma(s, x, v; 0, \zeta)dxdv\, ds \\ &+ e_s \int_{\mathbb{R}^2} \left(\partial_w f_s\right)(x, v, W_s)\Gamma(s, x, v; 0, \zeta)dxdv\, dW_s \\ &+ e_s \int_{\mathbb{R}^2} f_s(x, v, W_s)\left(\frac{a}{2}\partial_{vv} - v\partial_x\right)\Gamma(s, x, v; 0, \zeta)dxdv\, ds \\ &+ e_s\sigma \int_{\mathbb{R}^2} f_s(x, v, W_s)\partial_v\Gamma(s, x, v; 0, \zeta)dxdv\, dW_s \\ &+ e_s\sigma \int_{\mathbb{R}^2} \partial_w f_s(x, v, W_s)\partial_v\Gamma(s, x, v; 0, \zeta)dxdv\, ds. \end{aligned}$$

Integrating by parts, we find

$$dM_s = e_s \int_{\mathbb{R}^2} \left(\partial_w f_s - \sigma\partial_v f_s\right)(x, v, W_s)\Gamma(s, x, v; 0, \zeta)dxdv\, dW_s,$$

which shows that M is at least a local martingale.

To conclude, we recall the gradient estimates proved in Di Francesco and Pascucci (2007), Proposition 3.3: for any test function φ there exist two positive constants ε, C such that

$$|\partial_v f_s(x, v, w)| + |\partial_w f_s(x, v, w)| \leqslant \frac{C}{(t-s)^{\frac{1}{2}-\varepsilon}}, \qquad (s, x, v, w) \in [0, t) \times \mathbb{R}^3.$$

Thus, we have

$$\begin{aligned} &E\left[\int_0^t \left(\int_{\mathbb{R}^2} \left(\partial_w f_s - \sigma\partial_v f_s\right)(x, v, W_s)\Gamma(s, x, v; 0, \zeta)dxdv\right)^2 ds\right] \\ &\leqslant \int_0^t \frac{C}{(t-s)^{1-2\varepsilon}} E\left[\left(\int_{\mathbb{R}^2} \Gamma(s, x, v; 0, \zeta)dxdv\right)^2\right] ds \\ &= \int_0^t \frac{C}{(t-s)^{1-2\varepsilon}} ds < \infty \end{aligned}$$

and this proves that M is a true martingale. □

1.2. *Intrinsic vs Euclidean Hölder spaces for the deterministic Langevin equation* The parametrix method requires some assumption on the regularity of the coefficients of the PDE: in the uniformly parabolic case, it suffices to assume that the coefficients are bounded, Hölder continuous in the spatial variables and measurable in time (cf. Friedman (1964)).

In this paper, we apply the parametrix method assuming that the coefficients of the Langevin SPDE (2) are predictable processes that are Hölder continuous in (x, v) in the Euclidean sense. From the analytical perspective this is not the natural choice: indeed, it is well known that the natural framework for the study of Hörmander operators is the analysis on Lie groups (see, for instance, Folland and Stein (1982)). In this section, we motivate our choice to use Euclidean Hölder spaces rather than the intrinsic ones.

We recall that Lanconelli and Polidoro Lanconelli and Polidoro (1994) first studied the intrinsic geometry of the Langevin operator in (6) with $\sigma = 0$:

$$\mathcal{L}_a := \frac{a}{2}\partial_{vv} - v\partial_x - \partial_t.$$

They noticed that $\mathcal{L}_a$ is invariant with respect to the homogeneous Lie group $(\mathbb{R}^3, *, \delta)$ where the group law is given by

$$(\tau, \xi, \eta) * (t, x, v) = (t + \tau, x + \xi + t\eta, v + \eta), \qquad (\tau, \xi, \eta), (t, x, v) \in \mathbb{R}^3, \tag{16}$$

and $\delta = (\delta_\lambda)_{\lambda>0}$ is the ultra-parabolic dilation operator defined as

$$\delta_\lambda(t, x, v) = (\lambda^2 t, \lambda^3 x, \lambda v), \qquad (t, x, v) \in \mathbb{R}^3,\ \lambda > 0. \tag{17}$$

More precisely, $\mathcal{L}_a$ is invariant with respect to the left-$*$-translations $\ell_{(\tau,\xi,\eta)}(t, x, v) = (\tau, \xi, \eta) * (t, x, v)$, in the sense that

$$\mathcal{L}_a(f \circ \ell_{(\tau,\xi,\eta)}) = (\mathcal{L}_a f) \circ \ell_{(\tau,\xi,\eta)}, \qquad (\tau, \xi, \eta) \in \mathbb{R}^3,$$

and is δ-homogeneous of degree two, in that

$$\mathcal{L}_a(f \circ \delta_\lambda) = \lambda^2 (\mathcal{L}_a f) \circ \delta_\lambda, \qquad \lambda > 0.$$

It is natural to endow $(\mathbb{R}^3, *, \delta)$ with the δ-homogeneous norm

$$|(t, x, v)|_{\mathcal{L}} = |t|^{\frac{1}{2}} + |x|^{\frac{1}{3}} + |v|$$

and the distance

$$d_{\mathcal{L}}\left((t, x, v), (\tau, \xi, \eta)\right) = |(\tau, \xi, \eta)^{-1} * (t, x, v)|_{\mathcal{L}}. \tag{18}$$

The intrinsic Hölder spaces associated to $d_{\mathcal{L}}$ are particularly beneficial for the study of existence and regularity properties of solutions to the Langevin equation because they comply with the asymptotic properties of its fundamental solution Γ near the pole: let us recall that

$$\Gamma(t, x, v; \tau, \xi, \eta) = \Gamma_0\left((\tau, \xi, \eta)^{-1} * (t, x, v)\right), \qquad \tau < t,$$

where Γ_0 is the fundamental solution of $\mathcal{L}$ in (10) with $\sigma = 0$ and $(\tau, \xi, \eta)^{-1} = (-\tau, -\xi + \tau\eta, -\eta)$ is the $*$-inverse of (τ, ξ, η). Notice also that Γ is δ-homogeneous of degree four, where four is the so-called δ-homogeneous dimension of $\mathbb{R}^2$.

Based on the use of intrinsic Hölder spaces defined in terms of $d_{\mathcal{L}}$, a stream of literature has built a complete theory of existence and regularity, analogous to that for uniformly parabolic PDEs: we mention some of the main contributions Polidoro (1994), Polidoro (1997), Manfredini (1997), Lunardi (1997), Di Francesco and Pascucci (2005), Di Francesco and Polidoro (2006), Pagliarani, Pascucci and Pignotti (2016) and, in particular, Polidoro (1994), Di Francesco and Pascucci (2005), Konakov, Menozzi and Molchanov (2010) where the parametrix method for Kolmogorov PDEs was developed.

On the other hand, intrinsic Hölder regularity can be a rather restrictive property as shown by the following example.

Example 1.3. *For $x, \xi \in \mathbb{R}$ and $t \neq \tau$, let*

$$z = \left(x, -\frac{x-\xi}{t-\tau}\right), \qquad \zeta = \left(\xi, -\frac{x-\xi}{t-\tau}\right) \tag{19}$$

Then we have

$$(\tau, \zeta)^{-1} * (t, z) = (t - \tau, 0, 0),$$

and therefore

$$d_{\mathcal{L}}((t, z), (\tau, \zeta)) = |t - \tau|^{\frac{1}{2}}.$$

Since x and ξ are arbitrary real numbers, we see that points in $\mathbb{R}^3$ that are far from each other in the Euclidean sense, can be very close in the intrinsic sense. It follows that, if a function $f(t, x, v) = f(x)$ depends only on x and is Hölder continuous in the intrinsic sense (i.e. with respect to $d_{\mathcal{L}}$), then it must be constant: in fact, for z, ζ as in (19), we have

$$|f(x) - f(\xi)| = |f(t, z) - f(\tau, \zeta)| \leqslant C|t - \tau|^{\alpha}$$

for some positive constants C, α and for any $x, \xi \in \mathbb{R}$ and $t \neq \tau$.

When it comes to studying the stochastic Langevin equation, the use of Euclidean Hölder spaces seems unavoidable. The problem is that we have to deal with functions $f = f_t(x, v)$ that are:

- *Hölder continuous* with respect to the space variables (x, v) in order to apply the parametrix method;
- *measurable* with respect to the time variable t because f plays the role of a coefficient of the SPDE that is a predictable process (i.e. merely measurable in t).

As opposed to the standard parabolic case, in terms of the metric $d_{\mathcal{L}}$ there doesn't seem to be a clear way to separate regularity in (x, v) from regularity in t: indeed this is due to the definition of $*$-translation that mixes up spatial and time variables (see (16)). On the other hand, we may observe that the Euclidean- and $*$- differences of points at the same time level coincide:

$$(t,\xi,\eta)^{-1} * (t,x,v) = (0, x-\xi, v-\eta), \qquad x,v,\xi,\eta \in \mathbb{R}.$$

Thus, to avoid using $*$-translations, the idea is to combine this property with a suitable definition of *time-dependent parametrix* that makes the parametrix procedure work: this will be done in Section 5.

Concerning the use of the Euclidean or homogeneous norm in $\mathbb{R}^2$, let us denote by $bC^{\alpha}(\mathbb{R}^2)$ and $bC^{\alpha}_{\mathcal{L}}(\mathbb{R}^2)$ the space of bounded and Hölder continuous functions with respect to the Euclidean norm and the homogeneous norm $|x|^{\frac{1}{3}} + |v|$, respectively. Since $|(x,v)| \leqslant |x|^{\frac{1}{3}} + |v|$ for $|(x,v)| \leqslant 1$, we have the inclusion

$$bC^{\alpha}(\mathbb{R}^2) \subseteq bC^{\alpha}_{\mathcal{L}}(\mathbb{R}^2). \tag{20}$$

Preferring simplicity to generality, we shall use Hölder spaces defined in terms of the Euclidean norm (cf. Assumption 2.3): by (20), this results in a slightly more restrictive condition compared to the analogous one given in terms of the homogeneous norm. On the other hand, all the results of this paper can be proved using the homogeneous norm $|x|^{\frac{1}{3}} + |v|$ as in Polidoro (1994), Di Francesco and Pascucci (2005) and Konakov, Menozzi and Molchanov (2010): this would be more natural but would greatly increase the technicalities.

We close this section by giving some standard Gaussian estimates that play a central role in the parametrix construction. After the change of variables (13) with $\sigma = 0$, the Langevin operator $\mathcal{L}_a$ is transformed into

$$L_a = \frac{a}{2}\bar{\mathbf{V}}_t^2 - \partial_t, \qquad \bar{\mathbf{V}}_t := \partial_v - t\partial_x.$$

Since L_a is a heat operator with time dependent coefficients, its fundamental solution is the Gaussian function $\Gamma_a(t,z;s,\zeta) = \Gamma_a(t, z-\zeta; s, 0)$ where

$$\Gamma_a(t,x,y;s,0,0) = \frac{\sqrt{3}}{a\pi(t-s)^2} \exp\left(-\frac{2}{a(t-s)^3}\left(3x^2 + 3xy(t+s) + y^2(t^2+ts+s^2)\right)\right) \tag{21}$$

for $s < t$ and $x, y \in \mathbb{R}$.

LEMMA 1.4. *For every $\varepsilon > 0$ there exists a positive constant c such that*

$$\begin{aligned} \left|\bar{\mathbf{V}}_t \Gamma_a(t,x,y;s,0,0)\right| &\leqslant \frac{c}{\sqrt{t-s}}\Gamma_{a+\varepsilon}(t,x,y;s,0,0), \\ \left|\bar{\mathbf{V}}_t^2 \Gamma_a(t,x,y;s,0,0)\right| &\leqslant \frac{c}{t-s}\Gamma_{a+\varepsilon}(t,x,y;s,0,0), \end{aligned} \tag{22}$$

for every $0 \leqslant s < t \leqslant T$ and $x, y \in \mathbb{R}$.

PROOF. We remark that $\Gamma_a(t,x,y;s,0,0)$ has different asymptotic regimes as $t \to s^+$ depending on whether or not s is zero: in fact, if $s = 0$ then the quadratic form in the exponent of Γ_a is similar to that of the Langevin operator, that is

$$\frac{1}{a}\begin{pmatrix} \frac{6}{t^3} & \frac{3}{t^2} \\ \frac{3}{t^2} & \frac{2}{t} \end{pmatrix}.$$

On the other hand, if $s \neq 0$ we see in (21) that all the components of the quadratic form are $O((t-s)^{-3})$ as $t \to s^+$.

The thesis is a consequence of the following elementary inequality: for any $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists a positive constant $c_{\varepsilon,n}$ such that

$$|\lambda|^n e^{-\frac{\lambda^2}{\mu}} \leqslant c_{n,\varepsilon} e^{-\frac{\lambda^2}{\mu+\varepsilon}}, \qquad \lambda \in \mathbb{R}. \tag{23}$$

Indeed, we have

$$\left|\bar{\mathbf{V}}_t \Gamma_a(t,x,y;s,0,0)\right| = \frac{1}{\sqrt{t-s}}\left|\frac{6x+2v(t+2s)}{a(t-s)^{\frac{3}{2}}}\right| \Gamma_a(t,x,y;s,0,0) \leqslant$$

(by (23) with $n = 1$)

$$\leqslant \frac{C}{\sqrt{t-s}} \Gamma_{a+\varepsilon}(t,x,y;s,0,0).$$

The proof of (22) is similar, using that

$$\bar{\mathbf{V}}_t^2 \Gamma_a(t,x,y;s,0,0) = \frac{4}{a(t-s)}\left(\frac{(3x+v(t+2s))^2}{a(t-s)^3} - 1\right)\Gamma_a(t,x,y;s,0,0).$$

□

2. Assumptions and main results

We introduce the functional spaces used throughout the paper. For convenience, we give the definitions in the general multi-dimensional setting even if, except for Section 3, we will mainly consider dimension $d = 2$.

Let $k, d \in \mathbb{N}$, $0 < \alpha < 1$ and $0 \leqslant t < T$. Denote by $m\mathcal{B}_{t,T}$ the space of all real-valued Borel measurable functions $f = f_s(z)$ on $[t,T] \times \mathbb{R}^d$ and

- $C^{\alpha}_{t,T}$ (resp. $bC^{\alpha}_{t,T}$) is the space of (resp. bounded) functions $f \in m\mathcal{B}_{t,T}$ that are α-Hölder continuous in z uniformly with respect to s, that is
$$\sup_{\substack{s\in[t,T]\\ z\neq\zeta}} \frac{|f_s(z)-f_s(\zeta)|}{|z-\zeta|^{\alpha}} < \infty;$$
- $C^{k+\alpha}_{t,T}$ (resp. $bC^{k+\alpha}_{t,T}$) is the space of functions $f \in m\mathcal{B}_{t,T}$ that are k-times differentiable with respect to z with derivatives in $C^{\alpha}_{t,T}$ (resp. $bC^{\alpha}_{t,T}$).

We use boldface to denote the stochastic version of the previous functional spaces. More precisely, let $\mathcal{P}_{t,T}$ be the predictable σ-algebra on $[t,T] \times \Omega$.

DEFINITION 2.1. *We denote by* $\mathbf{C}^{k+\alpha}_{t,T}$ *the family of functions* $f = f_s(z,\omega)$ *on* $[t,T]\times\mathbb{R}^d\times\Omega$ *such that:*

i) $(z,x) \mapsto f_s(z,\omega) \in C^{k+\alpha}_{t,T}$ *for any* $\omega \in \Omega$;

ii) $(s,\omega) \mapsto f_s(z,\omega)$ *is* $\mathcal{P}_{t,T}$*-measurable for any* $z \in \mathbb{R}^d$.

Similarly, we define $\mathbf{bC}^{k+\alpha}_{t,T}$.

DEFINITION 2.2. *A stochastic fundamental solution* $\mathbf{\Gamma} = \mathbf{\Gamma}(t,x,v;\tau,\xi,\eta)$ *for the SPDE* (2) *is a function defined for* $0 \leqslant \tau < t \leqslant T$ *and* $x,v,\xi,\eta \in \mathbb{R}$, *such that for any* $(\tau,\zeta) \in [0,T) \times \mathbb{R}^2$ *we have:*

i) $\mathbf{\Gamma}(\cdot,\cdot,\cdot;\tau,\zeta)$ *belongs to* $\mathbf{C}_{t_0,T}(\mathbb{R}^2)$, *is twice continuously differentiable in* v *and with probability one satisfies*

$$\begin{aligned}\mathbf{\Gamma}(t,\gamma^B_t(x,v);\tau,\zeta) = \mathbf{\Gamma}(t_0,x,v;\tau,\zeta) &+ \int_{t_0}^t \frac{1}{2} a_s(\gamma^B_s(x,v))\left(\partial_{vv}\mathbf{\Gamma}\right)(s,\gamma^B_s(x,v);\tau,\zeta)ds \\ &+ \int_{t_0}^t \sigma_s(\gamma^B_s(x,v))\left(\partial_v\mathbf{\Gamma}\right)(s,\gamma^B_s(x,v);\tau,\zeta)dW_s\end{aligned} \tag{24}$$

for $\tau < t_0 \leqslant t \leqslant T$ *and* $x,v \in \mathbb{R}$, *with* $\gamma^B = \gamma^B_t(x,v)$ *as in* (3);

ii) *for any bounded and continuous function* φ *on* $\mathbb{R}^2$ *and* $z_0 \in \mathbb{R}^2$, *we have*

$$\lim_{\substack{(t,z)\to(\tau,z_0)\\ t>\tau}} \int_{\mathbb{R}^2} \mathbf{\Gamma}(t,z;\tau,\zeta)\varphi(\zeta)d\zeta = \varphi(z_0), \qquad P\text{-a.s.}$$

Next we state the standing assumptions on the coefficients of the SPDE (2).

ASSUMPTION 2.3 (**Regularity**). $a \in \mathbf{bC}^{\alpha}_{0,T}$ *for some* $\alpha \in (0,1)$ *and* $\sigma \in \mathbf{bC}^{3+\alpha}_{0,T}$.

ASSUMPTION 2.4 (**Coercivity**). *There exists a random, finite and positive constant* $\mathbf{m}$ *such that*

$$a_t(z) - \sigma_t^2(z) \geqslant \mathbf{m}, \qquad t \in [0,T],\ z \in \mathbb{R}^2,\ P\text{-a.s.}$$

One of the main tools in our analysis is the following Itô-Wentzell transform: for $\tau \in [0,T)$ and $(x,v) \in \mathbb{R}^2$, we consider the one-dimensional SDE

$$\gamma^{\mathrm{IW}}_{t,\tau}(x,v) = v - \int_\tau^t \sigma_s(x,\gamma^{\mathrm{IW}}_{s,\tau}(x,v))dW_s. \tag{25}$$

Assumption 2.3 ensures that (25) is solvable in the strong sense and the map $(x,v) \mapsto \left(x,\gamma^{\mathrm{IW}}_{t,\tau}(x,v)\right)$ is a stochastic flow of diffeomorphisms of $\mathbb{R}^2$ (see Theorem 3.1 below). In Section 4 we use this change of coordinates to transform the SPDE (2) into a PDE with random coefficients whose properties depend on the gradient of the stochastic flow: to have a control on it, we impose the following additional

ASSUMPTION 2.5. *There exist* $\varepsilon > 0$ *and a random variable* $M \in L^p(\Omega)$, *with* $p > \max\left\{2,\frac{1}{\varepsilon}\right\}$, *such that with probability one*

$$\sup_{\substack{t\in[0,T]\\ (x,v)\in\mathbb{R}^2}} (1+x^2+v^2)^{\varepsilon}\left|\partial_x^{\beta_1}\partial_v^{\beta_2}\sigma_t(x,v)\right| \leqslant M, \qquad \beta_1+\beta_2 = 1,$$

$$\sup_{\substack{t\in[0,T]\\(x,v)\in\mathbb{R}^2}} (1+x^2+v^2)^{\frac{1}{2}+\varepsilon}|\partial_x^{\beta_1}\partial_v^{\beta_2}\sigma_t(x,v)| \leqslant M, \qquad \beta_1+\beta_2 = 2,3.$$

Assumption 2.5 is the main ingredient in the estimates of Section 3: it requires that $\sigma_t(x,v)$ flattens as $(x,v)\to\infty$. In particular, this condition is clearly satisfied if $\sigma=\sigma_t$ depends only on t or, more generally, if the spatial gradient of σ has compact support.

We are now in position to state the main result of the paper.

THEOREM 2.6. *Let Assumptions 2.3, 2.4 and 2.5 be satisfied. Then the Fokker-Planck SPDE (2) has a fundamental solution $\mathbf{\Gamma}$ such that, for some positive random variables μ_1 and μ_2, with probability one we have*

$$\mu^{-1}\Gamma^{\text{heat}}\left(\mu^{-1}Q_{t-\tau}, g^{\text{IW},-1}(z)-\gamma_t^{\tau,\zeta}\right) \leqslant \mathbf{\Gamma}(t,z;\tau,\zeta) \leqslant \mu\Gamma^{\text{heat}}\left(\mu Q_{t-\tau}, g^{\text{IW},-1}(z)-\gamma_t^{\tau,\zeta}\right), \tag{26}$$

$$|\partial_v\mathbf{\Gamma}(t,x,v;\tau,\zeta)| \leqslant \frac{\mu}{\sqrt{t-\tau}}\Gamma^{\text{heat}}\left(\mu Q_{t-\tau}, g^{\text{IW},-1}(z)-\gamma_t^{\tau,\zeta}\right), \tag{27}$$

$$|\partial_{vv}\mathbf{\Gamma}(t,x,v;\tau,\zeta)| \leqslant \frac{\mu}{t-\tau}\Gamma^{\text{heat}}\left(\mu Q_{t-\tau}, g^{\text{IW},-1}(z)-\gamma_t^{\tau,\zeta}\right), \tag{28}$$

for every $0\leqslant\tau<t\leqslant T$ and $z=(x,v),\zeta\in\mathbb{R}^2$, where $g^{\text{IW},-1}$ denotes the inverse of the stochastic flow $(x,v)\mapsto\left(x,\gamma_{t,\tau}^{\text{IW}}(x,v)\right)$ defined by (25) and $\gamma_t^{\tau,\zeta}$ is the integral curve (see Theorem 3.1 below), starting from ζ, of the vector field

$$Y_{t,\tau} = \left(\gamma_{t,\tau}^{\text{IW}}, -\frac{\gamma_{t,\tau}^{\text{IW}}\partial_x\gamma_{t,\tau}^{\text{IW}}}{\partial_v\gamma_{t,\tau}^{\text{IW}}}\right),$$

Q_t is defined as in (8) and

$$\Gamma^{\text{heat}}(A,z) = \frac{1}{2\pi\sqrt{\det A}}e^{-\frac{1}{2}\langle A^{-1}z,z\rangle}$$

is the two-dimensional Gaussian kernel with symmetric and positive definite covariance matrix A.

The proof of Theorem 2.6 is postponed to Section 6.

3. Pointwise estimates for Itô processes In this section we prove some estimate for stochastic flows of diffeomorphisms that will play a central role in our analysis. Information about stochastic flows in a more general framework can be found in Kunita (1990). Since the following results are of a general nature and may be of independent interest, in this section we reset the notations and give the proofs in the more general multi-dimensional setting.

Specifically, until the end of the section, the point of $\mathbb{R}^d$ is denoted by $z=(z_1,\dots,z_d)$ and we set $\nabla_z=(\partial_{z_1},\dots,\partial_{z_d})$ and $\partial^\beta=\partial_z^\beta=\partial_{z_1}^{\beta_1}\cdots\partial_{z_d}^{\beta_d}$ for any multi-index β. We will also employ the notation

$$\langle z\rangle := \sqrt{1+|z|^2}, \qquad z\in\mathbb{R}^d.$$

First, we recall some basic facts about stochastic flows of diffeomorphisms. Let $k\in\mathbb{N}$. A $\mathbb{R}^d$-valued random field $\varphi_{\tau,t}(z)$, with $0\leqslant\tau\leqslant t\leqslant T$ and $z\in\mathbb{R}^d$, defined on $(\Omega,\mathcal{F},P)$, is called a (forward) stochastic flow of C^k-diffeomorphisms if there exists a set of probability one where:

i) $\varphi_{t,t}(z) = z$ for any $t \in [0,T]$ and $z \in \mathbb{R}^d$;

ii) $\varphi_{\tau,t} = \varphi_{s,t} \circ \varphi_{\tau,s}$ for $0 \leqslant \tau \leqslant s \leqslant t \leqslant T$;

iii) $\varphi_{\tau,t} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a C^k-diffeomorphism for all $0 \leqslant \tau \leqslant t \leqslant T$.

Stochastic flows can be constructed as solutions of stochastic differential equations. Let B a n-dimensional Brownian motion and consider the stochastic differential equation

$$\varphi_t(z) = z + \int_\tau^t b_s(\varphi_s(z))ds + \int_\tau^t \sigma_s(\varphi_s(z))dB_s \tag{29}$$

where $b = (b_t^i(z))$, $\sigma = (\sigma_t^{ij}(z))$ are a d-valued and $(d \times n)$-valued processes respectively, on $[0,T] \times \mathbb{R}^d \times \Omega$. The following theorem summarizes the results of Lemmas 4.5.3-7 and Theorems 4.6.4-5 in Kunita (1990).

Theorem 3.1. *Let b, $\sigma \in \mathbf{bC}^{k,\alpha}_{0,T}$ for some $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Then the solution of the stochastic differential equation (29) has a modification $\varphi_{\tau,t}$ that is a forward stochastic flow of C^k-diffeomorphisms. Moreover, $\varphi_{\tau,\cdot} \in \mathbf{C}^{k,\alpha'}_{\tau,T}$ for any $\alpha' \in [0,\alpha)$ and $\tau \in [0,T)$, and we have the following estimates: for each $p \in \mathbb{R}$ there exists a positive constant $\mathbf{c}_{1,p}$ such that*

$$E\left[\langle \varphi_{\tau,t}(z)\rangle^p\right] \leqslant \mathbf{c}_{1,p}\langle z\rangle^p, \qquad z \in \mathbb{R}^d, \tag{30}$$

and for each $p \geqslant 1$ there exists a positive constant $\mathbf{c}_{2,p}$ such that

$$E\left[\left|\partial^\beta \varphi_{\tau,t}(z)\right|^p\right] \leqslant \mathbf{c}_{2,p}, \qquad z \in \mathbb{R}^d,\ p \geqslant 1,\ 1 \leqslant |\beta| \leqslant k. \tag{31}$$

Now, consider $\varphi_{\tau,t}$ as in Theorem 3.1, $F_i = F_{i,t}(z;\zeta) \in \mathbf{C}^k_{0,T}(\mathbb{R}^{2d})$, $i = 1,2$, and a real Brownian motion W. The goal of this section is to prove some pointwise estimate for the Itô process

$$I_{\tau,t}(z) := \int_\tau^t F_{1,s}(z;\varphi_{\tau,s}(z))dW_s + \int_\tau^t F_{2,s}(z;\varphi_{\tau,s}(z))ds, \qquad 0 \leqslant \tau \leqslant t \leqslant T,\ z \in \mathbb{R}^d, \tag{32}$$

in terms of the usual Hölder norm in $\mathbb{R}^d$

$$|f|_\alpha = \sup_{z \in \mathbb{R}^d} |f(z)| + \sup_{\substack{z,\zeta \in \mathbb{R}^d \\ z \neq \zeta}} \frac{|f(z) - f(\zeta)|}{|z-\zeta|^\alpha}, \qquad \alpha \in (0,1),$$

under the following

Assumption 3.2. *There exist $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ with $\varepsilon := \varepsilon_1 + \varepsilon_2 > 0$ and a random variable $M \in L^{\bar p}(\Omega)$, with $\bar p > \left(2 \vee d \vee \frac{d}{\varepsilon}\right)$, such that*

$$\sum_{|\beta| \leqslant k} \sup_{\substack{t \in [0,T] \\ z,\zeta \in \mathbb{R}^d}} \langle z\rangle^{\varepsilon_1}\langle \zeta\rangle^{\varepsilon_2} |\partial^\beta_{z,\zeta} F_{i,t}(z;\zeta)| \leqslant M \qquad i = 1,2,\ P\text{-a.s.}$$

The main result of this section is the following theorem which provides global-in-space pointwise estimates for the process in (32).

THEOREM 3.3. *Let $\varphi_{\tau,t}$ be as in Theorem 3.1 and $F^{(i)} \in \mathbf{C}^k_{0,T}(\mathbb{R}^{2d})$, $i = 1, 2$, for some $k \in \mathbb{N}$. Let $I = I_{\tau,t}(z)$ be as in (32) and set*

$$I^{(\delta)}_{\tau,t}(z) := \langle z\rangle^{\delta} I_{\tau,t}(z).$$

Under Assumption 3.2, for any p, α and δ such that

$$\left(2 \vee d \vee \frac{d}{\varepsilon}\right) < p < \bar{p}, \qquad 0 \leqslant \alpha < \frac{1}{2} - \frac{1}{p}, \qquad 0 \leqslant \delta < \varepsilon - \frac{d}{p},$$

there exists a (random, finite) constant $\mathbf{m}$ such that

$$\sum_{|\beta| \leqslant k-1} |\partial^{\beta} I^{(\delta)}_{\tau,t}|_{1-\frac{d}{p}} \leqslant \mathbf{m}(t-\tau)^{\alpha} \qquad P\text{-a.s.} \tag{33}$$

PROOF. The proof is based on a combination of sharp L^p-estimates, Kolmogorov continuity theorem in Banach spaces and Sobolev embedding theorem.

Let us first consider the case $k = 1$. We prove some preliminary L^p-estimates for $I_{\tau,t}$ and $\partial^{\beta} I_{\tau,t}$ with $|\beta| = 1$. Below we denote by $\bar{c}$ various positive constants that depend only on p, d, T and the flow φ. By Burkölder's inequality we have

$$E\left[|I^{(\delta)}_{\tau,t}(z)|^p\right] \leqslant \bar{c}\langle z\rangle^{\delta p} E\left[\left(\int_{\tau}^{t} F^2_{1,s}(z;\varphi_{\tau,s}(z))ds\right)^{\frac{p}{2}}\right] + \bar{c}\langle z\rangle^{\delta p} E\left[\left(\int_{\tau}^{t} F_{2,s}(z;\varphi_{\tau,s}(z))ds\right)^{p}\right] \leqslant$$

(by Hölder's inequality)

$$\leqslant \bar{c}\langle z\rangle^{\delta p}(t-\tau)^{\frac{p-2}{2}} \int_{\tau}^{t} E\left[|F_{1,s}(z;\varphi_{\tau,s}(z))|^p\right] ds$$
$$+ \bar{c}\langle z\rangle^{\delta p}(t-\tau)^{p-1} \int_{\tau}^{t} E\left[|F_{2,s}(z;\varphi_{\tau,s}(z))|^p\right] ds \leqslant$$

(by Assumption 3.2)

$$\leqslant \bar{c}\langle z\rangle^{(\delta-\varepsilon_1)p}(t-\tau)^{\frac{p-2}{2}} \int_{\tau}^{t} E\left[M^p \langle\varphi_{\tau,s}(z)\rangle^{-\varepsilon_2 p}\right] ds \leqslant$$

(by Hölder's inequality with conjugate exponents $q := \frac{\bar{p}}{p}$ and r)

$$\leqslant \bar{c}\langle z\rangle^{(\delta-\varepsilon_1)p}(t-\tau)^{\frac{p-2}{2}} \|M\|^p_{L^{\bar{p}}(\Omega)} \int_{\tau}^{t} E\left[\langle\varphi_{\tau,s}(z)\rangle^{-\varepsilon_2 pr}\right]^{\frac{1}{r}} ds \leqslant$$

(by (30))

$$= \bar{c}\langle z\rangle^{(\delta-\varepsilon)p}(t-\tau)^{\frac{p}{2}}. \tag{34}$$

The same estimate holds for the gradient of $I_{\tau,t}$, that is

$$E\left[|\nabla I^{(\delta)}_{\tau,t}(z)|^p\right] \leqslant \bar{c}\langle z\rangle^{(\delta-\varepsilon)p}(t-\tau)^{\frac{p}{2}}. \tag{35}$$

Indeed, let us consider for simplicity only the case $\delta = 0$ since the general case is a straightforward consequence of the product rule: for $j = 1, \dots, d$, we have

$$\begin{aligned} E\left[|\partial_{z_j} I_{\tau,t}(z)|^p\right] &\leqslant \bar{c} E\left[\left|\int_\tau^t \Big((\partial_{z_j} F_{1,s})(z;\varphi_{\tau,s}(z)) + \langle \nabla_\zeta F_{1,s}(z;\varphi_{\tau,s}(z)), \partial_{z_j}\varphi_{\tau,s}(z)\rangle\Big) dW_s\right|^p\right] \\ &\quad + \bar{c} E\left[\left|\int_\tau^t \Big((\partial_{z_j} F_{2,s})(z;\varphi_{\tau,s}(z)) + \langle \nabla_\zeta F_{2,s}(z;\varphi_{\tau,s}(z)), \partial_{z_j}\varphi_{\tau,s}(z)\rangle\Big) ds\right|^p\right] \\ &\leqslant \bar{c}(t-\tau)^{\frac{p-2}{2}} \sum_{i=1}^{2} \int_\tau^t E\left[|(\partial_{z_j} F_{i,s})(z;\varphi_{\tau,s}(z))|^p + \left|\langle \nabla_\zeta F_{i,s}(z;\varphi_{\tau,s}(z)), \partial_{z_j}\varphi_{\tau,s}(z)\rangle\right|^p\right] ds. \end{aligned}$$

The terms containing $\partial_{z_j} F_{i,s}$ can be estimated as before, by means of Assumption 3.2. On the other hand, by Hölder's inequality with conjugate exponents q and r with $1 < q < \frac{\bar{p}}{p}$, for every $i, j = 1, \cdots, d$ we have

$$E\left[\left|\langle \nabla_\zeta F_{i,s}(z;\varphi_{\tau,s}(z)), \partial_{z_j}\varphi_{\tau,s}(z)\rangle\right|^p\right] \leqslant E\left[|\nabla_\zeta F_{i,s}(z;\varphi_{\tau,s}(z))|^{pq}\right]^{\frac{1}{q}} E\left[\left|\partial_{z_j}\varphi_{\tau,s}(z)\right|^{pr}\right]^{\frac{1}{r}} \leqslant$$

(by Assumption 3.2 and (31))

$$\leqslant \bar{c}_{2,pr}^{\frac{1}{r}} E\left[M^{pq} \langle \varphi_{\tau,s}(z)\rangle^{-\varepsilon_2 pq}\right]^{\frac{1}{q}} \langle z\rangle^{-\varepsilon_1 p} \leqslant$$

(by Hölder inequality with conjugate exponents $\bar{q} := \frac{\bar{p}}{pq} > 1$ and $\bar{r}$)

$$\leqslant \bar{c}_{2,pr}^{\frac{1}{r}} \|M\|^p_{L^{\bar{p}}(\Omega)} E\left[\langle \varphi_{\tau,s}(z)\rangle^{-\varepsilon_2 pq\bar{r}}\right]^{\frac{1}{q\bar{r}}} \langle z\rangle^{-\varepsilon_1 p} \leqslant$$

(by (30))

$$\leqslant \bar{c} \|M\|^p_{L^{\bar{p}}(\Omega)} \langle z\rangle^{-\varepsilon p}.$$

This proves (35) with $\delta = 0$.

Now, we have

$$E\left[\|I^{(\delta)}_{\tau,t}\|^p_{W^{1,p}(\mathbb{R}^d)}\right] = E\left[\int_{\mathbb{R}^d} \left(|I^{(\delta)}_{\tau,t}(z)|^p + |\nabla I^{(\delta)}_{\tau,t}(z)|^p\right) dz\right] \leqslant$$

(by (34) and (35))

$$\leqslant \bar{c}(t-\tau)^{\frac{p}{2}} \int_{\mathbb{R}^d} \langle z\rangle^{(\delta-\varepsilon)p} dz =$$

(since $(\varepsilon - \delta)p > d$ by assumption)

$$= \bar{c}(t-\tau)^{\frac{p}{2}}. \tag{36}$$

Estimate (36) and Kolmogorov's continuity theorem for processes with values in the Banach space $W^{1,p}(\mathbb{R}^d)$ (see, for instance, Kunita (1990), Theor.1.4.1) yield

$$\|I^{(\delta)}_{\tau,t}\|_{W^{1,p}(\mathbb{R}^d)} \leqslant \mathbf{m}(t-\tau)^\alpha, \qquad 0 \leqslant \tau \leqslant t \leqslant T, \ P\text{-a.s.}$$

for some positive and finite random variable $\mathbf{m}$ and for $\alpha \in [0, \frac{p-2}{2p})$. This is sufficient to prove (33) with $k = 1$: in fact, by the Sobolev embedding theorem, we have the following estimate of the Hölder norm

$$|I^{(\delta)}_{\tau,t}|_{1-\frac{d}{p}} \leqslant N \|I^{(\delta)}_{\tau,t}\|_{W^{1,p}(\mathbb{R}^d)} \tag{37}$$

where N is a positive constant that depends only on p and d. Thus, combining (33) and (37), we get the thesis with $k = 1$.

Noting that

$$\begin{aligned}\partial_{z_j} I_{\tau,t}(z) = &\int_\tau^t \Big((\partial_{z_j} F_{1,s})(z; \varphi_{\tau,s}(z)) + \langle \nabla_\zeta F_{1,s}(z; \varphi_{\tau,s}(z)), \partial_{z_j} \varphi_{\tau,s}(z)\rangle \Big) dW_s \\ &+ \int_\tau^t \Big((\partial_{z_j} F_{2,s})(z; \varphi_{\tau,s}(z)) + \langle \nabla_\zeta F_{2,s}(z; \varphi_{\tau,s}(z)), \partial_{z_j} \varphi_{\tau,s}(z)\rangle \Big) ds,\end{aligned}$$

for $j = 1, \dots, d$, the thesis with $k = 2$ can be proved repeating the previous arguments and using (33) for $k = 1$ and Assumption 3.2 with $k = 2$.

We omit the complete proof for brevity and since in the rest of the paper we will use (33) only for $k = 1, 2$. The general result can be proved by induction, using the multi-variate Faà di Bruno's formula. □

REMARK 3.4. *Let $I_{\tau,t}$ as in (32) with coefficients $\widetilde{F}_1, \widetilde{F}_2 \in b\mathbf{C}^1_{0,T}(\mathbb{R}^{2d})$ and let $\delta > 0$ and $\alpha \in [0, \frac{1}{2})$. Applying Theorem 3.3 with $F_{i,t}(z;\zeta) := \langle z\rangle^{-\delta} \widetilde{F}_{i,t}(z;\zeta)$, $i = 1, 2$, we get the existence of a (random, finite) constant $\mathbf{m}$ such that, with probability one,*

$$|I_{\tau,t}(z)| \leqslant \mathbf{m} \langle z\rangle^\delta (t-\tau)^\alpha, \qquad 0 \leqslant \tau \leqslant t \leqslant T, \; z \in \mathbb{R}^d.$$

4. Itô-Wentzell change of coordinates

We go back to the main SPDE (2) and suppose that Assumptions 2.3, 2.4 and 2.5 are satisfied. In this section we study the properties of a random change of variables which plays the same role as transformation (11) in Step 1 of Section 1.1 for the Langevin SPDE. The main result of this section is Theorem 4.3 which shows that this change of variables transforms SPDE (2) into a PDE with random coefficients.

We denote by $(x, \gamma^{\mathrm{IW}}_{t,\tau}(x,v))$ the stochastic flow of diffeomorphisms of $\mathbb{R}^2$ defined by equation (25), that is

$$\gamma^{\mathrm{IW}}_{t,\tau}(x,v) = v - \int_\tau^t \sigma_s(x, \gamma^{\mathrm{IW}}_{s,\tau}(x,v)) dW_s, \qquad 0 \leqslant \tau \leqslant t \leqslant T, \; (x,v) \in \mathbb{R}^2. \tag{38}$$

By Theorem 3.1, $\gamma^{\mathrm{IW}}_{\cdot,\tau} \in \mathbf{C}^{3,\alpha'}_{\tau,T}$ for any $\alpha' \in [0, \alpha)$. Global estimates for γ^{IW} and its derivatives are provided in the next:

LEMMA 4.1. *There exists $\varepsilon \in \left(0, \frac{1}{2}\right)$ and a (random, finite) constant $\mathbf{m}$ such that, with probability one,*

$$|\gamma^{\mathrm{IW}}_{t,\tau}(x,v)| \leqslant \mathbf{m}\sqrt{1 + x^2 + v^2}, \tag{39}$$

$$e^{-\mathbf{m}(t-\tau)^{\varepsilon}} \leqslant \partial_v \gamma^{\mathrm{IW}}_{t,\tau}(x,v) \leqslant e^{\mathbf{m}(t-\tau)^{\varepsilon}}, \tag{40}$$

$$|\partial_x \gamma^{\mathrm{IW}}_{t,\tau}(x,v)| \leqslant \mathbf{m}(t-\tau)^{\varepsilon}, \tag{41}$$

$$|\partial^{\beta} \gamma^{\mathrm{IW}}_{t,\tau}(x,v)| \leqslant \frac{\mathbf{m}(t-\tau)^{\varepsilon}}{\sqrt{1+x^2+v^2}}, \tag{42}$$

for any $(x,v) \in \mathbb{R}^2$, $0 \leqslant \tau \leqslant t \leqslant T$ *and* $|\beta| = 2$.

PROOF. Estimate (39) follows directly from Remark 3.4 (with $\delta = 1$). Differentiating (38), we find that $\partial_v \gamma^{\mathrm{IW}}_{t,\tau}$ solves the linear SDE

$$\partial_v \gamma^{\mathrm{IW}}_{t,\tau}(x,v) = 1 - \int_\tau^t (\partial_2 \sigma_s)(x, \gamma^{\mathrm{IW}}_{s,\tau}(x,v)) \partial_v \gamma^{\mathrm{IW}}_{s,\tau}(x,v) dW_s,$$

where $\partial_2 \sigma_t$ denotes the partial derivative of $\sigma_t(\cdot,\cdot)$ with respect to its second argument. Hence we have

$$\partial_v \gamma^{\mathrm{IW}}_{t,\tau}(x,v) = \exp\left(-\int_\tau^t (\partial_2 \sigma_s)(x, \gamma^{\mathrm{IW}}_{s,\tau}(x,v)) dW_s - \frac{1}{2} \int_\tau^t (\partial_2 \sigma_s)^2(x, \gamma^{\mathrm{IW}}_{s,\tau}(x,v)) ds \right).$$

Now we apply Theorem 3.3 with $\varphi_{\tau,t}(x,v) = (x, \gamma^{\mathrm{IW}}_{t,\tau}(x,v))$ and $F_{i,t}(z;x,V) = (\partial_2 \sigma_t(x,V))^i$, $i = 1,2$: thanks to Assumption 2.5, we get estimate (40). Incidentally, from Theorem 3.3 we also deduce that the first order derivatives of $\partial_v \gamma^{\mathrm{IW}}_t$ are bounded:

$$|\partial^{\beta} \partial_v \gamma^{\mathrm{IW}}_{t,\tau}(x,v)| \leqslant \mathbf{m}(t-\tau)^{\varepsilon}, \qquad |\beta| = 1. \tag{43}$$

This last estimate is used in the next step, for the proof of (41).

Similarly, we have

$$\partial_x \gamma^{\mathrm{IW}}_{t,\tau}(x,v) = -\int_\tau^t \big((\partial_1 \sigma_s)(x, \gamma^{\mathrm{IW}}_{s,\tau}(x,v)) + (\partial_2 \sigma_s)(x, \gamma^{\mathrm{IW}}_{s,\tau}(x,v)) \partial_x \gamma^{\mathrm{IW}}_{s,\tau}(x,v) \big) dW_s.$$

Thus, we have a linear SDE whose solution is given by

$$\begin{aligned} \partial_x \gamma^{\mathrm{IW}}_{t,\tau}(x,v) = & -\partial_v \gamma^{\mathrm{IW}}_{t,\tau}(x,v) \int_\tau^t \frac{(\partial_1 \sigma_s)(x, \gamma^{\mathrm{IW}}_{s,\tau}(x,v))}{\partial_v \gamma^{\mathrm{IW}}_{s,\tau}(x,v)} dW_s \\ & - \partial_v \gamma^{\mathrm{IW}}_{t,\tau}(x,v) \int_\tau^t \frac{(\partial_1 \sigma_s)(x, \gamma^{\mathrm{IW}}_{s,\tau}(x,v))(\partial_2 \sigma_s)(x, \gamma^{\mathrm{IW}}_{s,\tau}(x,v))}{\partial_v \gamma^{\mathrm{IW}}_{s,\tau}(x,v)} ds, \end{aligned}$$

Again, (41) follows from Theorem 3.3 thanks to Assumption 2.5 and estimates (40) and (43).

Eventually, the same argument can be used to prove (42): indeed, differentiating (38) we have that $\partial^{\beta} \gamma^{\mathrm{IW}}_t$ satisfies a linear SDE whose solution is explicit. Thus, for $|\beta| = 2$, $\partial^{\beta} \gamma^{\mathrm{IW}}_t$ can be expressed in the form (32) with the coefficients satisfying Assumption 3.2 for some $\varepsilon > 1$. Applying Theorem 3.3 with $\delta = 1$ we get estimate (42). □

Next, we provide a version of the Itô-Wentzell formula for an equation of the form

$$d_{\mathbf{Y}} u_{t,\tau}(x,v) = f_t(x,v) dt + g_t(x,v) dW_t, \qquad \mathbf{Y} = \partial_t + v \partial_x, \tag{44}$$

with $u, f, g \in \mathbf{C}_{\tau,T}$. Equation (44) is solved in the strong sense which means

$$u_{t,\tau}\left(\gamma^B_{t-\tau}(x,v)\right) = u_{\tau,\tau}(x,v) + \int_\tau^t f_s(\gamma^B_{s-\tau}(x,v))ds + \int_\tau^t g_s(\gamma^B_{s-\tau}(x,v))dW_s, \qquad t \in [\tau, T],$$

with probability one, where $\gamma^B_t(x,v) = (x+tv, v)$ is the integral curve in $\mathbb{R}^2$ of the vector field $v\partial_x$, starting from (x,v). The following lemma shows how (44) is modified by the Itô-Wentzell transform

$$\hat{u}_{t,\tau}(x,v) = u_{t,\tau}(x, \gamma^{\mathrm{IW}}_{t,\tau}(x,v)), \tag{45}$$

with $\gamma^{\mathrm{IW}}_{t,\tau}$ as in (38).

LEMMA 4.2 (**Itô-Wentzell formula**). *Let $\partial_2 u_{t,\tau}, \partial_{22} u_{t,\tau}, \partial_2 g_t \in \mathbf{C}_{\tau,T}$ and assume that (44) holds. Then $\hat{u}_{t,\tau}$ in (45) solves*

$$d_{\hat{\mathbf{Y}}}\hat{u}_{t,\tau}(x,v) = F_t(x,v)dt + G_t(x,v)dW_t, \tag{46}$$

where

$$\begin{aligned}
F_t(x,v) &= \hat{f}_t(x,v) + \frac{1}{2}\hat{\sigma}^2_t(x,v)\widehat{\partial_{22}u_{t,\tau}}(x,v) - \widehat{\partial_2 g_t}(x,v)\hat{\sigma}_t(x,v),\\
G_t(x,v) &= \hat{g}_t(x,v) - \hat{\sigma}_t(x,v)\widehat{\partial_2 u_{t,\tau}}(x,v),\\
\hat{\mathbf{Y}} &= \partial_t + \gamma^{\mathrm{IW}}_{t,\tau}\partial_x - \frac{\gamma^{\mathrm{IW}}_{t,\tau}\partial_x\gamma^{\mathrm{IW}}_{t,\tau}}{\partial_v\gamma^{\mathrm{IW}}_{t,\tau}}\partial_v.
\end{aligned} \tag{47}$$

Moreover, we have

$$\widehat{\partial_2 u_{t,\tau}} = \frac{\partial_v \hat{u}_{t,\tau}}{\partial_v \gamma^{\mathrm{IW}}_{t,\tau}}, \qquad \widehat{\partial_{22} u_{t,\tau}} = \frac{\partial_{vv}\hat{u}_{t,\tau}}{(\partial_v\gamma^{\mathrm{IW}}_{t,\tau})^2} - \frac{\partial_{vv}\gamma^{\mathrm{IW}}_{t,\tau}\partial_v\hat{u}_{t,\tau}}{(\partial_v\gamma^{\mathrm{IW}}_{t,\tau})^3}. \tag{48}$$

PROOF. We have to show that

$$\hat{u}_{t,\tau}\left(\gamma_{t,\tau}(x,v)\right) = \hat{u}_0(x,v) + \int_0^t F_s(\gamma_{s,\tau}(x,v))ds + \int_0^t G_s(\gamma_{s,\tau}(x,v))dW_s, \qquad t \in [0,T],$$

where $\gamma_{t,\tau}(x,v)$ is the integral curve, starting from (x,v), of the vector field

$$Y_{t,\tau} = \left(\gamma^{\mathrm{IW}}_{t,\tau}, -\frac{\gamma^{\mathrm{IW}}_{t,\tau}\partial_x\gamma^{\mathrm{IW}}_{t,\tau}}{\partial_v\gamma^{\mathrm{IW}}_{t,\tau}}\right).$$

Notice that, with the usual identification of vector fields with first order operators, we have $\hat{\mathbf{Y}} = \partial_t + Y_{t,\tau}$. Moreover, γ is well defined thanks to the estimates of Lemma 4.1.

If $u \in \mathbf{C}^2_{\tau,T}$ then (44) can be written in the usual Itô sense

$$du_{t,\tau}(x,v) = (f_t(x,v) - v\partial_x u_{t,\tau}(x,v))dt + g_t(x,v)dW_t.$$

Then, by the standard Itô-Wentzell formula (see, for instance, Theor. 3.3.1 in Kunita (1990)), we have

$$d\hat{u}_{t,\tau} = \left((\hat{f}_t - \gamma^{\mathrm{IW}}_{t,\tau}\widehat{\partial_1 u_{t,\tau}}) + \frac{1}{2}\hat{\sigma}^2_t\widehat{\partial^2_2 u_{t,\tau}} - \widehat{\partial_2 g_t}\hat{\sigma}_t\right)dt + \left(\hat{g}_t - \hat{\sigma}_t\widehat{\partial_2 u_{t,\tau}}\right)dW_t. \tag{49}$$

From the chain rule we easily derive equations (48) and also

$$\widehat{\partial_1 u_{t,\tau}} = \partial_x \hat{u}_{t,\tau} - \frac{\partial_x \gamma^{\mathrm{IW}}_{t,\tau}}{\partial_v \gamma^{\mathrm{IW}}_{t,\tau}} \partial_v \hat{u}_{t,\tau}.$$

Plugging these formulas into (49) we get (46).

In the general case, it suffices to apply what we have just proved to a smooth approximation in (x,v) of $u_{t,\tau}$ and then pass to the limit. □

Applying the Itô-Wentzell formula to SPDE (2) we get the following

THEOREM 4.3. *Let $u_{t,\tau}, \partial_2 u_{t,\tau}, \partial_{22} u_{t,\tau} \in \mathbf{C}_{\tau,T}$ and let Assumptions 2.3, 2.4 and 2.5 be satisfied. If $u_{t,\tau}$ solves the SPDE (2) then $\hat{u}_{t,\tau}$ in (45) is such that $\hat{u}_{t,\tau}, \partial_v \hat{u}_{t,\tau}, \partial_{vv} \hat{u}_{t,\tau} \in \mathbf{C}_{\tau,T}$ and*

$$d_{\hat{\mathbf{Y}}} \hat{u}_{t,\tau}(x,v) = \big(\bar{a}_{t,\tau}(x,v)\partial_{vv}\hat{u}_{t,\tau}(x,v) + \bar{b}_{t,\tau}(x,v)\partial_v \hat{u}_{t,\tau}(x,v)\big)dt, \tag{50}$$

with $\hat{\mathbf{Y}}$ as in (47) and

$$\bar{a}_{t,\tau} = \frac{\hat{a}_t - \hat{\sigma}_t^2}{2(\partial_v \gamma^{\mathrm{IW}}_{t,\tau})^2}, \qquad \bar{b}_{t,\tau} = -\frac{1}{(\partial_v \gamma^{\mathrm{IW}}_{t,\tau})^2}\left(\hat{\sigma}_t \partial_v \hat{\sigma}_t + \frac{\left(\hat{a}_t - \hat{\sigma}_t^2\right)\partial_{vv}\gamma^{\mathrm{IW}}_{t,\tau}}{2\partial_v \gamma^{\mathrm{IW}}_{t,\tau}}\right). \tag{51}$$

PROOF. The thesis follows from the Itô-Wentzell formula of Lemma 4.2 with $f_t = \frac{1}{2} a_t \partial_{22} u_{t,\tau}$ and $g_t = \sigma_t \partial_2 u_{t,\tau}$: the assumptions $\partial_2 u_{t,\tau}, \partial_{22} u_{t,\tau}, \partial_2 g_t \in \mathbf{C}_{\tau,T}$ are clearly satisfied. □

5. Time-dependent parametrix method In this section we study equation (50) for fixed $\omega \in \Omega$ and $0 \leqslant \tau < T < \infty$. More generally, we consider a deterministic equation of the form

$$\mathcal{K}_t u_t(z) = \mathcal{L}_t u_t(z) - \partial_t u_t(z) = 0 \tag{52}$$

where

$$\mathcal{L}_t u_t(z) := \frac{1}{2} a_t(z)\partial_{vv} u_t(z) + b_t(z)\partial_v u_t(z) - \langle Y_t(z), \nabla_z u_t(z)\rangle, \qquad t \in [\tau, T],\ z = (x,v) \in \mathbb{R}^2,$$

and $Y_t = (Y_{1,t}, Y_{2,t})$ is a generic vector field. We assume the following conditions on the coefficients.

ASSUMPTION 5.1. *There exist positive constants α, λ_1 such that $a, b \in C^\alpha_{\tau,T}$ with Hölder constant λ_1 and*

$$\lambda_1^{-1} \leqslant a_t(z) \leqslant \lambda_1, \quad |b_t(z)| \leqslant \lambda_1 \qquad (t,z) \in [\tau,T] \times \mathbb{R}^2. \tag{53}$$

ASSUMPTION 5.2. *$Y \in C_{\tau,T}$ and is uniformly Lipschitz continuous in the sense that*

$$\sup_{\substack{t \in [\tau,T] \\ z \neq \zeta}} \frac{|Y_t(z) - Y_t(\zeta)|}{|z - \zeta|} \leqslant \lambda_2$$

for some positive constant λ_2. Moreover $\partial_v Y_{1,t} \in C^\alpha_{\tau,T}$ and

$$\lambda_2^{-1} \leqslant \partial_v Y_{1,t}(z) \leqslant \lambda_2, \qquad (t,z) \in [\tau,T] \times \mathbb{R}^2. \tag{54}$$

REMARK 5.3. *When the coefficients are smooth, conditions (53) and (54) ensure the validity of the weak Hörmander condition: indeed the vector fields $\sqrt{a}\partial_v$ and Y, together with their commutator, span $\mathbb{R}^2$ at any point. In this case a smooth fundamental solution exists by Hörmander's theorem.*

Since the coefficients are assumed to be only measurable in time, a solution to (52) has to be understood in the integral sense according to the following definition.

DEFINITION 5.4. *A fundamental solution $\Gamma = \Gamma(t, z; s, \zeta)$ for equation (2) is a function defined for $\tau \leqslant s < t \leqslant T$ and $z, \zeta \in \mathbb{R}^2$, such that for any $(s, \zeta) \in [\tau, T) \times \mathbb{R}^2$ we have:*

i) for $s < t_0 \leqslant t \leqslant T$ and $z \in \mathbb{R}^2$, $\Gamma(\cdot, \cdot; s, \zeta)$ belongs to $C_{t_0,T}$, is twice continuously differentiable in v and satisfies

$$\mathbf{\Gamma}(t, \gamma_t^{t_0,z}; s, \zeta) = \Gamma(t_0, z; s, \zeta) + \int_{t_0}^{t} \left(\frac{1}{2} a_\varrho(\gamma_\varrho^{t_0,z}) \left(\partial_{vv}\Gamma\right)(\varrho, \gamma_\varrho^{t_0,z}; s, \zeta) + b_\varrho(\gamma_\varrho^{t_0,z}) \left(\partial_v\Gamma\right)(\varrho, \gamma_\varrho^{t_0,z}; s, \zeta) \right) d\varrho$$

where $\gamma_t^{t_0,z}$ stands for the integral curve of the field Y with initial datum $\gamma_{t_0}^{t_0,z} = z$;

ii) for any bounded and continuous function φ and $z_0 \in \mathbb{R}^2$, we have

$$\lim_{\substack{(t,z)\to(s,z_0)\\ t>s}} \int_{\mathbb{R}^2} \Gamma(t, z; s, \zeta)\varphi(\zeta) d\zeta = \varphi(z_0).$$

The main result of this section is the following

THEOREM 5.5. *Let Assumptions 5.1 and 5.2 be in force. Then the PDE (52) has a fundamental solution Γ such that, for any $z = (x, v), \zeta \in \mathbb{R}^2$ and $\tau \leqslant s < t \leqslant T$,*

$$\mu^{-1}\Gamma^{\text{heat}}\left(\mu^{-1}Q_{t-s}, z - \gamma_t^{s,\zeta}\right) \leqslant \Gamma(t, z; s, \zeta) \leqslant \mu\Gamma^{\text{heat}}\left(\mu Q_{t-s}, z - \gamma_t^{s,\zeta}\right), \tag{55}$$

$$|\partial_v\Gamma(t, x, v; s, \zeta)| \leqslant \frac{\mu}{\sqrt{t-s}}\Gamma^{\text{heat}}\left(\mu Q_{t-s}, z - \gamma_t^{s,\zeta}\right), \tag{56}$$

$$|\partial_{vv}\Gamma(t, x, v; s, \zeta)| \leqslant \frac{\mu}{t-s}\Gamma^{\text{heat}}\left(\mu Q_{t-s}, z - \gamma_t^{s,\zeta}\right). \tag{57}$$

where Q_t is as in (8), $\gamma_t^{s,\zeta}$ is as in Definition 5.4 and μ is a positive constant that depends only on λ_1, λ_2, α and T.

5.1. *Proof of Theorem 5.5* We prove the Gaussian bounds (55)-(57) under the assumption that the coefficients a and Y are smooth; by Remark 5.3, this guarantees the existence of a smooth fundamental solution. Our estimates extend and sharpen classical Gaussian bounds for Hörmander operators (e.g. Jerison and Sánchez-Calle (1986)). Moreover, our estimates are independent of the regularity of the coefficients and, by standard regularization arguments, lead to a priori Gaussian bounds for operators satisfying Assumptions 53-54.

5.1.1. *Parametrix expansion* For fixed $(s,\eta) \in [\tau, T) \times \mathbb{R}^2$, let

$$\gamma_t^{s,\eta} = \eta + \int_s^t Y_\varrho \left(\gamma_\varrho^{s,\eta}\right) d\varrho, \qquad t \in [\tau, T], \tag{58}$$

be the integral curve of Y_t starting from (s,η). Following Delarue and Menozzi (2010) we linearize $Y_t = Y_t(z)$ at (s,η) setting

$$\bar{Y}_t^{s,\eta}(z) = Y_t(\gamma_t^{s,\eta}) + (DY_t)\left(\gamma_t^{s,\eta}\right)\left(z - \gamma_t^{s,\eta}\right), \qquad t \in [s,T],\ z \in \mathbb{R}^2.$$

where DY_t stands for a reduced Jacobian defined as

$$DY_t := \begin{pmatrix} 0 & \partial_v Y_{1,t} \\ 0 & 0 \end{pmatrix}.$$

Then we consider the linear approximation of $\mathcal{L}_t$ defined as

$$\bar{\mathcal{L}}_t^{s,\eta} := \frac{1}{2} a_t(\gamma_t^{s,\eta})\partial_{vv} - \langle \bar{Y}_t^{s,\eta}(z), \nabla \rangle.$$

The diffusion coefficient of $\bar{\mathcal{L}}_t^{s,\eta}$ depends on t only (apart from s,η that are fixed parameters), while the drift coefficients depend on t and linearly on x, v. Notice that $\bar{\mathcal{L}}_t^{s,\eta} - \partial_t$ is the forward Kolmogorov operator of the system of linear SDEs

$$dH_t = \bar{Y}_t^{s,\eta}\left(H_t\right) dt + \sqrt{a_t(\gamma_t^{s,\eta})}\mathbf{e_2} dW_t. \tag{59}$$

Let $H_t^{t_0,\zeta}$ denote the solution of (59) starting from ζ at time $t_0 \in [s,T)$. Then $H_t^{t_0,\zeta}$ is a Gaussian process: the mean $\bar{\gamma}_{t,t_0}^{s,\eta}(\zeta) := E\left[H_t^{t_0,\zeta}\right]$ solves the ODE

$$\bar{\gamma}_{t,t_0}^{s,\eta}(\zeta) = \zeta + \int_{t_0}^t \bar{Y}_\varrho^{s,\eta}\left(\bar{\gamma}_{\varrho,t_0}^{s,\eta}(\zeta)\right) d\varrho, \qquad t \in [t_0, T],$$

and the covariance matrix is given by

$$\mathbf{A}_{t,t_0}^{s,\eta} = \int_{t_0}^t a_\varrho(\gamma_s^{\varrho,\eta})\left(E_{t,\varrho}^{s,\eta}\mathbf{e}_2\right)\left(E_{t,\varrho}^{s,\eta}\mathbf{e}_2\right)^* d\varrho,$$

where $E_{t,\varrho}^{s,\eta}$ is the fundamental matrix associated with $(DY_t)(\gamma_t^{s,\eta})$, that is the solution of

$$E_{t,\varrho}^{s,\eta} = \mathrm{Id} + \int_\varrho^t (DY_u)(\gamma_u^{s,\eta}) E_{u,\varrho}^{s,\eta} du, \qquad t \in [\varrho, T],$$

with Id equal to the (2×2)-identity matrix.

LEMMA 5.6. *For any $\eta \in \mathbb{R}^2$ and $\tau \leqslant s \leqslant t_0 < t \leqslant T$, we have* $\det \mathbf{A}_{t,t_0}^{s,\eta} > 0$.

PROOF. By Assumption 5.1 it is enough to prove the assertion for $a \equiv 1$. Suppose that there exist $\zeta \in \mathbb{R}^2 \setminus \{0\}$, $\eta \in \mathbb{R}^2$ and $\tau \leqslant s \leqslant t_0 < t \leqslant T$ such that $\langle \mathbf{A}_{t,t_0}^{s,\eta}\zeta, \zeta\rangle = 0$. Since $\mathbf{A}_{t,t_0}^{s,\eta}$ is positive semi-definite, this is equivalent to the condition

$$|(E_{t,\varrho}^{s,\eta}\mathbf{e}_2)^*\zeta|^2 = 0, \quad \text{a.e. } \varrho \in (t_0, t),$$

that is $((E^{s,\eta}_{t,\varrho})^*\zeta)_2 = 0$, for a.e. $\varrho \in (t_0, t)$. We have

$$\partial_\varrho (E^{s,\eta}_{t,\varrho})^*\zeta = -DY^*_\varrho(\gamma^{s,\eta}_\varrho)(E^{s,\eta}_{t,\varrho})^*\zeta,$$

and therefore

$$0 = \partial_\varrho((E^{s,\eta}_{t,\varrho})^*\zeta)_2 = \partial_v Y_{1,\varrho}(\gamma^{s,\eta}_\varrho)((E^{s,\eta}_{t,\varrho})^*\zeta)_1.$$

Since $((E^{s,\eta}_{t,\varrho})^*\zeta)_2 = 0$ and $\partial_v Y_{1,\varrho} \in [\lambda_2^{-1}, \lambda_2]$ by Assumption 5.2 we have $(E^{s,\eta}_{t,\varrho})^*\zeta \equiv 0$, for a.e. $\varrho \in (t_0, t)$, which is absurd. □

Lemma 5.6 ensures that the Gaussian process in (59) admits a transition density that is the fundamental solution of $\bar{\mathcal{L}}^{s,\eta}_t - \partial_t$. To be more precise, let us recall the notation $\Gamma^{\text{heat}}(A, z)$ for the two-dimensional Gaussian kernel with covariance matrix A (cf. Theorem 2.6).

PROPOSITION 5.7. *For any $0 \leqslant \tau \leqslant s \leqslant t_0 < t \leqslant T$ and $z, \zeta, \eta \in \mathbb{R}^2$, the function*

$$\Gamma_{s,\eta}(t, z; t_0, \zeta) := \Gamma^{\text{heat}}\left(\mathbf{A}^{s,\eta}_{t,t_0}, z - \bar{\gamma}^{s,\eta}_{t,t_0}(\zeta)\right)$$

is the fundamental solution of $\bar{\mathcal{L}}^{s,\eta}_t - \partial_t$, evaluated at (t, z) and with pole at (t_0, ζ).

We are now in position to define the parametrix Z for $\mathcal{K}_t$ in (52). We set

$$Z(t, z; s, \zeta) := \Gamma_{s,\zeta}(t, z; s, \zeta), \qquad \tau \leqslant s < t \leqslant T,\ z, \zeta \in \mathbb{R}^2.$$

Since

$$\gamma^{s,\zeta}_t = \zeta + \int_s^t Y_\varrho(\gamma^{s,\zeta}_\varrho)d\varrho = \zeta + \int_s^t \bar{Y}^{s,\zeta}_\varrho(\gamma^{s,\zeta}_\varrho)d\varrho$$

we have $\gamma^{s,\zeta}_t = \bar{\gamma}^{s,\zeta}_{t,s}(\zeta)$ and therefore the parametrix reads

$$Z(t, z; s, \zeta) = \Gamma^{\text{heat}}\left(\mathbf{A}^{s,\zeta}_{t,s}, z - \gamma^{s,\zeta}_t\right)$$

for $\tau \leqslant s < t \leqslant T$ and $z, \zeta \in \mathbb{R}^2$. The parametrix is an approximation of the fundamental solution Γ of $\mathcal{K}_t$: indeed, since $Z(s, \cdot; s, \zeta) = \delta_\zeta$ and $\Gamma(t, z; t, \cdot) = \delta_z$, we have

(60)

$$\begin{aligned}
\Gamma(t, z; s, \zeta) - Z(t, z; s, \zeta) &= \int_{\mathbb{R}^2} \left(\Gamma(t, z; s, \eta)Z(s, \eta; s, \zeta) - \Gamma(t, z; t, \eta)Z(t, \eta; s, \zeta)\right) d\eta \\
&= \int_s^t \int_{\mathbb{R}^2} -\partial_\varrho \left(\Gamma(t, z; \varrho, \eta)Z(\varrho, \eta; s, \zeta)\right) d\eta d\varrho \\
&= \int_s^t \int_{\mathbb{R}^2} \Big(\mathcal{L}^*_\varrho \Gamma(t, z; \varrho, \eta)Z(\varrho, \eta; s, \zeta) - \Gamma(t, z; \varrho, \eta)\bar{\mathcal{L}}^{s,\zeta}_\varrho Z(\varrho, \eta; s, \zeta)\Big) d\eta d\varrho = \\
&= \int_s^t \int_{\mathbb{R}^2} \Gamma(t, z; \varrho, \eta)(\mathcal{L}_\varrho - \bar{\mathcal{L}}^{s,\zeta}_\varrho)Z(\varrho, \eta; s, \zeta)d\eta d\varrho = \\
&= \int_s^t \int_{\mathbb{R}^2} \Gamma(t, z; \varrho, \eta)\mathcal{K}_\varrho Z(\varrho, \eta; s, \zeta)d\eta d\varrho.
\end{aligned}$$

Iterating the formula, for $N \geqslant 1$ we get the expansion

$$\begin{aligned}\Gamma(t,z;s,\zeta) = Z(t,z;s,\zeta) &+ \sum_{k=1}^{N-1} \int_s^t \int_{\mathbb{R}^2} Z(t,z;\varrho,\eta)(\mathcal{K}_\varrho Z)_k(\varrho,\eta;s,\zeta) d\eta d\varrho \\ &+ \int_s^t \int_{\mathbb{R}^2} \Gamma(t,z;\varrho,\eta)(\mathcal{K}_\varrho Z)_N(\varrho,\eta;s,\zeta) d\eta d\varrho\end{aligned} \tag{61}$$

where

$$\begin{aligned}(\mathcal{K}_t Z)_1(t,z;s,\zeta) &= \mathcal{K}_t Z(t,z;s,\zeta) \\ (\mathcal{K}_t Z)_{k+1}(t,z;s,\zeta) &= \int_s^t \int_{\mathbb{R}^2} \mathcal{K}_t Z(t,z;\varrho,\eta)(\mathcal{K}_\varrho Z)_k(\varrho,\eta;s,\zeta) d\eta d\varrho\end{aligned} \tag{62}$$

As N tends to infinity we formally obtain a representation of Γ as a series of convolution kernels. Unfortunately, as already noticed in Delarue and Menozzi (2010), such an argument cannot be made rigorous because of the transport term. The problem is that, using only the Gaussian estimates for the parametrix, it seems difficult to control the iterated kernels uniformly in k.

For this reason, we first prove some bound for expansion (61) and estimate the remainder via stochastic control techniques as in Delarue and Menozzi (2010). Once we have obtained the Gaussian bounds for the fundamental solution Γ, a posteriori we prove the convergence of the series and the bounds for the derivatives of Γ.

5.1.2. *Gaussian bounds for the parametrix*

PROPOSITION 5.8. *There exists a positive constant c, only dependent on λ_1, λ_2 and T, such that*

$$c^{-1}|\mathcal{D}_{\sqrt{t-s}}z|^2 \leqslant \langle \mathbf{A}_{t,s}^{s,\zeta} z, z\rangle \leqslant c|\mathcal{D}_{\sqrt{t-s}}z|^2, \qquad \tau \leqslant s < t \leqslant T,\ z,\zeta \in \mathbb{R}^2, \tag{63}$$

where, for $\lambda > 0$, $\mathcal{D}_\lambda$ is the diagonal matrix $diag(\lambda^3,\lambda)$ that is the spatial part of the ultra-parabolic dilation operator (17).

PROOF. By Assumptions 5.1 it is enough to prove the assertion for $a \equiv 1$. For $\lambda > 0$, let $\mathcal{U}_\lambda$ be the set of 2×2, time-dependent matrices $\mathcal{Y}_t$, with entries uniformly bounded by λ, and such that $(\mathcal{Y}_t)_{1,2} \in [\lambda^{-1},\lambda]$. Let $\mathcal{Y}_t \in \mathcal{U}_\lambda$ and

$$\mathcal{A}_{t,s} := \int_s^t (\mathcal{E}_{t,\varrho}\mathbf{e_2})(\mathcal{E}_{t,\varrho}\mathbf{e_2})^* d\varrho, \qquad \tau \leqslant s < t \leqslant T,$$

where $\mathcal{E}_{t,\varrho}$ denotes the resolvent associated with $\mathcal{Y}_t$. We split the proof in two steps.

Step 1. First we prove that

$$c^{-1}|z|^2 \leqslant \langle \mathcal{A}_{1,0}z, z\rangle \leqslant c|z|^2, \tag{64}$$

where c is a positive constant which depends only on λ. As in Delarue and Menozzi (2010) (see Proposition 3.4), we consider the map

$$\Psi : L^2([0,1],\mathcal{M}_2(\mathbb{R})) \longrightarrow \mathbb{R}, \qquad \Psi(\mathcal{Y}) := \det \mathcal{A}_{1,0},$$

where $\mathcal{M}_2(\mathbb{R})$ is the space of 2×2 matrices with real entries. Notice that $\mathcal{U}_\lambda$ is compact in the weak topology of $L^2([0,1], \mathcal{M}_2(\mathbb{R}))$ because it is bounded, convex and closed in the strong topology (cf., for instance, Brezis (1983), Corollary III.19). On the other hand, Ψ is continuous from $L^2([0,1], \mathcal{M}_2(\mathbb{R}))$, equipped with the weak topology, to $\mathbb{R}$. Therefore the image $\Psi(\mathcal{U}_\lambda)$ is a compact subset of $\mathbb{R}_{>0}$ by Lemma 5.6. Thus there exists $\bar\lambda > 0$ such that $\inf\{\det \mathcal{A}_{1,0} \mid \mathcal{Y} \in \mathcal{U}_\lambda\} \geqslant \bar\lambda^{-1}$ and $\sup\{\|\mathcal{A}_{1,0}\| \mid \mathcal{Y} \in \mathcal{U}_\lambda\} \leqslant \bar\lambda$. This suffices to prove (64).

Step 2. We use a scaling argument. For every $\tau \leqslant s < t \leqslant T$ we show that $\mathcal{D}_{\frac{1}{\sqrt{t-s}}} \mathcal{A}_{t,s} \mathcal{D}_{\frac{1}{\sqrt{t-s}}}$ coincides with some matrix $\hat{\mathcal{A}}_{1,0}$ to which we can apply the result of Step 1. We have

$$\begin{aligned}\mathcal{D}_{\frac{1}{\sqrt{t-s}}} \mathcal{A}_{t,s} \mathcal{D}_{\frac{1}{\sqrt{t-s}}} &= \int_s^t \left(\mathcal{D}_{\frac{1}{\sqrt{t-s}}} \mathcal{E}_{t,\varrho} \mathcal{D}_{\sqrt{t-s}} \mathbf{e_2}\right)\left(\mathcal{D}_{\frac{1}{\sqrt{t-s}}} \mathcal{E}_{t,\varrho} \mathcal{D}_{\sqrt{t-s}} \mathbf{e_2}\right)^* \frac{d\varrho}{t-s} \\ &= \int_0^1 \left(\hat{\mathcal{E}}^{t,s}_{1,\varrho} \mathbf{e_2}\right)\left(\hat{\mathcal{E}}^{t,s}_{1,\varrho} \mathbf{e_2}\right)^* =: \hat{\mathcal{A}}^{t,s}_{1,0}\end{aligned}$$

where

$$\hat{\mathcal{E}}^{t,s}_{\varrho_1,\varrho_2} = \mathcal{D}_{\frac{1}{\sqrt{t-s}}} \mathcal{R}_{s+\varrho_1(t-s), s+\varrho_1(t-s)} \mathcal{D}_{\sqrt{t-s}},$$

solves the differential system

$$\partial_{\varrho_1} \hat{\mathcal{E}}^{t,s}_{\varrho_1,\varrho_2} = (t-s) \mathcal{D}_{\frac{1}{\sqrt{t-s}}} \mathcal{Y}_{s+\varrho_1(t-s)} \mathcal{D}_{\sqrt{t-s}} \hat{\mathcal{E}}^{t,s}_{\varrho_1,\varrho_2} =: \hat{\mathcal{Y}}^{t,s}_{\varrho_1} \hat{\mathcal{E}}^{t,s}_{\varrho_1,\varrho_2}$$

with $\hat{\mathcal{E}}^{t,s}_{\varrho,\varrho} = I_2$. A direct computation shows that

$$(\hat{\mathcal{Y}}^{t,s}_\varrho)_{1,2} = (\mathcal{Y}_{s+\varrho(t-s)})_{1,2} \in [\lambda^{-1}, \lambda], \qquad \|\hat{\mathcal{Y}}^{t,s}_\varrho\|_\infty \leqslant (1+T^2)\|\mathcal{Y}_\varrho\|_\infty.$$

Therefore (64) holds for $\hat{\mathcal{A}}^{t,s}_{1,0}$, uniformly in t, s, with c dependent only on λ and T. □

REMARK 5.9. *Since, for $\tau \leqslant s < t \leqslant T$, $\mathbf{A}^{s,\zeta}_{t,s}$ is a symmetric and positive definite matrix,* (63) *also yields an analogous estimate for the inverse: we have*

$$c^{-1} \left|\mathcal{D}_{\frac{1}{\sqrt{t-s}}} z\right|^2 \leqslant \langle (\mathbf{A}^{s,\zeta}_{t,s})^{-1} z, z\rangle \leqslant c \left|\mathcal{D}_{\frac{1}{\sqrt{t-s}}} z\right|^2, \qquad \tau \leqslant s < t \leqslant T, \ z, \zeta \in \mathbb{R}^2. \tag{65}$$

The following result is a standard consequence of (63) and (65) (cf., for instance, Proposition 3.1 in Di Francesco and Pascucci (2005)).

PROPOSITION 5.10. *There exists a positive constant c, only dependent on λ_1, λ_2 and T, such that*

$$c^{-1} \Gamma^{\text{heat}} \left(c^{-1} \mathcal{D}_{t-s}, z - \gamma^{s,\zeta}_t\right) \leqslant Z(t, z; s, \zeta) \leqslant c\, \Gamma^{\text{heat}} \left(c \mathcal{D}_{t-s}, z - \gamma^{s,\zeta}_t\right), \tag{66}$$

for every $\tau \leqslant s < t \leqslant T$ and $z, \zeta \in \mathbb{R}^2$.

REMARK 5.11. *Since $Q_t = \mathcal{D}_{\sqrt{t}} Q_1 \mathcal{D}_{\sqrt{t}}$, where Q_1 is symmetric and positive definite, estimate* (66) *equally holds by replacing $\mathcal{D}_{t-s}$ with Q_{t-s}.*

Next we prove some estimate for the derivatives of $Z(t, z; s, \zeta)$. We start with the following

LEMMA 5.12. *We have*

$$(t-s)^{2-i}\left|\left((\mathbf{A}_{t,s}^{s,\zeta})^{-1}w\right)_i\right| \leqslant \frac{c}{\sqrt{t-s}}\left|\mathcal{D}_{\frac{1}{\sqrt{t-s}}}w\right|, \tag{67}$$

$$(t-s)^{4-i-j}\left|\left((\mathbf{A}_{t,s}^{s,\zeta})^{-1}\right)_{ij}\right| \leqslant \frac{c}{t-s} \tag{68}$$

for every $i,j\in\{1,2\}$, $\tau\leqslant s<t\leqslant T$ *and* $w,\zeta\in\mathbb{R}^2$.

PROOF. We have

$$\begin{aligned}(t-s)^{2-i}\left|\left((\mathbf{A}_{t,s}^{s,\zeta})^{-1}w\right)_i\right| &= \frac{1}{\sqrt{t-s}}\left|\left(\mathcal{D}_{\sqrt{t-s}}(\mathbf{A}_{t,s}^{s,\zeta})^{-1}\mathcal{D}_{\sqrt{t-s}}\mathcal{D}_{\frac{1}{\sqrt{t-s}}}w\right)_i\right| \\ &\leqslant \frac{1}{\sqrt{t-s}}\left\|\mathcal{D}_{\sqrt{t-s}}(\mathbf{A}_{t,s}^{s,\zeta})^{-1}\mathcal{D}_{\sqrt{t-s}}\right\|\left|\mathcal{D}_{\frac{1}{\sqrt{t-s}}}w\right|.\end{aligned}$$

In order to get (67) it suffice to notice that, by (65), we have

$$\left\|\mathcal{D}_{\sqrt{t-s}}(\mathbf{A}_{t,s}^{s,\zeta})^{-1}\mathcal{D}_{\sqrt{t-s}}\right\|\leqslant c.$$

Taking $w=\mathbf{e}_j$ we also get (68). □

We are ready to state the last result for this section, which is a standard consequence of estimates (67), (68) and Proposition 5.10 (cf., for instance, Proposition 3.6 in Di Francesco and Pascucci (2005)).

PROPOSITION 5.13. *There exists a positive constant* c*, only dependent on* λ_1*,* λ_2 *and* T *such that*

$$|\partial_x Z(t,z;s,\zeta)| \leqslant \frac{c}{(t-s)^{\frac{3}{2}}}\Gamma^{\text{heat}}\left(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}\right),$$

$$|\partial_v Z(t,x,v;s,\zeta)| \leqslant \frac{c}{\sqrt{t-s}}\Gamma^{\text{heat}}\left(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}\right), \tag{69}$$

$$|\partial_{vv} Z(t,x,v;s,\zeta)| \leqslant \frac{c}{t-s}\Gamma^{\text{heat}}\left(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}\right), \tag{70}$$

for every $\tau\leqslant s<t\leqslant T$ *and* $z=(x,v),\zeta\in\mathbb{R}^2$.

5.1.3. *Upper bound for the fundamental solution* In this section we assume $\tau=0$ for simplicity. We start with some preliminary lemmas.

LEMMA 5.14 (Reproduction formula). *For any* $c',c''>0$ *we have*

$$\begin{aligned}\Lambda(c',c'')^{-1}\Gamma^{\text{heat}}\left(\frac{c'\wedge c''}{2}\mathcal{D}_{t-s},\zeta''-\zeta'\right) &\leqslant \int_{\mathbb{R}^2}\Gamma^{\text{heat}}\left(c'\mathcal{D}_{t-\varrho},\zeta'-\eta\right)\Gamma^{\text{heat}}\left(c''\mathcal{D}_{\varrho-s},\eta-\zeta''\right)d\eta \\ &\leqslant \Lambda(c',c'')\Gamma^{\text{heat}}\left((c'\vee c'')\mathcal{D}_{t-s},\zeta''-\zeta'\right),\end{aligned}$$

for every $0\leqslant s<\varrho<t\leqslant T$, $\zeta',\zeta''\in\mathbb{R}^2$*, where* $\Lambda(c',c'')=\sqrt{\frac{2(c'\vee c'')}{c'\wedge c''}}$.

PROOF. It is a direct consequence (see also Delarue and Menozzi (2010), Lemma B.1) of the following trivial estimate

$$\frac{c' \wedge c''}{2}\mathcal{D}_{t-s} \leqslant c'\mathcal{D}_{t-\varrho} + c''\mathcal{D}_{\varrho-s} \leqslant (c' \vee c'')\mathcal{D}_{t-s}.$$

□

REMARK 5.15. *Let $\tau = 0$, $T = 1$. If $\hat{Y}$ is a vector field satisfying Assumption 5.2 and $\hat{\gamma}_t$ is the integral curve*

$$\hat{\gamma}_t(z) = z + \int_0^t \hat{Y}_s(\hat{\gamma}_s(z))ds, \qquad t \in [0,1],$$

then $\hat{\gamma}_1(\cdot)$ is a diffeomorphism of $\mathbb{R}^2$. Moreover, since $\hat{Y}$ is Lipschitz continuous, we have

$$m^{-1}|z - \hat{\gamma}_1(\zeta)| \leqslant |\hat{\gamma}_1^{-1}(z) - \zeta| \leqslant m|z - \hat{\gamma}_1(\zeta)|, \qquad z, \zeta \in \mathbb{R}^2, \tag{71}$$

for a constant m which depends only on λ_2.

LEMMA 5.16. *Let $\gamma_s^{t,z}$ be as in (58). There exists a positive constant m, only dependent on λ_2 and T, such that*

$$m^{-1}\left|\mathcal{D}_{\frac{1}{\sqrt{t-s}}}(z - \gamma_t^{s,\zeta})\right| \leqslant \left|\mathcal{D}_{\frac{1}{\sqrt{t-s}}}(\gamma_s^{t,z} - \zeta)\right| \leqslant m\left|\mathcal{D}_{\frac{1}{\sqrt{t-s}}}(z - \gamma_t^{s,\zeta})\right|,$$

for every $0 \leqslant s < t \leqslant T$ and $z, \zeta \in \mathbb{R}^2$.

PROOF. We use again a scaling argument: we set $z' = \mathcal{D}_{\sqrt{t-s}}z$ and

$$\hat{\gamma}_\varrho(z) = \mathcal{D}_{\frac{1}{\sqrt{t-s}}}\gamma_{s+\varrho(t-s)}^{s,z'}, \quad \hat{Y}_\varrho(z) = (t-s)\mathcal{D}_{\frac{1}{\sqrt{t-s}}}Y_{s+\varrho(t-s)}(z'), \qquad \varrho \in [0,1].$$

Then we have

$$\hat{\gamma}_\varrho(z) = z + \int_0^\varrho \hat{Y}_u(\hat{\gamma}_u(z))du, \qquad \varrho \in [0,1].$$

As in the proof of Proposition 5.8, we have that $\hat{Y}$ satisfies Assumption 5.2. By Remark 5.15, estimate (71) holds for $\hat{\gamma}_\varrho(z)$. To conclude, it suffices to substitute z and ζ with $\bar{z} = \mathcal{D}_{\frac{1}{\sqrt{t-s}}}z$ and $\bar{\zeta} = \mathcal{D}_{\frac{1}{\sqrt{t-s}}}\zeta$ in (71). □

LEMMA 5.17. *Let $(\mathcal{K}_t Z)_k$ be as in (62). There exists a constant $c > 0$, only dependent on λ_1, λ_2, α and T such that*

$$|(\mathcal{K}_t Z)_k(t,z;s,\zeta)| \leqslant \frac{M_k}{(t-s)^{1-\frac{k\alpha}{2}}}\Gamma^{\text{heat}}\left(c\,m^k\mathcal{D}_{t-s}, z - \gamma_t^{s,\zeta}\right), \qquad 0 \leqslant s < t \leqslant T,\ z,\zeta \in \mathbb{R}^2,$$

where m is the constant in Lemma 5.16 and $M_k = 2^{\frac{k-1}{2}}c^k m^{q_k}\frac{\Gamma_E^k(\frac{\alpha}{2})}{\Gamma_E(\frac{k\alpha}{2})}$, with $q_1 = 0$, $q_2 = \frac{1}{2}, q_k = q_{k-1} + \frac{k-2}{2}$ for $k \geqslant 2$.

PROOF. We give the proof for $k = 1$. The general case follows by induction, exploiting Lemmas 5.16 and 5.14 as in the proof of estimate (72).

$$
\begin{aligned}
(\mathcal{K}_t Z)_1(t,z;s,\zeta) &= (\mathcal{L}_t - \mathcal{L}_t^{s,\zeta})Z(t,z;s,\zeta)\\
&= \frac{1}{2}\left(a_t(z) - a_t(\gamma_t^{s,\zeta})\right)\partial_{vv}Z(t,z;s,\zeta) + b_t(z)\partial_v Z(t,z;s,\zeta) +\\
&\quad + \langle Y_t(z) - \bar{Y}_t^{s,\zeta}(z), \nabla Z(t,z;s,\zeta)\rangle\\
&=: E_1 + E_2 + E_3.
\end{aligned}
$$

By Assumption 5.1 and Proposition 5.13 we have

$$
\begin{aligned}
|E_1| &\leqslant \frac{c}{t-s}|z-\gamma_t^{s,\zeta}|^{\alpha}\Gamma^{\text{heat}}\left(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}\right)\\
&\leqslant \frac{c'}{(t-s)^{1-\frac{\alpha}{2}}}\left|\mathcal{D}_{\frac{1}{\sqrt{t-s}}}(z-\gamma_t^{s,\zeta})\right|^{\alpha}\Gamma^{\text{heat}}\left(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}\right) \leqslant
\end{aligned}
$$

(by (23))

$$
\leqslant \frac{c''}{(t-s)^{1-\frac{\alpha}{2}}}\Gamma^{\text{heat}}\left(c''\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}\right).
$$

By Assumption 5.1 and Proposition 5.13 we also have

$$
|E_2| \leqslant \frac{c}{\sqrt{t-s}}\Gamma^{\text{heat}}\left(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}\right).
$$

As for E_3, we have

$$
|(Y_t(z) - \bar{Y}_t^{s,\zeta}(z))_1| = |Y_{1,t}(z) - Y_{1,t}(\gamma_t^{s,\zeta}) - \partial_v Y_{1,t}(\gamma_t^{s,\zeta})(z-\gamma_t^{s,\zeta})| \leqslant c|z-\gamma_t^{s,\zeta}|^{1+\alpha},
$$

because $\partial_v Y_{1,t}$ is Hölder continuous by Assumption 5.2: here we use the elementary inequality

$$
\left|\int_0^1 (f'(y+t(x-y)) - f'(y))(x-y)dt\right| \leqslant c_\alpha |x-y|^{1+\alpha}.
$$

which is valid for $f \in C^{1+\alpha}$. On the other hand, we have

$$
|(Y_t(z) - \bar{Y}_t^{s,\zeta}(z))_2| \leqslant c|z-\gamma_t^{s,\zeta}|.
$$

Therefore, by Proposition 5.13, we have

$$
\begin{aligned}
|E_3| &\leqslant c\left(\frac{1}{(t-s)^{\frac{3}{2}}}|z-\gamma_t^{s,\zeta}|^{1+\alpha} + \frac{1}{(t-s)^{\frac{1}{2}}}|z-\gamma_t^{s,\zeta}|\right)\Gamma^{\text{heat}}\left(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}\right)\\
&\leqslant \frac{c'}{(t-s)^{1-\frac{\alpha}{2}}}\left(\left|\mathcal{D}_{\frac{1}{\sqrt{t-s}}}(z-\gamma_t^{s,\zeta})\right|^{1+\alpha} + \left|\mathcal{D}_{\frac{1}{\sqrt{t-s}}}(z-\gamma_t^{s,\zeta})\right|\right)\Gamma^{\text{heat}}\left(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}\right) \leqslant
\end{aligned}
$$

(by (23))

$$
\leqslant \frac{c''}{(t-s)^{1-\frac{\alpha}{2}}}\Gamma^{\text{heat}}\left(c''\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}\right).
$$

□

The following result is proved in Delarue and Menozzi (2010), Proposition 5.2.

LEMMA 5.18. *For any $\varepsilon > 0$ there exist a positive constant c, only dependent on $\lambda_1, \lambda_2, \alpha, T$ and ε, such that*

$$\int_{\mathbb{R}^2} \Gamma(t,z;\varrho,\eta)(\varrho - s)^2 \Gamma^{\text{heat}}(\varepsilon \mathcal{D}_{\varrho-s}, \eta - \gamma_{\varrho}^{s,\zeta}) d\eta \leqslant c\, \Gamma^{\text{heat}} \left(c\mathcal{D}_{t-s}, z - \gamma_t^{s,\zeta} \right),$$

for $s < \varrho < t \leqslant T$ and $z, \zeta \in \mathbb{R}^2$.

We close this section by proving the Gaussian upper bound in (55). Consider the parametrix expansion (61) with $0 < t - s \leqslant 1$. By Proposition 5.10, the first term in the RHS of (61) is bounded by $c\,\Gamma^{\text{heat}}(c\mathcal{D}_{t-s}, z - \gamma_t^{s,\zeta})$. On the other hand, if $N \geqslant \frac{6}{\alpha}$ then $(\varrho - s)^{1-\frac{N\alpha}{2}} \leqslant (\varrho - s)^2$ and therefore the last term in the RHS of (61) is bounded by the same quantity, by Lemmas 5.17 and 5.18.

Finally, denoting with c_k a positive constant dependent on $\lambda_1, \lambda_2, \alpha, T$ and k, we have

$$\int_s^t \int_{\mathbb{R}^2} Z(t,z;\varrho,\eta)(\mathcal{K}_\varrho Z)_k(\varrho,\eta;s,\zeta) d\eta d\varrho \leqslant$$

(by Lemmas 5.17 and 5.18)

$$\leqslant c_k \int_s^t (\varrho - s)^{\frac{k\alpha}{2}-1} \int_{\mathbb{R}^2} \Gamma^{\text{heat}}(c\mathcal{D}_{t-\varrho}, z - \gamma_t^{\varrho,\eta}) \Gamma^{\text{heat}}(c_k \mathcal{D}_{\varrho-s}, \eta - \gamma_\varrho^{s,\zeta}) d\eta d\varrho \leqslant$$

(by Lemma 5.16)

$$\leqslant c_k \int_s^t \varrho^{\frac{k\alpha}{2}-1} \int_{\mathbb{R}^2} \Gamma^{\text{heat}}(c' \mathcal{D}_{t-\varrho}, \gamma_\varrho^{t,z} - \eta) \Gamma^{\text{heat}}(c_k \mathcal{D}_{\varrho-s}, \eta - \gamma_\varrho^{s,\zeta}) d\eta d\varrho \leqslant$$

(by Lemma 5.14)

$$\leqslant c_k \int_s^t \varrho^{\frac{k\alpha}{2}-1} \Gamma^{\text{heat}}(c_k \mathcal{D}_{t-s}, \gamma_\varrho^{t,z} - \gamma_\varrho^{s,\zeta}) d\varrho \leqslant$$

(again by Lemma 5.16)

$$\leqslant c_k \Gamma^{\text{heat}}(c_k \mathcal{D}_1, z - \gamma_t^{s,\zeta}). \tag{72}$$

This proves the upper bound for $0 < t - s \leqslant 1$. The general case can be recovered by a scaling argument, similar to that of Proposition 5.8.

5.1.4. *Lower bound for the fundamental solution* We first derive a local bound, starting from the parametrix expansion (60) and exploiting the results of Section 5.1.3. We have

$$\Gamma(t,z;s,\zeta) \geqslant Z(t,z;s,\zeta) - \int_s^t \int_{\mathbb{R}^2} \Gamma(t,z;\varrho,\eta) \left| \mathcal{K}_\varrho Z(\varrho,\eta;s,\zeta) \right| d\eta d\varrho \geqslant$$

(by Lemmas 5.10 and 5.17 and the upper bound (55))

$$\geqslant c^{-1} \Gamma^{\text{heat}}(c^{-1} \mathcal{D}_{t-s}, z - \gamma_t^{s,\zeta})$$

$$-\int_s^t \frac{c}{(\varrho-s)^{1-\frac{\alpha}{2}}}\int_{\mathbb{R}^2}\Gamma^{\text{heat}}(c\mathcal{D}_{t-\varrho}, z-\gamma_t^{\varrho,\eta})\Gamma^{\text{heat}}(c\mathcal{D}_{\varrho-s}, z-\gamma_\varrho^{s,\zeta})d\eta d\varrho \geqslant$$

(by Lemma 5.14)

$$\geqslant c^{-1}\Gamma^{\text{heat}}(c^{-1}\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}) - \frac{c}{2}(t-s)^{\frac{\alpha}{2}}\Gamma^{\text{heat}}(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}).$$

Let $d_{t_2,t_1}(z_2,z_1) := |\mathcal{D}_{t_2-t_1}(z_2-\gamma_{t_2}^{t_1,z_1})|$ denote the "control metric" of the system. A direct computation shows that $\Gamma^{\text{heat}}(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}) \leqslant \Gamma^{\text{heat}}(c^{-1}\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta})$ if $d_{t,s}(z,\zeta) \leqslant \varrho_c$ where $\varrho_c = \sqrt{\frac{2c\ln c}{c^2-1}}$. Then we have

$$\Gamma(t,z;s,\zeta) \geqslant \left(\frac{1}{c^2} - \frac{(t-s)^{\frac{\alpha}{2}}}{2}\right)\Gamma^{\text{heat}}(c^{-1}\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}) \geqslant \frac{1}{2c}\Gamma^{\text{heat}}(c^{-1}\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}) \tag{73}$$

if $d_{t,s}(z,\zeta) \leqslant \varrho_c$ and $0 < t-s \leqslant T_c := c^{-\frac{4}{\alpha}}$.

In order to pass from the local to the global bound, we use a chaining procedure: we first need to define a sequence of points (t_k, z_k) such that $t_0 = s, z_0 = \zeta, t_{M+1} = t, z_{M+1} = z$ for some integer M (to be defined later), along which we can control the increments with respect to the control metric $d_{t_{k-1},t_k}(z_{k+1}, z_k)$. Let us consider the controlled version of the system (58):

$$\psi_\varrho^{s,\zeta} = \zeta + \int_s^\varrho \left(Y_\theta(\psi_\theta^{s,\zeta}) + v_s\mathbf{e}_2\right)d\theta, \qquad \varrho \in [s,t].$$

We have the following (see Polidoro (1997), Pascucci and Polidoro (2006) and Delarue and Menozzi (2010), Propositions 4.1 and 4.2):

LEMMA 5.19. *There exists a control $(v_\varrho)_{s\leqslant\varrho\leqslant t}$ with values in $\mathbb{R}^2$ such that*

i) the solution $\psi_\varrho^{s,\zeta}$ associated with v_ϱ reaches z at time t, that is $\psi_t^{s,\zeta} = z$;
ii) there exist two constants $m_1, m_2 > 0$, only dependent on the constants of Assumptions 5.1-5.2, such that

$$\int_s^t |v_\varrho|^2 d\varrho \geqslant m_1\left|\mathcal{D}_{\frac{1}{\sqrt{t-s}}}(z-\gamma_t^{s,\zeta})\right|^2, \qquad \sup_{s\leqslant\varrho\leqslant t}|v_\varrho|^2 \leqslant \frac{m_2}{t-s}\left|\mathcal{D}_{\frac{1}{\sqrt{t-s}}}(z-\gamma_t^{s,\zeta})\right|^2.$$

We set

$$t_i = s + i\frac{t-s}{M+1} = s + i\varepsilon, \quad z_k = \psi_{r_k}^{s,\zeta}, \qquad i = 1,\cdots,M,$$

where $\psi_\varrho^{s,\zeta}$ is the optimal path of Lemma 5.19 and M is the smallest integer greater than

$$\max\left\{\frac{K^2 d_{t,s}^2(z,\zeta)}{\varrho_c^2}, \frac{T}{T_c}\right\}.$$

with $K = \frac{12m^2m_2}{m_1}$, where m, m_1 and m_2 are the constants in Lemmas 5.16 and 5.19. Finally we define the sets

$$B_i(r) := \{z \in \mathbb{R}^2 \mid |\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(z-\gamma_{t_i}^{t_{i-1},z_{i-1}})| + |\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(z_{i+1}-\gamma_{t+i}^{t_i,z})| \leqslant r\},$$

and write

(74)

$$\Gamma(t,z;s,\zeta) \geqslant \int_{B_1(\varrho_c/3)} \cdots \int_{B_M(\varrho_c/3)} \Gamma(t,z;t_M,\zeta_M) \prod_{j=1}^{M-1} \Gamma(t_{j+1},\zeta_{j+1};t_j,\zeta_j)\Gamma(t_1,\zeta_1;s,\zeta)d\zeta_1 \dots d\zeta_M.$$

By definition of M we have

$$t_{j+1} - t_j = \frac{t-s}{M+1} \leqslant \frac{T}{M+1} \leqslant T_c.$$

On the other hand, if $\zeta_i \in B_i\left(\frac{\varrho_c}{3}\right)$ for $i = 1, \dots, M-1$ we have

$$\begin{aligned} d_{t_{i+1},t_i}(\zeta_{i+1},\zeta_i) &= |\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(\zeta_{i+1} - \gamma_{t_{i+1}}^{t_i,\zeta_i})| \\ &= |\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(\zeta_{i+1} - \gamma_{t_{i+1}}^{t_i,z_i})| + |\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(z_{i+1} - \gamma_{t_{i+1}}^{t_i,z_i})| + |\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(z_{i+1} - \gamma_{t_{i+1}}^{t_i,\zeta_i})| =: E_1 + E_2 + E_3, \end{aligned}$$

where $E_1 + E_3 \leqslant \frac{2}{3}\varrho_c$. By Lemma 5.19, we have

$$(75)\quad E_2 \leqslant m_1^{-1}\left(\int_{t_i}^{t_{i+1}} |v_\varrho|^2 d\varrho\right)^{\frac{1}{2}} \leqslant \frac{m_2}{m_1}\sqrt{\frac{\varepsilon}{t-s}}|\mathcal{D}_{\frac{1}{\sqrt{t-s}}}(z - \gamma_t^{s,\zeta})| = \frac{m_2}{m_1}\frac{d_{t,s}(z,\zeta)}{\sqrt{M+1}} \leqslant \frac{\varrho_c}{12m^2}.$$

Therefore $d_{t_{i+1},t_i}(\zeta_{i+1},\zeta_i) \leqslant \varrho_c$ and we can use (73) repeatedly in (74) to get

$$\Gamma(t,z;s,\zeta) \geqslant (2c)^{-(M+1)}\left|\prod_{i=1}^{M} B_i\left(\frac{\varrho_c}{3}\right)\right|\left(\frac{c(M+1)^2}{(t-s)^2}\right)^{M+1} \exp\left(-\frac{c}{2}\varrho_c^2(M+1)\right).$$

Assume for a moment the validity of the inequality

$$(76)\qquad \left|B_i\left(\frac{\varrho_c}{3}\right)\right| \geqslant C_0\pi\left(\frac{t-s}{M+1}\right)^2 \varrho_c^2$$

for some positive constant C_0 (only dependent on the constants of Assumptions 5.1-5.2). Then we have

$$\Gamma(t,z;s,\zeta) \geqslant C_1 C_2^M \frac{1}{2\pi\sqrt{\det \mathcal{D}_{t-s}}} \exp\left(-\frac{c}{2}\varrho_c^2 M\right) \geqslant \frac{C_3}{2\pi\sqrt{\det \mathcal{D}_{t-s}}} \exp\left(-\frac{C_4}{2}M\right),$$

for some positive constants $C_1, \dots, C_4$. Now, if $TT_c^{-1} \leqslant \frac{K^2 d_{t,s}^2(z,\zeta)}{\varrho_c^2}$ and $M < 2\frac{K^2 d_{t,s}^2(z,\zeta)}{\varrho_c^2}$, we have

$$\Gamma(t,z;s,\zeta) \geqslant \frac{C_3}{2\pi\sqrt{\det \mathcal{D}_{t-s}}} \exp\left(-\frac{C_5}{2}d_{t,s}^2(z,\zeta)\right) = C_6\Gamma^{\text{heat}}(C_5^{-1}\mathcal{D}_{t-s}, z - \gamma_t^{s,\zeta}).$$

On the other hand, if $M < 2TT_c^{-1}$ then

$$\Gamma(t,z;s,\zeta) \geqslant \frac{C_7}{2\pi\sqrt{\det \mathcal{D}_{t-s}}} \geqslant \frac{C_7}{2\pi\sqrt{\det \mathcal{D}_{t-s}}} \exp\left(-\frac{C_5}{2}d_{t,s}^2(z,\zeta)\right) = C_8\Gamma^{\text{heat}}(C_5^{-1}\mathcal{D}_{t-s}, z - \gamma_t^{s,\zeta}),$$

and this proves the lower bound.

We are left with the proof of (76). Let $\widetilde{B}_i(r) = \{z,\ |\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(z - z_i)| \leqslant r\}$: a direct computation shows $|\widetilde{B}_i(r)| = \pi\varepsilon^2 r^2$. Then it is enough to show that $B_i\left(\frac{\varrho_c}{3}\right) \supseteq \widetilde{B}_i\left(\frac{\varrho_c}{C}\right)$ for a positive constant C (only dependent on $\lambda_1, \lambda_2, \alpha$ and T). For any $z \in \widetilde{B}_i(r)$ we have

$$\begin{aligned}
&|\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(z - \gamma_{t_i}^{t_{i-1},z_{i-1}})| + |\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(z_{i+1} - \gamma_{t+i}^{t_i,z})| \leqslant \\
&\qquad \leqslant |\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(z - z_i)| + |\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(z_i - \gamma_{t_i}^{t_{i-1},z_{i-1}})| + m|\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(z - z_i)| + m^2|\mathcal{D}_{\frac{1}{\sqrt{\varepsilon}}}(z_{i+1} - \gamma_{t_{i+1}}^{t_{i1},z_{i1}})| \leqslant
\end{aligned}$$

(by (75))

$$\leqslant (1+m)r + \frac{\varrho_c}{6}.$$

Then it is sufficient to take $r \leqslant \frac{\varrho_c}{6(1+m)}$ and this concludes the proof.

5.1.5. *Gaussian bounds for $\partial_v\Gamma$ and $\partial_{vv}\Gamma$* The following lemma provides an alternative representation formula for Γ which will be used to prove the bounds for the derivatives. As a general rule, until the end of the section we will always denote with c a positive constant, only dependent on $\lambda_1, \lambda_2, \alpha$ and T in Assumptions 5.1-5.2.

LEMMA 5.20. *We have*

$$\Gamma(t,z;s,\zeta) = Z(t,z;s,\zeta) + \int_s^t \int_{\mathbb{R}^2} Z(t,z;\varrho,\eta)\varphi(r,\eta;s,\zeta)d\eta d\varrho, \qquad \tau \leqslant s < t \leqslant T,\ z,\zeta \in \mathbb{R}^2,$$

where

$$\varphi(\cdot,\cdot;s,\zeta) = \sum_{k\geqslant 1}(\mathcal{K}Z)_k(\cdot,\cdot;s,\zeta)$$

is uniformly convergent in $(s,T)\times\mathbb{R}^2$. Moreover, there exists a positive constant c such that

$$|\varphi(t,z;s,\zeta)| \leqslant \frac{c}{(t-s)^{1-\frac{\alpha}{2}}}\Gamma^{\text{heat}}(c\mathcal{D}_{t-s}, z - \gamma_t^{s,\zeta}), \tag{77}$$

(78)

$$|\varphi(t,z;s,\zeta) - \varphi(t,z';s,\zeta)| \leqslant c\frac{d_{\mathcal{L}}\left((t,z),(t,z')\right)^{\frac{\alpha}{2}}}{(t-s)^{1-\frac{\alpha}{2}}}\left(\Gamma^{\text{heat}}(c\mathcal{D}_{t-s}, z - \gamma_t^{s,\zeta}) + \Gamma^{\text{heat}}(c\mathcal{D}_{t-s}, z' - \gamma_t^{s,\zeta})\right),$$

for every $\tau \leqslant s < t \leqslant T$ and $z, z', \zeta \in \mathbb{R}^2$, where $d_{\mathcal{L}}$ is the intrinsic distance in (18).

PROOF. We start from the parametrix representation (61) and show that the remainder

$$R_N(t,z;s,\zeta) := \int_s^t \int_{\mathbb{R}^2} \Gamma(t,z;\varrho,\eta)(\mathcal{K}_\varrho Z)_N(\varrho,\eta;s,\zeta)d\eta d\varrho$$

converges uniformly to 0 as N tends to infinity. By the Gaussian upper bound (55), Lemmas 5.17 and 5.14, we have

$$|R_N(t,z;s,\zeta)| \leqslant cM_N \int_s^t \frac{1}{(t-\varrho)^{1-\frac{N\alpha}{2}}} \int_{\mathbb{R}^2} \Gamma^{\text{heat}}(c\mathcal{D}_{t-\varrho}, \gamma_\varrho^{t,z} - \eta)\Gamma^{\text{heat}}(c_N\mathcal{D}_{\varrho-s}, \eta - \gamma_s^{s,z})d\eta d\varrho$$

$$
\begin{aligned}
&\leqslant cM_N \int_s^t \frac{1}{(t-\varrho)^{1-\frac{N\alpha}{2}}} \Gamma^{\text{heat}}(c\mathcal{D}_{t-s}, \gamma_\varrho^{t,z} - \gamma_s^{s,\zeta}) d\varrho \\
&\leqslant cM_N (t-s)^{-2} \int_s^t \frac{1}{(t-\varrho)^{1-\frac{N\alpha}{2}}} d\varrho \\
&\leqslant c\frac{M_N}{N}(t-s)^{\frac{N\alpha}{2}-2},
\end{aligned}
$$

with $M_N = c^N m^{q_N} \frac{\Gamma_E^N(\frac{\alpha}{2})}{\Gamma_E(\frac{N\alpha}{2})}$, converges to zero by the properties of the Euler Gamma function Γ_E.

Next, exploiting the lower bound for Γ we can replace the Gaussian function Γ^{heat} in Propositions 5.10 and 5.13 by an appropriate fundamental solution satisfying an exact reproduction formula. Then, repeating the arguments in the proof of Lemma 5.17, we get

$$
|(\mathcal{K}_t Z)_k(t,z;s,\zeta)| \leqslant \frac{M_k}{(t-s)^{1-\frac{k\alpha}{2}}} \Gamma^{\text{heat}}\left(c\mathcal{D}_{t-s}, z - \gamma_t^{s,\zeta}\right), \qquad \tau \leqslant s < t \leqslant T,\ z,\zeta \in \mathbb{R}^2.
$$

Then estimate (77) easily follows. Estimate (78) can be proved by standard arguments (see, for instance, Lemma 6.1 in Di Francesco and Pascucci (2005)). $\square$

Now we show that

$$
\left| \int_s^t \int_{\mathbb{R}^2} \partial_v Z(t,z;\varrho,\eta)\varphi(r,\eta;s,\zeta) d\eta d\varrho \right| \leqslant \frac{c}{\sqrt{t-s}} \Gamma^{\text{heat}}\left(c\mathcal{D}_{t-s}, z - \gamma_t^{s,\zeta}\right), \tag{79}
$$

$$
\left| \int_s^t \int_{\mathbb{R}^2} \partial_{vv} Z(t,z;\varrho,\eta)\varphi(r,\eta;s,\zeta) d\eta d\varrho \right| \leqslant \frac{c}{t-s} \Gamma^{\text{heat}}\left(c\mathcal{D}_{t-s}, z - \gamma_t^{s,\zeta}\right), \tag{80}
$$

for $\tau \leqslant s < t \leqslant T$ and $z,\zeta \in \mathbb{R}^2$. Formula (79) is a standard consequence of Lemma 5.14 and estimates (69) and (24). Estimate (79) is less obvious. We have

$$
\begin{aligned}
\int_{\mathbb{R}^2} \partial_{vv} Z(t,z;\varrho,\eta)\varphi(\varrho,\eta;s,\zeta) s\eta = &\int_{\mathbb{R}^2} \partial_{vv} Z(t,z;\varrho,\eta)(\varphi(\varrho,\eta;s,\zeta) - \varphi(\varrho,w;s,\zeta)) d\eta \\
&+ \varphi(\varrho,w;s,\zeta) \int_{\mathbb{R}^2} \partial_{vv}(Z(t,z;\varrho,\eta) - \Gamma_{\varrho,w}(t,z;\varrho,\eta)) d\eta \\
&+ \varphi(\varrho,w;s,\zeta) \int_{\mathbb{R}^2} \partial_{vv} \Gamma_{\varrho,w}(t,z;\varrho,\eta) d\eta \\
=: &\, I_1 + I_2 + I_3.
\end{aligned}
$$

Then, by choosing $w = \gamma_\varrho^{t,z}$ we can rely on the Hölder regularity of φ and $\Gamma_{\varrho,y}$ to remove the singularity in $t = \varrho$. Here we show how to handle I_1 in detail: by estimates (70) and (78) we have

$$
\begin{aligned}
|I_1| \leqslant &\frac{c}{(\varrho-s)^{1-\frac{\alpha}{2}}} \int_{\mathbb{R}^2} \frac{d_{\mathcal{L}}\left((\varrho,\gamma_\varrho^{t,z}),(\varrho,\eta)\right)^\alpha}{t-\varrho} \Gamma^{\text{heat}}(c\mathcal{D}_{t-\varrho}, z - \gamma_t^{\varrho,\eta}) \times \\
&\times \underbrace{\left(\Gamma^{\text{heat}}(c\mathcal{D}_{\varrho-s}, \eta - \gamma_\varrho^{s,\zeta}) + \Gamma^{\text{heat}}(c\mathcal{D}_{\varrho-s}, \gamma_\varrho^{t,z} - \gamma_\varrho^{s,\zeta})\right)}_{=:J(\eta)} d\eta \\
\leqslant &\frac{c}{(t-\varrho)^{1-\frac{\alpha}{2}}(\varrho-s)^{1-\frac{\alpha}{2}}} \int_{\mathbb{R}^2} \left| \left(0, \mathcal{D}_{\frac{1}{\sqrt{t-\varrho}}}(z - \gamma_t^{\varrho,\eta})\right) \right|_{\mathcal{L}}^\alpha \Gamma^{\text{heat}}(c\mathcal{D}_{t-\varrho}, z - \gamma_t^{\varrho,\eta}) J(\eta) d\eta \leqslant
\end{aligned}
$$

(by (23))

$$
\begin{aligned}
&\leqslant \frac{c}{(t-\varrho)^{1-\frac{\alpha}{2}}(\varrho-s)^{1-\frac{\alpha}{2}}} \int_{\mathbb{R}^2} \Gamma^{\text{heat}}(c\mathcal{D}_{t-\varrho}, z-\gamma_t^{\varrho,\eta}) J(\eta) d\eta \\
&= \frac{c}{(t-\varrho)^{1-\frac{\alpha}{2}}(\varrho-s)^{1-\frac{\alpha}{2}}} (I_{11}+I_{12})
\end{aligned}
$$

where

$$I_{11} \leqslant c\,\Gamma^{\text{heat}}(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}),$$

by Lemma 5.14, and

$$I_{12} \leqslant \Gamma^{\text{heat}}(c\mathcal{D}_{\varrho-s}, z-\gamma_t^{s,\zeta}) \int_{\mathbb{R}^2} \Gamma^{\text{heat}}(c\mathcal{D}_{t-\varrho}, z-\gamma_t^{\varrho,\eta}) d\eta \leqslant c\,\Gamma^{\text{heat}}(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}),$$

because the integral is bounded by a constant and the matrix $\mathcal{D}_{\varrho-s}$ is increasing in ϱ. I_2 can be treated similarly, once we notice that

$$|\partial_{vv}\Gamma_{s,y}(t,z;s,\zeta) - \partial_{vv}\Gamma_{s,w}(t,z;s,\zeta)| \leqslant c\frac{d_{\mathcal{L}}\left((s,y),(s,w)\right)^{\alpha}}{t-s}\Gamma^{\text{heat}}(c\mathcal{D}_{t-s}, z-\gamma_t^{s,\zeta}),$$

for $\tau \leqslant s < t \leqslant T$ and $z,\zeta,y,w \in \mathbb{R}^2$ (see also Di Francesco and Pascucci (2005), Lemma 5.2). Lastly, $I_3 = 0$: indeed, for every $s < \varrho < t$ and $w \in \mathbb{R}^2$ we have $\int_{\mathbb{R}^2} \Gamma_{\varrho,w}(t,z;\varrho,\eta) d\eta = 1$ and therefore

$$\partial_{vv} \int_{\mathbb{R}^2} \Gamma_{\varrho,w}(t,z;\varrho,\eta) d\eta = 0.$$

Integrating in ϱ over the interval (s,t) we get estimate (80).

6. Finale: proof of Theorem 2.6

For any fixed $\tau \in [0,T)$ and $\omega \in \Omega$, let $\mathcal{K}_\tau$ the operator of the form (52), as defined by (50) and (51) through the random change of variable $\gamma^{\text{IW}}_{\tau,t}$. By Assumptions 2.3-2.4 and Lemma 4.1, $\mathcal{K}_\tau$ satisfies Assumptions 5.1-5.2 for a.e. $\omega \in \Omega$. Then, by Theorem 5.5, $\mathcal{K}_\tau$ admits a fundamental solution Γ_τ: we set

$$\mathbf{\Gamma}(t,x,v;\tau,\zeta) = \Gamma_\tau(t,x,\gamma^{\text{IW},-1}_{t,\tau}(x,v);\tau,\zeta), \qquad \tau < t \leqslant T,\ x,v \in \mathbb{R},\ z \in \mathbb{R}^2. \tag{81}$$

Combining Theorems 4.3, 5.5 and Lemma 4.1 we infer that $\mathbf{\Gamma}(\cdot,\cdot,\cdot;\tau,\zeta) \in \mathbf{C}^0_{t_0,T}$ for any $t_0 \in (\tau,T]$, is twice continuously differentiable in the variable v and satisfies (24) with probability one. Now, for any bounded and continuous function φ and $z_0 \in \mathbb{R}^2$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} \mathbf{\Gamma}(t,z;\tau,\zeta)\varphi(\zeta)d\zeta - \varphi(z_0) &= \int_{\mathbb{R}^2} \Gamma_\tau(t,z;\tau,\zeta)\varphi(\zeta)d\zeta - \varphi(z_0) + \\
&\quad + \int_{\mathbb{R}^2} \left(\Gamma_\tau(t,x,\gamma^{\text{IW},-1}_{t,\tau}(x,v);\tau,\zeta) - \Gamma_\tau(t,z;\tau,\zeta)\right)\varphi(\zeta)d\zeta = \\
&= I_1(t,z,\tau) + I_2(t,z,\tau).
\end{aligned}
$$

Now, by Theorem 5.5 and the dominated convergence theorem, we have

$$\lim_{\substack{(t,z)\to(\tau,z_0)\\ t>\tau}} I_i(t,z,\tau) = 0, \qquad i = 1,2.$$

This proves the first part of the thesis.

The Gaussian bounds (26) follow directly from the definition (81) and the analogous estimates (55) for Γ_τ in Theorem 5.5. Moreover, since

$$\partial_v \mathbf{\Gamma}(t,x;\tau,\xi) = (\partial_v \Gamma_\tau)\left(t,x,\gamma_{t,\tau}^{\mathrm{IW},-1}(x,v);\tau,\zeta\right)\partial_v \gamma_{t,\tau}^{\mathrm{IW},-1}(x,v),$$

the gradient estimate (27) follows from the analogous estimate (56) for Γ_τ and from Lemma 4.1. The proof of (28) is analogous.

References

Bossy, M., Jabir, J.-F. and Talay, D. (2011). On conditional McKean Lagrangian stochastic models. *Probab. Theory Related Fields* **151** 319–351. MR2834721

Brezis, H. (1983). *Analyse fonctionnelle. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]*. Masson, Paris. Théorie et applications. [Theory and applications]. MR697382 (85a:46001)

Cercignani, C. (1988). *The Boltzmann equation and its applications.* Springer-Verlag, New York. MR95i:82082

Chaleyat-Maurel, M. and Michel, D. (1984). Hypoellipticity theorems and conditional laws. *Z. Wahrsch. Verw. Gebiete* **65** 573–597. MR736147

Delarue, F. and Menozzi, S. (2010). Density estimates for a random noise propagating through a chain of differential equations. *J. Funct. Anal.* **259** 1577–1630. MR2659772

Desvillettes, L. and Villani, C. (2001). On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation. *Comm. Pure Appl. Math.* **54** 1–42. MR2001h:82079

Di Francesco, M. and Pascucci, A. (2005). On a class of degenerate parabolic equations of Kolmogorov type. *AMRX Appl. Math. Res. Express* **3** 77–116. MR2180050 (2006i:35202)

Di Francesco, M. and Pascucci, A. (2007). A continuous dependence result for ultraparabolic equations in option pricing. *J. Math. Anal. Appl.* **336** 1026–1041. MR2352998

Di Francesco, M. and Polidoro, S. (2006). Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form. *Adv. Differential Equations* **11** 1261–1320. MR2277064

Folland, G. B. and Stein, E. M. (1982). *Hardy spaces on homogeneous groups. Mathematical Notes* **28**. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo. MR657581

Friedman, A. (1964). *Partial differential equations of parabolic type.* Prentice-Hall Inc., Englewood Cliffs, N.J. MR31 ##6062

Hörmander, L. (1967). Hypoelliptic second order differential equations. *Acta Math.* **119** 147–171. MR36 ##5526

Jerison, D. S. and Sánchez-Calle, A. (1986). Estimates for the heat kernel for a sum of squares of vector fields. *Indiana Univ. Math. J.* **35** 835–854. MR865430

Kohatsu-Higa, A. and Yûki, G. (2018). Stochastic formulations of the parametrix method. *ESAIM Probab. Stat.* **22** 178–209. MR3891754

Kolmogorov, A. (1934). Zufallige Bewegungen. (Zur Theorie der Brownschen Bewegung.). *Ann. of Math., II. Ser.* **35** 116-117.

Konakov, V., Menozzi, S. and Molchanov, S. (2010). Explicit parametrix and local limit theorems for some degenerate diffusion processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 908—923. MR2744877

Krylov, N. V. (2015). Hörmander's theorem for stochastic partial differential equations. *Algebra i Analiz* **27** 157–182. MR3570961

Krylov, N. V. and Rozovskii, B. L. (1977). The Cauchy problem for linear stochastic partial differential equations. *Izv. Akad. Nauk SSSR Ser. Mat.* **41** 1329–1347, 1448. MR0501350

KRYLOV, N. V. and ZATEZALO, A. (2000). A direct approach to deriving filtering equations for diffusion processes. *Appl. Math. Optim.* **42** 315–332. MR1795614

KUNITA, H. (1982). Stochastic partial differential equations connected with nonlinear filtering. In *Nonlinear filtering and stochastic control (Cortona, 1981). Lecture Notes in Math.* **972** 100–169. Springer, Berlin. MR705933

KUNITA, H. (1990). *Stochastic flows and stochastic differential equations. Cambridge Studies in Advanced Mathematics* **24**. Cambridge University Press, Cambridge. MR1070361

LANCONELLI, E. and POLIDORO, S. (1994). On a class of hypoelliptic evolution operators. *Rend. Sem. Mat. Univ. Politec. Torino* **52** 29–63. MR95h:35044

LIONS, P. L. (1994). On Boltzmann and Landau equations. *Philos. Trans. Roy. Soc. London Ser. A* **346** 191–204. MR95d:82050

LUNARDI, A. (1997). Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbb{R}^N$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **24** 133–164. MR99a:35019

MANFREDINI, M. (1997). The Dirichlet problem for a class of ultraparabolic equations. *Adv. Differential Equations* **2** 831–866. MR2001a:35102

MENOZZI, S. (2011). Parametrix techniques and martingale problems for some degenerate Kolmogorov equations. *Electron. Commun. Probab.* **16** 234–250. MR2802040

MENOZZI, S. (2018). Martingale problems for some degenerate Kolmogorov equations. *Stochastic Process. Appl.* **128** 756–802. MR3758337

PAGLIARANI, S., PASCUCCI, A. and PIGNOTTI, M. (2016). Intrinsic Taylor formula for Kolmogorov-type homogeneous groups. *J. Math. Anal. Appl.* **435** 1054–1087. MR3429628

PARDOUX, E. (1979). Stochastic partial differential equations and filtering of diffusion processes. *Stochastics* **3** 127–167. MR553909

PASCUCCI, A. (2011). *PDE and martingale methods in option pricing. Bocconi&Springer Series.* Springer-Verlag, New York.

PASCUCCI, A. and PESCE, A. (2019). The parametrix method for parabolic SPDEs. *arXiv:1803.06543v3*.

PASCUCCI, A. and POLIDORO, S. (2006). Harnack inequalities and Gaussian estimates for a class of hypoelliptic operators. *Trans. Amer. Math. Soc.* **358** 4873–4893. MR2231876

POLIDORO, S. (1994). On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type. *Matematiche (Catania)* **49** 53–105. MR97a:35133

POLIDORO, S. (1997). A global lower bound for the fundamental solution of Kolmogorov-Fokker-Planck equations. *Arch. Rational Mech. Anal.* **137** 321–340. MR98g:35123

QIU, J. (2018). Hörmander-type theorem for Itô processes and related backward SPDEs. *Bernoulli* **24** 956–970. MR3706782

RISKEN, H. (1989). *The Fokker-Planck equation: Methods of solution and applications*, Second ed. Springer-Verlag, Berlin. MR90a:82002

SOWERS, R. B. (1994). Recent results on the short-time geometry of random heat kernels. *Math. Res. Lett.* **1** 663–675. MR1306012

ZAKAI, M. (1969). On the optimal filtering of diffusion processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **11** 230–243. MR0242552

DIPARTIMENTO DI MATEMATICA, PIAZZA DI PORTA SAN DONATO, 5 BOLOGNA (ITALY),
E-MAIL: andrea.pascucci@unibo.it; antonello.pesce2@unibo.it