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Multiple solutions for asymptotically q-linear (p,q)-Laplacian problems

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We investigate the existence and the multiplicity of solutions of the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth, bounded domain of $\mathbb{R}^N$, $1<p<q<\infty$, and the nonlinearity g behaves as u^{q-1} at infinity. We use variational methods and find multiple solutions as minimax critical points of the associated energy functional. Under suitable assumptions on the nonlinearity, we cover also the resonant case.

KEYWORDS

asymptotically q-linear problems, multiplicity of solutions, resonant problems, variational methods, (p, q)-Laplacian problems

MSC CLASSIFICATION

35J20; 35J62; 35P30; 35Q60; 47J30

1 | INTRODUCTION

We consider the following Dirichlet (p,q)-Laplacian problem:

$$\begin{cases} -\Delta_p u - \Delta_q u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, $1<p<q<\infty$, and the nonlinearity $g(x,\cdot)$ is asymptotically q-linear, meaning that $g(x,t) \sim \ell_\infty |t|^{q-2}t$ as $|t| \to \infty$ for some constant ℓ_∞.

Some of the difficulties arising in the study of this problem come from the nonhomogeneity of the operator $-\Delta_p - \Delta_q$. The interest in (p,q)-Laplacians and, in general, in nonhomogeneous operators has considerably increased since the seminal papers[1,2] by Marcellini, in late 1980s, on the regularity of minimizers of the so-called functionals with *nonstandard growth*. In this setting, both (p,q)- and $p(x)$-growth conditions have been widely considered; see Diening et al.[3] and the references therein for a comprehesive monograph on the Lebesgue and Sobolev spaces involved in the $p(x)$ variable exponent case. More recently, in Baroni et al.[4] and Colombo and Mingione,[5] many other progresses were achieved in the study of energy functionals related to the operators $-\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u)$. In these papers, the weight function $a(x) \geq 0$ switches two different elliptic behaviors, justifying the name of *double phase* functionals. These functionals were first introduced by Zhikov[6] to provide models for strongly anisotropic materials. In that setting, the exponents p and

q cannot be too far from each other, the bound $q/p < (1+\alpha/N)$ for some $\alpha \in (0,1]$ is needed both to develop a regularity theory and to prove some classical inequalities like a Poincaré-type one, in suitable Orlicz–Sobolev spaces; see Colasuonno and Squassina.[7, remark 2.19] On the other hand, $(2,q)$-Laplacian operators naturally arise in the Born–Infeld theory of nonlinear electromagnetism, where the leading operator, that is, the Minkowski curvature operator $-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)$, can be approximated by a truncated series of $2h$-Laplacians, $h \in \mathbb{N}$; comapre previous studies.[8–10] In this case, the highest exponent in the truncated series should morally go to infinity, and so no bounds on $q/2$ are admissible. In the same spirit, in the problem under consideration, the exponents p and q can be arbitrarily far. We observe that all the arguments in this paper can be adapted to operators of the form $-\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u)$, with the weight function $a(x)$ satisfying $a \in C^1(\bar{\Omega})$, $a > 0$ (it is enough to endow the functional space $W_0^{1,q}(\Omega)$ with the equivalent norm $\|a(x)\nabla u\|_q$). On the contrary, things change a lot for the more general case in which $a(x)$ can vanish somewhere; this case should be treated in suitable Orlicz–Sobolev spaces; the above mentioned bound on q/p should be required, and, in particular, being $p(1+1/N) < p^*$, the exponent q should be taken p-subcritical. While in our setting, since $a \equiv 1$ never vanishes, we are allowed not to require any relation between q and p (note that $q > p$ is not an assumption, since the roles of p and q are interchangeable).

Let us now introduce in details the hypotheses required on the nonlinearity of the problem. We assume that there exist $\ell_\infty \in \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$g(x,t) = \ell_\infty |t|^{q-2}t + f(x,t), \tag{1.2}$$

so that the problem can be written as

$$\begin{cases} -\Delta_p u - \Delta_q u = \ell_\infty |u|^{q-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In what follows, we denote by $\sigma(-\Delta_q)$ the spectrum of the Dirichlet q-Laplacian operator, namely, the set of $\lambda's$ in $\mathbb{R}$ for which the problem

$$\begin{cases} -\Delta_q u = \lambda |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a nontrivial weak solution. The nonlinearity f satisfies the following assumptions:

(f) f is a Carathéodory function (i.e., $f(\cdot,t)$ is measurable in Ω for all $t \in \mathbb{R}$ and $f(x,\cdot)$ is continuous in $\mathbb{R}$ for a.e. $x \in \Omega$) and $\max_{|t|\le R}|f(\cdot,t)| \in L^\infty(\Omega)$ for all $R > 0$;
(f_∞) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{q-2}t} = 0$ uniformly in $x \in \Omega$;
(f_{cpt}) one of the two assumptions holds:

(f_{nr}) $\ell_\infty \notin \sigma(-\Delta_q)$;
(f_{nr}) $\ell_\infty \in \sigma(-\Delta_q)$ and $\lim_{|t|\to\infty}(f(x,t)t - qF(x,t)) = +\infty$ uniformly in $x \in \Omega$, where $F(x,t) := \int_0^t f(x,s)ds$;
(f_0) there exists $\lim_{t\to 0} \frac{f(x,t)}{|t|^{q-2}t} =: \ell_0 \in \mathbb{R} \cup \{\pm\infty\}$ uniformly in $x \in \Omega$;
(f_{sym}) $f(x,\cdot)$ is odd for a.e. $x \in \Omega$.

Problem (1.3) has a variational structure, so the solutions are found as critical points of the associated energy functional $I : W_0^{1,q}(\Omega) \to \mathbb{R}$ defined as

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx - \frac{\ell_\infty}{q}\int_\Omega |u|^q dx - \int_\Omega F(x,u)dx,$$

compare Section 2.2. We observe that, beyond the nonhomogeneity of the operator, another feature of the problem that makes the analysis more interesting is the behavior of the nonlinearity at infinity. Indeed, the asymptotic q-linearity of $g(x,\cdot)$ in particular implies that the Ambrosetti–Rabinowitz condition—which is responsible for Palais–Smale sequences to be bounded—is not satisfied. This prevents the use of the classical Mountain Pass or Symmetric Mountain Pass Theorem for the existence of solutions to problem (1.1); see the introduction of Li and Zhou[11] for interesting comments on the topic. For existence results in the case of q-superlinear and subcritical nonlinearities in the (p,q)-setting, we refer to Perera and Squassina[12] in bounded domains and to Bartolo et al.[13] in the whole space $\mathbb{R}^N$.

Due to the behavior of $g(x,\cdot)$ at infinity, one can expect some interaction with the spectrum of the q-Laplacian; compare Li and Zhou[11] and Bartolo et al.[14] In particular, when ℓ_∞ is an eigenvalue of $-\Delta_q$, a stronger assumption on f is needed to get compactness; see (f_r). Furthermore, due to the symmetry condition (f_{sym}), the energy functional I is even, so that if u is a critical point of I at some critical value c, also $-u$ has the same property. This is the reason why in the statement of the main theorem below, we always refer to *pairs* of solutions. We believe that even without the symmetry condition (f_{sym}), it is possible to obtain existence results by applying a Linking Theorem as in Bartolo et al.[14]

In Section 2, we will introduce the definitions of two suitable sequences $(\eta_h^{(\alpha)})$, $(\nu_h^{(\alpha)})$ of quasi-eigenvalues of the problem

$$\begin{cases} -\alpha\Delta_p u - \Delta_q u = \lambda|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\alpha = 0$ or 1. For the exact definitions of $(\eta_h^{(\alpha)})$, $(\nu_h^{(\alpha)})$, we refer to (2.5) and (2.11), respectively. These sequences play an important role in the proof of the geometry conditions for the functional I.

We are now ready to state our main result.

Theorem 1.1. *Assume that (f), (f_∞), (f_{cpt}), (f_0), and (f_{sym}) hold. Suppose further that one of the following assumptions hold:*

(H_-) $-\infty \le \ell_0 < 0$ and there exist $h, k \in \mathbb{N}$, with $k \ge h$, such that $\ell_0 + \ell_\infty < \eta_h^{(0)}$ and $\ell_\infty > \nu_k^{(0)}$;
(H_+) $0 \le \ell_0 \le +\infty$ and there exist $h, k \in \mathbb{N}$, with $k \ge h$, such that $\ell_\infty < \eta_h^{(0)}$ and $\ell_\infty + \ell_0 > \frac{q}{p}\nu_k^{(1)}$.

Then, problem (1.3) has at least $k - h + 1$ distinct pairs of nontrivial solutions.

All the solutions found are minimax critical points of the associated energy functional. The proof of our main theorem is based on an abstract result proved in Bartolo et al.[14,15] by using the pseudo-index theory related to the Krasnosel'skiı genus; see Theorem 2.2. In order to apply this result, we prove that the energy functional I satisfies a compactness assumption, the Cerami's weaker variant of the Palais–Smale condition, both in the nonresonant case and in the resonant one under (f_r); compare Lemma 3.2. In the proof of this lemma, neither the symmetry nor the behavior near zero of f enter at all. Moreover, as usual, in order to find multiple minimax critical points, it is also needed to show that the energy functional I has the right geometry. This part of the proof involves the behavior of the nonlinearity both at infinity and at zero and is responsible for the assumptions on ℓ_∞ and ℓ_0 in (H_-) and (H_+).

Some remarks on the statement are now in order. In both cases (H_-) and (H_+), the larger the $|\ell_0|$, the higher the possibility of finding solutions. In fact, in the limit cases $|\ell_0| = \infty$, we get the highest number of solutions. If $\ell_0 = -\infty$, certainly $\ell_0 + \ell_\infty < \eta_1^{(0)}$; hence, (1.3) admits at least k pairs of distinct solutions. If $\ell_0 = \infty$, certainly $\ell_0 + \ell_\infty > \frac{q}{p}\nu_k^{(1)}$ is satisfied for every $k \in \mathbb{N}$. Moreover, being $(\eta_h^{(0)})$ divergent (see Proposition 2.5), there always exists $\bar{h} \in \mathbb{N}$ for which $\ell_\infty < \eta_{\bar{h}}^{(0)}$. Therefore, in this case, (1.3) admits infinitely many pairs of distinct solutions.

On the other hand, it is not clear whether the case $\ell_0 = 0$ is covered in our main result; see Remarks 3.4 and 3.8. In particular, the case $\ell_0 = 0$ includes the p-behavior at zero. We refer to previous studies,[16–18] for problems with $f \sim |t|^{p-2}t$ in a neighborhood of 0, and the so-called two-parameter eigenvalue problem for the (p,q)-Laplacian.

Finally, let λ_1 be the first eigenvalue of the Dirichlet q-Laplacian. Being $\nu_k^{(0)} \ge \nu_1^{(0)} = \lambda_1$ (see Section 2.3), condition (H_-) is never satisfied if $\ell_\infty \le \lambda_1$. Similarly, since $\nu_1^{(1)} \ge \lambda_1$, condition (H_+) is never satisfied if $\ell_\infty \le \frac{q}{p}\lambda_1 - \ell_0$ when $\ell_0 > 0$.

The paper is organized as follows. In Section 2, we present the abstract result that we will apply and introduce the variational setting and the two sequences of quasi-eigenvalues $(\eta_h^{(\alpha)})$ and $(\nu_h^{(\alpha)})$. In Section 3, we prove the multiplicity result through the intermediate steps of showing that the energy functional satisfies the compactness condition and has the right geometry.

2 | PRELIMINARY RESULTS

2.1 | Abstract results

Definition 2.1. Let X be a Banach space. A C^1-functional $I : X \to \mathbb{R}$ satisfies the Cerami–Palais–Smale condition (*(CPS)-condition* for short) if every sequence $(u_n) \subset X$ such that

$$I(u_n) \to c \in \mathbb{R} \text{ and } \|I'(u_n)\|_{X'}(1 + \|u_n\|_X) \to 0 \text{ as } n \to \infty$$

admits a convergent subsequence.

We will apply the following multiplicity result; see Bartolo et al.[15, theorem 2.9] for a proof in Hilbert spaces and Bartolo et al.[14, theorems 2.6 and 2.7] for Banach spaces. In particular, for the version that appears as *(resp. …)* in the statement, we refer to Bartolo et al.[14 remark 2.8]

Theorem 2.2. *Let X be a Banach space and for $\rho > 0$ denote $S_\rho := \{u \in X : \|u\|_X = \rho\}$. Suppose that the functional $I \in C^1(X, \mathbb{R})$ satisfies the following properties:*

i. I is even;
ii. I satisfies (CPS) in $(0, \infty)$, and $I(0) \geq 0$;
iii. there exist two closed subspaces $V, W \subset X$ such that $\dim V < \infty$ and $\dim W < \infty$, and two constants $c_\infty > c_0 > I(0)$ for which the following assumptions hold:

(a) $I(u) \geq c_0$ for every $u \in S_\rho \cap W$ (resp. for every $u \in S_\rho \cap V$);
(b) $I(u) \leq c_\infty$ for every $u \in V$ (resp. for every $u \in W$).

If furthermore $\dim V > \dim W$, then I possesses at least $m = \dim V - \dim W$ distinct pairs of critical points, whose corresponding critical values belong to $[c_0, c_\infty]$.

2.2 | Variational setting

Throughout the paper, for $1 \leq r \leq \infty$, we denote with $\|\cdot\|_r$, the usual norm in the Lebesgue space $L^r(\Omega)$. We look for solutions of (1.3) in the Sobolev space $W_0^{1,q}(\Omega)$ endowed with the equivalent norm

$$\|u\| := \|\nabla u\|_q.$$

Definition 2.3. A function $u \in W_0^{1,q}(\Omega)$ is a weak solution of (1.3) if for every $\varphi \in W_0^{1,q}(\Omega)$, the following distributional identity holds:

$$\int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \nabla \varphi dx = \ell_\infty \int_\Omega |u|^{q-2} u\varphi dx + \int_\Omega f(x,u)\varphi dx.$$

We observe that, due to the boundedness of Ω, all the integrals above are finite. Indeed, for every $r < q$, by Hölder's inequality, we have

$$\int_\Omega |v|^r dx \leq |\Omega|^{\frac{q-r}{q}} \|v\|_q^r \text{ for every } v \in L^q(\Omega). \tag{2.1}$$

The problem (1.3) has a variational structure, its associated energy functional $I : W_0^{1,q}(\Omega) \to \mathbb{R}$ is defined as follows for every $u \in W_0^{1,q}(\Omega)$:

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx - \frac{\ell_\infty}{q}\int_\Omega |u|^q dx - \int_\Omega F(x,u)dx.$$

It is straightforward to verify that I is of class C^1 and that $u \in W_0^{1,q}(\Omega)$ is a weak solution of (1.3) if and only if it is a critical point of I.

Moreover, for future use, we introduce the Sobolev critical exponent for the embedding $W^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$ to be

$$q^* := \begin{cases} \frac{Nq}{N-q} & \text{if } N > q, \\ +\infty & \text{if } N \leq q, \end{cases}$$

the conjugate exponent q' of q, and the dual space $(W_0^{1,q}(\Omega))' =: W^{-1,q'}(\Omega)$, with its operatorial norm denoted by $\|\cdot\|_{-1,q'}$.

2.3 | Two sequences of quasi-eigenvalues

Inspired by Candela and Palmieri[19] and Li and Zhou,[11] we define below two sequences, denoted by $(\eta_h^{(\alpha)})$ and $(\nu_h^{(\alpha)})$, of quasi-eigenvalues for a (p,q)-Laplacian-type operator. Compared with the arguments in Li and Zhou[11] and Candela and Palmieri,[19] for the classical p-Laplacian, here, the arguments are slightly more delicate due to the lack of homogeneity of the operator.

Since in this subsection $\alpha \geq 0$ is fixed, for simplicity in notation, throughout this subsection, we will drop the superscript (α) and denote the sequences simply by (η_h) and (ν_h).

For $\alpha \in [0,+\infty)$, let us define the C^1-functional $\Phi : W_0^{1,q}(\Omega) \to \mathbb{R}$ as

$$\Phi(u) = \alpha\|\nabla u\|_p^p + \|\nabla u\|_q^q \text{ for every } u \in W_0^{1,q}(\Omega).$$

We are now ready to construct the sequence (η_h) and its corresponding sequence of quasi-eigenfunctions (φ_h).

Let us define $\mathcal{S} := \{u \in W_0^{1,q}(\Omega) : \|u\|_q = 1\}$ and

$$\eta_1 := \inf_{u\in\mathcal{S}} \Phi(u).$$

Denoted by λ_1 the first eigenvalue of the Dirichlet q-Laplacian, we claim that $\eta_1 \geq \lambda_1 > 0$ is achieved by a function $\varphi_1 \in \mathcal{S}$.

- Proof of the claim. Let $(u_n) \subset \mathcal{S}$ be such that $\Phi(u_n) \to \eta_1$, then for every $n \in \mathbb{N}$

$$\|\nabla u_n\|_q^q \leq \alpha\|\nabla u_n\|_p^p + \|\nabla u_n\|_q^q = \eta_1 + o(1),$$

where $o(1) \to 0$ as $n \to \infty$. Then, (u_n) is bounded in the reflexive Banach space $W_0^{1,q}(\Omega)$ and so, up to a subsequence, $u_n \rightharpoonup u$ in $W_0^{1,q}(\Omega)$. Now, the function Φ is convex and continuous w.r.t. the strong topology in $W_0^{1,q}(\Omega)$; then, it is weakly lower semicontinuous in $W_0^{1,q}(\Omega)$. Hence,

$$\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n) = \eta_1. \tag{2.2}$$

On the other hand, by the compact embedding $W_0^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, $u_n \to u$ in $L^q(\Omega)$, and so $u \in \mathcal{S}$. Therefore, u is an admissible competitor for the infimum defining η_1, so that the only possibility for (2.2) to hold is that $\Phi(u) = \eta_1$.

We will denote by φ_1 a function in $\mathcal{S}$ where η_1 is attained. Hence, summarizing $\|\varphi_1\|_q = 1$, $\alpha\|\nabla\varphi_1\|_p^p + \|\nabla\varphi_1\|_q^q = \eta_1$, and

$$\eta_1 \leq \alpha\int_\Omega \left|\nabla\left(\frac{u}{\|u\|_q}\right)\right|^p dx + \int_\Omega \left|\nabla\left(\frac{u}{\|u\|_q}\right)\right|^q dx \text{ for every } u \in W_0^{1,q}(\Omega)\backslash\{0\}. \tag{2.3}$$

Remark 2.4. Being $p < q$, (2.3) implies

$$\eta_1\|u\|_q^q \leq \alpha\int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla u|^q dx \text{ for every } u \in W_0^{1,q}(\Omega) \cap \mathcal{B},$$
$$\eta_1\|u\|_q^p \leq \alpha\int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla u|^q dx \text{ for every } u \in W_0^{1,q}(\Omega)\backslash\mathcal{B},$$

where $\mathcal{B} := \{u \in W_0^{1,q}(\Omega) : \|u\|_q \leq 1\}$. In particular, if $\alpha = 0$ or equivalently $p = q$,

$$\eta_1\|u\|_q^q \leq (\alpha+1)\int_\Omega |\nabla u|^q dx \text{ for every } u \in W_0^{1,q}(\Omega).$$

We further remark that if $\alpha > 0$ and $q \leq p^*$, $\eta_1 (= \eta_1^{(\alpha)}) > \lambda_1 (= \eta_1^{(0)})$. Indeed, by the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ ($\|u\|_q \leq C_S\|\nabla u\|_p$, for some $C_S > 0$, for all $u \in W_0^{1,p}(\Omega)$) and the variational characterization of λ_1, for every $u \in \mathcal{S}$

$$\alpha\|\nabla u\|_p^p + \|\nabla u\|_q^q \geq \alpha C_S^{-p}\|u\|_q^p + \|\nabla u\|_q^q \geq \alpha C_S^{-p} + \lambda_1 > \lambda_1.$$

Related to φ_1, we can introduce the linear operator $\mathcal{L}_1 : L^q(\Omega) \to \mathbb{R}$ defined as

$$\mathcal{L}_1(u) := \int_\Omega |\varphi_1|^{q-2}\varphi_1 u\, dx \text{ for every } u \in L^q(\Omega).$$

We note that $\mathcal{L}_1(\varphi_1) = 1$, $\mathcal{L}_1 \in L^{q'}(\Omega)$ (in particular, by Hölder's inequality, $\|\mathcal{L}_1\|_{p'} = 1$) and $\mathcal{L}_1|_{W_0^{1,q}(\Omega)} \in W^{-1,q'}(\Omega)$ (in particular, by Hölder's and Poincaré's inequalities, $\|\mathcal{L}_1\|_{-1,q'} \le \lambda_1^{-1/q}$).

We introduce the new constraint

$$\mathcal{S}_1 := \{u \in \mathcal{S} : \mathcal{L}_1 u = 0\} = \ker(\mathcal{L}_1|_{\mathcal{S}})$$

and the corresponding constrained infimum

$$\eta_2 := \inf_{u \in \mathcal{S}_1} \Phi(u). \tag{2.4}$$

Since $\mathcal{S}_1 \subset \mathcal{S}$, we have $\eta_1 \le \eta_2$. Next, we claim that also the infimum in (2.4) is attained.

• Proof of the claim. The proof is the same as for the previous claim, with the only difference that one has to prove also that $\mathcal{L}_1(u) = 0$, u being the weak limit of the minimizing sequence $(u_n) \subset \mathcal{S}_1$. This is a consequence of $\mathcal{L}_1|_{W_0^{1,q}(\Omega)} \in W^{-1,q'}(\Omega)$, being $\mathcal{L}_1(u_n) = 0$ for every $n \in \mathbb{N}$.

By iterating this procedure, we introduce a sequence of positive numbers (η_h), a sequence of functions $(\varphi_h) \subset \mathcal{S}$ and, in correspondence, a sequence of linear operators $(\mathcal{L}_h) \subset L^{q'}(\Omega) \cap W^{-1,q'}(\Omega)$ defined by

$$\mathcal{L}_h u := \int_\Omega |\varphi_h|^{q-2}\varphi_h u\, dx \text{ for every } u \in L^q(\Omega) \text{ and } h \in \mathbb{N}.$$

More precisely, denoted $\mathcal{S}_0 := \mathcal{S}$, we define the following weakly closed subspaces of $W_0^{1,q}(\Omega)$

$$\mathcal{S}_h := \{u \in \mathcal{S} : \mathcal{L}_1 u = \ldots = \mathcal{L}_h u = 0\} = \bigcap_{i=1}^{h} \ker(\mathcal{L}_i|_{\mathcal{S}}),$$

and, for every $h \in \mathbb{N}$, the corresponding constrained infimum

$$\eta_h := \inf_{u \in \mathcal{S}_{h-1}} \Phi(u), \tag{2.5}$$

each one achieved on the corresponding function $\varphi_h \in \mathcal{S}_{h-1}$. From the definition, it easily follows that

$$0 < \eta_1 \le \ldots \le \eta_h \le \eta_{h+1} \le \ldots,$$

and

$$\mathcal{L}_h \varphi_h = 1 \text{ for every } h \in \mathbb{N} \text{ and } \mathcal{L}_k \varphi_h = 0 \text{ for } k = 1, \ldots, h-1. \tag{2.6}$$

In particular, $\mathrm{span}\{\varphi_h\} \cap \mathrm{span}\{\varphi_k\} = \{0\}$ if $h \ne k$. Since otherwise, for $k < h$, one could have for a suitable constant β, $1 = \mathcal{L}_k\varphi_k = \mathcal{L}_k(\beta\varphi_h) = \beta\mathcal{L}_k(\varphi_h) = 0$, which is absurd.

In the spirit of Candela and Palmieri,[19, lemma 5.2] we prove the following proposition.

Proposition 2.5. *The sequence (η_h) diverges positively.*

Proof. Suppose by contradiction that there exists a number $\bar\eta \in (0, \infty)$ such that

$$\eta_h \le \bar\eta \text{ for every } h \in \mathbb{N}.$$

As a consequence, for every $h \in \mathbb{N}$,

$$\|\nabla\varphi_h\|_q^q \le \Phi(\varphi_h) = \eta_h \le \bar\eta,$$

that is, (φ_h) is bounded in the reflexive Banach space $W_0^{1,q}(\Omega)$. Thus, there exist a subsequence, still denoted by (φ_h), and a function $\bar{\varphi} \in W_0^{1,q}(\Omega)$ such that $\varphi_h \rightharpoonup \bar{\varphi} \in W_0^{1,q}(\Omega)$ and $\varphi_h \to \bar{\varphi}$ in $L^q(\Omega)$. In particular, (φ_h) is a Cauchy sequence in $L^q(\Omega)$, thus for any positive $\varepsilon < 1$, there is $h_0 \in \mathbb{N}$ such that

$$\|\varphi_{h+k} - \varphi_h\|_q < \varepsilon \text{ for every } h \geq h_0, k \geq 1.$$

Therefore, (2.6) and the Hölder inequality imply

$$\begin{aligned} 1 = \mathcal{L}_h\varphi_h = \mathcal{L}_h\varphi_h - \mathcal{L}_h\varphi_{h+k} = |\mathcal{L}_h(\varphi_{h+k} - \varphi_h)| \leq \int_\Omega |\varphi_h|^{q-1}|\varphi_{h+k} - \varphi_h| dx \\ \leq \|\varphi_h\|_q \|\varphi_{h+k} - \varphi_h\|_q < \varepsilon, \end{aligned}$$

that is a contradiction. □

Let us recall that if $V \subseteq X$ is a closed subspace of a Banach space X, a subspace $W \subset X$ is a (topological) complement of V if W is closed and every $x \in X$ can be uniquely written as $v + w$, with $v \in V$ and $w \in W$; furthermore, the projection operators onto V and W are (linear and) continuous. When this happens and V has finite dimension, we say that W has finite codimension, with $\operatorname{codim} W = \dim V$.

Lemma 2.6. *Let $\alpha \geq 0$ be fixed. For every $h \in \mathbb{N}$, let us set*

$$V_h = V_h^{(\alpha)} := span\{\varphi_1, \dots, \varphi_h\}, \tag{2.7}$$

$$W_h = W_h^{(\alpha)} := \bigcap_{i=1}^{h} \ker(\mathcal{L}_i) = \{u \in W_0^{1,q}(\Omega) : \mathcal{L}_1 u = \dots = \mathcal{L}_h u = 0\}. \tag{2.8}$$

Then,

$$W_0^{1,q}(\Omega) = V_h \oplus W_h \text{ for every } h \in \mathbb{N}.$$

Proof. Reasoning as in Candela and Palmieri,[19, lemma 5.3] we point out that for any $u \in V_h$, by the linear independence of $\varphi_1, \dots, \varphi_h$, we can write uniquely

$$u = \sum_{i=1}^{h} c_i \varphi_i,$$

with $(c_1, \dots, c_h) \in \mathbb{R}^h$ and by (2.6), it results that

$$\mathcal{L}_1 u = c_1, \ \mathcal{L}_i u = \sum_{j=1}^{i-1} c_j \mathcal{L}_i \varphi_j + c_i \text{ for every } i \in \{2, \dots, h\}.$$

Therefore, given a function $u = \sum_{i=1}^{h} c_i \varphi_i$ in V_h, the following equivalences hold

$$\begin{aligned} u \in W_h &\iff \mathcal{L}_i u = 0 \text{ for every } i \in \{1, \dots, h\} \\ &\iff c_1 = \dots = c_h = 0 \iff u = 0. \end{aligned}$$

Which means that $V_h \cap W_h = \{0\}$. Now, fixed any $u \in W_0^{1,q}(\Omega)$, we put

$$c_1 := \mathcal{L}_1 u, \ c_i := \mathcal{L}_i u - \sum_{j=1}^{i-1} c_j \mathcal{L}_i \varphi_j \text{ for every } i \in \{2, \dots, h\}$$

and

$$v := \sum_{i=1}^{h} c_i \varphi_i \in V_h.$$

Taken $w := u - v$, we have that

$$\mathcal{L}_1 w = \mathcal{L}_1(u-v) = \mathcal{L}_1 u - c_1 = 0,$$

and for every $i \in \{2, \dots, h\}$,

$$\mathcal{L}_i w = \mathcal{L}_i(u-v) = \mathcal{L}_i u - \sum_{j=1}^{i-1} c_j \mathcal{L}_i \varphi_j - c_i - \sum_{j=i+1}^{h} c_j \mathcal{L}_i \varphi_j = 0.$$

Hence, $w \in W_h$, and we conclude the proof. □

We remark that, by definition, for each $h \in \mathbb{N}$, W_h is a closed subspace of $W_0^{1,q}(\Omega)$ of codimension h. Moreover, reasoning as in Remark 2.4, the following inequalities hold in W_{h-1}:

$$\begin{aligned} \eta_h \|u\|_q^q &\leq \alpha \int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla u|^q dx \text{ for every } u \in W_{h-1} \cap \mathcal{B}, \\ \eta_h \|u\|_q^p &\leq \alpha \int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla u|^q dx \text{ for every } u \in W_{h-1} \setminus \mathcal{B}, \end{aligned} \tag{2.9}$$

with $\mathcal{B} = \{u \in W_0^{1,q}(\Omega) : \|u\|_q \leq 1\}$, and in particular, if $\alpha = 0$ or $p = q$,

$$\eta_h \|u\|_q^q \leq (\alpha+1) \int_\Omega |\nabla u|^q dx \text{ for every } u \in W_{h-1}. \tag{2.10}$$

The sequence of quasi-eigenvalues (η_h) satisfies (2.9). In order to prove multiplicity results, it is useful to have also a reversed inequality on finite dimensional subspaces of $W_0^{1,q}(\Omega)$. Hence, we introduce (ν_h), another sequence of quasi-eigenvalues satisfying exactly this property.

For all $h \in \mathbb{N}$, we set

$$\mathbb{W}_h := \{V : V \text{ subspace of } W_0^{1,q}(\Omega), \varphi_1 \in V, \dim V \geq h\},$$

and

$$\nu_h := \inf_{V \in \mathbb{W}_h} \sup_{u \in V \setminus \{0\}} \frac{\alpha \|\nabla u\|_p^p + \|\nabla u\|_q^q}{\|u\|_q^q}, \tag{2.11}$$

with φ_1 defined above. Since $\mathbb{W}_{h+1} \subset \mathbb{W}_h$, $\nu_h \leq \nu_{h+1}$ for every h.

3 | MAIN RESULTS

We first observe that, by (f) and (f_∞), for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon |t|^{q-1} + A_\varepsilon \text{ for a.e. } x \in \Omega \text{ and every } t \in \mathbb{R}, \tag{3.1}$$

where $A_\varepsilon := \|\max_{|t| \leq R_\varepsilon} |f(\cdot, t)|\|_\infty \in (0, \infty)$.

Throughout this section, we will denote by the same symbol C various positive constants whose values are not important for the proof itself and may change from line to line.

3.1 | Compactness condition

We introduce here the following operator $A : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ defined as

$$\langle Au, v \rangle := \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \nabla v dx \text{ for every } u, v \in W_0^{1,q}(\Omega).$$

Lemma 3.1. *The operator $-\Delta_p - \Delta_q$ satisfies the (S_+)-property, that is, if $u_n \rightharpoonup u$ in $W_0^{1,q}(\Omega)$ and $\langle Au_n, u_n - u \rangle \to 0$, then $u_n \to u$ in $W_0^{1,q}(\Omega)$.*

Proof. This result is contained in Perera and Squassina,[12, proposition 2.2] but for the sake of clarity, we prefer to report here its proof. Since $\|\nabla u_n\|_p^{p-1} - \|\nabla u\|_p^{p-1}$ and $\|\nabla u_n\|_p - \|\nabla u\|_p$ have the same sign, and the same holds true for the q-norms,

$$(\|\nabla u_n\|_p^{p-1} - \|\nabla u\|_p^{p-1})(\|\nabla u_n\|_p - \|\nabla u\|_p) + (\|\nabla u_n\|_q^{q-1} - \|\nabla u\|_q^{q-1})(\|\nabla u_n\|_q - \|\nabla u\|_q) \geq 0. \tag{3.2}$$

On the other hand, for every $u, v \in W_0^{1,q}(\Omega)$, by Hölder's inequality

$$\langle Au, v\rangle \leq \|\nabla u\|_p^{p-1}\|\nabla v\|_p + \|\nabla u\|_q^{q-1}\|\nabla v\|_q.$$

Therefore, using the previous inequality,

$$\begin{aligned}
&(\|\nabla u_n\|_p^{p-1} - \|\nabla u\|_p^{p-1})(\|\nabla u_n\|_p - \|\nabla u\|_p) + (\|\nabla u_n\|_q^{q-1} - \|\nabla u\|_q^{q-1})(\|\nabla u_n\|_q - \|\nabla u\|_q)\\
&= (\|\nabla u_n\|_p^p + \|\nabla u_n\|_q^q) + (\|\nabla u\|_p^p + \|\nabla u\|_q^q) - (\|\nabla u_n\|_p^{p-1}\|\nabla u\|_p + \|\nabla u_n\|_q^{q-1}\|\nabla u\|_q)\\
&- (\|\nabla u\|_p^{p-1}\|\nabla u_n\|_p + \|\nabla u\|_q^{q-1}\|\nabla u_n\|_q)\\
&\leq \langle Au_n, u_n\rangle + \langle Au, u\rangle - \langle Au_n, u\rangle - \langle Au, u_n\rangle\\
&= \langle Au_n, u_n - u\rangle - \langle Au, u_n - u\rangle = o(1),
\end{aligned}$$

where $\langle Au_n, u_n - u\rangle = o(1)$ by assumption, and $\langle Au, u_n - u\rangle = o(1)$ because $u_n \rightharpoonup u_n$ both in $W_0^{1,q}(\Omega)$ and in $W_0^{1,p}(\Omega)$ (since $p < q$, $W^{-1,p'}(\Omega) \subset W^{-1,q'}(\Omega)$ and so $u_n \rightharpoonup u$ also in $W_0^{1,p}(\Omega)$). Definitely, taking into account also (3.2),

$$\begin{aligned}
&(\|\nabla u_n\|_p^{p-1} - \|\nabla u\|_p^{p-1})(\|\nabla u_n\|_p - \|\nabla u\|_p)\\
&+ (\|\nabla u_n\|_q^{q-1} - \|\nabla u\|_q^{q-1})(\|\nabla u_n\|_q - \|\nabla u\|_q) \to 0,
\end{aligned}$$

which implies in particular that $\|\nabla u_n\|_q \to \|\nabla u\|_q$. In conclusion, the convergence of the norms and the weak convergence yield the desired strong convergence $u_n \to u$ in the uniformly convex Banach space $W_0^{1,q}(\Omega)$. □

Lemma 3.2. *Under assumptions (f), (f_∞), and (f_{cpt}), the functional I satisfies the (CPS)-condition.*

Proof. Let $(u_n) \subset W_0^{1,q}(\Omega)$ be a sequence satisfying

$$I(u_n) \to c \in \mathbb{R} \text{ and } \|I'(u_n)\|_{-1,q}(1 + \|u_n\|) \to 0 \text{ as } n \to \infty. \tag{3.3}$$

We claim that it is enough to show that (u_n) is bounded in $W_0^{1,q}(\Omega)$. This is a quite standard consequence of Lemma 3.1; compare, for instance, Colasuonno and Pucci[20, lemma 3.1] and Dinca et al.[21, lemma 2] However, we prefer to write here all the details of the proof of this claim, because most of the arguments therein will be useful also in the rest of the proof of the present lemma. Suppose that $(\|\nabla u_n\|_q)$ is bounded. Since $W_0^{1,q}(\Omega)$ is a reflexive Banach space, there exist a subsequence, still denoted by (u_n), and a function $u \in W_0^{1,q}(\Omega)$ for which $u_n \rightharpoonup u$ in $W_0^{1,q}(\Omega)$. Moreover, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
\langle I'(u_n), u_n - u\rangle = &\int_\Omega (|\nabla u_n|^{p-2} + |\nabla u_n|^{q-2})\nabla u_n(\nabla u_n - \nabla u)dx\\
&- \ell_\infty \int_\Omega |u_n|^{q-2}u_n(u_n - u)dx - \int_\Omega f(x, u_n)(u_n - u)dx,
\end{aligned} \tag{3.4}$$

and by (3.3) and by the boundedness of (u_n) in $W_0^{1,q}(\Omega)$,

$$|\langle I'(u_n), u_n - u\rangle| \leq \|I'(u_n)\|_{-1,q'}\|u_n - u\| \leq \|I'(u_n)\|_{-1,q'} \sup_n \|u_n - u\| = o(1),$$

where $o(1) \to 0$ as $n \to \infty$. On the other hand, by the compact embedding $W_0^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$, $u_n \to u$ in $L^r(\Omega)$ for every $r \in [1, q^*)$. Therefore, by Hölder's inequality,

$$\left|\int_\Omega |u_n|^{q-2} u_n (u_n - u) dx\right| \le \int_\Omega |u_n|^{q-1} |u_n - u| dx \le \|u_n\|_q^{q-1} \|u_n - u\|_q = o(1), \tag{3.5}$$

and using (3.1) with $\varepsilon = 1$ and (3.5),

$$\begin{aligned}\left|\int_\Omega f(x, u_n)(u_n - u) dx\right| &\le \int_\Omega \left(|u_n|^{q-1}|u_n - u| + A_1 |u_n - u|\right) dx \\ &= o(1) + A_1 \|u_n - u\|_1 = o(1).\end{aligned} \tag{3.6}$$

Hence, inserting the last three estimates in (3.4), we get

$$\int_\Omega (|\nabla u_n|^{p-2} + |\nabla u_n|^{q-2}) \nabla u_n (\nabla u_n - \nabla u) dx = o(1). \tag{3.7}$$

Therefore, by Lemma 3.1, $u_n \to u$ in $W_0^{1,q}(\Omega)$.

It remains to prove that (u_n) is bounded in $W_0^{1,q}(\Omega)$. We will follow the guidelines of Bartolo et al.[14, proposition 3.1-(i)] and of Li and Zhou.[11, proposition 3.1-(ii)]

• We first consider the nonresonant case in which $(f_{\rm nr})$ holds. We argue by contradiction and suppose that there exists a subsequence, still denoted by (u_n), such that $\|u_n\| \to \infty$. Thus, without loss of generality, we can assume that $\|u_n\| > 0$ for every n and define $w_n := u_n / \|u_n\|$. Clearly, (w_n) is bounded in $W_0^{1,q}(\Omega)$, hence up to a subsequence, $w_n \rightharpoonup w$ in $W_0^{1,q}(\Omega)$ and $w_n \to w$ in $L^r(\Omega)$ for every $r \in [1, q^*)$ and some $w \in W_0^{1,q}(\Omega)$. We claim that $w \neq 0$.

Indeed, as a consequence of the second convergence in (3.3), the following relation holds

$$\int_\Omega (|\nabla u_n|^p + |\nabla u_n|^q) dx = \ell_\infty \int_\Omega |u_n|^q dx + \int_\Omega f(x, u_n) u_n dx + o(1). \tag{3.8}$$

Dividing (3.16) by $\|u_n\|^q = \|\nabla u_n\|_q^q$, we get

$$\frac{\|\nabla u_n\|_p^p}{\|\nabla u_n\|_q^q} + 1 = \ell_\infty \|w_n\|_q^q + \int_\Omega \frac{f(x, u_n) u_n}{\|\nabla u_n\|_q^q} dx + o(1). \tag{3.9}$$

Moreover, by Hölder's inequality,

$$\frac{\|\nabla u_n\|_p^p}{\|\nabla u_n\|_q^q} \le \frac{C}{\|\nabla u_n\|_q^{q-p}} = \frac{C}{\|u_n\|^{q-p}} = o(1). \tag{3.10}$$

Assume by contradiction that $w = 0$. Then, since $w_n \to w$ in $L^q(\Omega)$, by (3.16), we get

$$\int_\Omega \frac{f(x, u_n) u_n}{\|\nabla u_n\|_q^q} dx = 1 + o(1). \tag{3.11}$$

On the other hand, by (3.1), (2.1) with $r = 1$, and Poincaré's inequality, we have

$$\left|\int_\Omega \frac{f(x, u_n) u_n}{\|\nabla u_n\|_q^q} dx\right| \le \|w_n\|_q^q + A_1 \frac{\|u_n\|_1}{\|\nabla u_n\|_q^q} \le \|w_n\|_q^q + \frac{C}{\|\nabla u_n\|_q^{q-1}} = o(1).$$

This contradicts (3.11) and proves that $w \neq 0$.

Another consequence of (3.3) is the following:

$$\begin{aligned}&\int_\Omega (|\nabla u_n|^{p-2}+|\nabla u_n|^{q-2})\nabla u_n \nabla\varphi\, dx\\ &\quad=\ell_\infty\int_\Omega |u_n|^{q-2}u_n\varphi\, dx+\int_\Omega f(x,u_n)\varphi\, dx+o(1)\ \text{ for every }\ \varphi\in W_0^{1,q}(\Omega).\end{aligned}\tag{3.12}$$

We divide (3.13) by $\|u_n\|^{q-1}=\|\nabla u_n\|_q^{q-1}$ to get for every $\varphi\in W_0^{1,q}(\Omega)$

$$\begin{aligned}&\int_\Omega\left(\frac{|\nabla u_n|^{p-2}\nabla u_n}{\|\nabla u_n\|_q^{q-1}}+|\nabla w_n|^{q-2}\nabla w_n\right)\nabla\varphi\, dx\\ &\quad=\ell_\infty\int_\Omega |w_n|^{q-2}w_n\varphi\, dx+\int_\Omega\frac{f(x,u_n)\varphi}{\|\nabla u_n\|_q^{q-1}}dx+o(1).\end{aligned}\tag{3.13}$$

Now, by Hölder's inequality and (2.1),

$$\left|\int_\Omega\frac{|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi}{\|\nabla u_n\|_q^{q-1}}dx\right|\le\int_\Omega\frac{|\nabla u_n|^{p-1}|\nabla\varphi|}{\|\nabla u_n\|_q^{q-1}}dx\le C\frac{\|\nabla\varphi\|_p}{\|\nabla u_n\|_q^{q-p}}=o(1).\tag{3.14}$$

Thus, (3.13) evaluated at $\varphi=w$ can be written as

$$\int_\Omega|\nabla w_n|^{q-2}\nabla w_n\nabla w\, dx=\ell_\infty\int_\Omega |w_n|^{q-2}w_n w\, dx+\int_\Omega\frac{f(x,u_n)w}{\|\nabla u_n\|_q^{q-1}}dx+o(1).\tag{3.15}$$

On the other hand, taking into account (3.10) and being $\int_\Omega|\nabla w_n|^q dx=1$ for every n, (3.16) reads as

$$\int_\Omega|\nabla w_n|^{q-2}\nabla w_n\nabla w_n dx=\ell_\infty\int_\Omega |w_n|^{q-2}w_n w_n dx+\int_\Omega\frac{f(x,u_n)w_n}{\|\nabla u_n\|_q^{q-1}}dx+o(1).\tag{3.16}$$

Thus, subtracting (3.16) and (3.15), we get

$$\begin{aligned}&\int_\Omega|\nabla w_n|^{q-2}\nabla w_n(\nabla w_n-\nabla w)dx\\ &\quad=\ell_\infty\int_\Omega |w_n|^{q-2}w_n(w_n-w)dx+\int_\Omega\frac{f(x,u_n)}{\|\nabla u_n\|_q^{q-1}}(w_n-w)dx+o(1).\end{aligned}\tag{3.17}$$

Arguing as in (3.5) and (3.6), we get

$$\left|\int_\Omega |w_n|^{q-2}w_n(w_n-w)dx\right|\le\|w_n\|_q^{q-1}\|w_n-w\|_q=o(1),$$

$$\left|\int_\Omega\frac{f(x,u_n)}{\|\nabla u_n\|_q^{q-1}}(w_n-w)dx\right|\le\int_\Omega\left(|w_n|^{q-1}|w_n-w|+A_1\frac{|w_n-w|}{\|\nabla u_n\|_q^{q-1}}\right)dx=o(1),$$

Definitely, inserting the last two relations in (3.17), we have that also

$$\int_\Omega|\nabla w_n|^{q-2}\nabla w_n(\nabla w_n-\nabla w)dx=o(1).\tag{3.18}$$

Since $-\Delta_q$ satisfies the (S_+)-property (cf., for instance, Colasuonno et al.[22, lemma 2.5]) and $w_n \rightharpoonup w$ in $W_0^{1,q}(\Omega)$, (3.18) implies that $w_n \to w$ in $W_0^{1,q}(\Omega)$. Finally, by (3.13) and (3.14), for every $\varphi \in W_0^{1,q}(\Omega)$,

$$\int_\Omega |\nabla w_n|^{q-2}\nabla w_n \nabla\varphi\, dx = \ell_\infty \int_\Omega |w_n|^{q-2} w_n \varphi\, dx + \int_\Omega \frac{f(x,u_n)\varphi}{\|\nabla u_n\|_q^{q-1}} dx + o(1). \tag{3.19}$$

By (3.1), the fact that (u_n) is convergent and so bounded in $L^q(\Omega)$, and $\| \nabla u_n\|_q \to \infty$,

$$\int_\Omega \frac{f(x,u_n)\varphi}{\|\nabla u_n\|_q^{q-1}} dx \le \frac{\|u_n\|_q^{q-1}\|\varphi\|_q}{\|\nabla u_n\|_q^{q-1}} + o(1) \le \frac{C\|\varphi\|_q}{\|\nabla u_n\|_q^{q-1}} + o(1) = o(1).$$

And so, passing to the limit in (3.19), we get, by the Dominated Convergence Theorem,

$$\int_\Omega |\nabla w|^{q-2}\nabla w \nabla\varphi\, dx = \ell_\infty \int_\Omega |w|^{q-2} w\varphi\, dx \text{ for every } \varphi \in W_0^{1,q}(\Omega),$$

meaning that ℓ_∞ is an eigenvalue of $-\Delta_q$. This contradicts $(f_{\rm nr})$ and concludes the proof of this case.

• We now consider the resonant case and assume the validity of $(f_{\rm r})$. We observe that in this case $\ell_\infty > 0$. In this part of the proof, for brevity, we will write the nonlinear terms $\ell_\infty|t|^{q-2}t + f(x,t)$ as $g(x,t)$. Let $G(x,t) := \int_0^t g(x,s)ds$. It is straightforward to check that $g(x,t)t - qG(x,t) = f(x,t)t - qF(x,t)$ for a.e. $x \in \Omega$ and every $t \in \mathbb{R}$. Thus, the limit in $(f_{\rm r})$ can be rewritten equivalently in terms of g as follows:

$$\lim_{|t|\to\infty} (g(x,t)t - qG(x,t)) = +\infty \text{ uniformly in } x \in \Omega. \tag{3.20}$$

As a consequence, there exists $T_0 > 0$ such that

$$g(x,t)t - qG(x,t) \ge 0 \text{ for a.e.} x \in \Omega \text{ and for every } |t| \ge T_0. \tag{3.21}$$

By virtue of (f), there exists $C_0 > 0$ such that

$$\int_{\{|u_n|\le T_0\}} (g(x,u_n)u_n - qG(x,u_n))dx \ge -C_0, \tag{3.22}$$

where for simplicity, in notation, we have denoted $\{|u_n| \le T_0\} = \{x \in \Omega : |u_n(x)| \le T_0\}$. Now, by (3.3),

$$\begin{aligned} qc + o(1) &= qI(u_n) - \langle I'(u_n), u_n\rangle \\ &= \left(\frac{q}{p} - 1\right)\int_\Omega |\nabla u_n|^p dx + \int_\Omega (g(x,u_n)u_n - qG(x,u_n))dx. \end{aligned}$$

Hence, by (3.22) and being $q > p$,

$$qc + o(1) \ge \int_\Omega (g(x,u_n)u_n - qG(x,u_n))dx \ge \int_{\{|u_n|\ge T_0\}} (g(x,u_n)u_n - qG(x,u_n))dx - C_0. \tag{3.23}$$

Now, let $K > 0$ be a constant to be specified later. By (3.20), there exists $T_K \ge T_0 > 0$ such that

$$g(x,t)t - qG(x,t) \ge K \text{ for a.e. } x \in \Omega \text{ and for every } |t| \ge T_K. \tag{3.24}$$

Thus, continuing from (3.23), we have

$$qc + o(1) \ge K|\{|u_n| \ge T_K\}| - C_0. \tag{3.25}$$

On the other hand, for every $r > q > p$, by (3.3),

$$I(u_n) - \frac{1}{r}\langle I'(u_n), u_n\rangle = \left(\frac{1}{p} - \frac{1}{r}\right) \|\nabla u_n\|_p^p + \left(\frac{1}{q} - \frac{1}{r}\right) \|\nabla u_n\|_q^q - \int_\Omega [G(x, u_n) - \frac{1}{r} g(x, u_n)u_n]dx = c + o(1).$$

Since, by (*f*), $\left|\int_{\{|u_n| \le T_K\}} \left[G(x, u_n) - \frac{1}{r} g(x, u_n)u_n\right] dx\right| \le C_K$, for some $C_K > 0$,

$$\begin{aligned}
c + o(1) &\ge \left(\frac{1}{q} - \frac{1}{r}\right) \|\nabla u_n\|_q^q - \int_{\{|u_n| \ge T_K\}} \left[G(x, u_n) - \frac{1}{r} g(x, u_n)u_n\right] dx - C_K \\
&\ge \left(\frac{1}{q} - \frac{1}{r}\right) \|\nabla u_n\|_q^q - \int_{\{|u_n| \ge T_K\}} \left[\frac{g(x, u_n)u_n - K}{q} - \frac{1}{r} g(x, u_n)u_n\right] dx - C_K \\
&\ge \left(\frac{1}{q} - \frac{1}{r}\right) \|\nabla u_n\|_q^q - \left(\frac{1}{q} - \frac{1}{r}\right) \int_{\{|u_n| \ge T_K\}} g(x, u_n)u_n dx - C_K \\
&\ge \left(\frac{1}{q} - \frac{1}{r}\right) \left[\|\nabla u_n\|_q^q - \int_{\{|u_n| \ge T_K\}} [(\ell_\infty + 1)|u_n|^q + A_1|u_n|]dx\right] - C_K \\
&\ge \left(\frac{1}{q} - \frac{1}{r}\right) \left[\|\nabla u_n\|_q^q - \int_{\{|u_n| \ge T_K\}} (\ell_\infty + 1)|u_n|^q dx - A_1|\Omega|^{\frac{1}{q'}} \|u_n\|_q\right] - C_K \\
&\ge \left(\frac{1}{q} - \frac{1}{r}\right) \left[\|\nabla u_n\|_q^q - \int_{\{|u_n| \ge T_K\}} (\ell_\infty + 1)|u_n|^q dx - C\|\nabla u_n\|_q\right] - C_K,
\end{aligned}$$

where we have used (3.24), (3.1), and Hölder's and Poincaré's inequalities. Now, fix $s \in (q, q^*)$. By Hölder's inequality,

$$\int_{\{|u_n| \ge T_K\}} |u_n|^q dx \le |\{|u_n| \ge T_K\}|^{\frac{s-q}{s}} \|u_n\|_s^q.$$

In view of the embedding $W_0^{1,q}(\Omega) \hookrightarrow L^s(\Omega)$, let $C_S > 0$ be the best constant such that the following inequality holds for every $u \in W_0^{1,q}(\Omega)$:

$$\|u\|_s^q \le C_S \|\nabla u\|_q^q.$$

Thus, continuing the previous estimate, we get

$$c + o(1) \ge \left(\frac{1}{q} - \frac{1}{r}\right) \left(1 - (\ell_\infty + 1)C_S|\{|u_n| \ge T_K\}|^{\frac{s-q}{s}}\right) \|\nabla u_n\|_q^q - C\|\nabla u_n\|_q - C_K. \tag{3.26}$$

Now, it suffices to choose K in such a way that

$$1 - (\ell_\infty + 1)C_S|\{|u_n| \ge T_K\}|^{\frac{s-q}{s}} \ge \frac{1}{2} + o(1).$$

This is possible thanks to (3.25), taking $K = (qc + C_0)[2C_S(\ell_\infty + 1)]^{\frac{s}{s-q}}$. In conclusion, (3.26) gives

$$C\|\nabla u_n\|_q + c + C_K + o(1) \ge C'\|\nabla u_n\|_q^q$$

for a suitable positive constant C'. This proves the boundedness of $(\| u_n \|)$ and concludes the proof. □

3.2 | Geometry conditions and proof of Theorem 1.1

3.2.1 | The case $-\infty \le \ell_0 \le 0$.

In this part, we consider the sequences of quasi-eigenvalues $(\eta_h^{(\alpha)})$ and $(v_h^{(\alpha)})$ introduced in Section 2 related to the classical q-Laplacian case, namely, with $\alpha = 0$. Since $\alpha = 0$ is fixed, we will drop the superscript (0) throughout the present Section 3.2.1.

Lemma 3.3. *Assume that (f), (f_∞), and (f_0) hold. If $\ell_0 \le 0$, and*

$$\ell_\infty + \ell_0 < \eta_h \text{ for some } h \in \mathbb{N}, \tag{3.27}$$

there exist $\rho > 0$ sufficiently small and a constant $c_0 > 0$ such that

$$I(u) \ge c_0 \text{ for every } u \in S_\rho \cap W,$$

where $S_\rho = \{u \in W_0^{1,q}(\Omega) : \|\nabla u\|_q = \rho\}$ and $W \subset W_0^{1,q}(\Omega)$ is a closed subspace such that $\operatorname{codim} W = h - 1$.

Proof. • We first consider the case $\ell_0 > -\infty$. By (f), (f_∞), and (f_0), for every $\varepsilon > 0$ and every $s \in [0, q^* - q)$, there exists $a_\varepsilon > 0$ such that

$$F(x,t) \le \frac{\varepsilon + \ell_0}{q}|t|^q + a_\varepsilon |t|^{s+q} \text{ for a.e. } x \in \Omega \text{ and every } t \in \mathbb{R}. \tag{3.28}$$

Therefore, for every $u \in W_0^{1,q}(\Omega)$, by the Sobolev embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{s+q}(\Omega)$,

$$\begin{aligned} I(u) &\ge \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\ell_\infty + \varepsilon + \ell_0}{q}\|u\|_q^q - a_\varepsilon \|u\|_{s+q}^{s+q} \\ &\ge \frac{1}{q}\|\nabla u\|_q^q - \frac{\ell_\infty + \varepsilon + \ell_0}{q}\|u\|_q^q - C\|\nabla u\|_q^{s+q}. \end{aligned}$$

Moreover, we consider the closed subspace W_{h-1} defined in (2.8). By (2.10), the following inequality holds

$$\eta_h \|u\|_q^q \le \|\nabla u\|_q^q \text{ for every } u \in W_{h-1}.$$

Hence, fixed $\varepsilon \in (\max\{0, -\ell_\infty - \ell_0\}, \eta_h - \ell_\infty - \ell_0)$, by (3.27), there exists $\rho > 0$ so small that for every $u \in S_\rho \cap W_{h-1}$,

$$I(u) \ge \frac{1}{q}\left(1 - \frac{\ell_\infty + \varepsilon + \ell_0}{\eta_h}\right)\|\nabla u\|_q^q - C\|\nabla u\|_q^{s+q} \ge c_0,$$

for some $c_0 > 0$ and the conclusion follows with $W = W_{h-1}$ being, by Lemma 2.6, $\operatorname{codim} W_{h-1} = h - 1$.

• If $\ell_0 = -\infty$, clearly condition (3.27) is trivially verified also when $h = 1$. Again by the assumptions on f, for every $M > 0$, there exists $a_M > 0$ such that

$$F(x,t) \le -\frac{M}{q}|t|^q + a_M |t|^{s+q} \text{ for a.e. } x \in \Omega \text{ and every } t \in \mathbb{R}.$$

Hence, for every $u \in W_0^{1,q}(\Omega)$, taking $M > \max\{0, \ell_\infty\}$,

$$\begin{aligned} I(u) &\ge \frac{1}{q}\|\nabla u\|_q^q - \frac{\ell_\infty - M}{q}\|u\|_q^q - C\|\nabla u\|_q^{s+q} \\ &\ge \frac{1}{q}\|\nabla u\|_q^q - C\|\nabla u\|_q^{s+q}, \end{aligned}$$

and the conclusion follows as in the previous case in $S_\rho \cap W_0 = S_\rho \cap W_0^{1,q}(\Omega) = S_\rho$ with $\rho > 0$ small enough. □

Remark 3.4. A careful inspection in the proof of the previous lemma shows that the result holds true also under the assumption

$$-\infty < \ell_\infty + \ell_0 < \eta_h^{(1)}. \tag{3.29}$$

Indeed, in this case, by (2.9), for every $u \in \{u \in W_{h-1}^{(1)} : \|u\|_q \le 1\}$,

$$\begin{aligned} I(u) &\ge \frac{1}{q}(\|\nabla u\|_p^p + \|\nabla u\|_q^q) - \frac{\ell_\infty + \varepsilon + \ell_0}{q}\|u\|_q^q - C\|\nabla u\|_q^{s+q} \\ &\ge \frac{1}{q}\left(1 - \frac{\ell_\infty + \varepsilon + \ell_0}{\eta_h^{(1)}}\right)(\|\nabla u\|_p^p + \|\nabla u\|_q^q) - C\|\nabla u\|_q^{s+q} \\ &\ge \frac{1}{q}\left(1 - \frac{\ell_\infty + \varepsilon + \ell_0}{\eta_h^{(1)}}\right)\|\nabla u\|_q^q - C\|\nabla u\|_q^{s+q}. \end{aligned}$$

Now, if $\|\nabla u\|_q = \rho$ for $0 < \rho < \lambda_1^{1/q}$, by Poincaré's inequality $\|u\|_q \le 1$, and so $S_\rho \cap W_{h-1}^{(1)} \cap \{\|u\|_q \le 1\} = S_\rho \cap W_{h-1}^{(1)}$. Hence, for ρ small enough, $I(u) \ge c_0 > 0$ in $S_\rho \cap W_{h-1}^{(1)}$. We observe that, by Remark 2.4, at least for $h = 1$ and $q \le p^*$, $\eta_1^{(1)} > \eta_1^{(0)}$, and so condition (3.29) is weaker than $\ell_\infty + \ell_0 < \eta_h^{(0)}$.

Lemma 3.5. *Assume that (f), (f_∞), and (f_0) hold. If, for some $k \in \mathbb{N}$, $\ell_\infty > \nu_k$, there exist a k-dimensional closed subspace $V \subset W_0^{1,q}(\Omega)$, and a constant $c_\infty \in (c_0, \infty)$ such that*

$$I(u) \le c_\infty \text{ for every } u \in V.$$

Proof. By (3.1), for every $\varepsilon > 0$ and $u \in W_0^{1,q}(\Omega)$, we have

$$\begin{aligned} I(u) &\le \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\ell_\infty - \varepsilon/2}{q}\|u\|_q^q + A_{\varepsilon/2}\|u\|_1 \\ &\le C\|\nabla u\|_q^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\ell_\infty - \varepsilon/2}{q}\|u\|_q^q + C'\|u\|_q \end{aligned}$$

where in the last estimate, we have used Hölder's inequality, being $1 < p < q$. Now, by (2.11), there exists a k-dimensional subspace $V_k^\varepsilon \subset W_0^{1,q}(\Omega)$ such that for every $u \in V_k^\varepsilon$,

$$\|\nabla u\|_q < \left(\nu_k + \frac{\varepsilon}{2}\right)^{1/q}\|u\|_q.$$

Therefore, for every $u \in V_k^\varepsilon$,

$$I(u) \le \frac{\nu_k - \ell_\infty + \varepsilon}{q}\|u\|_q^q + C\left(\nu_k + \frac{\varepsilon}{2}\right)^{p/q}\|u\|_q^p + C'\|u\|_q.$$

Hence, being by assumption $\ell_\infty > \nu_k$, for ε sufficiently small $I|_{V_k^\varepsilon}(u) \to -\infty$ as $\|u\|_q \to \infty$. Being V_k^ε a finite dimensional linear space, of dimension k, the proof is concluded by taking $V = V_k^\varepsilon$ and enlarging if necessary c_∞ to make it greater than c_0. □

We are now ready to prove the first part of Theorem 1.1.

Proof of Theorem. In this case, we assume that condition (H_-) holds. In view of Lemmas 3.2,3.3, and 3.5, we can apply Theorem 2.2 with $X = W_0^{1,q}(\Omega)$, $W = W_{h-1}$, and $V = V_k^\varepsilon$. □

3.2.2 | The case $0 \le \ell_0 \le \infty$

In this part, we consider the sequences of quasi-eigenvalues $(\eta_h^{(0)})$ and $(\nu_h^{(1)})$ introduced in Section 2.

Lemma 3.6. *Assume that (f), (f_∞), and (f_0) hold. If $\ell_0 \ge 0$, and*

$$\ell_\infty + \ell_0 > \frac{q}{p}\nu_k^{(1)} \text{ for some } k \in \mathbb{N}, \tag{3.30}$$

there exist a k-dimensional closed subspace $V \subset W_0^{1,q}(\Omega)$, and two positive constants $\rho > 0$ and $c_0 > 0$ such that

$$-I(u) \ge c_0 \text{ for every } u \in S_\rho \cap V.$$

In particular, if $\ell_0 = \infty$, the conclusion holds for every $k \in \mathbb{N}$.

Proof. • We first consider the case $\ell_0 < \infty$. By (f), (f_∞), and (f_0), for every $\varepsilon > 0$ and every $s \in (0, \infty)$, there exists $a'\varepsilon > 0$ such that

$$F(x,t) \ge \frac{\ell_0 - \varepsilon}{q}|t|^q - a'_\varepsilon |t|^{s+q} \text{ for a.e. } x \in \Omega \text{ and every } t \in \mathbb{R}. \tag{3.31}$$

Therefore, for every $u \in W_0^{1,q}(\Omega)$, taking $s + q < q^*$ and using the embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{s+q}(\Omega)$, we obtain

$$\begin{aligned} -I(u) &\ge -\frac{1}{p}\left(\|\nabla u\|_p^p + \|\nabla u\|_q^q\right) + \frac{\ell_\infty + \ell_0 - \varepsilon/2}{q}\|u\|_q^q - a'_{\varepsilon/2}\|u\|_{s+q}^{s+q} \\ &\ge -\frac{1}{p}\left(\|\nabla u\|_p^p + \|\nabla u\|_q^q\right) + \frac{\ell_\infty + \ell_0 - \varepsilon/2}{q}\|u\|_q^q - C\|\nabla u\|_q^{s+q}. \end{aligned}$$

Now, from (3.30), we can take $\varepsilon > 0$ such that $\varepsilon < \min\left\{\frac{2q}{p}, \ell_\infty + \ell_0 - \frac{q}{p}\nu_k^{(1)}\right\}$. Then, by (2.11), there exists a k-dimensional closed subspace $V_k^{\varepsilon,(1)} \subset W_0^{1,q}(\Omega)$ such that for every $u \in V_k^{\varepsilon,(1)}$,

$$\|\nabla u\|_q^q \le \|\nabla u\|_p^p + \|\nabla u\|_q^q < \left(\nu_k^{(1)} + \frac{p}{2q}\varepsilon\right)\|u\|_q^q < \left(\nu_k^{(1)} + 1\right)\|u\|_q^q. \tag{3.32}$$

Therefore, for every $u \in V_k^{\varepsilon,(1)}$,

$$\begin{aligned} -I(u) &\ge \frac{1}{q}\left(\ell_\infty + \ell_0 - \frac{q}{p}\nu_k^{(1)} - \varepsilon\right)\|u\|_q^q - C\|\nabla u\|_q^{s+q} \\ &\ge \frac{1}{q}\left(\ell_\infty + \ell_0 - \frac{q}{p}\nu_k^{(1)} - \varepsilon\right)\frac{1}{\nu_k^{(1)} + 1}\|\nabla u\|_q^q - C\|\nabla u\|_q^{s+q}, \end{aligned}$$

where in the last inequality, we have used (3.30) and again (3.32). Hence, there exists $\rho > 0$ so small that for every $u \in S_\rho \cap V_k^{\varepsilon,(1)}$,

$$-I(u) \ge c_0 \text{ for some } c_0 > 0,$$

and the conclusion follows taking $V = V_k^{\varepsilon,(1)}$.

• If $\ell_0 = \infty$, clearly condition $\ell_\infty + \ell_0 > \frac{q}{p}\nu_k^{(1)}$ is satisfied for every $k \in \mathbb{N}$. Again, by the assumptions on f, for every $M > 0$ and for every $s \in (0, \infty)$, there exists $a'_M > 0$ such that

$$F(x,t) \ge \frac{M}{q}|t|^q - a'_M |t|^{s+q} \text{ for a.e. } x \in \Omega \text{ and every } t \in \mathbb{R}.$$

Hence, for every $u \in W_0^{1,q}(\Omega)$, taking $s+q<q^*$, we get

$$-I(u) \geq -\frac{1}{p}\left(\|\nabla u\|_p^p + \|\nabla u\|_q^q\right) + \frac{\ell_\infty + M}{q}\|u\|_q^q - C\|\nabla u\|_q^{s+q}.$$

Now, fixing any $\varepsilon \in (0,1)$, by (2.11), for every $k \in \mathbb{N}$, there exists a k-dimensional closed subspace $V_k^{\varepsilon,(1)} \subset W_0^{1,q}(\Omega)$ such that

$$\|\nabla u\|_q^q \leq \|\nabla u\|_p^p + \|\nabla u\|_q^q < \left(\nu_k^{(1)} + \varepsilon\right)\|u\|_q^q < \left(\nu_k^{(1)} + 1\right)\|u\|_q^q \text{ for every } u \in V_k^{\varepsilon,(1)}. \tag{3.33}$$

Therefore, for every $k \in \mathbb{N}$ and $u \in V_k^{\varepsilon,(1)}$,

$$\begin{aligned} -I(u) &\geq \frac{1}{q}\left(\ell_\infty + M - \frac{q}{p}\nu_k^{(1)} - \varepsilon\right)\|u\|_q^q - C\|\nabla u\|_q^{s+q} \\ &\geq \frac{1}{q}\left(\ell_\infty + M - \frac{q}{p}\nu_k^{(1)} - \varepsilon\right)\frac{1}{\nu_k^{(1)}+1}\|\nabla u\|_q^q - C\|\nabla u\|_q^{s+q}. \end{aligned}$$

Then, for every k, we can choose $M > \frac{q}{p}\nu_k^{(1)} + 1 - \ell_\infty$, thus making positive the coefficient of $\|\nabla u\|_q^q$. Therefore, arguing as in the previous part, for some $\rho > 0$ sufficiently small, $-I(u) \geq c_0 > 0$ in $S_\rho \cap V_k^{\varepsilon,(1)}$. Since this argument can be repeated for every $k \in \mathbb{N}$, the proof is concluded. □

Lemma 3.7. *Assume that (f), (f_∞), and (f_0) hold. If, for some $h \in \mathbb{N}$, $\ell_\infty < \eta_h^{(0)}$, there exist a closed subspace $W \subset W_0^{1,q}(\Omega)$ of codimension $h-1$, and a constant $c_\infty \in (c_0, \infty)$ such that*

$$-I(u) \leq c_\infty \text{ for every } u \in W.$$

Proof. By (3.1), for every $\varepsilon > 0$ and $u \in W_0^{1,q}(\Omega)$, we have

$$\begin{aligned} -I(u) &\leq -\frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\|\nabla u\|_q^q + \frac{\ell_\infty + \varepsilon}{q}\|u\|_q^q + A_\varepsilon\|u\|_1 \\ &\leq -\frac{1}{q}\|\nabla u\|_q^q + \frac{\ell_\infty + \varepsilon}{q}\|u\|_q^q + C\|u\|_q. \end{aligned}$$

Now, by (2.10), for every $u \in W_{h-1}^{(0)}$,

$$-I(u) \leq -\frac{1}{q}\left(\eta_h^{(0)} - \ell_\infty - \varepsilon\right)\|u\|_q^q + C\|u\|_q.$$

Hence, being by assumption $\ell_\infty < \eta_h^{(0)}$, for $0 < \varepsilon < \eta_h^{(0)} - \ell_\infty$, we have that $-I|_{W_{h-1}^{(0)}}(u) \to -\infty$ as $\|u\|_q \to \infty$. Being $W_{h-1}^{(0)}$ a closed subspace of codimension $h-1$, the proof is concluded by taking $W := W_{h-1}^{(0)}$ and enlarging if necessary c_∞ to make it greater than c_0.

We are now ready to prove the second part of Theorem 1.1. □

Proof of Theorem. In this case, we assume that condition (H_+) holds. In view of Lemmas 3.2,3.6, and 3.7, we can apply Theorem 2.2 to the functional $-I$, with $X = W_0^{1,q}(\Omega)$, $W = W_{h-1}^{(0)}$, and $V = V_k^{\varepsilon,(1)}$. □

Remark 3.8. It has been proved in Bartolo et al.[14] that the sequences $(\eta_h^{(0)})$ and $(\nu_h^{(0)})$ are increasing and divergent and that for every $h \in \mathbb{N}$, $\eta_h^{(0)} \leq \nu_h^{(0)}$. Therefore, the two conditions required in (H_-) when $\ell_0 < 0$ can be written as the following chain of inequalities:

$$\ell_\infty + \ell_0 < \eta_h^{(0)} \leq \eta_k^{(0)} \leq \nu_k^{(0)} < \ell_\infty,$$

from which it becomes apparent that for $\ell_0 = 0$, they are never compatible. On the other hand, if we take into account Remark 3.4, under the weaker condition (3.29), in general, we cannot exclude that the intersection of the two conditions in (H_-), when $\ell_0 = 0$, is empty. Similarly, it is not known whether or not there is some order relation between $\eta_h^{(0)}$ and $\nu_h^{(1)}$. Thus, we cannot exclude that the intersection of the two conditions in (H_+) when $\ell_0 = 0$ (i.e., $\ell_\infty < \eta_h^{(0)}$ and $\ell_\infty > \frac{q}{p}\nu_k^{(1)}$) is not empty.

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CONFLICT OF INTEREST

There are no conflicts of interest to this work.

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