

RECEIVED: July 22, 2021

REVISED: November 11, 2021

ACCEPTED: November 25, 2021

PUBLISHED: December 6, 2021

# Worldline description of a bi-adjoint scalar and the zeroth copy

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**ABSTRACT:** Bi-adjoint scalars are helpful in studying properties of color/kinematics duality and the double copy, which relates scattering amplitudes of gauge and gravity theories. Here we study bi-adjoint scalars from a worldline perspective. We show how a global  $G \times \tilde{G}$  symmetry group may be realized by worldline degrees of freedom. The worldline action gives rise to vertex operators, which are compared to similar ones describing the coupling to gauge fields and gravity, thus exposing the color/kinematics interplay in this framework. The action is quantized by path integrals to find a worldline representation of the one-loop QFT effective action of the bi-adjoint scalar cubic theory. As simple applications, we recover the one-loop beta function of the theory in six dimensions, verifying its vanishing, and compute the self-energy correction to the propagator. The model is easily extendable to that of a particle carrying an arbitrary representation of direct products of global symmetry groups, including the multi-adjoint particle, whose one-loop beta function we reproduce as well.

**KEYWORDS:** Scattering Amplitudes, Duality in Gauge Field Theories

**ARXIV EPRINT:** [2107.10130](https://arxiv.org/abs/2107.10130)

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## 1 Introduction

The duality between color and kinematics [1, 2] reveals a deep structure relating amplitudes of gauge and gravity theories, see [3] for a review. An interesting model, used for studying color/kinematic relations, is that of a bi-adjoint scalar [4, 5]. Its basic structure is that of a  $\phi^3$  theory, a standard QFT toy model which is super-renormalizable in 4 dimensions and renormalizable in 6 dimensions (where it becomes asymptotically free), but with the scalar field extended to be symmetric under the action of a compact group of the form  $G \times \tilde{G}$ . In particular, the scalar is taken to transform in the adjoint representation for each factor, so that its Feynman rules carry corresponding symmetry factors (named color factors). The latter may be substituted by kinematical factors having the same symmetries, thus producing Feynman rules and amplitudes of non-abelian gauge fields and gravity.

In a purely classical setting, the bi-adjoint scalar has also served as the basis to find maps between classical solutions in gauge and gravity at both non-perturbative [6–9] and perturbative [10–13] levels. The perturbative solutions of the bi-adjoint scalar theory coupled with point sources (parametrized by their worldline coordinates) are obtained by iteratively solving the equations of motion. The kinematic structure of these solutions is crucial for a systematic amplitude-like implementation of the double copy [13]. At classical perturbative level, the form of the action leading to equations of motion is unimportant as long as it reproduces the equations of motion, but its precise form is central to undertake quantization.

Seeds of color/kinematic relations, and corresponding double copy structure that reproduces gravity amplitudes from the gauge ones, can be traced back to the origin of string theory, when it was noticed that the Veneziano amplitude (an open-string scattering amplitude) could be related in a simple way to the Virasoro-Shapiro amplitude (a closed-string amplitude). Open strings give rise to gauge fields while closed strings give rise to gravitons, so one notices a signal of the gauge/gravity relations [14]. The string results were obtained and interpreted in a first quantized picture. Similarly, in this paper we wish to investigate the color/kinematic relations using a first quantized approach to particle theory [15].

Thus, after some preliminary considerations, we construct a worldline action for the bi-adjoint particle, paying attention to the realization of its color degrees of freedom. The traditional approach is to use a matrix-valued particle action, whose matrix indices act directly on the indices of the wave-function, defined ab-initio to have multiple components which correspond to the color degrees of freedom of the particle.<sup>1</sup> This approach requires a path ordering prescription for the path integral quantization, as in [16] where worldline methods were employed to compute trace anomalies.

An alternative approach is to introduce color variables, using the set-up described in [17–19], where color variables are coupled to a worldline U(1) gauge field which carries an additional Chern-Simons term. The gauge field implements a projection to the desired representation of the symmetry group assigned to the particle. The choice of the representation is obtained by tuning the Chern-Simons coupling to a suitable discrete value. A form of this projection was described also in [20], and later recognized to arise from the coupling to a U(1) gauge field. The projection mechanism was seen at work in the worldline path integral treatments of differential forms [21–23], and extended shortly afterwards to color degrees of freedom [17]. Further extensions have been discussed in [24, 25]. On the other hand, worldline color variables had been introduced originally much earlier [26, 27].

While the two methods are equivalent, the color variable approach may turn out to be more useful for studying the classical limit of the color charge, as it leaves open the option of projecting to different sectors of the Hilbert space, and in particular to those containing a large number of copies of the original elementary charge. We will not address this issue here, though.

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<sup>1</sup>Strictly speaking, as long as the color is not gauged, one should name them flavor degrees of freedom, but we keep with the current terminology.

After having identified the correct worldline action for the bi-adjoint particle, we use it to represent the one-loop effective action of the bi-adjoint scalar field. Then, we extract the one-loop beta function of the theory in 6 dimensions, which was recently shown to vanish [28]. We also study the one-loop self-energy correction to the bi-adjoint propagator. Eventually, we indicate how our model can be extended to that of a particle carrying an arbitrary representation of direct products of global symmetry groups, and compute the one-loop beta function of a multi-adjoint scalar particle in six dimensions.

The rest of the paper is organized as follows: in section 2, after reviewing our motivations, we derive two alternative forms of the action that describes a bi-adjoint scalar particle. In section 3 we discuss the corresponding one-loop effective action using these two alternative but equivalent approaches, and give examples. Section 4 discusses extensions of the bi-adjoint model to more general situations. Our Conclusions are given in section 5.

## 2 Construction of worldline actions

### 2.1 Motivation: from gauge to scalar and gravity

A recurrent topic in the worldline approach to quantum field theory is the derivation of parameter integrals which sum up whole classes of Feynman diagrams. At 1-loop one is sometimes interested in obtaining a 1PI effective action induced by a scalar loop in scalar/gauge/graviton field backgrounds,  $\Psi^{(s)}(x)$ , where  $s = 0, 1, 2$ . Denoting by  $\chi$  a set of auxiliary variables for the particle, the effective action has the schematic form

$$\Gamma[\Psi^{(s)}] = \int_{S^1} \frac{Dx D\chi}{\text{Vol}(\text{Gauge})} e^{-S[x, \chi; \Psi^{(s)}(x)]}, \quad (2.1)$$

where  $S[x, \chi; \Psi^{(s)}(x)]$  is the Euclidean action for a scalar particle coupled to a background field  $\Psi^{(s)}$ , and  $S^1$  is the circle (the loop). The auxiliary variables  $\chi$  may include the einbein  $e$ , color variables, or ghosts. After setting appropriate boundary conditions and performing a gauge-fixing, the path integral can be solved perturbatively by considering Wick contractions of vertex operators as follows

$$\langle V^{(s)}[k_1, \varepsilon_1; x_1, \chi_1] \cdots V^{(s)}[k_n, \varepsilon_n; x_n, \chi_n] \rangle, \quad (2.2)$$

where the total number  $n$  corresponds to the total number of plane waves of the external background fields ( $k_i$  and  $\varepsilon_i$  denote their momenta and polarizations,  $x_i \equiv x(\tau_i)$ , etc.).

For instance, the gluon vertex operator corresponding to  $s = 1$  is given by [17]

$$V^{(1)}[k, \varepsilon, a; x, c, \bar{c}] = \int_0^1 d\tau \varepsilon \cdot \dot{x}(\tau) \bar{c}_{a'}(\tau) (T^a)^{a' b'} c^{b'}(\tau) e^{ik \cdot x(\tau)}, \quad (2.3)$$

where  $c^{a'}(\tau)$  and  $\bar{c}_{a'}(\tau)$  are auxiliary worldline color variables transforming in the fundamental and antifundamental representations of the gauge group  $G$ , and  $(T^a)^{a' b'}$  are the generators in the fundamental representation. Other representations could be considered as well, of course. In the spirit of the double copy, we may define the kinematic factor

$K(\tau) = \varepsilon \cdot \dot{x}(\tau)$  and the color factor  $C^a(\tau) = \bar{c}_{a'}(\tau)(T^a)^{a' b'} c^{b'}(\tau)$ , and consider in (2.3) the replacement

$$C^a(\tau) \rightarrow \tilde{K}(\tau), \tag{2.4}$$

where the tilde indicates dependence on another polarization vector  $\tilde{\varepsilon}$ . We obtain the following vertex operator

$$V^{(2)}[k, \varepsilon; x] = \int_0^1 d\tau \varepsilon \cdot \dot{x}(\tau) \tilde{\varepsilon} \cdot \dot{x}(\tau) e^{ik \cdot x(\tau)}, \tag{2.5}$$

which should correspond to a vertex operator for the emission/absorption of a graviton [29]. This is indeed correct, after identifying the graviton polarization by  $\varepsilon_\mu \tilde{\varepsilon}_\nu \rightarrow \varepsilon_{\mu\nu}$ , and eventually taking into account prescriptions for regulating UV ambiguities on the worldline (this is typical for the coupling of a particle to gravity [30, 31]).

Now, inspired by the zeroth copy, let us consider instead the replacement

$$K(\tau) \rightarrow \tilde{C}^\alpha(\tau), \tag{2.6}$$

with a color factor associated to a different symmetry group  $\tilde{G}$ , carrying its own color variables  $d^{\alpha'}(\tau)$  and  $\bar{d}_{\alpha'}(\tau)$  taken in the fundamental and antifundamental representation, respectively. This replacement produces the vertex operator

$$V^{(0)}[k, a, \alpha; x, c, \bar{c}, d, \bar{d}] = \int_0^1 d\tau \bar{c}_{a'}(\tau)(T^a)^{a' b'} c^{b'}(\tau) \bar{d}_{\alpha'}(\tau)(T^\alpha)^{\alpha' \beta'} d^{\beta'}(\tau) e^{ik \cdot x(\tau)}, \tag{2.7}$$

where  $a$  and  $\alpha$  are indices of the adjoint representation of  $G$  and  $\tilde{G}$ , respectively. In the case of scattering amplitudes it is well-known that the replacement of kinematic numerators by color ones leads to the bi-adjoint scalar theory [32, 33]. Here, we see that this replacement produces a vertex operator for the coupling of the particle to the plane wave of a bi-adjoint scalar, reminding the classical double copy of [6]. This vertex operator can be used to obtain an effective action of the form (2.1) for a particle coupled to a background bi-adjoint scalar field  $\Psi^{(0)} \sim \Phi^{a\alpha}$ .

## 2.2 Worldline matrix-valued action

The previous considerations motivate us to derive in more details the worldline action for a particle coupling to the bi-adjoint scalar  $\Phi^{a\alpha}$ . We consider directly the special case of a particle interpreted as the quantum of the bi-adjoint scalar field itself. For that purpose, we start from the QFT of a bi-adjoint scalar field  $\Phi^{a\alpha}$ , charged under a  $G \times \tilde{G}$  global symmetry group and transforming in the adjoint for each factor.<sup>2</sup> The Euclidean Lagrangian of the model is [7, 9]

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi^{a\alpha} \partial^\mu \Phi^{a\alpha} + \frac{m^2}{2} \Phi^{a\alpha} \Phi^{a\alpha} + \frac{y}{3!} f^{abc} \tilde{f}^{\alpha\beta\gamma} \Phi^{a\alpha} \Phi^{b\beta} \Phi^{c\gamma}, \tag{2.8}$$

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<sup>2</sup>The Lie algebra for each factor has the form  $[T^a, T^b] = if^{ab} T^c$  and the adjoint representation is given by the structure constants  $(T_\Lambda^a)^b_c = -if^{ab}_c = -if^{abc}$ . The Killing metric is normalized to  $\delta^{ab}$ , so that upper and lower adjoint indices are equivalent. We use greek indices to label generators of the second group  $\tilde{G}$ .

where  $m$  is the mass and  $y$  the coupling constant. This model is the massive version of a massless one, whose amplitudes in the Cachazo-He-Yuan (CHY) representation were considered in [4, 5]. The massive generalization of these amplitudes was considered e.g., in [34–36]. In a quantum/background split ( $\Phi^{a\alpha} \rightarrow \phi^{a\alpha} + \Phi^{a\alpha}$ ) one finds, after a partial integration, a quadratic action for the quantum field  $\phi^{a\alpha}$

$$\mathcal{L}_q^{(2)} = \frac{1}{2} \phi^{a\alpha} \left( \delta^{ab} \delta^{\alpha\beta} (-\partial^2 + m^2) + y f^{abc} \tilde{f}^{\alpha\beta\gamma} \Phi^{c\gamma} \right) \phi^{b\beta}, \quad (2.9)$$

which is coupled to the background  $\Phi^{a\alpha}$ . It contains a quadratic differential operator of the form (setting  $p^2 = -\partial^2$ )

$$2H = \delta^{ab} \delta^{\alpha\beta} (p^2 + m^2) + y f^{abc} \tilde{f}^{\alpha\beta\gamma} \Phi^{c\gamma}, \quad (2.10)$$

which is interpreted as a particle Hamiltonian  $H$  — the factor 2 being conventional — which acts on wave functions  $\phi^{a\alpha}(x)$  that carry indices of the bi-adjoint. The inverse of this Hamiltonian gives the bi-adjoint propagator in the  $\Phi^{a\alpha}$  background, and the functional trace of the corresponding heat kernel is related to the one-loop effective action [37]. As it stands, this Hamiltonian has a matrix-valued potential, and it produces a matrix-valued action. In a path integral, a path ordering must be used to properly define its exponential.

To present the particle Hamiltonian in a more compact way, let us introduce a matrix notation to cast (2.9) in the form

$$\mathcal{L}_q^{(2)} = \frac{1}{2} \phi^T \left( -\partial^2 + m^2 + y \hat{\Phi} \right) \phi, \quad (2.11)$$

where  $\phi = \phi^A = \phi^{a\alpha}$  has to be considered as a column vector with components labeled by the multi-index  $A = (a\alpha)$ , while  $\hat{\Phi}$  is a matrix-valued background potential

$$\hat{\Phi} = \hat{\Phi}^{A,B} = \hat{\Phi}^{a\alpha,b\beta} = f^{abc} \tilde{f}^{\alpha\beta\gamma} \Phi^{c\gamma}. \quad (2.12)$$

The matrix  $\hat{\Phi}$  is symmetric in the indices  $A$  and  $B$ . The corresponding matrix-valued Hamiltonian, eq. (2.10), is then cast as

$$2H = p^2 + m^2 + y \hat{\Phi}(x). \quad (2.13)$$

It acts on wave functions of the form  $\phi(x) = \phi^A(x)$ , which have components labeled by the multi-index  $A$ . The Hamiltonian leads classically to an action that is also matrix-valued. The latter, after some simple redefinitions, can be written in the form

$$S[x] = \int_0^1 d\tau \left[ \frac{1}{4T} \dot{x}^2 + Tm^2 + Ty \hat{\Phi}(x) \right], \quad (2.14)$$

where  $T$  is the Fock-Schwinger proper-time. The corresponding path integral needs a time ordering prescription, that makes sure that it solves the correct Schrödinger equation (just as for time dependent Hamiltonians). Then, following Schwinger [37], we find that the worldline path integral for the effective action (3.23) takes the form

$$\Gamma[\Phi] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \text{tr} \int_{\mathcal{P}} Dx \mathbf{T} e^{-S[x]}, \quad (2.15)$$

where  $\mathbf{T}$  denotes the time ordering, ‘tr’ the additional trace on the finite color degrees of freedom, and with the constant mass term explicitly taken out of the worldline action  $S$ . The symbol  $\mathcal{P}$  indicates that the coordinates  $x$  must have periodic boundary conditions in order to produce the trace in the corresponding Hilbert space.

### 2.3 Worldline action with auxiliary color variables

A way to get rid of path ordering is to use color variables. The quantization of the latter gives rise to creation/annihilation operators that create color degrees of freedom in the Hilbert space of the particle. At the same time their worldline propagator reconstructs the path ordering. Eventually, a projection on a fixed occupation number of the color degrees of freedom selects the precise representation of the symmetry group (the color charge) that one is willing to assign to the particle. This last step is achieved by coupling the color variables to a  $U(1)$  worldline gauge field with an additional discrete Chern-Simons coupling, that fixes the chosen color occupation number in the Hilbert space [17]. Thus, recalling that for a relativistic particle the Hamiltonian must be gauged,<sup>3</sup> one is led to the following worldline action in phase space (subscript ‘ps’) for the bi-adjoint scalar in real time  $\tau$

$$S_{\text{ps}} = \int d\tau \left[ p_\mu \dot{x}^\mu + i\bar{c}_a \dot{c}^a + i\bar{d}_\alpha \dot{d}^\alpha - eH - aJ - \tilde{a}\tilde{J} \right], \tag{2.16}$$

where  $H, J, \tilde{J}$  denote first class constraints

$$H = \frac{1}{2}(p^2 + m^2 - y C^a \Phi^{a\alpha}(x) \tilde{C}^\alpha), \quad C^a = -if^{abc} \bar{c}^b c^c, \quad \tilde{C}^\alpha = -i\tilde{f}^{\alpha\beta\gamma} \bar{d}^\beta d^\gamma, \tag{2.17}$$

$$J = \bar{c}_a c^a - s, \tag{2.18}$$

$$\tilde{J} = \bar{d}_\alpha d^\alpha - \tilde{s}, \tag{2.19}$$

gauged by the fields  $e, a, \tilde{a}$ , i.e. the einbein and the two independent worldline  $U(1)$  gauge fields, respectively, while the constants  $s$  and  $\tilde{s}$  are suitable Chern-Simons couplings. The color variables can be taken to be either commuting or anticommuting. Here we choose the first option, which as a bonus may allow to study a classical limit of the color charge, as described by Wong’s equations [38]. The kinetic term defines the phase space symplectic form, leading to the Poisson brackets. The nonvanishing Poisson brackets are given by  $\{x^\mu, p_\nu\} = \delta_\nu^\mu$ ,  $\{c^a, \bar{c}_b\} = -i\delta_b^a$ ,  $\{d^\alpha, \bar{d}_\beta\} = -i\delta_\beta^\alpha$ , and used to verify that the constraints are indeed first class

$$\{H, J\} = \{H, \tilde{J}\} = \{J, \tilde{J}\} = 0. \tag{2.20}$$

Thus, the constraints can be gauged consistently. Notice also the following Poisson brackets

$$\{C^a, C^b\} = f^{abc} C^c, \quad \{\tilde{C}^\alpha, \tilde{C}^\beta\} = \tilde{f}^{\alpha\beta\gamma} \tilde{C}^\gamma, \quad \{C^a, \tilde{C}^\alpha\} = 0, \quad \{J, C^a\} = 0, \quad \text{etc.} \tag{2.21}$$

The precise value of the Chern-Simons couplings  $s$  and  $\tilde{s}$  will be specified after transition to the quantum theory, which we discuss next using canonical quantization.

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<sup>3</sup>The corresponding gauge field (the einbein) will eventually lead to the Fock-Schwinger proper time of the previous section.

The quantum theory of the particle has an associated Hilbert space of “wave-functions” on which the fundamental quantum operators act. These operators are the particle position and momentum,  $\hat{x}^\mu, \hat{p}_\mu$ , and two pairs of color variables,  $\hat{c}^a, \hat{c}_a^\dagger$  and  $\hat{d}^\alpha, \hat{d}_\alpha^\dagger$ . The latter act as sets of creation and annihilation operators. Considering only the first set of color variables,  $\hat{c}^a, \hat{c}_a^\dagger$ , they satisfy

$$[\hat{c}^a, \hat{c}_b^\dagger] = \delta_b^a, \quad [\hat{c}^a, \hat{c}^b] = 0 = [\hat{c}_a^\dagger, \hat{c}_b^\dagger] \quad (2.22)$$

with indices running up to  $N_A = \dim G$ , the dimension of the adjoint representation. They are naturally represented by  $\hat{c}_a^\dagger \sim \bar{c}_a$  and  $\hat{c}^a \sim \partial/\partial\bar{c}_a$  when acting on wave-functions of the form  $\phi(x, \bar{c})$ . The Taylor expansion of the latter reads

$$\phi(x, \bar{c}) = \phi(x) + \phi^a(x)\bar{c}_a + \frac{1}{2!}\phi^{ab}(x)\bar{c}_a\bar{c}_b + \dots, \quad (2.23)$$

and exposes sectors with different occupation numbers for the color variables, as measured by the number operator  $\hat{N} = \hat{c}_a^\dagger\hat{c}^a \sim \bar{c}^a\partial/\partial\bar{c}^a$ . If one is interested only on wave-functions transforming in the adjoint, and not on tensor products of the adjoint, one must impose a constraint that fixes the occupation number to 1, i.e.,

$$(\hat{N} - 1)\phi(x, \bar{c}) = \left(\bar{c}^a\frac{\partial}{\partial\bar{c}^a} - 1\right)\phi(x, \bar{c}) = 0. \quad (2.24)$$

This is precisely the restriction that the quantum version of the first class constraint (2.18) should impose for meeting our purposes. The quantization of this constraint must resolve ordering ambiguities, which we fix as in the harmonic oscillator by requiring a symmetric ordering

$$J = \bar{c}_a c^a - s = \frac{1}{2}(\bar{c}_a c^a + c^a \bar{c}_a) - s, \quad (2.25)$$

leading to

$$\hat{J} = \frac{1}{2}(\hat{c}_a^\dagger\hat{c}^a + \hat{c}^a\hat{c}_a^\dagger) - s = \hat{c}_a^\dagger\hat{c}^a + \frac{N_A}{2} - s = \bar{c}^a\frac{\partial}{\partial\bar{c}^a} + \frac{N_A}{2} - s. \quad (2.26)$$

It is now simple to fix the Chern-Simons coupling  $s$  to achieve occupation number 1

$$\frac{N_A}{2} - s = -1 \quad \rightarrow \quad s = 1 + \frac{N_A}{2}. \quad (2.27)$$

A similar coupling with  $\tilde{s} = 1 + \frac{\tilde{N}_A}{2}$  must be used for the other color sector as well, so to have physical wave-functions transforming in the bi-adjoint, as desired. Thus, the particle wave-function  $\phi(x, \bar{c}, \bar{d})$  that satisfies the quantum constraints corresponding to  $J$  and  $\tilde{J}$  has the form

$$\phi(x, \bar{c}, \bar{d}) = \phi^{a\alpha}(x)\bar{c}_a\bar{d}_\alpha \quad (2.28)$$

and describes a wave-function carrying indices of the bi-adjoint. The remaining quantum constraint is the one corresponding to the Hamiltonian  $H$  in (2.17), whose operatorial version takes the form

$$\hat{H} = \frac{1}{2}\left(-\partial^2 + m^2 + y\Phi^{a\alpha}(x)f^{abc}\bar{c}^b\frac{\partial}{\partial\bar{c}^c}\tilde{f}^{\alpha\beta\gamma}\bar{d}^\beta\frac{\partial}{\partial\bar{d}^\gamma}\right). \quad (2.29)$$



The constraint equation  $\hat{H}\phi(x, \bar{c}, \bar{d}) = 0$  reproduces precisely the equations of motion of the bi-adjoint field obtained from the Lagrangian (2.9). Note that simple Lie groups have traceless generators, so that in this case the ordering of the color operators in the quantum version of  $C^a$  and  $\tilde{C}^\alpha$  is inconsequential. We have thus completed the derivation of the worldline model for the bi-adjoint field theory.

From the phase-space particle action (2.16), one may pass to configuration space (subscript ‘cs’) by eliminating the momenta, finding

$$S_{cs} = \int d\tau \left[ \frac{1}{2} e^{-1} \dot{x}^2 + i\bar{c}_a (\partial_\tau + ia) c^a + i\bar{d}_\alpha (\partial_\tau + i\tilde{a}) d^\alpha - \frac{e}{2} (m^2 - y C^a \Phi^{a\alpha}(x) \tilde{C}^\alpha) + sa + \tilde{s}\tilde{a} \right] \quad (2.30)$$

with  $C^a$  and  $\tilde{C}^\alpha$  already defined in (2.17). In the following, we are going to use an Euclidean version of this action, obtained by a Wick rotation<sup>4</sup> ( $iS_{cs} \rightarrow -S_E$ ) producing

$$S_E = \int d\tau \left[ \frac{1}{2} e^{-1} \dot{x}^2 + \bar{c}_a (\partial_\tau + ia) c^a + \bar{d}_\alpha (\partial_\tau + i\tilde{a}) d^\alpha + \frac{e}{2} (m^2 - y C^a \Phi^{a\alpha}(x) \tilde{C}^\alpha) - isa - i\tilde{s}\tilde{a} \right]. \quad (2.31)$$

A covariant gauge fixing can be imposed by setting  $(e(\tau), a(\tau), \tilde{a}(\tau)) = (2T, \theta, \phi)$ , with  $T$  identified with the usual Fock-Schwinger proper time, and  $(\theta, \phi)$  two additional moduli, corresponding to two angles, related to the projection on occupation number 1 for each color sector. This gauge fixing is valid both for worldlines with the topology of a circle, as appropriate for the one-loop effective action, and for worldlines with the topology of an interval, as appropriate for the propagator. The difference is just on the measure on the moduli space. The gauge-fixed Euclidean action simplifies further. We now drop the subscript ‘E’ and consider the parametrization of the whole worldline with  $\tau \in [0, 1]$ , so that

$$S = Tm^2 - is\theta - i\tilde{s}\phi + \int_0^1 d\tau \left[ \frac{1}{4T} \dot{x}^2 + \bar{c}_a (\partial_\tau + i\theta) c^a + \bar{d}_\alpha (\partial_\tau + i\phi) d^\alpha - Ty C^a \Phi^{a\alpha}(x) \tilde{C}^\alpha \right], \quad (2.32)$$

which appears in the path integral as  $e^{-S}$ .

Let us comment on the vertex operators it produces. The background  $\Phi^{a\alpha}(x)$  can be taken to be a plane wave with fixed colors  $a', \alpha'$  and incoming momentum  $k^\mu$

$$\Phi^{a\alpha}(x) = \delta^{aa'} \delta^{\alpha\alpha'} e^{ik \cdot x}. \quad (2.33)$$

Treating it as a perturbation, one may expand the exponential of the interaction part  $e^{-S_{\text{int}}}$  on which it sits. Then, the leading term of the expansion identifies the vertex operator

$$yT \int_0^1 d\tau C^{a'}(\tau) \tilde{C}^{\alpha'}(\tau) e^{ik \cdot x(\tau)} = yTV^{(0)}[k, a', \alpha'], \quad (2.34)$$

which reproduces the one we conjectured earlier on in (2.7), with the correct power of proper time  $T$  and coupling constant  $y$ , which must go along with it.

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<sup>4</sup>This is achieved by sending  $\tau \rightarrow -i\tau$ , accompanied by  $x^0 \rightarrow -ix^4$  as well as  $a \rightarrow ia$  and  $\tilde{a} \rightarrow i\tilde{a}$  for the worldline U(1) gauge fields.

### 3 Effective action

We have now the required tools to quantize the theory using path integrals. We wish to find a worldline representation of the one-loop effective action  $\Gamma[\Phi]$  of the bi-adjoint particle. As we did in the previous section, we first consider the method with the matrix-valued action and path ordering, and then the one with the auxiliary color variables.

#### 3.1 The matrix-valued approach

As already discussed in section 2.2, the matrix-valued action requires a path ordering prescription to define properly the corresponding path integral. The Schwinger proper-time representation of the one-loop effective action of the bi-adjoint field has already been anticipated in (2.15), where the trace of the heat kernel is computed by the path integral with periodic boundary conditions. The periodic paths  $x^\mu(\tau)$  with  $x^\mu(0) = x^\mu(1)$  can be split as

$$x^\mu(\tau) = x_0^\mu + q^\mu(\tau), \tag{3.1}$$

where  $x_0^\mu$  is any point on the loop traced by  $x^\mu(\tau)$ , while  $q^\mu(\tau)$  are the remaining fluctuations that must satisfy vanishing Dirichlet boundary conditions (i.e.  $q^\mu(0) = q^\mu(1) = 0$ ). Then, one integrates over  $x_0^\mu$  and  $q^\mu(\tau)$  separately, and the effective action takes the form

$$\Gamma[\Phi] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int d^D x_0 \operatorname{tr} \int_{\mathcal{D}} Dq \mathbf{T} e^{-S[x]}, \tag{3.2}$$

where  $S[x]$  is the action in (2.14), and  $\mathcal{D}$  indicates the Dirichlet boundary conditions on  $q^\mu(\tau)$  mentioned above. Other methods for extracting the constant  $x_0$  are possible, and will be discussed in the next section.

The effective action (3.2) may now be computed perturbatively in terms of the Wick contractions of the  $q$ -propagators (see appendix A for further details)

$$\Gamma[\Phi] = \int d^D x_0 \left[ -\frac{1}{2} \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{(4\pi T)^{\frac{D}{2}}} \operatorname{tr} \langle \mathbf{T} e^{-S_{\text{int}}} \rangle \right] \tag{3.3}$$

with

$$S_{\text{int}} = T y \int_0^1 d\tau \hat{\Phi}(x(\tau)). \tag{3.4}$$

The perturbative computation gives an answer of the form

$$\operatorname{tr} \langle \mathbf{T} e^{-S_{\text{int}}} \rangle = \sum_{n=0}^\infty \operatorname{tr} a_n(x_0) T^n, \tag{3.5}$$

where  $a_n(x_0)$  are the so-called Seeley-DeWitt (or heat kernel) coefficients. Inserting this expansion in (3.3), and renaming  $x_0 \rightarrow x$ , gives

$$\begin{aligned} \Gamma[\Phi] &= \int d^D x \left[ -\frac{1}{2} \sum_{n=0}^\infty \frac{\operatorname{tr} a_n(x)}{(4\pi)^{\frac{D}{2}}} \int_0^\infty \frac{dT}{T} e^{-m^2 T} T^{n-\frac{D}{2}} \right] \\ &= \int d^D x \left[ -\frac{1}{2} \sum_{n=0}^\infty \frac{\operatorname{tr} a_n(x)}{(4\pi)^{\frac{D}{2}}} \frac{1}{(m^2)^{n-\frac{D}{2}}} \int_0^\infty \frac{dT}{T} e^{-T} T^{n-\frac{D}{2}} \right] \\ &= \int d^D x \left[ -\frac{1}{2} \sum_{n=0}^\infty \frac{\operatorname{tr} a_n(x)}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma[n-\frac{D}{2}]}{(m^2)^{n-\frac{D}{2}}} \right]. \end{aligned} \tag{3.6}$$

We have integrated the proper time to obtain the usual gamma function. At  $D = 6$ , we see divergences for  $n = 0, 1, 2, 3$  in the effective action, as captured by the poles of the gamma function. The one for  $n = 3$  is mass independent and corresponds to the usual logarithmic divergence seen in dimensional regularization. The specific calculation of the coefficients in (3.5) can be done as in [16, 39–41], and gives the answer

$$a_0 = 1, \quad a_1 = -y\hat{\Phi}, \quad a_2 = \frac{y^2}{2}\hat{\Phi}^2, \quad a_3 = -\frac{y^2}{6}\hat{\Phi}^3 + \frac{y^2}{12}\hat{\Phi}\partial^2\hat{\Phi}, \quad (3.7)$$

which we have simplified by adding total derivatives, and neglecting matrix orderings that are inconsequential under the trace. The calculation of these coefficients is briefly outlined in appendix A.

### 3.1.1 Application: the beta function at $D = 6$

The  $n = 3$  pole in the effective Lagrangian at  $D = 6 - \epsilon$  is

$$\mathcal{L}_{\text{eff, div}} = -\frac{1}{2} \frac{\text{tr } a_3(x)}{(4\pi)^3} \Gamma\left[\frac{\epsilon}{2}\right] = -\frac{\text{tr } a_3(x)}{(4\pi)^3} \frac{1}{\epsilon}, \quad (3.8)$$

Then, using (3.7) the diverging part of the effective Lagrangian (3.8) becomes

$$\mathcal{L}_{\text{eff, div}} = \frac{1}{(4\pi)^3} \frac{1}{\epsilon} \left( \frac{y^3}{6} \text{tr } \hat{\Phi}^3 - \frac{y^2}{12} \text{tr } \hat{\Phi} \partial^2 \hat{\Phi} \right). \quad (3.9)$$

Let us now relate the terms in (3.9) to the one in (2.8), and recognize the counterterms needed to renormalize the theory at one-loop. We find

$$\begin{aligned} \text{tr } \hat{\Phi}^2 &= T(A) \tilde{T}(A) (\Phi^{a\alpha})^2 \\ \text{tr } \hat{\Phi}^3 &= \frac{1}{4} T(A) \tilde{T}(A) f^{abc} \tilde{f}^{\alpha\beta\gamma} \Phi^{a\alpha} \Phi^{b\beta} \Phi^{c\gamma}, \end{aligned} \quad (3.10)$$

where  $T(A)$  indicates the index in the adjoint representation.<sup>5</sup> These divergences ask for wave function and coupling constant renormalizations, obtained by adding to (2.8) the counterterms

$$\mathcal{L}_{\text{ct}} = (Z_\Phi - 1) \frac{1}{2} \Phi^{a\alpha} (-\partial^2) \Phi^{a\alpha} + (Z_y - 1) \frac{y}{3!} f^{abc} \tilde{f}^{\alpha\beta\gamma} \Phi^{a\alpha} \Phi^{b\beta} \Phi^{c\gamma} \quad (3.11)$$

with

$$Z_\Phi - 1 = -\frac{1}{6} T(A) \tilde{T}(A) \frac{y^2}{(4\pi)^3} \frac{1}{\epsilon}, \quad Z_y - 1 = -\frac{1}{4} T(A) \tilde{T}(A) \frac{y^2}{(4\pi)^3} \frac{1}{\epsilon}. \quad (3.12)$$

These counterterms produce a vanishing one-loop beta function, as recently discovered in [28]. In more details, parameterizing the counterterms with coefficients  $a_1$  and  $c_1$  as

$$\begin{aligned} Z_\Phi - 1 &= a_1 \frac{y^2}{(4\pi)^3} \frac{1}{\epsilon}, & a_1 &= -\frac{1}{6} T(A) \tilde{T}(A), \\ Z_y - 1 &= c_1 \frac{y^2}{(4\pi)^3} \frac{1}{\epsilon}, & c_1 &= -\frac{1}{4} T(A) \tilde{T}(A), \end{aligned} \quad (3.13)$$

<sup>5</sup>The index  $T(R)$  of a representation  $R$  is defined by  $\text{tr}(T_R^a T_R^b) = T(R) \delta^{ab}$ . It coincides with its quadratic Casimir if  $R$  is the adjoint representation. In particular, one has  $T(A) = C_2(A) = N$  for the adjoint representation of  $\text{SU}(N)$ .

the one-loop beta function is computed by

$$\beta(y) = \left(c_1 - \frac{3}{2}a_1\right) \frac{y^3}{(4\pi)^3} = 0. \quad (3.14)$$

In [28] the calculation of the beta function has been pushed to four loops (and actually to five loops in [42]), with the two-loop result indicating the asymptotic freedom of the theory. The latter would furnish a nice test on the structure of worldline methods at higher loops [15, 43].

There is also a mass renormalization that we have ignored so far. To complete the one-loop renormalization of the theory, let us identify the counterterm needed for renormalizing the mass. To achieve that, we have to treat the mass perturbatively in (3.3). Expanding the term  $e^{-m^2 T}$  we find (up to total derivatives and up to a constant that renormalizes the vacuum energy) an extra contribution to  $\text{tr } a_3(x)$  of the form

$$\Delta \text{tr } a_3(x) = -\frac{1}{2}m^2 y^2 \text{tr } \hat{\Phi}^2 = -\frac{1}{2}m^2 y^2 T(A) \tilde{T}(A) (\Phi^{a\alpha})^2 \quad (3.15)$$

giving an extra divergence of the effective Lagrangian

$$\Delta \mathcal{L}_{\text{eff, div}} = -\frac{\Delta \text{tr } a_3(x)}{(4\pi)^3} \frac{1}{\epsilon} = \frac{1}{(4\pi)^3} \frac{1}{\epsilon} \left( \frac{y^2 m^2}{2} T(A) \tilde{T}(A) (\Phi^{a\alpha})^2 \right). \quad (3.16)$$

This is canceled by adding to (2.8) the additional counterterm

$$\Delta \mathcal{L}_{\text{ct}} = (Z_m - 1) \frac{1}{2} m^2 (\Phi^{a\alpha})^2 \quad (3.17)$$

with

$$Z_m - 1 = -T(A) \tilde{T}(A) \frac{y^2}{(4\pi)^3} \frac{1}{\epsilon}, \quad (3.18)$$

which leads to the mass anomalous dimension  $\gamma_m$  (defined as usual by  $\gamma_m = \frac{1}{m} \frac{dm}{d \ln \mu}$ )

$$\gamma_m(y) = -\frac{5}{12} T(A) \tilde{T}(A) \frac{y^2}{(4\pi)^3}. \quad (3.19)$$

### 3.2 The color variables approach

Let us now proceed with the auxiliary color variables approach. The general structure of the one-loop effective action was anticipated in (2.1), with the role of  $\Psi^{(0)}$  now taken by  $\Phi^{a\alpha}$ . Our auxiliary variables  $\chi$  are the einbein  $e$ , the gauge fields  $a$ ,  $\tilde{a}$ , and the color variables  $c$ ,  $\bar{c}$ ,  $d$ ,  $\bar{d}$ . Hence, the one-loop effective action induced by a self-coupled bi-adjoint scalar particle is

$$\Gamma[\Phi] = \int_{S^1} \frac{Dx D e D a D \tilde{a} D c D \bar{c} D d D \bar{d}}{\text{Vol}(\text{Gauge})} e^{-S_E} \quad (3.20)$$

with  $S_E$  given in eq. (2.31), with  $\tau \in [0, 1]$  describing the circle. To bring it in a computable form, we must gauge-fix it. Let us review the main steps adapted to the circle  $S^1$ .

As anticipated, on the circle one may fix the worldline gauge fields to constants  $(e(\tau), a(\tau), \tilde{a}(\tau)) = (2T, \theta, \phi)$ , with the latter playing the role of moduli, i.e. gauge invariant configurations that must be integrated over a suitable moduli space [21]. The

measure on the moduli space is given by Faddeev-Popov determinants, which are constant in our case (i.e. they do not depend on the moduli), except that there is a factor  $\frac{1}{T}$  that takes into account the symmetry generated by constant translations on the circle: this factor avoids relative overcountings of paths with different proper time  $T$ . Then, fixing the overall normalization to match that of a real scalar, we have

$$\Gamma[\Phi] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{\mathcal{P}} Dx Dc D\bar{c} Dd D\bar{d} e^{-S}, \quad (3.21)$$

where the gauge-fixed action  $S$  is given in (2.32), and  $\mathcal{P}$  denotes periodic boundary conditions for the remaining path integration variables. The integration over the moduli space is set to cover the moduli space only once. The remaining path integral is normalized to that of a free particle, arising when the interactions are set to vanish (more on this later).

Let us further manipulate the remaining path integral to unearth the zero modes that may be present on the circle. Let us first address the zero modes of the coordinates  $x^\mu$ . They are present when the potential is treated perturbatively. These zero modes satisfy  $\partial_\tau^2 x^\mu(\tau) = 0$  and are therefore constant configurations. Their separation is generically achieved by splitting

$$x^\mu(\tau) = x_0^\mu + q^\mu(\tau), \quad (3.22)$$

where  $x_0^\mu$  are the constant zero modes, and  $q^\mu(\tau)$  are the remaining quantum fluctuations without the zero modes. There is a variety of ways of achieving this split, with the most common ones corresponding to setting Dirichlet boundary conditions on  $q^\mu(\tau)$  (i.e. imposing  $q(0) = q(1) = 0$ ) or using the “string-inspired” boundary conditions (that is requiring  $\int_0^1 d\tau q^\mu(\tau) = 0$ ). They give equivalent results, see [41] for a recent application and comparison of the two methods. The integration of the zero modes factorizes, as perturbatively there is no action for them, and there remains the path integral over the quantum fluctuations  $q^\mu$ . At this stage, we may extract the zero modes  $x_0$  to rewrite the path integral (3.21) as

$$\Gamma[\Phi] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \int d^D x_0 \int_{\bar{\mathcal{P}}} Dq \int_{\mathcal{P}} Dc D\bar{c} Dd D\bar{d} e^{-S}, \quad (3.23)$$

where now  $\bar{\mathcal{P}}$  denotes periodic boundary conditions with a constraint that eliminates the zero modes (it depends on the chosen method). We have considered arbitrary dimensions  $D$ , also in view of the application of dimensional regularization, though the case  $D = 6$  is the most interesting one as in such dimensions the model is renormalizable.

Let us now address the color variables  $c$  and  $d$ . For non-vanishing constant gauge fields  $\theta$  and  $\phi$ , and periodic boundary conditions, the color variables do not have any zero mode. Indeed,  $\theta$  and  $\phi$  act as a kind of mass term, i.e. a Wick rotated frequency-squared term for these harmonic-like oscillators, see their action in (2.32). Zero modes appear only for  $\theta = 0$  and  $\phi = 0$ , and we will consider their effect later on. It is also possible to remove the couplings to  $\theta$  and  $\phi$  by shifting those couplings to the boundary conditions by field redefinitions

$$c(\tau) \rightarrow e^{-i\theta\tau} c(\tau), \quad \bar{c}(\tau) \rightarrow e^{i\theta\tau} \bar{c}(\tau), \quad d(\tau) \rightarrow e^{-i\phi\tau} d(\tau), \quad \bar{d}(\tau) \rightarrow e^{i\phi\tau} \bar{d}(\tau), \quad (3.24)$$

which implies that the new fields  $c$  and  $d$  thus obtained satisfy twisted boundary conditions (which we denote by  $\mathcal{T}$ )

$$c(1) = e^{i\theta}c(0), \quad \bar{c}(1) = e^{-i\theta}\bar{c}(0), \quad d(1) = e^{i\phi}d(0), \quad \bar{d}(1) = e^{-i\phi}\bar{d}(0). \quad (3.25)$$

The effective action then takes the form

$$\Gamma[\Phi] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{is\theta} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\bar{s}\phi} \int d^D x_0 \int_{\bar{\mathcal{P}}} Dq \int_{\mathcal{T}} Dc D\bar{c} Dd D\bar{d} e^{-S_0}, \quad (3.26)$$

where the worldline action  $S_0$  now reads

$$S_0 = \int_0^1 d\tau \left[ \frac{1}{4T} \dot{q}^2 + \bar{c}_a \dot{c}^a + \bar{d}_\alpha \dot{d}^\alpha - Ty C^a \Phi^{a\alpha}(x) \tilde{C}^\alpha \right] \quad (3.27)$$

since we have extracted the mass term  $e^{-m^2 T}$  and the Chern-Simons couplings  $e^{is\theta+i\bar{s}\phi}$ .

From this expression we see that in a perturbative expansion we may treat the potential as a perturbation, while recognizing from the kinetic term the free propagators of the particle coordinates  $q^\mu$  and auxiliary color variables  $c, \bar{c}, d, \bar{d}$ . Extracting the normalization due to the free path integral we find

$$\Gamma[\Phi] = \int d^D x_0 \left[ -\frac{1}{2} \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{(4\pi T)^{\frac{D}{2}}} \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{e^{is\theta}}{\left(2i \sin \frac{\theta}{2}\right)^{N_A}} \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{e^{i\bar{s}\phi}}{\left(2i \sin \frac{\phi}{2}\right)^{\tilde{N}_A}} \langle e^{-S_{\text{int}}} \rangle \right] \quad (3.28)$$

with the term in square bracket representing the QFT one-loop effective Lagrangian  $\mathcal{L}_{\text{eff}}$  (that, of course, must be renormalized, as for example by treating the arbitrary dimension  $D$  as in dimensional regularization). The perturbation is here given by

$$S_{\text{int}} = -Ty \int_0^1 d\tau C^a \Phi^{a\alpha}(x) \tilde{C}^\alpha. \quad (3.29)$$

The free propagators that go along with this representation, and needed to compute the average  $\langle e^{-S_{\text{int}}} \rangle$  by Wick contractions, take into account the boundary conditions. They are given by

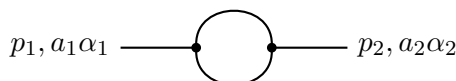
$$\begin{aligned} \langle q^\mu(\tau) q^\nu(\sigma) \rangle &= -2T \delta^{\mu\nu} \Delta(\tau, \sigma), \\ \langle c^a(\tau) \bar{c}_b(\sigma) \rangle &= \delta_b^a \Delta_\tau(\tau, \sigma; \theta), \\ \langle d^\alpha(\tau) \bar{d}_\beta(\sigma) \rangle &= \delta_\beta^\alpha \Delta_\tau(\tau, \sigma; \phi), \end{aligned} \quad (3.30)$$

where<sup>6</sup>

$$\begin{aligned} \Delta(\tau, \sigma) &= \frac{1}{2} |\tau - \sigma| - \frac{1}{2} (\tau - \sigma)^2 - \frac{1}{12}, \\ \Delta_\tau(\tau, \sigma; \theta) &= \frac{1}{2i \sin \frac{\theta}{2}} \left[ e^{i\frac{\theta}{2}} \Theta(\tau - \sigma) + e^{-i\frac{\theta}{2}} \Theta(\sigma - \tau) \right], \end{aligned} \quad (3.31)$$

where  $\Theta(x)$  is the standard Heaviside step-function with  $\Theta(0) = \frac{1}{2}$ .

<sup>6</sup>We use the string inspired method to factor out the zero modes from  $q$ . The constant term  $-\frac{1}{12}$  could also be dropped from the propagator  $\Delta(\tau, \sigma)$  [15]. As for the propagator  $\Delta_\tau(\tau, \sigma; \theta)$ , one may check that it satisfies the Green's equation  $\partial_\tau \Delta_\tau(\tau, \sigma; \theta) = \delta(\tau - \sigma)$ , and the twisted boundary conditions  $\Delta_\tau(1, \sigma; \theta) = e^{i\theta} \Delta_\tau(0, \sigma; \theta)$  and  $\Delta_\tau(\tau, 1; \theta) = e^{-i\theta} \Delta_\tau(\tau, 0; \theta)$ .



**Figure 1.** Self-energy contribution to the bi-adjoint propagator at one-loop.

Eq. (3.28) is our final form of the worldline representation of the one-loop effective action of the bi-adjoint scalar, with the additional color variables. It is readily calculable in perturbation theory. One may compare it with (3.3), which used instead a matrix valued action.

### 3.2.1 Application: the self-energy

Let us now consider a simple application of (3.28), the calculation of the self-energy of the bi-adjoint scalar, as described by figure 1.

To start with, let us first check the normalization of the effective action (3.28), by setting  $S_{\text{int}}$  to vanish, and verifying that it contains the correct number of degrees of freedom circulating in the loop. To do that, and to perform in a more convenient way the integration over the angular moduli, we change variables from  $\theta$  and  $\phi$  to the Wilson loop variables  $z = e^{-i\theta} = e^{-i \int_0^1 d\tau a(\tau)}$  and  $w = e^{-i\phi} = e^{-i \int_0^1 d\tau \tilde{a}(\tau)}$ , so that the original integration is mapped to a contour integration, the unit circle  $\gamma$  of the complex plane for each variable, e.g. for  $\theta$

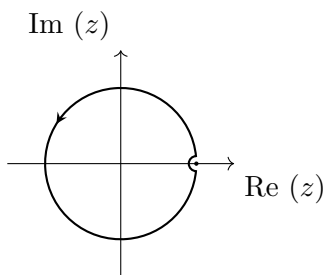
$$\int_0^{2\pi} \frac{d\theta}{2\pi} = \oint_{\gamma} \frac{dz}{2\pi i z}, \quad 2i \sin \frac{\theta}{2} = \frac{1-z}{\sqrt{z}}, \quad \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{e^{is\theta}}{\left(2i \sin \frac{\theta}{2}\right)^{N_A}} = \oint_{\gamma} \frac{dz}{2\pi i} \frac{1}{z^2} \frac{1}{(1-z)^{N_A}}, \quad (3.32)$$

where we used the value of the Chern-Simons coupling  $s = 1 + \frac{N_A}{2}$ . One may notice that at  $\theta = 0$  the auxiliary color variables  $c, \bar{c}$  develop zero modes, which correspond to the pole at  $z = 1$ . This divergence is due to the determinant of the free path integral that vanishes at such a point because of the zero modes. Previously we postponed the discussion of these zero modes, but now we see their effect and the way we should treat them. They cause a singularity on the integration contour at  $z = 1$ , which we push out of the integration contour (indicated by  $\gamma_-$ , see figure 2). This prescription gives the expected answer. With it, only the poles at  $z = 0$  are responsible for the projection and we get

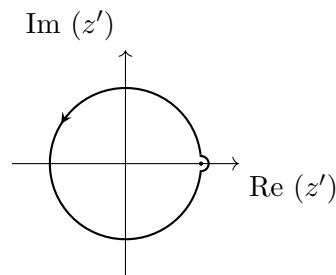
$$\oint_{\gamma_-} \frac{dz}{2\pi i} \frac{1}{z^2} \frac{1}{(1-z)^{N_A}} = \frac{d}{dz} (1-z)^{-N_A} \Big|_{z=0} = N_A, \quad (3.33)$$

which is the expected number.

Note that inclusion of the pole  $z = 1$  inside the contour would give a total vanishing result, which means that the previous answer can be obtained also by viewing the complex plane as the Riemann sphere, and considering only the poles outside the contour  $\gamma_-$ . This fact is related to a redefinition of the Wilson loop variable  $z \rightarrow z' = \frac{1}{z} = e^{i\theta}$ , (or equivalently to a different gauge fixing for  $a(\tau)$ , namely  $\theta \rightarrow -\theta$ ), that must give equivalent results. This change of variables maps the contour  $\gamma_-$  of the  $z$ -plane to the contour  $\gamma_+$  of the  $z'$ -plane,



**Figure 2.** Integration contour  $\gamma_-$  in  $z$ -plane. Pole at  $z = 1$  is excluded.



**Figure 3.** Integration contour  $\gamma_+$  in  $z'$ -plane. Pole at  $z' = 1$  is included.

which now includes the pole at  $z' = 1$ , see figure 3 (a change of sign has been taken care of by reversing the orientation of the contour).

Then eq. (3.28) reduces to

$$\Gamma[\Phi] = -\frac{1}{2} \int d^D x_0 \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{(4\pi T)^{\frac{D}{2}}} \left( N_A \tilde{N}_A + \dots \right), \quad (3.34)$$

which reproduces the expected degrees of freedom of the bi-adjoint scalar.<sup>7</sup>

After this check, we are ready to study the self-energy of figure 1. For that we consider a background given by the sum of two-plane waves with quantum numbers  $a_1, \alpha_1, p_1$  and  $a_2, \alpha_2, p_2$

$$\Phi^{a\alpha}(x) = \delta^{aa_1} \delta^{\alpha\alpha_1} e^{ip_1 \cdot x(\tau)} + \delta^{aa_2} \delta^{\alpha\alpha_2} e^{ip_2 \cdot x(\tau)}. \quad (3.35)$$

Expanding  $S_{\text{int}}$  in (3.28) and keeping the contribution linear in each plane wave, we find the contribution of two vertex operators of the form given in (2.34)

$$\langle e^{-S_{\text{int}}} \rangle \rightarrow y^2 T^2 \langle V(p_1, a_1, \alpha_1) V(p_2, a_2, \alpha_2) \rangle. \quad (3.36)$$

The integration over the zero modes produce a delta function for momentum conservation and one finds the self-energy correction

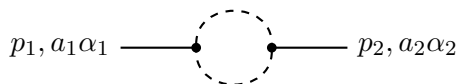
$$\Gamma[\Phi] \rightarrow (2\pi)^D \delta^D(p_1 + p_2) \Pi^{a_1\alpha_1, a_2\alpha_2}(p), \quad (3.37)$$

where  $p \equiv p_1 = -p_2$  and

$$\begin{aligned} \Pi^{a_1\alpha_1, a_2\alpha_2}(p) = & -\frac{y^2}{2} \int_0^\infty dT \frac{e^{-m^2 T} T}{(4\pi T)^{\frac{D}{2}}} \oint_{\gamma_-} \frac{dz}{2\pi i} \frac{z^{-2}}{(1-z)^{N_A}} \oint_{\gamma_-} \frac{dw}{2\pi i} \frac{w^{-2}}{(1-w)^{\tilde{N}_A}} \\ & \times \langle V_q(p, a_1, \alpha_1) V_q(-p, a_2, \alpha_2) \rangle \end{aligned} \quad (3.38)$$

<sup>7</sup>A factor of 1 instead of  $N_A \tilde{N}_A$  corresponds to a free real scalar.





**Figure 4.** Self-energy contribution to the bi-adjoint propagator at one-loop due to a virtual scalar particle. The charge of the particle in the loop is specified by occupation numbers  $r$  and  $\tilde{r}$ .

with the vertex operators  $V_q$  depending only on the  $q$  coordinates without the zero modes. Carrying out the Wick contractions, one finds

$$\begin{aligned}
 & \langle V_q(p, a_1, \alpha_1) V_q(-p, a_2, \alpha_2) \rangle \\
 &= \int_0^1 d\tau \int_0^1 d\sigma \langle C^{a_1}(\tau) C^{a_2}(\sigma) \rangle \langle \tilde{C}^{\alpha_1}(\tau) \tilde{C}^{\alpha_2}(\sigma) \rangle \langle e^{ip \cdot q(\tau)} e^{-ip \cdot q(\sigma)} \rangle \\
 &= f^{a_1 bc} f^{a_2 bc} \tilde{f}^{\alpha_1 \beta \gamma} \tilde{f}^{\alpha_2 \beta \gamma} \frac{zw}{(1-z)^2(1-w)^2} \int_0^1 d\tau \int_0^1 d\sigma e^{-2Tp^2 \Delta_0(\tau, \sigma)} \\
 &= \delta^{a_1 a_2} \delta^{\alpha_1 \alpha_2} T(A) \tilde{T}(A) \frac{zw}{(1-z)^2(1-w)^2} \int_0^1 d\tau e^{-Tp^2(\tau - \tau^2)},
 \end{aligned} \tag{3.39}$$

where  $\Delta_0(\tau, \sigma) = \Delta(\tau, \sigma) - \Delta(\tau, \tau)$ , and we used translational invariance of the string inspired propagator to get the last line. Next, using that the complex integration gives one, we find that the proper time integration yields

$$\Pi^{a_1 \alpha_1, a_2 \alpha_2}(p) = -\frac{y^2}{2(4\pi)^{\frac{D}{2}}} \delta^{a_1 a_2} \delta^{\alpha_1 \alpha_2} T(A) \tilde{T}(A) (P^2)^{\frac{D}{2}-2} \Gamma\left(2 - \frac{D}{2}\right), \tag{3.40}$$

where we defined

$$(P^2)^x \equiv \int_0^1 d\tau \left(m^2 + p^2(\tau - \tau^2)\right)^x. \tag{3.41}$$

This is the unrenormalized contribution to the self-energy. As a check, one may extract the UV divergences related to wave-function and mass renormalizations using dimensional regularization, which are easily seen to match the ones calculated in the previous section.

We have used the Chern-Simons couplings to select occupation number 1 in each colored sector, thus making sure that there is precisely a bi-adjoint scalar particle circulating in the loop. However, one could modify our worldline theory by selecting different occupation numbers, say  $r$  and  $\tilde{r}$ , for the color variables. This corresponds to a differently charged particle circulating in the loop, but coupled to the same background bi-adjoint field  $\Phi^{a\alpha}$ , see figure 4.

This is achieved by setting the Chern-Simons couplings to

$$s = r + \frac{N_A}{2}, \quad \tilde{s} = \tilde{r} + \frac{\tilde{N}_A}{2}, \tag{3.42}$$

as compared to our previous eq. (2.27). For example, setting  $(r, \tilde{r}) = (0, 0)$  gives an uncharged particle that should decouple from the loop, setting  $(r, \tilde{r}) = (2, 0)$  would give a particle that transforms in the symmetric products of two adjoints for the group  $G$  and scalar under  $\tilde{G}$ , which should also decouple, while setting  $(r, \tilde{r}) = (2, 1)$  would give a

particle that transforms in the symmetric products of two adjoints for the group  $G$  and in the adjoint for  $\tilde{G}$ , which should give a nontrivial contribution to the self-energy, as depicted in figure 4. Thus, using the above Chern-Simons couplings, which modify eq. (3.32), we find the following contribution to the self-energy

$$\Pi_{r\tilde{r}}^{a_1\alpha_1, a_2\alpha_2}(p) = -\frac{y_{r\tilde{r}}^2}{2(4\pi)^{\frac{D}{2}}}\delta^{a_1a_2}\delta^{\alpha_1\alpha_2}T(A)\tilde{T}(A)\binom{N_A+r}{N_A+1}\binom{\tilde{N}_A+\tilde{r}}{\tilde{N}_A+1}(P^2)^{\frac{D}{2}-2}\Gamma\left(2-\frac{D}{2}\right), \tag{3.43}$$

where we have denoted by  $y_{r\tilde{r}}$  the coupling constant of the new particle to the bi-adjoint field, and where the definition of  $(P^2)^x$  in eq. (3.41) should contain a different mass  $m_{r\tilde{r}}$  instead of  $m$ . The binomial coefficients are defined to vanish for  $r = 0$  and  $\tilde{r} = 0$ , as usual, as one may check that in those cases there are no poles in the Cauchy integrals over the moduli. The particle with charge  $r, \tilde{r}$  corresponds to a scalar field  $\varphi^{i,x}$  of mass  $m_{r\tilde{r}}$  that would couple to the bi-adjoint  $\Phi^{a\alpha}$  with a potential of the form

$$\mathcal{L}_{\text{int}} = -\frac{y_{r\tilde{r}}}{2}(T_r^a)^{ij}(\tilde{T}_{\tilde{r}}^\alpha)^{xy}\Phi^{a\alpha}\varphi^{ix}\varphi^{jy}, \tag{3.44}$$

where by  $T_r^a$  we indicate the generators in the symmetric tensor product of  $r$  adjoint representations, and so on, compare with eq. (2.8). Note that such a representation has an index  $T(r) = T(A)\binom{N_A+r}{N_A+1}$ , which indeed is what appears in (3.43). To make our formulas more explicit for the cases of  $SU(N)$  groups, we recall that  $T(A) = N$  and  $N_A = N^2 - 1$  in those cases.

The methods we have developed in this section can be generalized to higher point correlation functions at 1-loop, taking (3.28) as starting point. The factorized structure of the two point function (3.38) is preserved at higher points so it is easy to see that our prescription (2.4) is enough to reproduce the  $N$ -point correlation functions contributing to the 1-loop Yang-Mills effective action starting from the  $N$ -point functions for the bi-adjoint particle. In appendix B we present a Bern-Kosower-type of formula for an arbitrary number of vertex insertions.

## 4 Multi-colored scalars and coupling to gauge fields

The action of the bi-adjoint particle suggests extensions that might also be of some interest. We will again proceed with extensions using first the formulation with a matrix valued action, and then the one with auxiliary variables. The choice of which formalism to use is a matter of taste, as both methods are equivalent, as previously described.

### 4.1 Path ordering and multi-adjoint scalars

#### 4.1.1 Example: beta function for the multi-adjoint particle

As an extension, one may consider the  $n$ -adjoint scalar particle, quantum of the  $n$ -adjoint scalar field  $\Phi^A = \Phi^{a_1a_2\dots a_n}$  where the multi-index  $A$  contains an adjoint index  $a_i$  for each factor of the symmetry group  $G_1 \times G_2 \dots \times G_n$ . A suitable interacting QFT Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\Phi^A(-\partial^2 + m^2)\Phi^A + \frac{y}{3!}F^{ABC}\Phi^A\Phi^B\Phi^C, \tag{4.1}$$

where

$$F^{ABC} = f^{a_1 b_1 c_1} f^{a_2 b_2 c_2} \dots f^{a_n b_n c_n} . \quad (4.2)$$

The theory is nontrivial for  $n$  even, otherwise the potential vanishes as the  $F^{ABC}$  are then completely antisymmetric. In a quantum-background split  $\Phi \rightarrow \phi + \Phi$ , the quadratic part of the quantum field Lagrangian takes the same form of (2.11), but with a matrix-like background potential

$$\hat{\Phi}^{AB} = F^{ABC} \Phi^C . \quad (4.3)$$

The formula for the one-loop logarithmic divergences (3.9) still applies, now with

$$\begin{aligned} \text{tr } \hat{\Phi}^2 &= \alpha \Phi^A \Phi^A \\ \text{tr } \hat{\Phi}^3 &= \frac{\alpha}{2^n} F^{ABC} \Phi^A \Phi^B \Phi^C , \end{aligned} \quad (4.4)$$

where

$$\alpha = \prod_{i=1}^n T_i(A) \quad (4.5)$$

is the product of the indices of the adjoint representation for each factor  $G_i$ . Here we have taken  $n$  to be an even integer. Note that  $\alpha$  is always a positive number. These divergences are renormalized by adding to the Lagrangian (4.1) the counterterms

$$\mathcal{L}_{\text{ct}} = (Z_\Phi - 1) \frac{1}{2} \Phi^A (-\partial^2) \Phi^A + (Z_y - 1) \frac{y}{3!} F^{ABC} \Phi^A \Phi^B \Phi^C \quad (4.6)$$

with

$$Z_\Phi - 1 = -\frac{\alpha}{6} \frac{y^2}{(4\pi)^3} \frac{1}{\epsilon} , \quad Z_y - 1 = -\frac{\alpha}{2^n} \frac{y^2}{(4\pi)^3} \frac{1}{\epsilon} , \quad (4.7)$$

producing the one-loop beta function

$$\beta(y) = \frac{\alpha}{4} \left( 1 - \frac{1}{2^{n-2}} \right) \frac{y^3}{(4\pi)^3} . \quad (4.8)$$

Evidently, the beta function for  $n > 2$  is positive, while it vanishes for  $n = 2$ , and it is negative for  $n = 0$  (setting  $\alpha = 1$  in such a case), consistently with the asymptotic freedom of the latter case. These results are in agreement with those of ref. [28].

#### 4.1.2 Example: symmetric multi-adjoint scalar

As a final example, let us consider a different theory for the  $n$ -adjoint scalar by taking only groups of the type  $SU(N)$  with  $N \geq 3$ , which admit the completely symmetric invariant tensor  $d^{abc}$ . Models for a single symmetric tensor were introduced long time ago to study critical phenomena in condensed matter physics, see, e.g., refs. [44, 45] and refs. [42, 46, 47] for more recent discussions. Thus, let us consider the replacement in (4.1)

$$F^{ABC} \quad \rightarrow \quad D^{ABC} = d^{a_1 b_1 c_1} d^{a_2 b_2 c_2} \dots d^{a_n b_n c_n} , \quad (4.9)$$

which is consistent for all integers  $n \in \mathbb{N}$ , as  $D^{ABC}$  is now always totally symmetric. The corresponding one-loop effective action is again represented as in (2.15), but now with

$$\hat{\Phi}^{AB} = D^{ABC} \Phi^C \quad (4.10)$$

which produces

$$\begin{aligned}\text{tr } \hat{\Phi}^2 &= x_1 \delta^{AB} \Phi^A \Phi^B, \\ \text{tr } \hat{\Phi}^3 &= x_2 D^{ABC} \Phi^A \Phi^B \Phi^C,\end{aligned}\tag{4.11}$$

with coefficients  $x_1$  and  $x_2$  that depend on the chosen groups. They are computed by

$$x_1 = \prod_{i=1}^n \frac{N_i^2 - 4}{N_i}, \quad x_2 = \prod_{i=1}^n \frac{N_i^2 - 12}{2N_i},\tag{4.12}$$

with  $N_i$  being the dimension of the fundamental representation of each copy of the  $SU(N_i)$  group, see [48] whose normalization for the  $d^{abc}$  tensor we adopt. From the one-loop logarithmic divergences of the effective action (3.9) we extract the counterterms

$$\mathcal{L}_{\text{ct}} = (Z_\Phi - 1) \frac{1}{2} \Phi^A (-\partial^2) \Phi^A + (Z_y - 1) \frac{y}{3!} D^{ABC} \Phi^A \Phi^B \Phi^C\tag{4.13}$$

with

$$Z_\Phi - 1 = -\frac{x_1}{6} \frac{y^2}{(4\pi)^3} \frac{1}{\epsilon}, \quad Z_y - 1 = -x_2 \frac{y^2}{(4\pi)^3} \frac{1}{\epsilon},\tag{4.14}$$

producing the one-loop beta function

$$\beta(y) = \frac{y^3}{(4\pi)^3} \left( \frac{x_1}{4} - x_2 \right)\tag{4.15}$$

that is

$$\beta(y) = \frac{y^3}{(4\pi)^3} \left\{ \frac{1}{4} \left( \prod_{i=1}^n \frac{N_i^2 - 4}{N_i} \right) \left[ 1 - \frac{1}{2^{n-2}} \prod_{i=1}^n \frac{N_i^2 - 12}{N_i^2 - 4} \right] \right\}.\tag{4.16}$$

This formula includes the well-known asymptotic freedom case for  $n = 0$  (with  $x_1 = x_2 = 1$ ), while for  $n = 1$  the result is consistent with [46] upon mapping different conventions, showing a positive beta function for  $N \leq 4$ , and a negative beta function (and thus asymptotic freedom) for  $N \geq 5$ . Further, the bi-adjoint particle ( $n = 2$ ) has beta function

$$\beta(y) = \frac{y^3}{(4\pi)^3} \frac{2}{N_1 N_2} (N_1^2 + N_2^2 - 16),\tag{4.17}$$

which is non-vanishing and always positive, independently of the dimension of the  $SU(N)$  groups ( $N_i \geq 3$ , as otherwise the  $d^{abc}$  symbols would vanish). Quite generally, the one-loop beta function is positive for  $n \geq 2$  in these models.

## 4.2 Multi-adjoint scalar using auxiliary variables

Let us consider again the case of a particle symmetric under the product of an arbitrary number of simple Lie groups,  $G_1 \times G_2 \times \dots \times G_n$ , with the wave-function transforming in an arbitrary representation  $R_i$  with generators  $(T_{R_i}^{a_i})^{\alpha_i}_{\beta_i}$  for each factor  $G_i$ . For that purpose we introduce a set of color variables  $c_i^{\alpha_i}(\tau)$  and complex conjugate  $\bar{c}_{\alpha_i}^i(\tau)$ , with  $i = 1, \dots, n$ , and indices that match those of the chosen representation, to construct the color charges (no sum over the index  $i$ )

$$C_i^{a_i}(\tau) = \bar{c}_{\alpha_i}^i(\tau) (T_{R_i}^{a_i})^{\alpha_i}_{\beta_i} c_i^{\beta_i}(\tau)\tag{4.18}$$

and write down the action for the phase space coordinates  $(x, p, c_i, \bar{c}_i)$  and worldline gauge variables  $(e, a_i)$  as follows

$$S = \int_0^1 d\tau \left[ p_\mu \dot{x}^\mu - \frac{e}{2} (p^2 + m^2 + U(\Phi(x), C_i)) + \sum_{i=1}^n (i\bar{c}_{\alpha_i}^i (\partial_\tau + i a_i) c_i^{\alpha_i} + s_i a_i) \right]. \quad (4.19)$$

The potential  $U(\Phi(x), C_i)$  is taken to depend on suitable scalar background fields, collectively denoted by  $\Phi$ , and color charges  $C_i$  as given above. Of course, the potential is required to be invariant under the global symmetry group. Our previous bi-adjoint particle (2.16) is a special case of this more general model with  $n = 2$ , color variables  $c_i, \bar{c}_i$  transforming in the adjoint representation of the corresponding group, and scalar potential  $U(\Phi(x), C_i)$  taken as in (2.17).

Introducing gauge fields to the theory is straightforward. Starting with the phase space action (4.19) we add gauge field interactions by the minimal substitution

$$p_\mu \rightarrow p_\mu - \sum_i g_i A_\mu^i \quad (4.20)$$

in the constraint, with the non-abelian gauge fields  $A_\mu^i$  now given by (no sum over  $i$ )

$$A_\mu^i = A_\mu^{a_i, i}(x) C_i^{a_i} = A_\mu^{a_i, i}(x) \bar{c}_{\alpha_i}^i(\tau) (T_{R_i}^{a_i})^{\alpha_i}_{\beta_i} c_i^{\beta_i}(\tau), \quad (4.21)$$

which also uses the composite color charge (4.18). In order to gauge only a subset of the symmetry groups one may simply set the remaining gauge couplings to zero. For instance, considering again the bi-adjoint case given by the action (4.19) with  $n = 2$  and generators in the adjoint, the coupling to a single gauge field can be implemented by, say, setting  $g_1 \equiv g$  and  $g_2 = 0$ . Eliminating the momenta through their equations of motion, using  $(T_A^a)^b_c = -i f^{abc}$ , and Wick rotating, produces the action

$$S[x, c, \bar{c}, d, \bar{d}, e, a, \tilde{a}; \Phi, A_\mu] = \int_0^1 d\tau \left[ \frac{1}{2} e^{-1} \dot{x}^2 - g f^{abc} \dot{x}_\mu A^{\mu a}(x) \bar{c}^b c^c + \frac{e}{2} (m^2 - y C^a \Phi^{a\alpha}(x) \tilde{C}^\alpha) + \bar{c}_a (\partial_\tau + i a) c^a + \bar{d}_\alpha (\partial_\tau + i \tilde{a}) d^\alpha - i s (a + \tilde{a}) \right], \quad (4.22)$$

where we used again the notation of previous sections. It is easy to see that the action (4.22) leads to two types of vertices, namely the spin one vertex  $V^{(1)}[k, \varepsilon, a; x, c, \bar{c}]$  and the spin zero vertex  $V^{(0)}[k, a, \alpha; x, c, \bar{c}, d, \bar{d}]$  which must be considered separately (here  $a$  denotes an adjoint color index, not the worldline gauge field). Using dimensional reduction, similar types of interactions and corresponding amplitudes have been considered in the context of the CHY representation [49] and the double copy for supersymmetric theories [50, 51]. There, quartic scalar potentials are also included. Starting from our general approach these potentials may be easily added to the theory. Also, from the double copy dictionary [3], one expects that the double copy procedure  $C \rightarrow K$  at the level of the vertex operators will lead to the Yang-Mills-Einstein theory. To introduce the gravitational couplings in the worldline action one should covariantize it by replacing  $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$  and implement the path integral for curved spaces [30, 31].

### 4.2.1 Example: self-energy for the multi-adjoint scalar

One can now repeat the exercise of section 3.2.1, and calculate the one-loop self-energy correction to the propagator of the  $n$ -adjoint scalar due to particles that sit in the  $r_i$  representation (meaning the symmetric tensor product of  $r_i$  copies of the adjoint) of the group  $G_i$ . Denoting by  $N_{A,i}$  the dimension of the group  $G_i$ , we can write the self-energy as

$$\Pi_r^{BC}(p) = -\frac{y_r^2}{2(4\pi)^{\frac{D}{2}}} \delta^{BC} (P^2)^{\frac{D}{2}-2} \Gamma\left(2 - \frac{D}{2}\right) \prod_{i=1}^n T_i(A) \binom{N_{A,i} + r_i}{N_{A,i} + 1}, \quad (4.23)$$

where by  $r$  we indicate the vector with components  $r_i$ . The definition of  $(P^2)^x$  given in eq. (3.41) now depends on a new mass  $m_r$ . The case of the pure  $n$ -adjoint field propagating in the loop is obtained by setting  $r_i = 1$  for all  $i$ . Again, the group theory factor  $\prod_{i=1}^n T_i(A) \binom{N_{A,i} + r_i}{N_{A,i} + 1}$  can be rewritten as the product of the indices of the  $r_i$  representations, namely  $\prod_{i=1}^n T_i(r_i)$ . Similarly one could proceed as before to compute the beta functions for the couplings  $y$  and  $y_r$ , showing in this case a nontrivial mixing.

## 5 Conclusions

In this paper we have employed worldline methods to study the bi-adjoint scalar particle. In order to apply worldline methods we derived a suitable particle action and used it to represent the 1-loop effective action of the corresponding QFT. We have employed two different approaches to deal with color degrees of freedom. On the one hand the simplicity of the double copy approach suggested the introduction of auxiliary bosonic color variables in order to keep track of path ordering. These auxiliary variables can be used also in coupling to gauge fields. On the other hand we have considered as well a direct path ordering, which we have exemplified by computing the beta function of the theory. We have verified the vanishing of the beta function at 1-loop in the bi-adjoint scalar theory, as noted recently in [28]. More generally, we have obtained the 1-loop beta function for the theory of a scalar field with an arbitrary number of adjoint indices and with an interaction potential dictated by the structure constants of the symmetry groups, finding again agreement with [28].

Extensions of the model in the worldline approach are straightforward, and follow from the basic structure of the bi-adjoint case. It would be interesting to study the case of gravity and color together in order to explore more deeply the double copy from the worldline perspective.

Using the auxiliary variables prescription, in section 2 we have proposed a double copy prescription to lift vertex operators of the bi-adjoint scalar to those of gauge fields and gravity. This prescription leads to the effective action induced by a scalar particle coupled to a non-abelian gauge field. It is closer in spirit to the formal substitution rules in the classical double copy [6]. Indeed, one of our motivations towards a worldline approach to the bi-adjoint is aimed at applications at the classical level where the worldline formalism, in the context of Wong's equations, can be useful to describe the dynamics of classical scattering [52]. In particular, it would be interesting to derive worldline Feynman rules for Yang-Mills theory as done in refs. [53–55] for gravity.

We believe that our formulation of the multi-adjoint theory can be useful to study the color-kinematics duality and the double copy from a worldline perspective, with some properties already noticed in [56] and investigated more recently in [57, 58], perhaps pointing towards a potential first principle understanding of its origin. In addition, the multi-adjoint particle could be used as a toy model to study finite temperature systems with color [59–61]. Another interesting direction worth pursuing would be to study the worldline meaning of a novel color-kinematics relation, suggested in [62], where the momentum of the particle appearing in the plane wave of the vertex operator is substituted by color matrices.

## Acknowledgments

LDLC acknowledges financial support from the *Open Physics Hub* at the Physics and Astronomy Department in Bologna. Some of our figures were produced with the help of TikZ-Feynman [63].

## A Seeley-DeWitt coefficients

Let us briefly review the perturbative calculation of the Seeley-DeWitt coefficients in (3.7), starting from the representation of the effective action in (3.3), which contains the time ordering. At the perturbative order needed to obtain eq. (3.7), the effect of the time ordering is inconsequential, but it is important at higher orders.

Using the split (3.1), we Taylor expand the interaction term  $S_{\text{int}}$  in (3.4) around  $x_0$

$$S_{\text{int}} = Ty \int_0^1 d\tau \left( \hat{\Phi}(x_0) + q^\mu(\tau) \partial_\mu \hat{\Phi}(x_0) + \dots \right), \quad (\text{A.1})$$

insert it into the exponential in (3.5), and expand the exponential. Then, collecting all terms of the same power in  $T$ , and taking into account that the  $q$ -propagator with the vanishing Dirichlet boundary conditions goes like  $T$

$$\langle q^\mu(\tau) q^\nu(\sigma) \rangle = -2T \delta^{\mu\nu} \left( \frac{1}{2} |\tau - \sigma| - \frac{1}{2} (\tau + \sigma) + \tau\sigma \right), \quad (\text{A.2})$$

we easily find from the constant  $\hat{\Phi}(x_0)$  term

$$a_0 = 1, \quad a_1 = -y \hat{\Phi}(x_0), \quad a_2 = \frac{y^2}{2} \hat{\Phi}^2(x_0), \quad a_3 = -\frac{y^2}{6} \hat{\Phi}^3(x_0) + \dots \quad (\text{A.3})$$

where the dots denote three additional terms arising from using the  $q$ -propagator. They are as follows. A first term is

$$\begin{aligned} & \left\langle \mathbf{T} \frac{1}{2} \left( Ty \int_0^1 d\tau q^\mu(\tau) \partial_\mu \hat{\Phi}(x_0) \right) \left( Ty \int_0^1 d\sigma q^\nu(\sigma) \partial_\nu \hat{\Phi}(x_0) \right) \right\rangle \\ &= \frac{1}{2} T^2 y^2 \partial_\mu \hat{\Phi}(x_0) \partial_\nu \hat{\Phi}(x_0) \int_0^1 d\tau \int_0^1 d\sigma \langle q^\mu(\tau) q^\nu(\sigma) \rangle \\ &= \frac{1}{12} T^3 y^2 (\partial_\mu \hat{\Phi}(x_0))^2 \end{aligned} \quad (\text{A.4})$$

where the time ordering  $\mathbf{T}$  is again inconsequential as the matrices at different times commute. A second term reads

$$\begin{aligned}
 & \left\langle \mathbf{T} \frac{1}{2} \left( T y \hat{\Phi}(x_0) \right) \left( T y \int_0^1 d\tau \frac{1}{2} q^\mu(\tau) q^\nu(\tau) \partial_\mu \partial_\nu \hat{\Phi}(x_0) \right) \right\rangle \\
 &= \frac{1}{4} T^2 y^2 \hat{\Phi}(x_0) \int_0^1 d\tau \langle q^\mu(\tau) q^\nu(\tau) \rangle \partial_\mu \partial_\nu \hat{\Phi}(x_0) \\
 &= \frac{1}{12} T^3 y^2 \hat{\Phi}(x_0) \partial^2 \hat{\Phi}(x_0) .
 \end{aligned} \tag{A.5}$$

A third term is similar, but with the matrices  $\hat{\Phi}(x_0)$  and  $\partial^2 \hat{\Phi}(x_0)$  interchanged. These matrices commute under the trace, so that this last term just doubles the previous one. This completes the derivation of  $a_3$ , which is equivalent to the one in (3.7) by adding a total derivative, that anyway drops out from the effective action.

## B Effective vertices

Let us now discuss briefly how to get a general formula for the 1PI higher-point correlation functions at one-loop in momentum space, by taking (3.28) as starting point. The generalization of the plane wave expansion of the scalar field (3.35) is given by

$$\Phi^{a\alpha}(x_0 + q) = \sum_{\ell=1}^N \delta^{a a_\ell} \delta^{\alpha \alpha_\ell} e^{i p_\ell \cdot x_0} e^{i p_\ell \cdot q(\tau)} , \tag{B.1}$$

which can be inserted in (3.28), with the constraint that each plane wave should appear only once. The integration over the loop base point produces a momentum conservation delta function which enables us to rewrite the 1-loop effective action as

$$\Gamma[\Phi] = (2\pi)^D \delta^D \left( \sum_{\ell=1}^N p_\ell \right) \Gamma_{r\bar{r}}^{a_1 \alpha_1 \dots a_N \alpha_N} (p_1 \dots p_n) . \tag{B.2}$$

Stripping off the momentum conservation Dirac delta function we define the 1PI  $N$ -point function as

$$\begin{aligned}
 \Gamma_{r\bar{r}}^{a_1 \alpha_1 \dots a_N \alpha_N} (p_1 \dots p_n) &= -\frac{1}{2} y_{r\bar{r}}^N \oint_{\gamma_-} \frac{dz}{2\pi i} \frac{1}{z^{r+1} (1-z)^{N_A}} \oint_{\gamma_-} \frac{dw}{2\pi i} \frac{1}{w^{\bar{r}+1} (1-w)^{\bar{N}_A}} \\
 & \int_0^\infty \frac{dT}{T} \frac{e^{-m_{r\bar{r}}^2 T}}{(4\pi T)^{\frac{D}{2}}} T^N \left( \prod_{\ell=1}^N \int_0^1 d\tau_\ell \right) \langle e^{i \sum_{k=1}^N p_k \cdot q(\tau_k)} \rangle \langle \prod_{k=1}^N C^{a_k}(\tau_k) \rangle \langle \prod_{\ell=1}^N \tilde{C}^{\alpha_\ell}(\tau_\ell) \rangle ,
 \end{aligned} \tag{B.3}$$

where we also switched to the Wilson loop variables defined in (3.32). The v.e.v of the kinematical part of the vertex operator can be evaluated using the identity

$$\langle e^{i \sum_{k=1}^N p_k \cdot q(\tau_k)} \rangle = e^{T \sum_{k,\ell=1}^N p_k \cdot p_\ell \Delta(\tau_k, \tau_\ell)} . \tag{B.4}$$

The integral over proper time  $T$  can be performed exactly

$$\begin{aligned}
 & \int_0^\infty dT e^{-T \left( m_{r\bar{r}}^2 - \sum_{k,\ell=1}^N p_k \cdot p_\ell \Delta(\tau_k, \tau_\ell) \right)} T^{N - \frac{D}{2} - 1} \\
 &= \Gamma \left( N - \frac{D}{2} \right) \left( m_{r\bar{r}}^2 - \sum_{k,\ell=1}^N p_k \cdot p_\ell \Delta(\tau_k, \tau_\ell) \right)^{\frac{D}{2} - N} .
 \end{aligned} \tag{B.5}$$



Finally, putting all of the pieces together we get our desired master formula

$$\Gamma_{r\tilde{r}}^{a_1\alpha_1\dots a_N\alpha_N}(p_1, p_2, \dots, p_N) = -\frac{1}{2} \frac{y_{r\tilde{r}}^N}{(4\pi)^{\frac{D}{2}}} \oint_{\gamma_-} \frac{dz}{2\pi i} \frac{1}{z^{r+1}(1-z)^{N_A}} \oint_{\gamma_-} \frac{dw}{2\pi i} \frac{1}{w^{\tilde{r}+1}(1-w)^{N_{\bar{A}}}}$$

$$\Gamma\left(N - \frac{D}{2}\right) \left(\prod_{\ell=1}^N \int_0^1 d\tau_\ell\right) \left(m_{r\tilde{r}}^2 - \sum_{k,\ell=1}^N p_k \cdot p_\ell \Delta(\tau_k, \tau_\ell)\right)^{\frac{D}{2}-N} \left\langle \prod_{k=1}^N C^{a_k}(\tau_k) \right\rangle \left\langle \prod_{\ell=1}^N \tilde{C}^{\alpha_\ell}(\tau_\ell) \right\rangle, \quad (\text{B.6})$$

which holds in the case where the particle propagating in the loop has charge  $r, \tilde{r}$  with respect to both Lie groups. The (amputated) external lines correspond instead to the plane waves of the bi-adjoint field  $\Phi^{\alpha\alpha}$ .

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