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ON THE MOTIVE OF THE QUOT SCHEME OF FINITE QUOTIENTS OF A LOCALLY FREE SHEAF

ANDREA T. RICOLFI

ABSTRACT. Let X be a smooth variety, E a locally free sheaf on X . We express the generating function of the motives $[\mathrm{Quot}_X(E, n)]$ in terms of the power structure on the Grothendieck ring of varieties. This extends a recent result of Bagnarol, Fantechi and Perroni for curves, and a result of Gusein-Zade, Luengo and Melle-Hernández for Hilbert schemes. We compute this generating function for curves and we express the relative motive $[\mathrm{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r}) \rightarrow \mathrm{Sym} \mathbb{A}^d]$ as a plethystic exponential.

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0. INTRODUCTION

Let X be a smooth quasi-projective variety over \mathbb{C} , and let E be a locally free sheaf of rank r on X . The Quot scheme $\mathrm{Quot}_X(E, n)$ parameterises quotients $E \twoheadrightarrow Q$ such that Q is a zero-dimensional sheaf of length n . Recently Bagnarol, Fantechi and Perroni [1] have shown that if C is a smooth proper curve, the class

$$[\mathrm{Quot}_C(E, n)] \in K_0(\mathrm{Var}_{\mathbb{C}})$$

in the Grothendieck ring of varieties does not depend on E . We use the theory of *power structures* [9] to extend their result to arbitrary dimension. Roughly speaking, a power structure on a ring R is a way of making sense of expressions $A(t)^m$, where $A(t) = 1 + A_1 t + A_2 t^2 + \dots$ is a power series with coefficients in R and $m \in R$.

For (X, E) as above, we form the generating function

$$Z_E(t) = \sum_{n \geq 0} [\mathrm{Quot}_X(E, n)] t^n,$$

and we denote by $P_{r,n} \in K_0(\mathrm{Var}_{\mathbb{C}})$ the motive of the *punctual Quot scheme*, namely the closed subscheme $P_{r,n} \subset \mathrm{Quot}_X(E, n)$ parameterising quotients that are entirely supported at a single (fixed) point in X .

Our first main result (proved in Theorem 2.3) is the following.

Key words and phrases. Quot schemes, Moduli spaces of sheaves, Grothendieck ring of varieties.

Theorem A. *There is an identity*

$$Z_E(t) = \left(\sum_{n \geq 0} P_{r,n} t^n \right)^{[X]}.$$

Since the punctual Quot scheme only depends on r , n and $\dim X$, it follows that the same holds true for the motive of $\text{Quot}_X(E, n)$. Note that this was proved for $r = 1$ (the Hilbert scheme case) by Gusein-Zade, Luengo and Melle-Hernández [10].

Our second main result is of *relative* nature and concerns $X = \mathbb{A}^d$. The Quot-to-Chow morphism

$$\text{Quot}_X(E, n) \rightarrow \text{Sym}^n X$$

sends a quotient $E \rightarrow Q$ to the support of Q , viewed as a zero-cycle with multiplicities. We consider the relative motive

$$Z^{\text{rel}}(\mathbb{A}^d, r) = \sum_{n \geq 0} [\text{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r}, n) \rightarrow \text{Sym}^n \mathbb{A}^d] \in K_0(\text{Var}_{\text{Sym} \mathbb{A}^d})$$

over the symmetric product of \mathbb{A}^d . We define classes $\Omega_{r,n} \in K_0(\text{Var}_{\mathbb{C}})$ by

$$\sum_{n \geq 0} P_{r,n} t^n = \text{Exp} \left(\sum_{n > 0} \Omega_{r,n} t^n \right)$$

where Exp is the *motivic exponential* (see Section 1.5) induced by the lambda ring structure on $K_0(\text{Var}_{\mathbb{C}})$. For \mathbb{A}^d , we refine Theorem A by showing (see Theorem 2.9) that $Z^{\text{rel}}(\mathbb{A}^d, r)$ is generated on the small diagonal by the absolute motives $\Omega_{r,n}$.

Theorem B. *There is an identity*

$$Z^{\text{rel}}(\mathbb{A}^d, r) = \text{Exp}_{\cup} \left(\sum_{n > 0} \Omega_{r,n} \boxtimes [\mathbb{A}^d \xrightarrow{\Delta_n} \text{Sym}^n \mathbb{A}^d] \right).$$

See [6, 14] for analogues of this result in the context of motivic Donaldson–Thomas theory and [3] for the calculation of the (absolute) *virtual motive* of $\text{Hilb}^n(\mathbb{A}^3)$.

Finally, our last result (see Section 3.1) is the full “solution” of the motivic theory of the Quot scheme of a smooth curve, which can be summarised by the identities

$$\Omega_{r,n} = \begin{cases} [\mathbb{P}^{r-1}] & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Theorem C. *If E is a locally free sheaf on a smooth curve C , there is an identity*

$$Z_E(t) = \text{Exp}([C \times \mathbb{P}^{r-1}]t).$$

Moreover, in $K_0(\text{Var}_{\text{Sym} \mathbb{A}^1})$ there is an identity

$$Z^{\text{rel}}(\mathbb{A}^1, r) = \text{Exp}_{\cup}([[\mathbb{P}^{r-1}]] \boxtimes [\mathbb{A}^1 \xrightarrow{\text{id}} \mathbb{A}^1]).$$

We use the first relation to compute the Hodge–Deligne polynomial of the smooth space $\text{Quot}_C(E, n)$ for a proper curve C (Proposition 3.5). We stress that the formula for Z_E in the proper case was already implicit in the calculation of [1, Prop. 4.5].

In Section 3.3 we discuss the case $r = 1$ on a surface, where we find $\Omega_{1,n} = \mathbb{L}^{n-1}$ according to Göttsche’s formula [7]. Finally, we conclude by proposing a geometric open problem related to punctual Quot schemes on curves.

We work over the field of complex numbers throughout.

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1. MOTIVIC PRELIMINARIES

In this section we recall a few motivic constructions that will be needed later. Most of this material is a simplified version of [6, Section 1], adapted to suit the purposes of this paper.

1.1. The Grothendieck ring of varieties. Fix a complex scheme S locally of finite type over \mathbb{C} . The Grothendieck ring of S -varieties

$$K_0(\text{Var}_S)$$

is the free abelian group generated by isomorphism classes $[X \rightarrow S]$ of S -varieties modulo the *scissor relations*, namely the identities

$$[X \xrightarrow{f} S] = [Y \xrightarrow{f|_Y} S] + [X \setminus Y \xrightarrow{f|_{X \setminus Y}} S]$$

imposed whenever $Y \subset X$ is a closed S -subvariety of X . The ring structure is given on generators by fibre product over S ,

$$(1.1) \quad [X \rightarrow S] \cdot [Y \rightarrow S] = [X \times_S Y \rightarrow S].$$

The element

$$\mathbb{L} = [\mathbb{A}^1 \times_{\mathbb{C}} S \rightarrow S] \in K_0(\text{Var}_S)$$

is called the *Lefschetz motive* (over S). If S' is another complex scheme, there is an external product

$$(1.2) \quad K_0(\text{Var}_S) \times K_0(\text{Var}_{S'}) \xrightarrow{\boxtimes} K_0(\text{Var}_{S \times S'})$$

defined on generators by sending $([f: X \rightarrow S], [g: X' \rightarrow S']) \mapsto [f \times g: X \times X' \rightarrow S \times S']$.

A morphism $f: S \rightarrow T$ induces a ring homomorphism $f^*: K_0(\text{Var}_T) \rightarrow K_0(\text{Var}_S)$ by base change and a $K_0(\text{Var}_T)$ -linear map $f_!: K_0(\text{Var}_S) \rightarrow K_0(\text{Var}_T)$ defined on generators by composition with f .

Definition 1.1. We denote by $S_0(\text{Var}_S)$ the semigroup of *effective* motives, i.e. the semigroup generated by isomorphism classes $[X \rightarrow S]$ of complex quasi-projective S -varieties modulo the scissor relations. The product (1.1) turns $S_0(\text{Var}_S)$ into a semiring. There is a natural semiring map $S_0(\text{Var}_S) \rightarrow K_0(\text{Var}_S)$, and we say that $\alpha \in K_0(\text{Var}_S)$ is *effective* if it lies in the image of this map.

1.2. Equivariant motives and the quotient map. Recall that if S is a scheme with a *good* action by a finite group G (i.e. an action such that every point of S has an affine G -invariant open neighborhood), the quotient S/G exists as a scheme. For instance, finite group actions on quasi-projective varieties are good.

Definition 1.2. Let G be a finite group, S a scheme with good G -action. We denote by $\tilde{K}_0^G(\text{Var}_S)$ the free abelian group generated by isomorphism classes $[X \rightarrow S]$ of G -equivariant S -varieties with good action, modulo the G -equivariant scissor relations. We denote by $K_0^G(\text{Var}_S)$ the quotient of $\tilde{K}_0^G(\text{Var}_S)$ by the relations

$$[V \rightarrow X \rightarrow S] = [\mathbb{A}_X^r \rightarrow S],$$

where $V \rightarrow X$ is a G -equivariant vector bundle of rank r over a G -equivariant S -variety X .

There is a natural ring structure on $\tilde{K}_0^G(\text{Var}_S)$, where the product of two classes $[X \rightarrow S]$ and $[Y \rightarrow S]$ is given by taking the diagonal action on $X \times_S Y$. The structures f^* , $f_!$ and \boxtimes naturally extend to the equivariant setting, along with their basic compatibilities. For instance, if $f: S \rightarrow T$ (resp. $g: S' \rightarrow T'$) is a G -equivariant (resp. G' -equivariant) map, and u, v are equivariant motives over S, S' , then

$$(1.3) \quad (f \times g)_!(u \boxtimes v) = f_! u \boxtimes g_! v$$

in the $(G \times G')$ -equivariant K -group over $T \times T'$.

One can define a $K_0(\text{Var}_{S/G})$ -linear map (cf. [6, Lemma 1.5])

$$(1.4) \quad \pi_G: \tilde{K}_0^G(\text{Var}_S) \rightarrow K_0(\text{Var}_{S/G})$$

given on generators by taking the orbit space,

$$\pi_G[X \rightarrow S] = [X/G \rightarrow S/G].$$

This map does not always extend to $K_0^G(\text{Var}_S)$. It does when G acts freely on S , by [5, Lemma 3.2].

1.3. Lambda ring structures. Let $n > 0$ be an integer, and let \mathfrak{S}_n be the symmetric group of n elements. By [6, Lemma 1.6], namely the relative version of [3, Lemma 2.4], there exist “ n -th power” maps

$$(1.5) \quad (\cdot)^{\otimes n}: K_0(\text{Var}_S) \rightarrow \tilde{K}_0^{\mathfrak{S}_n}(\text{Var}_{S^n})$$

where $S^n = S \times \cdots \times S$ is endowed with the natural \mathfrak{S}_n -action. The power map takes $[f: X \rightarrow S]$ to the class of the equivariant function $f^n: X^n \rightarrow S^n$. For $A \in K_0(\text{Var}_S)$, consider the classes

$$\pi_{\mathfrak{S}_n}(A^{\otimes n}) \in K_0(\text{Var}_{S^n/\mathfrak{S}_n}).$$

The *lambda ring* operations on $K_0(\text{Var}_{\mathbb{C}})$ are defined by

$$A \mapsto \sigma^n(A) = \pi_{\mathfrak{S}_n}(A^{\otimes n}) \in K_0(\text{Var}_{\mathbb{C}})$$

for effective classes $A \in K_0(\text{Var}_{\mathbb{C}})$, and then taking the unique extension to a lambda ring structure on $K_0(\text{Var}_{\mathbb{C}})$, determined by the relation

$$(1.6) \quad \sum_{i=0}^n \sigma^i([X] - [Y]) \sigma^{n-i}[Y] = \sigma^n[X].$$

If S comes with a commutative associative map $\nu: S \times S \rightarrow S$, we likewise define

$$\sigma_\nu^n(A) = \bar{\nu}_! \pi_{\mathfrak{S}_n}(A^{\otimes n}) \in K_0(\text{Var}_S)$$

on effective classes $A = [X \rightarrow S]$, where $\bar{\nu}$ is the map $S^n/\mathfrak{S}_n \rightarrow S$. One then uses the analogue of the relation (1.6) to find a unique set of lambda ring operators σ_ν^n restricting to the previous identity on effective motives.

As a special case, one can consider $(S, \nu) = (\mathbb{N}, +)$, viewed as a symmetric monoid in the category of schemes. We obtain lambda operations $\sigma^n = \sigma_+^n$ on $K_0(\text{Var}_{\mathbb{C}})[[t]]$ via the isomorphism

$$(1.7) \quad K_0(\text{Var}_{\mathbb{C}})[[t]] \xrightarrow{\sim} K_0(\text{Var}_{\mathbb{N}})$$

defined by sending $\sum_{n \geq 0} [Y_n] t^n \mapsto [\coprod_{n \in \mathbb{N}} Y_n \rightarrow \{n\}]$.

1.4. Power structures. The main references for power structures are [9, 10].

Definition 1.3 ([9]). A *power structure* on a (semi)ring R is a map

$$\begin{aligned} (1 + tR[[t]]) \times R &\rightarrow 1 + tR[[t]] \\ (A(t), m) &\mapsto A(t)^m \end{aligned}$$

satisfying the following conditions:

- (1) $A(t)^0 = 1$,
- (2) $A(t)^1 = A(t)$,
- (3) $(A(t) \cdot B(t))^m = A(t)^m \cdot B(t)^m$,
- (4) $A(t)^{m+m'} = A(t)^m \cdot A(t)^{m'}$,
- (5) $A(t)^{mm'} = (A(t)^m)^{m'}$,
- (6) $(1+t)^m = 1 + mt + O(t^2)$,
- (7) $A(t)^m|_{t \rightarrow t^e} = A(t^e)^m$.

Throughout we use the following:

Notation 1.4. Partitions $\alpha \vdash n$ are written as $\alpha = (1^{\alpha_1} \dots i^{\alpha_i} \dots s^{\alpha_s})$, meaning that there are α_i parts of size i . In particular we recover $n = \sum_i i\alpha_i$. The *automorphism group* of α is the product of symmetric groups $G_\alpha = \prod_i \mathfrak{S}_{\alpha_i}$.

Example 1.5. If $R = \mathbb{Z}$, $A(t) = 1 + \sum_{n>0} A_n t^n \in \mathbb{Z}[[t]]$ and $m \in \mathbb{N}$, the known formula [16, p. 40]

$$A(t)^m = 1 + \sum_{n \geq 0} \sum_{\alpha \vdash n} \left(\prod_{i=0}^{||\alpha||-1} (m-i) \cdot \frac{\prod_i A_i^{\alpha_i}}{\prod_i \alpha_i!} \right) t^n$$

defines a power structure on \mathbb{Z} , where we have set $||\alpha|| = \sum_i \alpha_i$.

Gusein-Zade, Luengo and Melle-Hernández have proved [9, Thm. 2] that there is a unique power structure

$$(A(t), m) \mapsto A(t)^m$$

on $K_0(\text{Var}_{\mathbb{C}})$ extending the one defined in *loc. cit.* on the semiring $S_0(\text{Var}_{\mathbb{C}})$ of effective motives. The latter is given by the formula

$$(1.8) \quad A(t)^{[X]} = 1 + \sum_{n \geq 0} \sum_{\alpha \vdash n} \pi_{G_\alpha} \left(\left[\prod_i X^{\alpha_i} \setminus \Delta \right] \cdot \prod_i A_i^{\otimes \alpha_i} \right) t^n.$$

Here, $\Delta \subset \prod_i X^{\alpha_i}$ is the “big diagonal” (the locus in the product where at least two entries are equal), and the product in big round brackets is a G_α -equivariant motive in $\tilde{K}_0^{G_\alpha}(\text{Var}_{\mathbb{C}})$, thanks to the power map (1.5).

Remark 1.6. We will not encounter non-effective coefficients in this paper, so we will have direct access to Formula (1.8).

1.5. Motivic exponential. The *motivic exponential* is a group isomorphism

$$\text{Exp}: t K_0(\text{Var}_{\mathbb{C}})[[t]] \xrightarrow{\sim} 1 + t K_0(\text{Var}_{\mathbb{C}})[[t]],$$

converting sums into products and preserving effectiveness. If $A = \sum_{n>0} A_n t^n$ is an effective power series, one has by definition

$$\text{Exp}\left(\sum_{n>0} A_n t^n\right) = \prod_{n>0} (1 - t^n)^{-A_n},$$

and if A and B are effective, one sets

$$(1.9) \quad \text{Exp}(A - B) = \prod_{n>0} (1 - t^n)^{-A_n} \cdot \left(\prod_{n>0} (1 - t^n)^{-B_n} \right)^{-1}.$$

More generally, if $(S, \nu: S \times S \rightarrow S)$ is a commutative monoid in the category of schemes, with a submonoid $S_+ \subset S$ such that the induced map $\coprod_{n \geq 1} S_+^{\times n} \rightarrow S$ is of finite type, we similarly define

$$\text{Exp}_{\nu}(A) = \sum_{n \geq 0} \sigma_{\nu}^n(A)$$

on effective classes, and for A and B two effective classes, we define $\text{Exp}_{\nu}(A - B)$ by the analogue of (1.9), i.e. by $\text{Exp}_{\nu}(A) \cdot \text{Exp}_{\nu}(B)^{-1}$.

1.6. Motives over symmetric products. The machinery described so far will be applied to the following situation. For a variety X , we will consider $(\text{Sym}(X), \cup)$, where

$$\text{Sym}(X) = \coprod_{n \geq 0} \text{Sym}^n(X)$$

can be viewed as a monoid via the morphism

$$\text{Sym}(X) \times \text{Sym}(X) \xrightarrow{\cup} \text{Sym}(X)$$

sending two zero-cycles (with multiplicities) on X to their union. The submonoid $\text{Sym}(X)_+ = \coprod_{n>0} \text{Sym}^n(X)$ allows one to construct the map Exp_{\cup} as in Section 1.5.

In order to recover a formal power series in $K_0(\text{Var}_{\mathbb{C}})[[t]]$ from a relative motive over $\text{Sym}(X)$, we consider the operation

$$(1.10) \quad \#_! \left(\sum_{n \geq 0} [Y_n \rightarrow \text{Sym}^n X] \right) = \sum_{n \geq 0} [Y_n] t^n.$$

In other words we take the direct image along the “tautological” map $\#: \text{Sym}(X) \rightarrow \mathbb{N}$ which collapses $\text{Sym}^n(X)$ onto the point n . In the right hand side of (1.10), we use the isomorphism (1.7) to identify relative motives over \mathbb{N} and formal power series with coefficients in $K_0(\text{Var}_{\mathbb{C}})$.

The following result, a special case of [6, Prop. 1.12], will be needed in the proof of Theorem 2.9.

Lemma 1.7. *Let U be a variety and let $\Delta_n: U \rightarrow \text{Sym}^n U$ be the small diagonal. Let $A = \sum_{n>0} A_n$ be an effective motive over $\mathbb{N}_{>0}$ and set $B = \text{Exp}(A) = 1 + \sum_{n>0} B_n$. Define*

$$Z = \sum_{n \geq 0} \sum_{\alpha \vdash n} \cup_! \pi_{G_{\alpha}} j_{\alpha}^* \left(\boxtimes_{i|\alpha_i \neq 0} \Delta_i! \left([U \xrightarrow{\text{id}} U] \boxtimes B_i \right)^{\otimes \alpha_i} \right) \in K_0(\text{Var}_{\text{Sym } U}),$$

where j_α is the G_α -equivariant open immersion $\prod_i \mathrm{Sym}^i(U)^{\alpha_i} \setminus \Delta \hookrightarrow \prod_i \mathrm{Sym}^i(U)^{\alpha_i}$. Then there is an identity

$$Z = \mathrm{Exp}_\cup \left(\sum_{n>0} A_n \boxtimes [U \xrightarrow{\Delta_n} \mathrm{Sym}^n U] \right)$$

Moreover,

$$\#_! \mathrm{Exp}_\cup \left(\sum_{n>0} A_n \boxtimes [U \xrightarrow{\Delta_n} \mathrm{Sym}^n U] \right) = B^{[U]} \in K_0(\mathrm{Var}_\mathbb{C})[[t]].$$

We briefly explain how to read the right hand side of the first equation of the lemma. First of all, we view \cup as a map $\mathrm{Sym}(U)^b \rightarrow \mathrm{Sym}(U)$ for any $b > 0$. The map π_{G_α} appearing in the definition of Z sends a G_α -equivariant relative motive over $\prod_i \mathrm{Sym}^i(U)^{\alpha_i} \setminus \Delta$ to a relative motive over $\prod_i \mathrm{Sym}^{i\alpha_i}(U) \setminus \Delta$, therefore we can apply the direct image $\cup_!$ to get a relative motive over $\mathrm{Sym}^n U$, where $n = \sum_i i\alpha_i$.

2. THE MOTIVE OF THE QUOT SCHEME

2.1. Main characters. Let X be a smooth quasi-projective variety of dimension d . Let E be a rank r locally free sheaf on X . For a given integer $n \geq 0$, the Quot scheme

$$\mathrm{Quot}_X(E, n)$$

parameterises quotients $E \twoheadrightarrow Q$ such that

$$\dim(\mathrm{Supp} Q) = 0, \quad \chi(Q) = n.$$

The Quot-to-Chow map

$$\sigma_n: \mathrm{Quot}_X(E, n) \rightarrow \mathrm{Sym}^n X$$

constructed in [8, Section 6] (see also [15, Cor. 7.15] for a modern treatment) takes a quotient $E \twoheadrightarrow Q$ to the zero-cycle (with multiplicities) determined by the set-theoretic support of Q . We define the *punctual Quot scheme* to be the preimage

$$\mathrm{Quot}_X(E, n)_p = \sigma_n^{-1}(n \cdot p)$$

of the cycle $n \cdot p \in \mathrm{Sym}^n X$, where $p \in X$ is a point. This is easily seen to only depend on a formal neighborhood of $p \in X$ (but not on p, X or E). In particular, one has isomorphisms

$$(2.1) \quad \mathrm{Quot}_X(E, n)_p \cong \mathrm{Quot}_X(\mathcal{O}_X^{\oplus r}, n)_p \cong \mathrm{Quot}_{\mathbb{A}^d}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)_0$$

where 0 is the origin in \mathbb{A}^d . This scheme will be denoted $P_{r,n}$ from now on, and

$$P_{r,n} = [P_{r,n}] \in K_0(\mathrm{Var}_\mathbb{C})$$

will denote its motive.

We pause for a second to explain how to prove the second isomorphism in (2.1). Using smoothness of X , we can fix étale coordinates around $p \in X$. This means we can find a pair (U, φ) where $p \in U \subset X$ is an open neighborhood and $\varphi: U \rightarrow \mathbb{A}^d$ is an étale map such that $\varphi(p) = 0 \in \mathbb{A}^d$. As in the proof of [2, Lemma A.1], we can further shrink U until $U \cap \varphi^{-1}(0)$ is the single point p . Then, we consider the open subscheme $W \subset \mathrm{Quot}_U(\mathcal{O}_U^{\oplus r}, n) \subset \mathrm{Quot}_X(\mathcal{O}_X^{\oplus r}, n)$ consisting of quotients $\mathcal{O}_U^{\oplus r} \twoheadrightarrow Q$ such that $\varphi|_{\mathrm{Supp} Q}$ is injective. Note that W contains $\mathrm{Quot}_U(\mathcal{O}_U^{\oplus r}, n)_p = \mathrm{Quot}_X(\mathcal{O}_X^{\oplus r}, n)_p$ as a closed subscheme. By [2, Proposition A.3], sending

$$(\mathcal{O}_U^{\oplus r} \twoheadrightarrow Q) \mapsto (\mathcal{O}_{\mathbb{A}^d}^{\oplus r} \rightarrow \varphi_* \varphi^* \mathcal{O}_U^{\oplus r} = \varphi_* \mathcal{O}_U^{\oplus r} \twoheadrightarrow \varphi_* Q)$$

defines an étale morphism $\Phi: W \rightarrow \text{Quot}_{\mathbb{A}^d}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)$. Its restriction

$$(2.2) \quad \Phi^{-1}(\text{Quot}_{\mathbb{A}^d}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)_0) \rightarrow \text{Quot}_{\mathbb{A}^d}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)_0$$

to the punctual Quot scheme of \mathbb{A}^d is étale and bijective, hence an isomorphism. For surjectivity, use that p is the only point in $U \cap \varphi^{-1}(0)$, and for injectivity use that $\varphi|_U$ is an immersion around p , so that $\varphi^* \varphi_* Q \xrightarrow{\sim} Q$ is an isomorphism for all Q supported entirely at p . Finally, again by our choice of U , the source of the morphism (2.2) is naturally identified with $\text{Quot}_U(\mathcal{O}_U^{\oplus r}, n)_p$.

Remark 2.1. The *punctual motives* $P_{r,n}$ clearly depend on the dimension $d = \dim X$, but we omit d from the notation.

Example 2.2. On a curve (i.e. if $d = 1$), by [1, Prop. 2.6] we have

$$(2.3) \quad P_{r,1} = [\mathbb{P}^{r-1}].$$

2.2. Absolute motives. Let X and E be as in the previous section. Define the generating functions

$$\begin{aligned} P_r(t) &= \sum_{n \geq 0} P_{r,n} t^n, \\ Z_E(t) &= \sum_{n \geq 0} [\text{Quot}_X(E, n)] t^n \end{aligned}$$

in the power series ring $K_0(\text{Var}_{\mathbb{C}})[[t]]$. The following result (namely Theorem A from the Introduction) is the higher rank analogue of the corresponding statement for the Hilbert scheme of points [10, Thm. 1], obtained by setting $r = 1$.

Theorem 2.3. *Let X be a smooth quasi-projective variety. Let E be a rank r locally free sheaf on X . There is an identity*

$$(2.4) \quad Z_E(t) = P_r(t)^{[X]}.$$

Proof. For α a partition of n , let $\text{Sym}^\alpha X \subset \text{Sym}^n X$ be the locally closed subvariety parameterising zero-cycles whose support is distributed according to α . We get a motivic decomposition

$$(2.5) \quad [\text{Quot}_X(E, n)] = \sum_{\alpha \vdash n} [\text{Quot}_X(E, n)_\alpha],$$

where we have set $\text{Quot}_X(E, n)_\alpha = \sigma_n^{-1}(\text{Sym}^\alpha X)$. By standard arguments (see e.g. [4, Sec. 4] and [13, Sec. 3]), one sees that the deepest stratum of the Quot-to-Chow map

$$\sigma_{(n)}: \text{Quot}_X(E, n)_{(n)} \rightarrow X$$

is a Zariski locally trivial fibration with fibre $P_{r,n}$. This relies on the local case $X = \mathbb{A}^d$, where one has a global decomposition

$$\text{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r}, n)_{(n)} \cong \mathbb{A}^d \times P_{r,n}$$

under which $\sigma_{(n)}$ is identified with the first projection.

For a fixed partition $\alpha \vdash n$, let

$$V_\alpha \hookrightarrow \prod_i \text{Quot}_X(E, i)^{\alpha_i}$$

be the open subscheme parameterising finite quotients with disjoint supports. By [2, Prop. A.3] (but see also [4, Lemma 4.10] for the Hilbert scheme version), taking the union of points gives an étale map

$$u_\alpha: V_\alpha \rightarrow \text{Quot}_X(E, n)$$

and we let U_α denote its image. The stratum $\text{Quot}_X(E, n)_\alpha$ sits inside U_α as a closed subscheme. We let the cartesian diagram

$$(2.6) \quad \begin{array}{ccc} Z_\alpha & \hookrightarrow & V_\alpha \\ \tilde{u}_\alpha \downarrow & \square & \downarrow u_\alpha \\ \text{Quot}_X(E, n)_\alpha & \hookrightarrow & U_\alpha \end{array}$$

define the scheme Z_α . The map \tilde{u}_α is a finite étale cover with Galois group G_α , in particular we have

$$(2.7) \quad \text{Quot}_X(E, n)_\alpha = Z_\alpha / G_\alpha.$$

In fact, Z_α can also be realised as the fibre product

$$(2.8) \quad \begin{array}{ccc} Z_\alpha & \hookrightarrow & \prod_i \text{Quot}_X(E, i)_{(i)}^{\alpha_i} \\ f_\alpha \downarrow & \square & \downarrow \\ \prod_i X^{\alpha_i} \setminus \Delta & \hookrightarrow & \prod_i X^{\alpha_i} \end{array}$$

where the bottom open immersion is the complement of the big diagonal and the map f_α is a G_α -equivariant piecewise trivial fibration with fibre $\prod_i P_{r,i}^{\alpha_i}$. This implies the identity

$$[Z_\alpha] = \left[\prod_i X^{\alpha_i} \setminus \Delta \right] \cdot \prod_i P_{r,i}^{\otimes \alpha_i}$$

in $K_0^{G_\alpha}(\text{Var}_{\mathbb{C}})$. Using (2.7), it follows that

$$\begin{aligned} [\text{Quot}_X(E, n)_\alpha] &= \pi_{G_\alpha}[Z_\alpha] \\ &= \pi_{G_\alpha} \left(\left[\prod_i X^{\alpha_i} \setminus \Delta \right] \cdot \prod_i P_{r,i}^{\otimes \alpha_i} \right), \end{aligned}$$

where $\pi_{G_\alpha}: K_0^{G_\alpha}(\text{Var}_{\mathbb{C}}) \rightarrow K_0(\text{Var}_{\mathbb{C}})$ is the quotient map extending (1.4). Since the classes $P_{r,i}$ are effective, combining the decomposition (2.5) with the power structure formula (1.8) and summing over n proves the result. \square

The following is a generalisation of [1, Thm. 4.1] to arbitrary varieties.

Corollary 2.4. *The series $Z_E(t)$ does not depend on E . In particular, the identity*

$$[\text{Quot}_X(E, n)] = [\text{Quot}_X(\mathcal{O}_X^{\oplus r}, n)]$$

holds in $K_0(\text{Var}_{\mathbb{C}})$ for all locally free sheaves E of rank r on X .

Definition 2.5. Define absolute classes $\Omega_{r,n} \in K_0(\text{Var}_{\mathbb{C}})$ via

$$(2.9) \quad \text{Exp} \left(\sum_{n \geq 0} \Omega_{r,n} t^n \right) = P_r(t).$$

Remark 2.6. In terms of the motivic exponential, we can rephrase Equation (2.4) as

$$(2.10) \quad Z_E(t) = \text{Exp} \left([X] \sum_{n \geq 0} \Omega_{r,n} t^n \right).$$

It is then clear that to determine the series Z_E one has to compute the fully punctual classes $\Omega_{r,n}$. We will do this in the case of curves (for arbitrary r) in Section 3.1, and for surfaces (only for $r = 1$) in Section 3.3.

2.3. Relative motives. Let (X, E) be as in the previous sections. Consider the relative motive

$$Z_E^{\text{rel}} = \sum_{n \geq 0} \left[\text{Quot}_X(E, n) \xrightarrow{\sigma_n} \text{Sym}^n X \right] \in K_0(\text{Var}_{\text{Sym} X}).$$

In other words, $Z_E^{\text{rel}} = [\text{Quot}_X(E) \rightarrow \text{Sym} X]$, the class of $\text{Quot}_X(E) = \coprod_n \text{Quot}_X(E, n)$ over $\text{Sym} X$. Note that Z_E^{rel} is a refinement of Z_E , in the sense that

$$\#_! Z_E^{\text{rel}} = Z_E(t),$$

where $\#_!$ is the operation introduced in (1.10).

We simply write

$$Z^{\text{rel}}(X, r) = Z_{\mathcal{O}^{\oplus r}}^{\text{rel}}$$

when $E = \mathcal{O}^{\oplus r}$ is the trivial bundle over X . We will show below (Theorem 2.9) that the relative motive $Z^{\text{rel}}(\mathbb{A}^d, r) \in K_0(\text{Var}_{\text{Sym} \mathbb{A}^d})$ is generated under Exp_{\cup} by the motives $\Omega_{r,n}$ defined in (2.9), extended on the small diagonal

$$\mathbb{A}^d \xrightarrow{\Delta_n} \text{Sym}^n \mathbb{A}^d.$$

Example 2.7. Set $r = 1$, $d = 1$ (i.e. we consider line bundles on curves). Then $\text{Quot}_X(L, n) = \text{Hilb}^n X = \text{Sym}^n X$ for all line bundles L on X , and

$$Z^{\text{rel}}(X, 1) = Z_{\mathcal{O}_X}^{\text{rel}} = [\text{Sym} X \xrightarrow{\text{id}} \text{Sym} X] = \mathbb{1} \in K_0(\text{Var}_{\text{Sym} X}).$$

Pushing this forward via $\#$ yields

$$Z_{\mathcal{O}_X}(t) = \sum_{n \geq 0} [\text{Sym}^n X] t^n = \zeta_X(t),$$

the Kapranov *motivic zeta function* of the curve X .

Remark 2.8. By definition of the power structure and of the motivic exponential, one has

$$\zeta_Y(t) = (1 - t)^{-[Y]} = \text{Exp}([Y]t),$$

for every variety Y . Moreover, the identities

$$(2.11) \quad \zeta_Y(\mathbb{L}^s t) = \zeta_{\mathbb{A}^s \times Y}(t) = \text{Exp}(\mathbb{L}^s [Y]t)$$

hold in $K_0(\text{Var}_{\mathbb{C}})[[t]]$ for every $s \in \mathbb{N}$.

We now prove Theorem B from the Introduction.

Before we begin, let us observe that for a morphism of varieties $f: S \rightarrow T$ and an integer $n > 0$, there is a commutative diagram

$$(2.12) \quad \begin{array}{ccc} K_0(\text{Var}_S) & \xrightarrow{f_!} & K_0(\text{Var}_T) \\ (\cdot)^{\otimes n} \downarrow & \circlearrowleft & \downarrow (\cdot)^{\otimes n} \\ \tilde{K}_0^{\mathfrak{S}_n}(\text{Var}_{S^n}) & \xrightarrow{f_!^n} & \tilde{K}_0^{\mathfrak{S}_n}(\text{Var}_{T^n}) \end{array}$$

where $(\cdot)^{\otimes n}$ is the power map (1.5).

Theorem 2.9. *There is an identity*

$$Z^{\text{rel}}(\mathbb{A}^d, r) = \text{Exp}_{\cup} \left(\sum_{n \geq 0} \Omega_{r,n} \boxtimes [\mathbb{A}^d \xrightarrow{\Delta_n} \text{Sym}^n \mathbb{A}^d] \right) \in K_0(\text{Var}_{\text{Sym} \mathbb{A}^d}).$$

Proof. For a partition $\alpha \vdash n$, set $Q_{\alpha}^n = \text{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r}, n)_{\alpha}$. One has a decomposition

$$Z^{\text{rel}}(\mathbb{A}^d, r) = \sum_{n \geq 0} \sum_{\alpha \vdash n} [Q_{\alpha}^n \rightarrow \text{Sym}^n \mathbb{A}^d].$$

Let us consider the G_{α} -equivariant cartesian diagram

$$(2.13) \quad \begin{array}{ccc} Z_{\alpha} & \hookrightarrow & \prod_i Q_{(i)}^{\alpha_i} \\ \downarrow & \square & \downarrow \\ \prod_i (\mathbb{A}^d)^{\alpha_i} \setminus \Delta & \xhookrightarrow{\iota_{\alpha}} & \prod_i (\mathbb{A}^d)^{\alpha_i} \\ \downarrow \Delta & \square & \downarrow \Delta \\ \prod_i \text{Sym}^i(\mathbb{A}^d)^{\alpha_i} \setminus \Delta & \xhookrightarrow{j_{\alpha}} & \prod_i \text{Sym}^i(\mathbb{A}^d)^{\alpha_i} \end{array}$$

where the top square is Diagram (2.8), the horizontal maps are open immersions (the complements of the big diagonals) and the vertical inclusions are products of small diagonals. We have a base change identity

$$(2.14) \quad j_{\alpha}^* \Delta_{!} = \Delta_{!} \iota_{\alpha}^*.$$

On the deepest stratum, we have a commutative diagram

$$\begin{array}{ccc} Q_{(n)}^n & \xrightarrow{\sim} & \mathbb{A}^d \times P_{r,n} \\ \sigma_{(n)} \downarrow & \swarrow \text{pr}_1 & \\ \mathbb{A}^d & & \end{array}$$

inducing an identity

$$(2.15) \quad [Q_{(n)}^n \rightarrow \mathbb{A}^d] = [\mathbb{A}^d \xrightarrow{\text{id}} \mathbb{A}^d] \boxtimes P_{r,n} \in K_0(\text{Var}_{\mathbb{A}^d}).$$

For a general partition α of n , consider the equivariant motives

$$[Q_{(i)}^i \rightarrow \mathbb{A}^d]^{\otimes \alpha_i} \in \tilde{K}_0^{\mathfrak{S}_{\alpha_i}}(\text{Var}_{(\mathbb{A}^d)^{\alpha_i}}).$$

If ι_{α} is as in Diagram (2.13), one has

$$\Delta_{!} \left[Z_{\alpha} \rightarrow \prod_i (\mathbb{A}^d)^{\alpha_i} \setminus \Delta \right] = \Delta_{!} \iota_{\alpha}^* \left(\boxtimes_{i|\alpha_i \neq 0} [Q_{(i)}^i \rightarrow \mathbb{A}^d]^{\otimes \alpha_i} \right) \in \tilde{K}_0^{G_{\alpha}}(\text{Var}_{\prod_i \text{Sym}^i(\mathbb{A}^d)^{\alpha_i} \setminus \Delta}).$$

Applying the quotient map π_{G_α} to the last identity, followed by the pushforward along the union of points map, we obtain

$$\begin{aligned}
[Q_\alpha^n \rightarrow \text{Sym}^n \mathbb{A}^d] &= \cup_! \pi_{G_\alpha} \Delta_! \iota_\alpha^* \left(\boxtimes_{i|\alpha_i \neq 0} [Q_{(i)}^i \rightarrow \mathbb{A}^d]^{\otimes \alpha_i} \right) \\
&= \cup_! \pi_{G_\alpha} J_\alpha^* \Delta_! \left(\boxtimes_{i|\alpha_i \neq 0} [Q_{(i)}^i \rightarrow \mathbb{A}^d]^{\otimes \alpha_i} \right) && \text{by (2.14)} \\
&= \cup_! \pi_{G_\alpha} J_\alpha^* \left(\boxtimes_{i|\alpha_i \neq 0} (\Delta_i^{\alpha_i})_! [Q_{(i)}^i \rightarrow \mathbb{A}^d]^{\otimes \alpha_i} \right) && \text{by (1.3)} \\
&= \cup_! \pi_{G_\alpha} J_\alpha^* \left(\boxtimes_{i|\alpha_i \neq 0} (\Delta_i)_! [Q_{(i)}^i \rightarrow \mathbb{A}^d]^{\otimes \alpha_i} \right) && \text{by (2.12)} \\
&= \cup_! \pi_{G_\alpha} J_\alpha^* \left(\boxtimes_{i|\alpha_i \neq 0} \Delta_i! \left([\mathbb{A}^d \xrightarrow{\text{id}} \mathbb{A}^d] \boxtimes P_{r,i} \right)^{\otimes \alpha_i} \right) && \text{by (2.15)}
\end{aligned}$$

so that summing these classes over all partitions of integers and noting that $\Omega_{r,n}$ are effective (because $P_{r,n}$ are effective) yields precisely

$$\text{Exp}_\cup \left(\sum_{n>0} \Omega_{r,n} \boxtimes [\mathbb{A}^d \xrightarrow{\Delta_n} \text{Sym}^n \mathbb{A}^d] \right)$$

by an application of Lemma 1.7. \square

Remark 2.10. By the last part of Lemma 1.7, the theorem implies the formula

$$Z_{\mathcal{Q}^{\oplus r}}(t) = P_r(t)^{\mathbb{L}^d}$$

of Theorem 2.3 for $X = \mathbb{A}^d$.

2.4. Related work on more general Quot schemes. The theory developed so far relies crucially on the locally free assumption on E . Indeed, the isomorphisms (2.1) fail even if E is, say, reflexive but not locally free. However, the geometry of the Quot scheme can be interesting also in the non-locally free case. For instance, the Quot scheme of finite quotients of the ideal sheaf $\mathcal{I}_C \subset \mathcal{O}_Y$ of a smooth curve in a 3-fold Y has been studied in [13], where essential local triviality statements on the Quot-to-Chow morphism were proved (see e.g. Corollary 3.2 in *loc. cit.*). Moreover, in [12, Thm. 2.1] it is proven that $\text{Quot}_Y(\mathcal{I}_C, n)$ appears as the typical (scheme-theoretic) fibre of the Hilbert–Chow morphism $\text{Hilb}(Y) \rightarrow \text{Chow}(Y)$ in a neighborhood of the cycle of the smooth curve C . (This holds in all dimensions, not just 3-folds.) This was used to prove the C -local DT/PT correspondence for Calabi–Yau 3-folds [12, Thm. 1.1]. The (virtual) motivic theory of $\text{Quot}_Y(\mathcal{I}_C, n)$ was developed in [6].

The enumerative geometry of $\text{Quot}_X(E, n)$, for E a sheaf of homological dimension at most one on a 3-fold, was studied in [2] and related to the local Pandharipande–Thomas theory of X . The Appendix in *loc. cit.* develops the abstract theory comparing various Quot schemes of smooth quasi-projective varieties, and implicitly shows that the singularities of $\text{Quot}_X(E, n)$ only depend on n and $\dim X$. The (virtual) motivic theory in the locally free case for 3-folds was developed in [14], along with a construction of a virtual fundamental class on the Quot scheme.

3. CALCULATIONS: CURVES AND SURFACES

In this section we compute the fully punctual motives

$$\Omega_{r,n}$$

in the case of curves, for all $r > 0$ and $n > 0$, and in the case of surfaces for $r = 1$ and all $n > 0$.

3.1. The class of the Quot scheme on a curve. We fix a locally free sheaf E of rank r on a smooth quasi-projective curve C .

Lemma 3.1. *On a curve, we have*

$$\Omega_{r,1} = [\mathbb{P}^{r-1}].$$

Proof. By the properties of the power structure, one has

$$\begin{aligned} P_r(t) &= \prod_{n \geq 1} (1 - t^n)^{-\Omega_{r,n}} \\ &= \prod_{n \geq 1} (1 - t)^{-\Omega_{r,n}} \Big|_{t \rightarrow t^n} \\ &= \prod_{n \geq 1} (1 + \Omega_{r,n} t + \cdots) \Big|_{t \rightarrow t^n} \\ &= \prod_{n \geq 1} (1 + \Omega_{r,n} t^n + \cdots), \end{aligned}$$

which immediately implies

$$\Omega_{r,1} = P_{r,1}.$$

On the other hand, the equality $P_{r,1} = [\mathbb{P}^{r-1}]$ holds by (2.3). \square

We now reformulate (and generalise to the quasi-projective case) the main formula proved in [1, Prop. 4.5]. The following is Theorem C from the Introduction.

Theorem 3.2. *There is an identity*

$$(3.1) \quad Z_E(t) = \text{Exp}([C \times \mathbb{P}^{r-1}] t)$$

in $K_0(\text{Var}_{\mathbb{C}})[[t]]$. Moreover, in $K_0(\text{Var}_{\text{Sym } \mathbb{A}^1})$ there is an identity

$$Z^{\text{rel}}(\mathbb{A}^1, r) = \text{Exp}_{\cup}([\mathbb{P}^{r-1}] \boxtimes [\mathbb{A}^1 \xrightarrow{\text{id}} \mathbb{A}^1]).$$

Proof. By [1, Prop. 4.5], for projective C one has

$$(3.2) \quad [\text{Quot}_C(E, n)] = \sum_{n_1 + \cdots + n_r = n} [\text{Sym}^{n_1} C] \cdots [\text{Sym}^{n_r} C] \cdot \mathbb{L}^{\sum_{i=0}^{r-1} (i-1)n_i}.$$

and it is clear that the generating function $Z_E(t)$ of these motives can be expanded as a product of shifted motivic zeta functions. More precisely, one has

$$\begin{aligned} \sum_{n \geq 0} [\text{Quot}_C(E, n)] t^n &= \prod_{i=1}^r \zeta_C(\mathbb{L}^{i-1} t) \\ &= \prod_{i=1}^r \text{Exp}([C] \mathbb{L}^{i-1} \cdot t) \\ &= \text{Exp}\left([C] \sum_{i=1}^r \mathbb{L}^{i-1} \cdot t\right) \\ &= \text{Exp}([C \times \mathbb{P}^{r-1}] t), \end{aligned}$$

where the second equality follows by (2.11). So the statement is true when C is projective. In this case, comparing (3.1) with Equation (2.10) and using the injectivity of Exp , we obtain the identities

$$(3.3) \quad [C] \cdot \Omega_{r,n} = \begin{cases} [C] \cdot [\mathbb{P}^{r-1}] & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

By Equation (2.10), to prove the statement on an arbitrary C it is enough to show that

$$(3.4) \quad \Omega_{r,n} = \begin{cases} [\mathbb{P}^{r-1}] & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

By Lemma 3.1, we already know that $\Omega_{r,1} = [\mathbb{P}^{r-1}]$. Finally, the equation $\Omega_{r,n} = 0$ holds for $n > 1$ because $\Omega_{r,n}$ is effective. Indeed, write $\Omega_{r,n} = [Y]$ for a variety Y , so that $0 = [C] \cdot \Omega_{r,n} = [C \times Y]$. But the class of a variety vanishes if and only if the variety is empty, and this happens if and only if $Y = \emptyset$.

To prove the last assertion, it is enough to combine Theorem 2.9 with the relations (3.4). \square

Remark 3.3. The formula (3.2) is proved in [1] over a field k of arbitrary characteristic.

Remark 3.4. By Equation (3.4), the generating function of the punctual motives can be computed as

$$(3.5) \quad P_r(t) = \text{Exp}([\mathbb{P}^{r-1}])t = \zeta_{\mathbb{P}^{r-1}}(t) = \prod_{i=0}^{r-1} \frac{1}{1 - \mathbb{L}^i t}.$$

3.2. The Hodge numbers of the Quot scheme on a curve. The Hodge–Deligne polynomial (also called the E-polynomial) of a smooth complex projective variety Y is given by

$$E(Y; u, v) = \sum_{p,q} (-1)^{p+q} h^{p,q}(Y) u^p v^q$$

where $h^{p,q}(Y) = \dim_{\mathbb{C}} H^q(Y, \Omega_Y^p)$ are the Hodge numbers of Y . For instance, one has

$$(3.6) \quad \begin{aligned} E(\mathbb{P}^{r-1}; u, v) &= \sum_{i=0}^{r-1} u^i v^i, \\ E(C; u, v) &= 1 - gu - gv + uv, \end{aligned}$$

where C is a smooth projective curve of genus g . Sending $[Y] \mapsto E(Y; u, v)$ defines a motivic measure

$$K_0(\text{Var}_{\mathbb{C}}) \xrightarrow{E} \mathbb{Z}[u, v]$$

which is in fact a homomorphism of rings with power structure. The power structure on the polynomial ring $\mathbb{Z}[u, v]$ is determined by the formula

$$(1-t)^{-f(u,v)} = \prod_{i,j} (1 - u^i v^j t)^{-p_{ij}},$$

where we have written $f(u, v) = \sum_{i,j} p_{ij} u^i v^j$ for integers p_{ij} . This implies (cf. [10, Prop. 4]) the basic relation

$$(3.7) \quad E((1-t)^{-[Y]}) = (1-t)^{-E(Y;u,v)}.$$

Let C be a smooth projective curve of genus g , and let E be a rank r locally free sheaf on C . We compute the generating function

$$E_r(C, t) = \sum_{n \geq 0} E(\text{Quot}_C(E, n); u, v) t^n.$$

We already know this series does not depend on E .

Proposition 3.5. *There is an identity*

$$(3.8) \quad E_r(C, t) = \prod_{i=0}^{r-1} \frac{(1 - u^i v^{i+1} t)^g (1 - u^{i+1} v^i t)^g}{(1 - u^i v^i t)(1 - u^{i+1} v^{i+1} t)}$$

in the ring $\mathbb{Z}[u, v][[t]]$.

Proof. We have

$$\begin{aligned} E_r(C, t) &= E(\text{Exp}([C \times \mathbb{P}^{r-1}]t)) && \text{by (3.1)} \\ &= E((1-t)^{-[C \times \mathbb{P}^{r-1}]}) && \text{by definition of Exp} \\ &= (1-t)^{-E([C \times \mathbb{P}^{r-1}])} && \text{by (3.7)} \\ &= (1-t)^{-(1-g)u - g v + u v \sum_{i=0}^{r-1} u^i v^i}. \end{aligned}$$

We have used that E is a ring homomorphism and the identities (3.6) in the last step. The result now follows from direct computation and by definition of the power structure on $\mathbb{Z}[u, v]$. \square

Remark 3.6. Setting $u = v$ in Formula (3.8) one recovers the generating function of (signed) Poincaré polynomials computed in [1, Remark 4.6], namely

$$\sum_{n \geq 0} P(\text{Quot}_C(F, n), -u) t^n = \prod_{i=0}^{r-1} \frac{(1 - u^{2i+1} t)^{2g}}{(1 - u^{2i} t)(1 - u^{2i+2} t)}.$$

3.3. The Hilbert scheme of points on a surface. Let S be a smooth quasi-projective surface, and set $r = 1$, so that $\text{Quot}_S(L, n) = \text{Hilb}^n S$ for every line bundle L . We know by Formula (2.10) that

$$Z_{\mathcal{O}_S}(t) = \text{Exp} \left([S] \sum_{n > 0} \Omega_{1,n} t^n \right).$$

On the other hand, by Göttsche's formula [7],

$$Z_{\mathcal{O}_S}(t) = \text{Exp} \left(\frac{[S]t}{1 - \mathbb{L}t} \right) = \text{Exp} \left([S] \sum_{n > 0} \mathbb{L}^{n-1} t^n \right).$$

By the injectivity of Exp , we conclude that on a surface S one has

$$(\Omega_{1,n} - \mathbb{L}^{n-1})[S] = 0.$$

However, this relation holds universally for *every* quasi-projective surface, in particular for $S = \mathbb{P}^2$ and $S = \mathbb{A}^1 \times \mathbb{P}^1$. Therefore

$$(\Omega_{1,n} - \mathbb{L}^{n-1})(1 + \mathbb{L} + \mathbb{L}^2) = 0 = (\Omega_{1,n} - \mathbb{L}^{n-1})(\mathbb{L} + \mathbb{L}^2),$$

showing that

$$(3.9) \quad \Omega_{1,n} = \mathbb{L}^{n-1}, \quad n > 0.$$

In particular, we recover the known generating function of the motives of punctual Hilbert schemes, given by the formula

$$(3.10) \quad P_1(t) = \sum_{n \geq 0} [\mathrm{Hilb}^n(\mathbb{A}^2)_0] t^n = \prod_{n \geq 1} (1 - \mathbb{L}^{n-1} t^n)^{-1}.$$

Finally, we obtain the following relative statement.

Theorem 3.7. *There is an identity*

$$\sum_{n \geq 0} [\mathrm{Hilb}^n \mathbb{A}^2 \xrightarrow{\sigma_n} \mathrm{Sym}^n \mathbb{A}^2] = \mathrm{Exp}_{\cup} \left(\sum_{n > 0} \mathbb{L}^{n-1} \boxtimes [\mathbb{A}^2 \xrightarrow{\Delta_n} \mathrm{Sym}^n \mathbb{A}^2] \right)$$

in $K_0(\mathrm{Var}_{\mathrm{Sym} \mathbb{A}^2})$.

Proof. Combine Theorem 2.9 with Equation (3.9). \square

Remark 3.8. The relation (3.10) was already proved in [9], and it was exploited in [11] to provide a motivic check of the classification of modules of length 3 and 4 over the polynomial ring $k[x, y]$.

4. A MOTIVIC-TO-GEOMETRIC OPEN PROBLEM

Let C be a smooth quasi-projective curve. The punctual Quot scheme

$$P_{r,n} \subset \mathrm{Quot}_C(\mathcal{O}^{\oplus r}, n)$$

parameterises quotients $\mathcal{O}^{\oplus r} \rightarrow Q$ entirely supported at a single (fixed) point $p \in C$. As proved in [1, Prop. 2.6], one has

$$P_{r,1} = \mathbb{P}^{r-1}.$$

How can one describe $P_{r,n}$ for $n > 1$? The relation

$$P_r(t) = \mathrm{Exp}([\mathbb{P}^{r-1}]t) = \zeta_{\mathbb{P}^{r-1}}(t)$$

established in Equation (3.5) translates into the motivic identity

$$(4.1) \quad [P_{r,n}] = [\mathrm{Sym}^n \mathbb{P}^{r-1}] = [\mathrm{Sym}^n P_{r,1}].$$

It thus makes sense to ask the following:

Question 4.1. What is the geometric meaning of the relation (4.1)? Can one geometrically compare the schemes $P_{r,n}$ and $\mathrm{Sym}^n P_{r,1}$?

REFERENCES

- [1] Massimo Bagnarol, Barbara Fantechi, and Fabio Perroni, *On the motive of zero-dimensional Quot schemes on a curve*, New York J. Math. **26** (2020), no. 2020, 138–148.
- [2] Sjoerd Beentjes and Andrea T. Ricolfi, *Virtual counts on Quot schemes and the higher rank local DT/PT correspondence*, To appear in Math. Res. Lett., 2018.
- [3] Kai Behrend, Jim Bryan, and Balázs Szendrői, *Motivic degree zero Donaldson–Thomas invariants*, Invent. Math. **192** (2013), no. 1, 111–160.
- [4] Kai Behrend and Barbara Fantechi, *Symmetric obstruction theories and Hilbert schemes of points on three-folds*, Algebra Number Theory **2** (2008), 313–345.
- [5] Franziska Bittner, *On motivic zeta functions and the motivic nearby fiber*, Math. Z. **249** (2005), no. 1, 63–83.
- [6] Ben Davison and Andrea T. Ricolfi, *The local motivic DT/PT correspondence*, Preprint <https://arxiv.org/abs/1905.12458>, 2019.

- [7] Lothar Göttsche, *On the motive of the Hilbert scheme of points on a surface*, Math. Res. Lett. **8** (2001), no. 5-6, 613–627.
- [8] Alexander Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert*, Séminaire Bourbaki, Vol. 6, Soc. Math. France, Paris, 1995, pp. Exp. No. 221, 249–276.
- [9] Sabir M. Gusein-Zade, Ignacio Luengo, and Alejandro Melle-Hernández, *A power structure over the Grothendieck ring of varieties*, Math. Res. Lett. **11** (2004), no. 1, 49–57.
- [10] Sabir M. Gusein-Zade, Ignacio Luengo, and Alejandro Melle-Hernández, *Power structure over the Grothendieck ring of varieties and generating series of Hilbert schemes of points*, Mich. Math. J. **54** (2006), no. 2, 353–359.
- [11] Riccardo Moschetti and Andrea T. Ricolfi, *On coherent sheaves of small length on the affine plane*, Journal of Algebra **516** (2018), 471–489.
- [12] Andrea T. Ricolfi, *The DT/PT correspondence for smooth curves*, Math. Z. **290** (2018), no. 1-2, 699–710.
- [13] Andrea T. Ricolfi, *Local contributions to Donaldson–Thomas invariants*, Int. Math. Res. Not. IMRN **2018** (2018), no. 19, 5995–6025.
- [14] Andrea T. Ricolfi, *Virtual classes and virtual motives of Quot schemes on threefolds*, To appear in Adv. Math., 2019.
- [15] David Rydh, *Families of cycles and the Chow scheme*, Ph.D. thesis, KTH, Stockholm, 2008.
- [16] Richard P. Stanley, *Enumerative combinatorics*, vol. 2, Cambridge Univ. Press, Cambridge, 1999.

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