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THE HILBERT SCHEME OF HYPERELLIPTIC JACOBIANS AND MODULI OF PICARD SHEAVES

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ABSTRACT. Let C be a hyperelliptic curve embedded in its Jacobian J via an Abel–Jacobi map. We compute the scheme structure of the Hilbert scheme component of Hilb_J containing the Abel–Jacobi embedding as a point. We relate the result to the ramification (and to the fibres) of the Torelli morphism $\mathcal{M}_g \to \mathcal{A}_g$ along the hyperelliptic locus. As an application, we determine the scheme structure of the moduli space of Picard sheaves (introduced by Mukai) on a hyperelliptic Jacobian.

CONTENTS

0. INTRODUCTION

Main result. In this paper we study the deformation theory of a smooth *hyperelliptic* curve C of genus $g \geq 2$, embedded in its Jacobian $J = (\mathrm{Pic}^0 C, \Theta_C)$ via an Abel–Jacobi map

$$\mathsf{aj}\colon C \hookrightarrow J.$$

We work over an algebraically closed field k of characteristic different from 2. Our aim is to compute the scheme structure of the Hilbert scheme component

$$\mathrm{Hilb}_{C/J} \subset \mathrm{Hilb}_J$$

containing the point defined by aj. It is well known that the embedded deformations of C into J are parametrised by translations of C, and that they are *obstructed* as long as $g \geq 3$ (see the next section for more details). In other words $\mathrm{Hilb}_{C/J}$ is *singular*, with reduced underlying variety isomorphic to J. The tangent space dimension to the Hilbert scheme has been computed in [12, 8]. The result is

$$\dim_k H^0(C, N_C) = 2g-2.$$

Therefore, as $\dim J = g$, the non-reduced structure of $\mathrm{Hilb}_{C/J}$ along J is accounted for (up to first order) by $g-2$ extra tangents. By homogeneity of the Jacobian, it is natural to expect a decomposition

$$\mathrm{Hilb}_{C/J} = J \times R_g$$

for some artinian scheme R_g with embedding dimension $g-2$. As we shall see, this is precisely what happens, and R_g turns out to be the "smallest" (in the sense of Lemma 3.3) artinian scheme with the required embedding dimension. More precisely, let

$$R_g = \mathrm{Spec}\, k[s_1, \ldots, s_{g-2}]/\mathfrak{m}^2, \tag{0.1}$$

where $\mathfrak{m} = (s_1, \ldots, s_{g-2})$ is the maximal ideal of the origin. The main result of this paper (proved in Theorem 3.6 in the main body) is the following.

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THEOREM 1. *Let C be a hyperelliptic curve of genus $g \geq 2$ over a field k of characteristic different from 2, and let J be its Jacobian. Then there is an isomorphism of k-schemes*

$$\mathrm{Hilb}_{C/J} \cong J \times R_g,$$

where R_g is the artinian scheme (0.1)*.*

Interpretation. Let $\mathcal{M}_g$ be the moduli stack of smooth curves of genus g, and let $\mathcal{A}_g$ be the moduli stack of principally polarised abelian varieties of dimension g. The Torelli morphism

$$\tau_g : \mathcal{M}_g \to \mathcal{A}_g$$

sends a curve C to its Jacobian $J = \mathrm{Pic}^0 C$, principally polarised by the Theta divisor Θ_C. One can interpret the artinian scheme R_g as the fibre of τ_g over a hyperelliptic point $[J, \Theta_C] \in \mathcal{A}_g$. This makes explicit the link between the *ramification* of τ_g along the hyperelliptic locus (in other words, the failure of the infinitesimal Torelli property) and the singularities of the Hilbert scheme $\mathrm{Hilb}_{C/J}$ (in other words, the *obstructions* to deform C in J). We come back to this in Section 3.2.

Moduli of Picard sheaves. As an application of our result, in Section 4 we compute the scheme structure of certain moduli spaces of *Picard sheaves* on a hyperelliptic Jacobian J. Mukai introduced these spaces as an application of his Fourier transform; he completed their study in the non-hyperelliptic case [13, 14], leaving open the hyperelliptic one.

Let F be the Fourier–Mukai transform of a line bundle $\xi = \mathscr{O}_C(d p_0)$, where $p_0 \in C$ and we assume $1 \leq d \leq g-1$ to ensure that F is a simple sheaf on J. Let $M(F)$ be the connected component of the moduli space of simple sheaves containing the point $[F]$. Mukai proved that $M(F)_{\mathrm{red}} = \widehat{J} \times J$, the isomorphism being given by the family of twists and translations of F [14, Example 1.15]. Under the same assumptions of Theorem 1, we prove the following (cf. Theorem 4.2 in the main body of the text).

THEOREM 2. *There is an isomorphism of k-schemes*

$$M(F) \cong \widehat{J} \times J \times R_g.$$

Enumerative Geometry of abelian varieties. A motivation for understanding the *scheme structure* of classical moduli spaces such as the Hilbert scheme (Theorem 1) and the moduli space of Picard sheaves (Theorem 2) comes from the subject of Enumerative Geometry of abelian varieties.

For instance, the Hilbert scheme of curves (in a 3-fold) is the main player in Donaldson–Thomas theory — see, for instance, [4] for an exhaustive treatment (including several interesting conjectures) of the Enumerative Geometry of curves on abelian surfaces and 3-folds. Understanding the scheme structure (or even the closed points!) of the Hilbert scheme of curves on a 3-fold is very often a hopeless problem. Of course, Donaldson–Thomas theory has developed several sophisticated tools to deal with the lack of an explicit description of the Hilbert scheme; however, this paper shows that, at least for an arbitrary *Abel–Jacobi curve*, the Hilbert scheme can be described completely. Thus an immediate corollary of Theorem 1 is the explicit description of the Donaldson–Thomas theory of an Abel–Jacobi curve, cf. Section 3.3.

On the other hand, it is conceivable that the theory of *Picard sheaves*, arising as a direct application of the Fourier–Mukai transform, could be exploited to aim for a deeper understanding of the intersection theory and cohomology of Jacobians, and possibly their compactifications. Having at one's disposal global results such as Theorem 2 might allow one to treat the *whole* moduli space (the universal Jacobian over the moduli space of curves) at once in developing a theory of *tautological rings* for (possibly compactified, universal) Jacobians, by combining Fourier–Mukai techniques with suitable analogues of the intersection theoretic calculations carried out in [17].

Conventions. We work over an algebraically closed field k of characteristic $p \neq 2$. All curves are smooth and proper over k, they are (geometrically) connected, and their Jacobians are principally polarised by the Theta divisor.

1. RAMIFICATION OF TORELLI AND THE HILBERT SCHEME

In this section we provide the framework where the problem tackled in this paper naturally lives in.

1.1. **Deformations of Abel–Jacobi curves.** The following theorem was proved in the stated form by Lange–Sernesi, but see also the work of Griffiths [8].

THEOREM 1.1 ([12, Theorem 1.2]). *Let C be a smooth curve of genus $g \geq 3$.*

(i) *If C is non-hyperelliptic, then $\mathrm{Hilb}_{C/J}$ is smooth of dimension g.*
(ii) *If C is hyperelliptic, then $\mathrm{Hilb}_{C/J}$ is irreducible of dimension g and everywhere non-reduced, with Zariski tangent space of dimension $2g-2$.*

In both cases, the only deformations of C in J are translations.

The statement of Theorem 1.1 is proved over $\mathbb{C}$ in [12], but it holds over algebraically closed fields k of arbitrary characteristic. To see this, we need Collino's extension of the Ran–Matsusaka criterion for Jacobians to an arbitrary field, which we state here for completeness.

THEOREM 1.2 ([6]). *Let X be an abelian variety of dimension g over an algebraically closed field k. Let D be an effective 1-cycle generating X and let $\Theta \subset X$ be an ample divisor such that $D \cdot \Theta = g$. Then (X, Θ, D) is a Jacobian triple.*

Proof of Theorem 1.1. Let $C \to \operatorname{Spec} k$ be a smooth curve of genus g and fix an Abel–Jacobi map $C \hookrightarrow J$. Consider the normal bundle exact sequence

$$0 \to T_C \to T_J|_C \to N_C \to 0.$$

Since we have a canonical identification $T_J|_C = H^1(C, \mathscr{O}_C) \otimes_k \mathscr{O}_C$, the induced cohomology sequence is

$$0 \to H^1(C, \mathscr{O}_C) \to H^0(C, N_C) \xrightarrow{\partial} H^1(C, T_C) \xrightarrow{\sigma} H^1(C, \mathscr{O}_C)^{\otimes 2}. \tag{1.1}$$

Since $H^0(C, N_C)$ is the tangent space to the Hilbert scheme, and $\dim_k H^1(C, \mathscr{O}_C) = g$, it is clear that $\mathrm{Hilb}_{C/J}$ is smooth of dimension g if and only if $\partial = 0$, if and only if σ is injective. The map σ factors through the subspace $\mathrm{Sym}^2 H^1(C, \mathscr{O}_C)$, and its dual is the multiplication map

$$\mu_C \colon \mathrm{Sym}^2 H^0(C, K_C) \to H^0(C, K_C^2),$$

where K_C is the canonical line bundle of C. For a modern, fully detailed proof of the identification $\sigma^\vee = \mu_C$, we refer the reader to [11, Theorem 4.3]. By a theorem of Max Noether [2, Chapter III § 2], the map μ_C is surjective if and only if C is non-hyperelliptic (see also [8, 1] for different proofs). If C is hyperelliptic, the quotient $H^0(C, N_C)/H^1(C, \mathscr{O}_C) = \operatorname{Im} \partial$ has dimension $g-2$, as shown directly in [16, Section 2] by choosing appropriate bases of differentials. This proves part (i) of Theorem 1.1, along with the count $h^0(C, N_C) = 2g-2$ (and the non-reducedness statement) of part (ii). So in the non-hyperelliptic case, $\mathrm{Hilb}_{C/J}$ is smooth of dimension g.

To finish the proof of part (ii), suppose C is hyperelliptic, and let $D \subset J$ be a closed 1-dimensional k-subscheme defining a point of $\mathrm{Hilb}_{C/J}$. Then D is represented by the *minimal cohomology class*

$$\frac{\Theta_C^{g-1}}{(g-1)!}$$

on J. This implies at once that D generates J, and that $D \cdot \Theta_C = g$. Therefore, by Theorem 1.2, $(\mathrm{Pic}^0 D, \Theta_D)$ and (J, Θ_C) are isomorphic as principally polarised abelian varieties. By Torelli's theorem, this implies (using also that C is hyperelliptic) that D is a translate of C. Thus $\mathrm{Hilb}_{C/J}$ is irreducible of dimension g, and its k-points coincide with those of J. The result follows. □

COROLLARY 1.3. *Let J be the Jacobian of a non-hyperelliptic curve C. Then the family of translations of C inside J induces an isomorphism*

$$J \xrightarrow{\sim} \mathrm{Hilb}_{C/J}.$$

Proof. The natural morphism $h \colon J \to \mathrm{Hilb}_{C/J}$ is proper (since J is proper and the Hilbert scheme is proper, hence separated), injective on points and tangent spaces — since the tangent map at $0 \in J$ is the map $\mathrm{d}h \colon H^1(C, \mathscr{O}_C) \hookrightarrow H^0(C, N_C)$ in the sequence (1.1). Thus h is a closed immersion, in particular it is unramified. However, the proof of Theorem 1.1 shows that $h \colon J \to \mathrm{Hilb}_{C/J}$ is bijective and, since C is non-hyperelliptic, $\mathrm{d}h$ is an isomorphism. Thus h is an isomorphism. □

Remark 1.4. If C is a generic complex curve of genus at least 3, its 1-cycle on J is not algebraically equivalent to the cycle of $-C$ by a famous theorem of Ceresa [5]. Here $-C$ is the image of C under the automorphism $-1\colon J \to J$. Therefore the Hilbert scheme Hilb_J contains another component $\mathrm{Hilb}_{-C/J}$, disjoint from $\mathrm{Hilb}_{C/J}$ and still isomorphic to J.

1.2. **Torelli problems.** Consider the Torelli morphism

$$\tau_g\colon \mathcal{M}_g \to \mathcal{A}_g$$

from the stack of nonsingular curves of genus g to the stack of principally polarised abelian varieties, sending a curve to its (canonically polarised) Jacobian. The *infinitesimal Torelli problem* asks whether the Torelli morphism is an immersion. It is well known that τ_g is ramified along the hyperelliptic locus: this is again Noether's theorem, stating that μ_C, the *codifferential* of τ_g at $[C] \in \mathcal{M}_g$, is not surjective. So, even though τ_g is injective on geometric points by Torelli's theorem, it is not an immersion.

To sum up, we have the following. Let C be an arbitrary smooth curve of genus $g \geq 3$, and let J be its Jacobian. Then the following conditions are equivalent:

(i) C is hyperelliptic,
(ii) $\mathrm{Hilb}_{C/J}$ is singular at $[\mathsf{aj}\colon C \hookrightarrow J]$,
(iii) the embedded deformations of C into J are obstructed,
(iv) $\tau_g\colon \mathcal{M}_g \to \mathcal{A}_g$ is ramified at $[C]$,
(v) infinitesimal Torelli fails at C.

The *local Torelli problem* for curves, studied by Oort and Steenbrink in [16], asks whether the morphism

$$t_g\colon M_g \to A_g$$

between the coarse moduli spaces is an immersion. These schemes do not represent the corresponding moduli functors, so the local structure of t_g is not (directly) linked with deformation theory of curves and their Jacobians. However, introducing suitable level structures, one replaces the normal varieties M_g and A_g with smooth varieties

$$M_g^{(n)}, \quad A_g^{(n)}$$

that are *fine* moduli spaces for the corresponding moduli problem, and are étale over $\mathcal{M}_g$ and $\mathcal{A}_g$, respectively.

Let $p \geq 0$ be the characteristic of the base field. Oort and Steenbrink show that t_g is an immersion if $p = 0$. The answer to the local Torelli problem is also affirmative if $p > 2$, at almost all points of M_g. More precisely, t_g is an immersion at those points in M_g representing curves C such that $\mathrm{Aut}\, C$ has no elements of order p [16, Cor. 3.2]. Finally, t_g is *not* an immersion if $p = 2$ and $g \geq 5$ [16, Cor. 5.3].

2. Moduli spaces with level structures

In this section we introduce the moduli spaces of curves and abelian varieties we will be working with throughout.

2.1. **Level structures.** Let S be a scheme. An abelian scheme over S is a group scheme $X \to S$ which is smooth and proper and has geometrically connected fibres. We let $\widehat{X} \to S$ denote the dual abelian scheme. A polarisation on $X \to S$ is an S-morphism $\lambda\colon X \to \widehat{X}$ such that its restriction to every geometric point $s \in S$ is of the form

$$\phi_{\mathscr{L}}\colon X_s \to \widehat{X}_s, \quad x \mapsto \mathsf{t}_x^*\mathscr{L} \otimes \mathscr{L}^\vee,$$

for some ample line bundle $\mathscr{L}$ on X_s. Here and in what follows, t_x is the translation $y \mapsto x + y$ by the element $x \in X_s$. We say λ is *principal* if it is an isomorphism.

Fix an integer $n > 0$ and an abelian scheme $X \to S$ of relative dimension g. Multiplication by n is an S-morphism of group schemes

$$[n]\colon X \to X,$$

and we denote its kernel by X_n. Assuming n is not divisible by p, we have that X_n is an étale group scheme over S, locally isomorphic in the étale topology to the constant group scheme $(\mathbb{Z}/n\mathbb{Z})^{2g}$.

One has $\widehat{X}_n = X_n^D$, where the superscript D denotes the Cartier dual of a finite group scheme. Then any principal polarisation λ on X induces a skew-symmetric bilinear form

$$E_n\colon X_n \times_S X_n \xrightarrow{\mathrm{id}\times\lambda} X_n \times_S X_n^D \xrightarrow{e_n} \mu_n,$$

where e_n is the Weil pairing. The group $\mathbb{Z}/n\mathbb{Z}$ is Cartier dual to μ_n. We endow $(\mathbb{Z}/n\mathbb{Z})^g \xrightarrow{\sim} \mu_n^g$ with the standard symplectic structure, given by the $2g\times 2g$ matrix

$$\begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}.$$

Definition 2.1 ([16])**.** A (symplectic) level-n structure on a principally polarised abelian scheme $(X/S,\lambda)$ is a symplectic isomorphism

$$\alpha\colon (X_n, E_n) \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

A level-n structure on a smooth proper curve $\mathcal{C}\to S$ is a level structure on its Jacobian $\mathrm{Pic}^0(\mathcal{C}/S)\to S$.

Curves with level structure are represented by pairs (C,α). We consider (C,α) and (C',α') as being isomorphic if there is an isomorphism $u\colon C \xrightarrow{\sim} C'$ such that the induced isomorphism $J(u)\colon J' \xrightarrow{\sim} J$ between the Jacobians takes α' to α. An isomorphism between (X,λ,α) and (X',λ',α') is an isomorphism $(X',\lambda') \xrightarrow{\sim} (X,\lambda)$ of principally polarised abelian schemes, taking α' to α.

Remark 2.2. If C is a curve of genus $g\geq 3$ with trivial automorphism group, and α is a level structure on C, then (C,α) is not isomorphic to $(C,-\alpha)$. On the other hand, if J denotes the Jacobian of C, one has that (J,Θ_C,α) and $(J,\Theta_C,-\alpha)$ are isomorphic, because the automorphism $-1\colon J\to J$, defined globally on J, identifies the two pairs.

2.1.1. *Choice of level.* As indicated by Theorem 2.3 below, moduli spaces of curves and abelian varieties with level structure are well behaved when the condition $(p,n)=1$ is met. For later purposes, we need to strengthen the condition $(p,n)=1$. Note that $p=\mathrm{char}\,k$ is fixed, as well as the genus g. However, we are free to choose $n\geq 3$, and the condition we require is that the order of the symplectic group

$$|\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})| = n^{g^2}\cdot\prod_{i=1}^{g}(n^{2i}-1)$$

is not divisible by p. In particular, this implies $(p,n)=1$. From now on,

$$n \text{ is fixed in such a way that } p \text{ does not divide } |\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})|. \tag{2.1}$$

This condition implies that the symplectic group $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$ acts freely and transitively on the set of symplectic level-n structures on a smooth curve defined over k. This will be used in the proof of Lemma 2.5.

2.2. **Moduli spaces.** Let $\mathscr{M}_g^{(n)}$ be the functor $\mathrm{Sch}_k^{\mathrm{op}}\to\mathrm{Sets}$ sending a k-scheme S to the set of S-isomorphism classes of curves of genus g with level-n structure. Similarly, let $\mathscr{A}_g^{(n)}$ be the functor sending S to the set of S-isomorphism classes of principally polarised abelian schemes of relative dimension g over S equipped with a level-n structure.

THEOREM 2.3. *If $n\geq 3$ and $(p,n)=1$, the functors $\mathscr{M}_g^{(n)}$ and $\mathscr{A}_g^{(n)}$ are represented by smooth quasi-projective varieties $M_g^{(n)}$ and $A_g^{(n)}$ of dimensions $3g-3$ and $g(g+1)/2$ respectively.*

Proof. For the statement about $\mathscr{M}_g^{(n)}$ we refer to [18], whereas the one about $\mathscr{A}_g^{(n)}$ is [15, Theorem 7.9]. □

Consider the morphism

$$j_n\colon M_g^{(n)} \to A_g^{(n)} \tag{2.2}$$

sending a curve with level structure to its Jacobian, as usual principally polarised by the Theta divisor. The map j_n is generically of degree two onto its image, essentially because of Remark 2.2. To link it back to $t_g\colon M_g\to A_g$, Oort and Steenbrink form the geometric quotient

$$V^{(n)} = M_g^{(n)}/\Sigma,$$

where

$$\Sigma\colon M_g^{(n)} \to M_g^{(n)} \tag{2.3}$$

is the involution sending $[D,\beta] \mapsto [D,-\beta]$. Note that Σ is the identity if $g \le 2$. The map j_n factors through a morphism

$$\iota\colon V^{(n)} \to A_g^{(n)},$$

which turns out to be injective on geometric points [16, Lemma 1.11]. In fact, we need the following stronger statement.

THEOREM 2.4 ([16, Theorem 3.1]). *If $g \ge 2$ and $\operatorname{char} k \ne 2$ then ι is an immersion.*

Oort and Steenbrink use this result crucially to solve the local Torelli problem as we recalled in Section 1.2. For us, it is not important to have the statement of local Torelli (which strictly speaking only holds globally in characteristic 0): all we need in our argument is Theorem 2.4, which is why we require the base field k to have characteristic $p \ne 2$.

The following result was proven in [7, Prop. 5.8] in greater generality. We give a short proof here for the sake of completeness.

LEMMA 2.5. *The maps $\varphi\colon M_g^{(n)} \to \mathcal{M}_g$ and $\psi\colon A_g^{(n)} \to \mathcal{A}_g$ forgetting the level structure are étale.*

Proof. We start by showing that φ is flat. Choose an atlas for $\mathcal{M}_g$, that is, an étale surjective map $a\colon U \to \mathcal{M}_g$ from a scheme. Form the fibre square

$$\begin{tikzcd} V \arrow[r, "b"] \arrow[d] \arrow[dr, phantom, "\square"] & M_g^{(n)} \arrow[d, "\varphi"] \\ U \arrow[r, "a"'] & \mathcal{M}_g \end{tikzcd}$$

and pick a point $u \in U$, with image $y = a(u) \in \mathcal{M}_g$. The fibre $V_u \subset V$ is contained in $b^{-1}\varphi^{-1}(y)$, which is étale over $\varphi^{-1}(y)$ because b is étale. In particular, since $\varphi^{-1}(y)$ is finite, the same is true for V_u. Therefore $V \to U$ is a map of smooth varieties with fibres of the same dimension (zero); by "miracle flatness" [9, Prop. 15.4.2], it is flat; therefore φ is flat. On the other hand, the geometric fibres of φ are the symplectic groups $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$, and they are reduced by our choice of n (cf. (2.1) in Section 2.1.1). Hence φ is smooth of relative dimension zero, that is, étale. The same argument applies to the map ψ, with the symplectic group replaced by $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})/\pm\mathbb{1}$. $\square$

Remark 2.6. The maps $M_g^{(n)} \to M_g$ and $A_g^{(n)} \to A_g$ down to the coarse moduli schemes are still finite Galois covers, but they are not étale.

By Lemma 2.5, we can identity the tangent space to a point $[C,\alpha] \in M_g^{(n)}$ with the tangent space to its image $[C] \in \mathcal{M}_g$ under φ, and similarly on the abelian variety side. Moreover, the cartesian diagram

$$\begin{tikzcd} M_g^{(n)} \arrow[r, "j_n"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\square"] & A_g^{(n)} \arrow[d, "\psi"] \\ \mathcal{M}_g \arrow[r, "\tau_g"'] & \mathcal{A}_g \end{tikzcd} \tag{2.4}$$

allows us to identify the map

$$\sigma\colon H^1(C,T_C) \to \operatorname{Sym}^2 H^1(C,\mathscr{O}_C),$$

already appeared in (1.1), with the tangent map of j_n at a point $[C,\alpha]$. As we already mentioned, in [16, Section 2] it is shown that if C is hyperelliptic the kernel of σ has dimension $g-2$.

3. Proof of the main theorem

3.1. **Proof of Theorem 1.** Let C be a hyperelliptic curve of genus $g \geq 3$ and let J be its Jacobian. Fix an Abel–Jacobi embedding $C \hookrightarrow J$ and let

$$H := \mathrm{Hilb}_{C/J}$$

be the Hilbert scheme component containing such embedding as a point. Let

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook, "\iota"] \arrow[d] & H \times J \arrow[dl, "\mathrm{pr}_1"] \\ H & \end{tikzcd}$$

be the universal family over the Hilbert scheme.

Lemma 3.1. *The restriction morphism*

$$\iota^*\colon \mathrm{Pic}^0(H \times J/H) \to \mathrm{Pic}^0(\mathcal{Z}/H)$$

is an isomorphism of abelian schemes over H.

Proof. We use the *critère de platitude par fibres* [9, Théorème 11.3.10] in the following special case: suppose given a scheme S and an S-morphism $f\colon X \to Y$ such that: (a) X/S is finitely presented and flat, (b) Y/S is locally of finite type, and (c) $f_s\colon X_s \to Y_s$ is flat for each $s \in S$. Then f is flat. Applying this to $(S, f) = (H, \iota^*)$, we conclude that ι^* is flat. But $\mathrm{Pic}^0(H \times J/H)$ is isomorphic, over H, to the constant abelian scheme $H \times J$, and ι^* is an isomorphism on each fibre over H. Therefore it is a flat, unramified and bijective morphism, hence an isomorphism. $\square$

Let α be a fixed level-n structure on J, with $n \geq 3$ chosen as in Section 2.1.1. Form the constant level structure α_H on the abelian scheme $H \times J \to H$. Transferring the level structure α_H from $H \times J$ to $\mathrm{Pic}^0(\mathcal{Z}/H)$ using the isomorphism ι^* of Lemma 3.1, we can now regard $\mathcal{Z} \to H$ as a family of Abel–Jacobi curves with level-n structure. Since $M_g^{(n)}$ is a *fine* moduli space for these objects, we obtain a morphism

$$f\colon H \to M_g^{(n)}. \tag{3.1}$$

Note that the topological image of f is just the point $x \in M_g^{(n)}$ corresponding to $[C, \alpha]$. The tangent map $\mathrm{d}f$ at the point $[C] \in H$ is the connecting homomorphism

$$\partial\colon H^0(C, N_C) \to H^1(C, T_C),$$

already appeared in (1.1).

Our next goal is to view the Hilbert scheme H over a suitable artinian scheme R_g. Recall the Torelli type morphism j_n introduced in (2.2). We define

$$R_g \subset M_g^{(n)}$$

to be the scheme-theoretic fibre of j_n over the moduli point $[J, \alpha] \in A_g^{(n)}$. Let $y \in V^{(n)}$ be the image of the point $x = [C, \alpha]$ under the quotient map

$$M_g^{(n)} \to V^{(n)} = M_g^{(n)}/\Sigma,$$

where Σ is the involution first appeared in (2.3). During the proof of [16, Cor. 3.2] it is shown that one can choose local coordinates $t_1, \ldots, t_{3g-3}$ around x such that $\Sigma^* t_i = t_i$ if $i = 1, \ldots, 2g-1$ and $\Sigma^* t_i = -t_i$ if $i = 2g, \ldots, 3g-3$. Oort–Steenbrink deduce that

$$\widehat{\mathscr{O}}_y = \widehat{\mathscr{O}}_x^{\Sigma} = k[\![t_1, \ldots, t_{2g-1}, t_{2g}^2, t_{2g}t_{2g+1}, \ldots, t_{3g-3}^2]\!]. \tag{3.2}$$

Since we have a factorisation

$$j_n\colon M_g^{(n)} \to V^{(n)} \overset{\iota}{\hookrightarrow} A_g^{(n)}$$

where ι is an *immersion* by Theorem 2.4, we deduce from (3.2) that

$$R_g = \mathrm{Spec}\, k[s_1, \ldots, s_{g-2}]/\mathfrak{m}^2,$$

where $\mathfrak{m} = (s_1, \ldots, s_{g-2}) \subset k[s_1, \ldots, s_{g-2}]$. For instance, R_3 is the scheme of dual numbers $k[s]/s^2$, and if $g = 4$ we get the triple point $k[s,t]/(s^2, st, t^2)$.

Recall the cohomology sequence

$$0 \to H^1(C, \mathscr{O}_C) \to H^0(C, N_C) \xrightarrow{\partial} H^1(C, T_C) \xrightarrow{\sigma} H^1(C, \mathscr{O}_C)^{\otimes 2}, \tag{3.3}$$

where σ factors through $\operatorname{Sym}^2 H^1(C, \mathscr{O}_C)$, the tangent space of $\mathcal{A}_g$ at $[J, \Theta_C]$. Since C is hyperelliptic, the image of ∂ has dimension $g-2>0$. In other words, the differential $\partial = \mathrm{d}f$, where f was defined in (3.1), does not vanish at the point $[C] \in H$. Thus f is not scheme-theoretically constant, although $x = [C, \alpha] \in M_g^{(n)}$ is the only point in the image. On the other hand, the composition

$$j_n \circ f : H \to M_g^{(n)} \to A_g^{(n)}$$

is the constant morphism since its differential is identically zero. Indeed the composition

$$\sigma \circ \partial : H^0(C, N_C) \to H^1(C, T_C) \to \operatorname{Sym}^2 H^1(C, \mathscr{O}_C)$$

vanishes by exactness of (3.3). So the image point $[J, \alpha]$ does not deform even at first order, and we conclude that f factors through the scheme-theoretic fibre of j_n. This gives us a morphism

$$\pi : H \to R_g. \tag{3.4}$$

We will exploit the following technical lemma.

LEMMA 3.2 ([10, Lemma 1.10.1]). *Let R be the spectrum of a local ring, $p: U \to V$ a morphism over R, with $U \to R$ flat and proper. If the restriction $p_0: U_0 \to V_0$ of p over the closed point $0 \in R$ is an isomorphism, then p is an isomorphism.*

Recall that $J = H_{\mathrm{red}}$, so we have a closed immersion $J \hookrightarrow H$ (with empty complement). Consider the closed point $0 \in J$ corresponding to C. Let us fix a regular sequence $f_1, \ldots, f_g$ in the maximal ideal of $\mathscr{O}_{J,0}$. Choose lifts $\widetilde{f}_i \in \mathscr{O}_{H,0}$ along the natural surjection $\mathscr{O}_{H,0} \twoheadrightarrow \mathscr{O}_{J,0}$, for $i = 1, \ldots, g$. Then we consider the zero scheme

$$i : S_g = Z(\widetilde{f}_1, \ldots, \widetilde{f}_g) \hookrightarrow H, \tag{3.5}$$

the largest artinian scheme supported at $0 \in H$. We next show that the composition

$$\rho = \pi \circ i : S_g \hookrightarrow H \to R_g \tag{3.6}$$

is an isomorphism, where π is defined in (3.4). We will need the following lemma.

LEMMA 3.3. *Let $\ell : k[x_1, \ldots, x_d]/\mathfrak{m}^2 \twoheadrightarrow B$ be a surjection of local Artin k-algebras such that the differential $\mathrm{d}\ell$ is an isomorphism. Then ℓ is an isomorphism.*

Proof. Since $\mathrm{d}\ell$ is an isomorphism by assumption, B has embedding dimension d, hence it can be written as a quotient $k[x_1, \ldots, x_d]/I$, so that its maximal ideal is $\mathfrak{m}_B = \mathfrak{m}/I$. Starting from the surjection ℓ, it is then clear that $\mathfrak{m}^2 \subset I$, and we have to show the other inclusion. This follows from the chain of isomorphisms

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \mathfrak{m}_B/\mathfrak{m}_B^2 = \frac{\mathfrak{m}/I}{(\mathfrak{m}/I)^2} = \frac{\mathfrak{m}/I}{\mathfrak{m}^2/I \cap \mathfrak{m}^2} = \frac{\mathfrak{m}/\mathfrak{m}^2}{I/\mathfrak{m}^2},$$

where the first isomorphism is $(\mathrm{d}\ell)^\vee$. □

LEMMA 3.4. *The tangent map $\mathrm{d}\rho : T_{S_g} \to T_{R_g}$ is an isomorphism.*

Proof. The kernel of $H^1(C, T_C) \to H^1(C, \mathscr{O}_C)^{\otimes 2}$, namely the image of $\partial : H^0(C, N_C) \to H^1(C, T_C)$, is the tangent space T_{R_g} to the artinian scheme R_g, as the latter is by definition the fibre of j_n. We then have a direct sum decomposition $T_0 H = T_0 J \oplus T_{R_g}$. The intersection of S_g and J inside H is the reduced origin $0 \in J$, so the linear subspace $T_{S_g} \subset T_0 H$ intersects $T_0 J$ trivially, which implies that the tangent map

$$\mathrm{d}\rho : T_{S_g} \subset T_0 J \oplus T_{R_g} \to T_{R_g}$$

is injective. On the other hand, the inclusion $T_{S_g} \subset T_0 H$ is cut out by independent linear functions, again because $T_{S_g} \cap T_0 J = (0)$. It follows that the linear inclusion $T_{S_g} \subset T_0 H$ has codimension equal to $\dim T_0 J = g$, thus

$$\dim T_{S_g} = \dim T_0 H - g = g - 2 = \dim T_{R_g}.$$

The result follows. □

COROLLARY 3.5. *The map $\rho: S_g \to R_g$ of (3.6) is an isomorphism.*

Proof. The map ρ is proper, injective on points and, by Lemma 3.4, injective on tangent spaces. Then it is a closed immersion; in fact, by Lemma 3.4 again, it is an isomorphism on tangent spaces, so by Lemma 3.3 it is an isomorphism. □

The corollary yields a section of π,

$$s = i \circ \rho^{-1}: R_g \xrightarrow{\sim} S_g \hookrightarrow H,$$

which finally allows us to prove the main result of this paper.

THEOREM 3.6. *Let C he a hyperelliptic curve of genus $g \geq 2$, and let J be its Jacobian. Then there is an isomorphism of schemes*

$$J \times R_g \xrightarrow{\sim} H.$$

Proof. If $g = 2$, the Hilbert scheme is nonsingular because $\partial: H^0(C, N_C) \to H^1(C, T_C)$, the connecting homomorphism in (1.1), vanishes. If $g \geq 3$, consider the translation action $\mu: J \times H \to H$ by J on the Hilbert scheme and the composition

$$J \times R_g \xhookrightarrow{\mathrm{id}_J \times s} J \times H \xrightarrow{\mu} H,$$

viewed as a morphism over the artinian scheme R_g. Since it restricts to the identity id_J over the closed point of R_g, by Lemma 3.2 it must be an isomorphism. □

3.2. **Relation between Hilbert scheme and Torelli.** Let $z = [J, \Theta_C]$ be a point in the image of the Torelli morphism $\tau_g: \mathcal{M}_g \to \mathcal{A}_g$. The fibre of τ_g over $\operatorname{Spec} k(z) \to \mathcal{A}_g$ is, topologically, just a point, by Torellli's theorem. This point is scheme-theoretically reduced if C is non-hyperelliptic. However, thanks to the cartesian diagram (2.4), what we can observe is that $\tau_g^{-1}(z) = \mathcal{M}_g \times_{\mathcal{A}_g} \operatorname{Spec} k(z) \subset \mathcal{M}_g$ is the artinian scheme R_g when z represents a hyperelliptic Jacobian. Theorem 3.6 thus fully develops in a qualitative form the idea already present in [12], namely that understanding the ramification (the fibres) of the Torelli morphism is equivalent to understanding the singularities of the Hilbert scheme: what the present work shows is that these singularities are controlled by the artinian scheme R_g.

The results proved so far essentially show the following.

PROPOSITION 3.7. *Let C be a smooth curve of genus $g \geq 2$, and let J be its Jacobian. Then $\tau_g^{-1}([J, \Theta_C])$ is isomorphic to the largest closed subscheme of $\mathrm{Hilb}_{C/J}$ supported at $[\mathsf{aj}: C \hookrightarrow J]$.*

Proof. In the non-hyperelliptic case, we have $\tau_g^{-1}([J, \Theta_C]) \cong \operatorname{Spec} k$, because τ_g is unramified at $[C]$. The result then follows because $J \to \mathrm{Hilb}_{C/J}$ is an isomorphism (by Corollary 1.3). In the hyperelliptic case we get, using Lemma 3.5,

$$S_g \xrightarrow{\sim} R_g = \tau_g^{-1}([J, \Theta_C]),$$

where $S_g \subset \mathrm{Hilb}_{C/J}$, introduced in (3.5), is precisely the largest subscheme of the Hilbert scheme supported at $[\mathsf{aj}: C \hookrightarrow J]$. □

3.3. **Donaldson–Thomas invariants for Jacobians.** Let C be a smooth complex projective curve of genus 3. One can study the "C-local Donaldson–Thomas invariants" of the abelian 3-fold $J = \operatorname{Pic}^0 C$. As explained in [20, 19], these invariants are completely determined by the "BPS number" of the curve,

$$n_C = \nu_H(\mathscr{I}_C) \in \mathbb{Z},$$

in the sense that their generating function is equal to the rational function

$$n_C \cdot q^{-2}(1+q)^4.$$

Here $\nu_H: \mathrm{Hilb}_{C/J} \to \mathbb{Z}$ is the Behrend function of the Hilbert scheme. The Behrend function attached to a general finite type $\mathbb{C}$-scheme X is an invariant of the singularities of X. It was introduced in [3] and is now a key tool in Donaldson–Thomas theory. For a smooth scheme Y one has that ν_Y is the constant $(-1)^{\dim Y}$, and moreover $\nu_{X\times Y} = \nu_X \cdot \nu_Y$ for two complex schemes X and Y. While for non-hyperelliptic C we have $n_C = -1$ (because the Hilbert scheme is a copy of the smooth 3-fold J), the structure result

$$\mathrm{Hilb}_{C/J} = J \times \operatorname{Spec} \mathbb{C}[s]/s^2$$

in the hyperelliptic case yields $n_C = -2$, because the scheme of dual numbers has Behrend function $\nu_{R_3} = 2$.

4. An application to moduli spaces of Picard sheaves

Mukai introduced in [13] his celebrated Fourier transform, and gave an application to the moduli space of Picard sheaves on Jacobians of curves. We now review his results on non-hyperelliptic Jacobians and extend them to the hyperelliptic case. We assume that the base field k is, as ever, algebraically closed of characteristic different from 2.

We let $\Phi\colon \mathrm{D}^b(\widehat{J}) \to \mathrm{D}^b(J)$ be the Fourier transform with kernel the Poincaré line bundle $\mathscr{P} \in \operatorname{Pic}(\widehat{J} \times J)$. If $\widehat{\mathsf{p}}\colon \widehat{J} \times J \to \widehat{J}$ and $\mathsf{p}\colon \widehat{J} \times J \to J$ are the projections, by definition one has

$$\Phi(\mathscr{E}) = R\mathsf{p}_*(\widehat{\mathsf{p}}^*\mathscr{E} \otimes \mathscr{P}).$$

We will denote by $\Phi^i(\mathscr{E})$ the i-th cohomology sheaf of the complex $\Phi(\mathscr{E})$.

Let $p_0 \in C$ be a point on a smooth curve of genus $g \geq 2$. Let us form the line bundle $\xi = \mathscr{O}_C(d\,p_0)$. From now on we view it as a sheaf on $\widehat{J}$ by pushing it forward along the Abel–Jacobi map $\mathsf{aj}\colon C \hookrightarrow J$ followed by the identification of J with its dual. Applying his Fourier transform, Mukai constructs

$$F = \Phi^1(\mathsf{aj}_*\xi), \tag{4.1}$$

a *Picard sheaf* of rank $g - d - 1$ living on J. Assume $1 \leq d \leq g-1$, so that by [13, Lemma 4.9] we know that F is simple (that is, $\operatorname{End}_{\mathscr{O}_J}(F) = k$), and

$$\dim \operatorname{Ext}^1_{\mathscr{O}_J}(F,F) = \begin{cases} 2g & \text{if } C \text{ is not hyperelliptic} \\ 3g-2 & \text{if } C \text{ is hyperelliptic.} \end{cases} \tag{4.2}$$

Let Spl_J be the moduli space of simple coherent sheaves on J, and let $M(F) \subset \mathrm{Spl}_J$ be the connected component containing the point corresponding to F. It is shown in [13, Theorem 4.8] that if $g = 2$ or C is non-hyperelliptic, the morphism

$$f\colon \widehat{J} \times J \to M(F), \quad (\eta, x) \mapsto \mathsf{t}_x^* F \otimes \mathscr{P}_\eta, \tag{4.3}$$

is an isomorphism. By (4.2), the space $M(F)$ is reduced precisely when C has genus 2 or is non-hyperelliptic. For C hyperelliptic, f turns out to be an isomorphism onto the reduction $M(F)_{\mathrm{red}} \subsetneq M(F)$, as Mukai showed in [14, Example 1.15].

Remark 4.1. The moduli space $M(F)$ is a priori only an algebraic space. But an algebraic space is a scheme if and only if its reduction is a scheme. Therefore $M(F)$ is a scheme because of the isomorphism $\widehat{J} \times J \cong M(F)_{\mathrm{red}}$.

The following result, which can be seen as a corollary of Theorem 3.6, completes the study of Picard sheaves on Jacobians considered by Mukai, namely those of rank $g - d - 1$, with $d \leq g - 1$.

THEOREM 4.2. *Let C be a hyperelliptic curve of genus $g \geq 2$. Let J be its Jacobian and F a Picard sheaf as above. Then, as schemes,*

$$M(F) = \widehat{J} \times J \times R_g.$$

Proof. The case $g = 2$ is already covered by Mukai's tangent space calculation. By Theorem 3.6, it is enough to exhibit an isomorphism $\widehat{J} \times H \xrightarrow{\sim} M(F)$, where as usual $H \subset \mathrm{Hilb}_J$ is the Hilbert scheme component containing the Abel–Jacobi point $[C]$. We will do this by extending the morphism (4.3) defined by Mukai, that is, completing the diagram

$$\begin{tikzcd} \widehat{J} \times J \arrow[r, "\sim"] \arrow[d, hook] & M(F)_{\mathrm{red}} \arrow[d, hook] \\ \widehat{J} \times H \arrow[r, dashed, "\phi"] & M(F) \end{tikzcd} \tag{4.4}$$

and showing that the extension ϕ is an isomorphism. Recall that via the identification $J = H_{\mathrm{red}}$ we can identify a k-valued point $x \in J(k)$ with a k-valued point of H. Also, for any such point $x \in J \subset H$, we will use the notation $x + p_0$ for the point on the Abel–Jacobi curve $\mathrm{t}_x C \subset J$ obtained by translating $p_0 \in C \subset J$ via the automorphism $\mathrm{t}_x : J \to J$. Let

$$\mathcal{Z} \overset{\iota}{\hookrightarrow} H \times J \to H$$

be the universal family of the Hilbert scheme: the fibre of $\mathcal{Z} \to H$ over $\operatorname{Spec} k(x) \hookrightarrow H$ is the subscheme $\mathrm{t}_x C \subset J$, and ι, the universal Abel–Jacobi map, restricts to $\mathsf{aj} \circ \mathrm{t}_{-x} : \mathrm{t}_x C \to C \hookrightarrow \{x\} \times J$ over the point $x \in H$. We now construct a section σ of $\mathcal{Z} \to H$ restricting to the divisor $d p_0$ on C (in other words: a "universal" version of ξ). If $q : H \to J$ denotes the projection (forgetting the non-reduced structure) and $u : J \to J$ is the composition $\mathrm{t}_{dp_0} \circ [d]$, the section σ is the map

$$\sigma : H \xrightarrow{(1_H, q)} H \times J \xrightarrow{1_H \times u} H \times J, \quad x \mapsto (x, d(x + p_0)).$$

Here we view $d(x + p_0)$ as a degree d divisor on the translated Abel–Jacobi curve $\mathrm{t}_x C \subset J$, in particular the image of σ clearly lands inside $\mathcal{Z}$. Let $\mathscr{L} = \mathscr{O}_{\mathcal{Z}}(\sigma)$ be the associated line bundle on the total space $\mathcal{Z}$. Then, by construction, restricting $\mathscr{L}$ to a fibre of $\mathcal{Z} \to H$ we get

$$\mathscr{L}|_{\mathrm{t}_x C} = \mathscr{O}_{\mathrm{t}_x C}(d(x + p_0)) = \mathrm{t}_{-x}^* \xi. \tag{4.5}$$

If we consider the pushforward $\iota_* \mathscr{L}$ to $H \times J$, using Equation (4.5) we obtain

$$(\iota_* \mathscr{L})|_{x \times J} = (\mathsf{aj} \circ \mathrm{t}_{-x})_* (\mathscr{L}|_{\mathrm{t}_x C}) = \mathsf{aj}_* \xi. \tag{4.6}$$

Note that $\mathscr{L}$ is flat over H (because $\mathcal{Z} \to H$ is flat), therefore the same is true for $\iota_* \mathscr{L}$. Since taking the Fourier–Mukai transform commutes with base change, Equation (4.6) yields

$$\Phi^1(\iota_* \mathscr{L})|_{x \times J} = \Phi^1(\mathsf{aj}_* \xi) = F. \tag{4.7}$$

Now we consider the following diagram:

$$\begin{tikzcd}
 & (\widehat{J} \times J) \times J \arrow[r, "\sim"] \arrow[d, hook, "i"] & (J \times J) \times J \arrow[r, "m \times \mathrm{id}_J"] \arrow[d, hook] & J \times J \arrow[r, "\mathrm{pr}_1"] \arrow[d, hook] & J \\
\widehat{J} \times J & (\widehat{J} \times H) \times J \arrow[l, "\mathrm{pr}_{13}"'] \arrow[r, "\sim"] & (J \times H) \times J \arrow[r, "\mu \times \mathrm{id}_J"] & H \times J &
\end{tikzcd}$$

where m and μ are the translation actions by J on J and H respectively. The Fourier–Mukai transform $\Phi^1(\iota_* \mathscr{L})$ lives on $H \times J$ and is flat over H, by flatness of $\iota_* \mathscr{L}$. By (4.7), we know that the families of sheaves $\Phi^1(\iota_* \mathscr{L})|_{J \times J}$ and $\mathrm{pr}_1^* F$ (both flat over J) define the same morphism $J \to M(F)$, namely the constant morphism hitting the point $[F]$. Since Mukai's morphism $\widehat{J} \times J \to M(F)$, defined in (4.3), corresponds (after identifying J with its dual) to the family of sheaves

$$(m \times \mathrm{id}_J)^* \mathrm{pr}_1^* F \otimes (\mathrm{pr}_{13} \circ i)^* \mathscr{P},$$

it follows that the family

$$(\mu \times \mathrm{id}_J)^* \Phi^1(\iota_* \mathscr{L}) \otimes \mathrm{pr}_{13}^* \mathscr{P}$$

defines an extension $\phi : \widehat{J} \times H \to M(F)$, completing diagram (4.4). We know that ϕ is an isomorphism around $[\xi] \mapsto [F]$. Indeed, ϕ is precisely the morphism constructed by Mukai in [14, Prop. 1.12], where he proves that $M(\xi)$ and $M(F)$ are isomorphic along a Zariski open subset. The construction is homogeneous, in the sense that ϕ does not depend on the initial point $[\xi] \in M(\xi)$. Therefore ϕ is globally an isomorphism, as claimed. □

Remark 4.3. The connected component $M(\xi)$ of the moduli space of simple sheaves containing the point $[\xi]$ is the relative Picard variety $\mathrm{Pic}^d(\mathcal{Z}/H)$, which can be identified with $\widehat{J} \times H$ by Lemma 3.1. It is possible to adapt the proof of [14, Prop. 1.12] to show that the birational map

$$\mathrm{Pic}^d(\mathcal{Z}/H) \dashrightarrow M(F)$$

is everywhere defined (and an isomorphism), giving an immediate proof of Corollary 4.2. We preferred to present the argument above, because the construction makes the isomorphism $\phi : \widehat{J} \times H \to M(F)$ arise directly, as a "thickening" of Mukai's isomorphism $\widehat{J} \times J \to M(F)_{\mathrm{red}}$. Moreover the argument makes explicit use of (the properties of) the Fourier–Mukai transform.

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