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Optimal Control of Infinite-Dimensional Piecewise Deterministic Markov Processes: A BSDE Approach.
Application to the Control of an Excitable Cell Membrane

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Optimal control of infinite-dimensional Piecewise Deterministic Markov Processes: a BSDE approach. Application to the control of an excitable cell membrane.

Elena BANDINI*

Michèle THIEULLEN †

Abstract

In this paper we consider the optimal control of Hilbert space-valued infinite-dimensional Piecewise Deterministic Markov Processes (PDMP) and we prove that the corresponding value function can be represented via a Feynman-Kac type formula through the solution of a constrained Backward Stochastic Differential Equation. A fundamental step consists in showing that the corresponding integro-differential Hamilton-Jacobi-Bellman equation has a unique viscosity solution, by proving a suitable comparison theorem. We apply our results to the control of a PDMP Hodgkin-Huxley model with spatial component, previously studied in [23], [22] and inspired by optogenetics.

Keywords: infinite-dimensional PDMPs, constrained backward stochastic differential equations, integro-differential Hamilton-Jacobi-Bellman equation, viscosity solutions in infinite dimensions, spatio-temporal Hodgkin-Huxley models.

MSC 2010: 93E20, 60H10, 60J25.

1 Introduction

In this paper we consider optimal control problems for Hilbert space-valued infinite-dimensional Piecewise Deterministic Markov Processes, and we prove that the corresponding value function can be represented through a Feynman-Kac formula by means of the solution of a constrained Backward Stochastic Differential Equation (BSDE). As an intermediate step, we also show that the corresponding Hamilton-Jacobi-Bellmann (HJB) has a unique viscosity solution by providing a comparison theorem for suitable Integro Partial Differential Equations (IPDE). We apply our theoretical results to the control of a PDMP Hodgkin-Huxley model with spatial component, previously considered in [23], [22] and inspired by optogenetics.

The Feynman-Kac type representation for the value function is obtained by implementing the randomization procedure introduced in [20] for jump-diffusions, later extended in [5] and [4] respectively to the case of finite-dimensional pure jump Markov processes and of finite-dimensional PDMPs. The control randomization method is particularly useful to probabilistically represent the value function associated to stochastic control problems, where the laws of the family of controlled processes are not dominated by a common measure. Roughly speaking, the randomization principle consists in enlarging the state space by an additional independent piecewise constant component corresponding to the control, and in subsequently generating a family of dominated laws and an

*Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Via R. Cozzi 55, 20125 Milano, Italy; e-mail: elena.bandini@unimib.it

†Laboratoire de Probabilités, Statistique et Modélisation (LPSM, UMR 8001), Sorbonne Université - Campus Pierre et Marie Curie, Boite 158, 4 Place Jussieu, 75252 Paris Cedex 05, France; e-mail: michele.thieullen@upmc.fr

auxiliary control problem, where the cost is optimized with respect to the intensity of the extended pure jump component. The value function of this latter (randomized) control problem can be represented by means of the solution of a constrained BSDE, namely a backward equation driven by a random measure with a sign constraint on its martingale part. In order to be able to relate this backward equation to the HJB equation associated to the primal problem, one has to show that the randomized value function does not depend on the additional component, and that it provides a solution to the above-mentioned HJB equation. Afterwards, the Feynman-Kac representation formula for the original value function comes from the uniqueness of the viscosity solution to the corresponding HJB equation. We refer the reader to the introduction of [20] for an extended exposition of the issues involved. Note that the randomization procedure is a very general methodology which applies even if the laws of the controlled processes are dominated. The Feynman-Kac representation formula can be used to design algorithms based on the numerical approximation of the solution to the corresponding constrained BSDE, and therefore to get probabilistic numerical approximations for the value function of the addressed optimal control problem, see e.g. [21].

In our infinite-dimensional setting, we provide existence and uniqueness (in a suitable sense) of the solution of such a constrained BSDE and its independence with respect to the additional component. We also prove a randomized dynamic principle which enables us to establish that the value function of the randomized problem is a viscosity solution of the HJB-IPDE on the Hilbert space. Viscosity solutions for partial differential equations in infinite dimension with unbounded linear terms have been first studied in [10] and [11], where the notions of B -upper/lower-semicontinuity are introduced, and subsequently considered by many other authors, see e.g. [14] for a modern and detailed exposition on this topic. Recently the papers [27] and [28] have addressed respectively existence and uniqueness for an HJB-IPDE resulting from the control of an Hilbert space-valued SDE driven by a Lévy process. Notice that in our framework we do not ask that our PDMP is a strong solution to some SDE. Our approach is instead based on the study of the local characteristics of the PDMP in the spirit of the theory developed in [12]. We prove a comparison theorem which implies the uniqueness of the viscosity solution of our HJB-IPDE. The appropriate definition of viscosity solution, on which the comparison theorem relies, is derived suitably extending the one provided in [28].

Our theoretical results are applied to the control of a PDMP Hodgkin-Huxley model with spatial component. Hilbert space-valued PDMP models describing the spatio-temporal evolution of a neuron with a finite number of ion channels (or more general excitable membranes) have been rigorously settled in [7]. In particular it was proved in [24] that such PDMP converge to the spatio-temporal Hodgkin-Huxley model proposed in [15] when the number of channels goes to infinity, see also [2]. Inspired by optogenetics, optimal control of general infinite-dimensional PDMP has been previously considered in [23], [22]. In particular the results in [23] were applied to a tracking problem for a Hilbert space-valued Hodgkin-Huxley type PDMP. In that paper, as in the present one, piecewise open loop controls (see e.g. [29]) were considered, and the control acted on the three characteristics of the PDMP. However, the main tools were relaxed controls and the optimal control theory of Markov Decision Processes, see [6]. Moreover, even if an HJB-IPDE were written down in that paper, no study was conducted about existence or uniqueness of its solutions. We also mention the more recent paper [8], which exploits Markov Decision Processes in infinite dimension in the framework of stochastic filtering.

Many generalizations of the present work may be possible. For instance, it would be interesting to treat the general case with infinite-dimensional PDMPs on a state space with boundary, from which additional instantaneous jumps into the interior of the domain may occur (in the finite-dimensional case, this feature has been recently considered in [3]). Moreover, in our application section we have considered the classical case of a Laplacian operator, but other operators could be addressed as well. Finally, a challenging future development would consist in applying our results

to the infinite-dimensional PDMP that naturally arise in filtering problems.

The paper is organized as follows. In Section 2 we construct our infinite-dimensional controlled PDMP and we define the related optimal control problem. In particular, inspired by [18], we provide a canonical construction of the PDMP state process in infinite dimension, by suitably extending the finite-dimensional construction implemented in [4], [5]. We then set the associated control problem, and we establish in Theorem 2.11 that the corresponding value function is a viscosity solution of the HJB equation (2.19)-(2.20). In Section 3 we describe the control randomization method in our setting, and we introduce the randomized optimal control problem. Then in Section 4 we define and study the related constrained BSDE, and we address the Feynman-Kac representation. As described above, the first step of the randomization approach consists in proving that the randomized value function does not depend on the additional component, and that satisfies a suitable randomized dynamic programming principle, see respectively Proposition 4.2 and Theorem 4.3. Then in Theorem 4.4 we show that also the randomized value function is a viscosity solution to the HJB equation. The last step towards the Feynman-Kac representation consists in the comparison Theorem 4.5, which provides uniqueness of the viscosity solutions to our HJB-IPDE equation. Section 5 is devoted to the application of our results to the control of a spatio-temporal Hodgkin-Huxley type model. Finally, Sections 6 and 7 are devoted to the proofs of the results provided respectively in Sections 2 and 4.

2 Optimal control of infinite-dimensional PDMPs

In the present section we are going to formulate an optimal control problem for infinite-dimensional piecewise deterministic Markov processes, and to discuss its solvability. The PDMP state space E is a real separable Hilbert space, equipped with the norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$, with corresponding Borel σ -field \mathcal{E} . In addition, we introduce a compact Polish space A , endowed with its Borel σ -field \mathcal{A} , called the space of control actions. The other data of the problem consist in four functions f, b, λ on $E \times A$, g on E , a probability transition kernel Q from $(E \times A, \mathcal{E} \otimes \mathcal{A})$ to (E, \mathcal{E}) , and an operator L satisfying the following conditions.

(HL)

(i) L is a linear, densely defined, maximal monotone operator in E . Moreover, there exists an operator B on E bounded, linear, positive (i.e., $\langle Bx, x \rangle > 0$ for every $x \in E$, $x \neq 0$) and self-adjoint, such that L^*B is bounded on E , and, for some $c_0 \geq 0$,

$$\langle (L^*B + c_0B)x, x \rangle \geq 0 \quad \forall x \in E. \quad (2.1)$$

We define the space E_{-1} to be the completion of E under the norm $\|x\|_{-1} = \|B^{1/2}x\|$. E_{-1} is an Hilbert space equipped with the inner product $\langle x, x \rangle_{-1} = \langle B^{1/2}x, B^{1/2}x \rangle$. Moreover,

$$\|x\|_{-1} \leq \|B^{1/2}\| \|x\|, \quad x \in E. \quad (2.2)$$

(ii) $-L$ generates a strongly continuous semigroup $(S(u))_{u \geq 0}$ such that, for any $u > 0$, $S(u)$ is a contraction on E with respect to $\|\cdot\|_{-1}$.

Remark 2.1. $-L$ is the generator of a strongly continuous semigroup of contractions $(S(u))_{u \geq 0}$ on E , see e.g. Theorem B.45 in [14].

Definition 2.2. We say that a function $u : W \rightarrow \mathbb{R}$ is B -upper-semicontinuous (resp., B -lower-semicontinuous) on $W \subset [0, T] \times E$ if, whenever $t_n \rightarrow t$, $x_n \rightarrow x$, $Bx_n \rightarrow Bx$, $(t, x) \in W$, then $\limsup_{n \rightarrow \infty} u(t_n, x_n) \leq u(t, x)$ (resp. $\liminf_{n \rightarrow \infty} u(t_n, x_n) \geq u(t, x)$). The function u is B -continuous on W if it is B -upper-semicontinuous and B -lower-semicontinuous on W .

In the assumptions below C is a generic constant which may vary from line to line.

(HbλQ)

(i) $b : E \times A \mapsto E$, $\lambda : E \times A \mapsto R_+$ are bounded continuous functions such that

$$\begin{cases} \|b(x, a) - b(x', a)\| \leq C \|x - x'\|_{-1}, & x, x' \in E, a \in A \\ |\lambda(x, a) - \lambda(x', a)| \leq C \|x - x'\|_{-1}, & x, x' \in E, a \in A. \end{cases}$$

(ii) Q maps $E \times A$ into the set of probability measures on (E, \mathcal{E}) , and is a continuous stochastic kernel (see e.g. Proposition 7.30 in [6]). Moreover, for any real function φ continuous on $(\varepsilon, T - \varepsilon) \times E$ for any $\varepsilon > 0$ and bounded, and for every $R > 0$, we have, for all $s, s' \in (\varepsilon, T - \varepsilon)$,

$$\left| \int_E \varphi(s, y) Q(z, a, dy) - \int_E \varphi(s, y) Q(z', a, dy) \right| \leq C \omega(\|z - z'\|_{-1}), \quad z, z' \in E, a \in A, \quad (2.3)$$

$$\left| \int_E [\varphi(s, y) - \varphi(s', y)] Q(z, a, dy) \right| \leq C \sigma_R(|s - s'|), \quad z \in E : \|z\| \leq R, a \in A. \quad (2.4)$$

where ω is a modulus of continuity, and $\sigma_R(\cdot)$ is a modulus of continuity depending on R .

(Hfg) $f : E \times A \mapsto \mathbb{R}_+$, $g : E \mapsto \mathbb{R}_+$ are continuous and bounded functions, such that

$$|f(x, a) - f(x', a)| + |g(x) - g(x')| \leq C \omega(\|x - x'\|_{-1}), \quad a \in A,$$

for all $x, x' \in E$, where ω is a modulus of continuity.

2.1 The optimal control problem

We construct the controlled process X in a canonical way. We start by fixing $(t, x) \in [0, T] \times E$, and we set $\Omega^t = [0, T] \times E \times \bar{\Omega}^t$, where $\omega = (t, x, \bar{\omega})$, $\bar{\Omega}^t$ being the set of sequences $\bar{\omega} = (t_n, e_n)_{n \geq 1}$ contained in $((t, \infty) \times E \cup \{(\infty, \Delta)\})$, where $\Delta \notin E$ is an isolated point adjoined to E , such that $t_n \leq t_{n+1}$, and $t_n < t_{n+1}$ if $t_n < \infty$. On the sample space Ω^t we define the canonical functions $T_0 : \Omega^t \rightarrow [0, \infty)$, $E_0 : \Omega^t \rightarrow E$ and, for $n \geq 1$, $T_n^t : \Omega^t \rightarrow (t, \infty]$, $E_n : \Omega^t \rightarrow E \cup \{\Delta\}$, as follows: $T_0(\omega) = t$, $E_0(\omega) = x$, $T_n^t(\omega) = t_n$, $E_n(\omega) = e_n$, with $T_\infty^t(\omega) = \lim_{n \rightarrow \infty} t_n$. We also introduce the associated integer-valued counting measure on $(t, \infty) \times E$ given by $p(t; ds dy) = \sum_{n \in \mathbb{N}} \delta_{(T_n^t, E_n)}(ds, dy)$.

The class of admissible control laws \mathcal{A}_{ad}^t is the set of all predictable processes α with values in A of the form

$$\alpha_s(\omega) = \alpha_0(s - t, x) \mathbb{1}_{(t, T_1^t(\omega)]}(s) + \sum_{n=1}^{\infty} \alpha_n(s - T_n^t(\omega), E_n(\omega)) \mathbb{1}_{(T_n^t(\omega), T_{n+1}^t(\omega)]}(s), \quad s \in [t, T], \quad (2.5)$$

where $(\alpha_n)_n$, $\alpha_n : \mathbb{R}_+ \times E \rightarrow A$, is a sequence of Borel-measurable functions, see for instance [12], [9], [1]. In other words, at each jump time T_n^t , we choose an open loop control α_n depending on the initial condition E_n and on the time elapsed up to T_n^t , to be used until the next jump time. We define the controlled process $X : \Omega^t \times [t, \infty) \rightarrow E \cup \{\Delta\}$ setting

$$X_s = \begin{cases} \phi^{\alpha_0}(s - t, x) & \text{if } s \in [t, T_1^t), \\ \phi^{\alpha_n}(s - T_n^t, E_n) & \text{if } s \in [T_n^t, T_{n+1}^t), \quad n \in \mathbb{N}, \\ \Delta & \text{if } s \geq T_\infty^t, \end{cases} \quad (2.6)$$

where $\phi^\beta(s, x)$ is the unique mild solution to the parabolic partial differential equation

$$\dot{x}(s) = -Lx(s) + b(x(s), \beta(s)), \quad x(0) = x \in E, \quad (2.7)$$

with $\beta(s)$ an \mathcal{A} -measurable function, namely

$$\phi^\beta(s, x) = S(s)x + \int_0^s S(s-r)b(\phi^\beta(r, x), \beta(r))dr. \quad (2.8)$$

One can easily prove the following result, see e.g. Lemma 3.5 in [23].

Proposition 2.3. *Let (HL) and (Hb λ Q) hold. Then, for every $R > 0$, $t \in [0, T]$, $t < s' < s$, $\alpha \in \mathcal{A}_{ad}^t$, there exists a constant C , only depending on T , such that*

$$\|\phi^\alpha(s-t, x) - \phi^\alpha(s-t, x')\| \leq C \omega(\|x - x'\|), \quad x, x' \in E, \quad (2.9)$$

$$\|\phi^\alpha(s-t, x) - \phi^\alpha(s'-t, x)\| \leq C \sigma_R(s-s'), \quad x \in E : \|x\| \leq R, \quad (2.10)$$

$$\|\phi^\alpha(s-t, x)\| \leq C(1 + \|x\|), \quad x \in E, \quad (2.11)$$

$$\|\phi^\alpha(s-t, x) - \phi^\alpha(s-t, x')\|_{-1} \leq C \omega(\|x - x'\|_{-1}) \quad x, x' \in E, \quad (2.12)$$

$$\|\phi^\alpha(s-t, x) - \phi^\alpha(s'-t, x)\|_{-1} \leq C \sigma_R(s-s'), \quad x \in E : \|x\| \leq R. \quad (2.13)$$

where ω is a modulus of continuity, and σ_R is a modulus of continuity depending on R .

Set $\mathcal{F}_0 = \mathcal{B}([0, T]) \otimes \mathcal{E} \otimes \{\emptyset, \Omega'\}$ and, for all $s \geq t$, $\mathcal{G}_s^t = \sigma(p((t, r] \times B) : r \in (t, s], B \in \mathcal{E})$. For all $s \geq t$, let \mathcal{F}_s^t be the σ -algebra generated by \mathcal{F}_0 and \mathcal{G}_s^t . In the following all the concepts of measurability for stochastic processes will refer to the right-continuous, natural filtration $\mathbb{F}^t = (\mathcal{F}_s^t)_{s \geq t}$. By the symbol \mathcal{P}^t we will denote the σ algebra of \mathbb{F}^t -predictable subsets of $[t, \infty) \times \Omega$.

For every initial time and starting point $(t, x) \in [0, T] \times E$ and for each $\alpha \in \mathcal{A}_{ad}^t$, by Theorem 3.6 in [18] there exists a unique probability measure on $(\Omega^t, \mathcal{F}_\infty^t)$, denoted by $\mathbb{P}_\alpha^{t,x}$, such that its restriction to \mathcal{F}_t^t is δ_x , and the \mathbb{F}^t -compensator under $\mathbb{P}_\alpha^{t,x}$ of the measure $p(t; ds dy)$ is

$$\tilde{p}^\alpha(t; ds dy) = \sum_{n=1}^{\infty} \mathbb{1}_{[T_n^t, T_{n+1}^t)}(s) \lambda(X_s, \alpha_n(s - T_n^t, E_n)) Q(X_s, \alpha_n(s - T_n^t, E_n), dy) ds.$$

We will denote by $\mathbb{E}_\alpha^{t,x}$ the expectation under $\mathbb{P}_\alpha^{t,x}$. The following proposition can be obtained by suitably extending the analogous finite-dimensional result, see Theorem 1.2 in [23].

Proposition 2.4. *Assume that Hypotheses (HL) and (Hb λ Q) hold. For any $(t, x) \in [0, T] \times E$ and $\alpha \in \mathcal{A}_{ad}^t$, let $s \mapsto \phi^\alpha(s, x)$ be the unique mild solution to (2.7) with $\beta = \alpha$, and X be the process in (2.6) with law $\mathbb{P}_\alpha^{t,x}$. Then X is an homogeneous strong Markov process.*

Moreover, let \mathcal{D} be the set of all measurable functions $\psi : \mathbb{R}_+ \times E \rightarrow \mathbb{R}$ which are absolutely continuous on \mathbb{R}_+ as maps $s \mapsto \psi(s, \phi^\alpha(s-t, x))$, for all $x \in E$, and such that the map $(x, s, \omega) \mapsto \psi(s, y) - \psi(s, X_{s-})$ is a valid integrand for the random measure Q , and set

$$\bar{\mathcal{D}} := \{\psi \in \mathcal{D}, \psi \in C^1(\mathbb{R} \times E) :$$

$$D\psi(s, x) \in E \text{ if } x \in E, D\psi(s, x), \frac{\partial \psi}{\partial s}(s, x) \text{ bounded if } x \text{ bounded}\},$$

where $D\psi$ is the unique element of E such that $\frac{d\psi}{dx}[s, x](y) = \langle y, D\psi(s, x) \rangle$, $y \in E$, where $\frac{d\psi}{dx}[s, x]$ denotes the Fréchet-derivative of ψ w.r.t. $x \in E$ evaluated at $(s, x) \in [0, T] \times E$. Let $t < \bar{T} < T$, $\hat{\tau}$ be a stopping time such that $\hat{\tau} \in [t, \bar{T}]$, let τ_R be the exit time of X from $\{y : \|y\| \leq R\}$, $R > 0$, and set $\tau = \hat{\tau} \wedge \tau_R$. Then, for every $\psi \in \bar{\mathcal{D}}$,

$$\begin{aligned} \mathbb{E}_\alpha^{t,x} [\psi(\tau, X_\tau)] &= \psi(t, x) + \mathbb{E}_\alpha^{t,x} \left[\int_t^\tau \left(\frac{\partial \psi}{\partial t}(r, X_r) + \langle b(X_r, \alpha_r), D\psi(r, X_r) \rangle \right) dr \right] \\ &\quad - \mathbb{E}_\alpha^{t,x} \left[\int_t^\tau \langle L X_r, D\psi(r, X_r) \rangle dr \right] + \mathbb{E}_\alpha^{t,x} \left[\int_t^\tau \int_E (\psi(r, y) - \psi(r, X_r)) \lambda(X_r, \alpha_r) Q(X_r, \alpha_r, dy) dr \right]. \end{aligned} \quad (2.14)$$

At this point, we define for any $(t, x) \in [0, T] \times E$ and $\alpha \in \mathcal{A}_{ad}^t$, the cost functional

$$J(t, x, \alpha) = \mathbb{E}_\alpha^{t,x} \left[\int_t^T f(X_s, \alpha_s) ds + g(X_T) \right] \quad (2.15)$$

and the value function of the control problem

$$V(t, x) = \inf_{\alpha \in \mathcal{A}_{ad}^t} J(t, x, \alpha). \quad (2.16)$$

Proposition 2.5. *Assume that Hypotheses **(HL)**, **(HbλQ)** and **(Hfg)** hold. Then the value function V in (2.16) is bounded and uniformly continuous in the $|\cdot| \times \|\cdot\|_{-1}$ norm. Moreover, V satisfies the following dynamic programming principle (DPP):*

$$V(t, x) = \inf_{\alpha \in \mathcal{A}_{ad}^t} \mathbb{E}_\alpha^{t,x} \left[\int_t^{T_1^t \wedge T} f(X_s, \alpha_s) ds + V(T_1^t \wedge T, X_{T_1^t \wedge T}) \right] \quad t \in [0, T], x \in E. \quad (2.17)$$

Proof. See Section 6.1. □

One can prove that formula (2.17) also holds with $h \wedge T \wedge T_1^t$, for any deterministic time $h > t$, in place of $T \wedge T_1^t$. More generally, the previous result can be extended as follows.

Proposition 2.6. *Under the same hypotheses of Proposition 2.5, the (DPP) (2.17) can be extended to the form*

$$V(t, x) = \inf_{\alpha \in \mathcal{A}_{ad}^t} \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta f(X_s, \alpha_s) ds + V(\theta, X_\theta) \right] \quad t \in [0, T], x \in E, \quad (2.18)$$

with

$$\theta := \tau \wedge T_1^t \wedge T, \quad \tau := \inf \{s \geq t : (s, X_s) \notin B((t, x); \rho)\},$$

where $B((t, x); \rho) := \{(s, y) \in (t, T) \times E : \|y - x\| < \rho, |s - t| < \rho\}$, $(t, x) \in [0, T] \times E$, $\rho > 0$.

Proof. See Section 6.2. □

2.2 The related HJB equation

Let us now consider the HJB-IPDE associated to the optimal control problem: this is the following parabolic nonlinear equation on $[0, T] \times E$:

$$\frac{\partial v}{\partial t}(t, x) - \langle Lx, Dv(t, x) \rangle + \inf_{a \in A} \{ \mathcal{L}^a v(t, x) + f(x, a) \} = 0, \quad (2.19)$$

$$v(T, x) = g(x), \quad (2.20)$$

where \mathcal{L}^a is the time-homogeneous operator depending on $a \in A$ defined as

$$\mathcal{L}^a \psi(t, x) := \langle b(x, a), D\psi(t, x) \rangle + \lambda(x, a) \int_E (\psi(t, y) - \psi(t, x)) Q(x, a, dy). \quad (2.21)$$

Remark 2.7. *The HJB equation (2.19)-(2.20) can be rewritten as*

$$H^v(x, v, Dv) = 0 \quad (2.22)$$

$$v(T, x) = g(x), \quad (2.23)$$

where

$$H^\psi(z, v, p) = \frac{\partial v}{\partial t} - \langle Lz, p \rangle + \inf_{a \in A} \left\{ b(z, a) \cdot p + \int_E (\psi(y) - \psi(z)) \lambda(z, a) Q(z, a, dy) + f(z, a) \right\}.$$

Definition 2.8. We say that a function ψ is a test function if $\psi(t, x) = \varphi(t, x) + \delta(t, x) h(\|x\|)$, where

(i) $\psi, \frac{\partial \varphi}{\partial t}, D\varphi, L^*D\varphi, \frac{\partial \delta}{\partial t}, D\delta, L^*D\delta$ are uniformly continuous on $(\varepsilon, T - \varepsilon) \times E$ for every $\varepsilon > 0$, $\delta \geq 0$ is B -continuous and bounded, φ is B -lower semicontinuous and bounded.

(ii) $h \in C^2(\mathbb{R})$ with h', h'' uniformly continuous, h is even and bounded, $h'(r) \geq 0$ for $r \in (0, +\infty)$.

Definition 2.9. Viscosity solution to (2.19)-(2.20).

(i) A bounded B -upper-semicontinuous function $u : (0, T) \times E \rightarrow \mathbb{R}$ is a viscosity subsolution of (2.19) if, whenever $u - \psi$ has a global maximum at a point (t, x) for a test function ψ , then

$$\begin{aligned} & \frac{\partial \psi}{\partial t}(t, x) - \langle x, L^* D\varphi(t, x) + h(\|x\|) L^* D\delta(t, x) \rangle \\ & + \inf_{a \in A} \left\{ \langle b(x, a), D\psi(t, x) \rangle + \int_E (\psi(t, y) - \psi(t, x)) \lambda(x, a) Q(x, a, dy) + f(x, a) \right\} \geq 0. \end{aligned}$$

(ii) A bounded B -lower-semicontinuous function $w : (0, T) \times E \rightarrow \mathbb{R}$ is a viscosity supersolution of (2.19) if, whenever $w + \psi$ has a global minimum at a point (t, x) for a test function ψ , then

$$\begin{aligned} & - \frac{\partial \psi}{\partial t}(t, x) + \langle x, L^* D\varphi(t, x) + h(\|x\|) L^* D\delta(t, x) \rangle \\ & + \inf_{a \in A} \left\{ \langle b(x, a), -D\psi(t, x) \rangle - \int_E (\psi(t, y) - \psi(t, x)) \lambda(x, a) Q(x, a, dy) + f(x, a) \right\} \leq 0. \end{aligned}$$

(iii) A viscosity solution of (2.19)-(2.20) is a function which is both a viscosity subsolution and a viscosity supersolution.

The following lemma will play a fundamental role in the following.

Lemma 2.10. Let $\psi(s, y) = \varphi(s, y) + \delta(s, y) h(\|y\|)$ be a test function of the type introduced in Definition 2.8. For any $a \in A$, define

$$G_a^\psi(s, z) := -\frac{\partial \psi}{\partial s}(s, z) + \langle z, L^* D\varphi(s, z) + h(\|z\|) L^* D\delta(s, z) \rangle + f(z, a) - \mathcal{L}^a \psi(s, z) \quad (2.24)$$

where \mathcal{L}^a is defined in (2.21). Then, for any $t \in (\varepsilon, T - \varepsilon)$, $\varepsilon > 0$, $x \in E$, and any measurable function $\alpha_0 : \mathbb{R}_+ \times E \rightarrow A$, the map

$$r \mapsto G_a^\psi(r, \phi^{\alpha_0}(r - t, x))$$

is continuous on $[t, T - \varepsilon)$, $\varepsilon > 0$, uniformly in a and α_0 . In particular, for any $t \in (\varepsilon, T - \varepsilon)$, $\varepsilon > 0$, $x \in E$, and any measurable function $\alpha_0 : \mathbb{R}_+ \times E \rightarrow A$, the map

$$r \mapsto \mathcal{G}^{\alpha_0}(r) := \inf_{a \in A} G_a^\psi(r, \phi^{\alpha_0}(r - t, x))$$

is continuous on $[t, T - \varepsilon)$, $\varepsilon > 0$, uniformly in α_0 , and uniformly on $B_R(x) := \{x \in E : \|x\| \leq R\}$, $R > 0$.

Proof. See Section 6.3. □

We end this section with the following important result. We recall that, by Proposition 2.5, the value function V is bounded and B -continuous.

Theorem 2.11. Let **(HL)**, **(HbλQ)** and **(Hfg)** hold. Then the value function V provides a viscosity solution to (2.19)-(2.20).

Proof. See Section 6.4. □

3 Control randomization

In this section we start to implement the control randomization method. As a first step, for an initial time $t \geq 0$ and a starting point $x \in E$, we construct an (uncontrolled) PDMP (X, I) with values in $E \times A$ by specifying its local characteristics, see (3.3)-(3.4)-(3.5) below. Next we formulate an auxiliary optimal control problem where, roughly speaking, we optimize a cost functional by modifying the intensity of the process I over a suitable family of probability measures.

3.1 Construction of randomized state systems

Let E still denote a real separable Hilbert space Borel σ -field \mathcal{E} , and A be a Polish space with corresponding Borel σ -field \mathcal{A} . Let moreover b, λ and Q be respectively two real functions on $E \times A$ and a probability transition from $(E \times A, \mathcal{E} \otimes \mathcal{A})$, satisfying **(Hb λ Q)** as before. We denote by $\phi(s, x, a)$ the unique mild solution to the parabolic partial differential equation

$$\dot{x}(s) = -Lx(s) + b(x(s), a), \quad x(0) = x \in E, \quad a \in A. \quad (3.1)$$

In particular, $\phi(s, x, a)$ corresponds to the function $\phi^\beta(s, x)$ introduced in Section 2 when $\beta(s) \equiv a$ and, for every $x, x' \in E$, $0 < s' < s < T$, $a \in A$, satisfies

$$\|\phi(s, x, a) - \phi(s', x', a)\|_{-1} \leq C\omega(\|x - x'\|_{-1} + (s - s')) \quad (3.2)$$

with C a constant only depending on T , and ω some modulus of continuity by Proposition 2.3. This fact will be of great use in the sequel.

Let us now introduce another finite measure λ_0 on (A, \mathcal{A}) satisfying the following assumption:

(H λ_0) λ_0 is a finite measure on (A, \mathcal{A}) with full topological support.

The existence of such a measure is guaranteed by the fact that A is a separable space with metrizable topology. We define

$$\tilde{\phi}(t, x, a) := (\phi(t, x, a), \quad a), \quad (3.3)$$

$$\tilde{\lambda}(x, a) := \lambda(x, a) + \lambda_0(A), \quad (3.4)$$

$$\tilde{Q}(x, a, dy db) := \frac{\lambda(x, a) Q(x, a, dy) \delta_a(db) + \lambda_0(db) \delta_x(dy)}{\tilde{\lambda}(x, a)}. \quad (3.5)$$

We wish to construct a PDMP (X, I) with enlarged state space $E \times A$ and local characteristics $(\tilde{\phi}, \tilde{\lambda}, \tilde{Q})$. Firstly, we need to introduce a suitable sample space to describe the jump mechanism of the process (X, I) on $E \times A$. Accordingly, we fix $(t, x, a) \in [0, T] \times E \times A$, and, proceeding as in Section 2.1, we set $\tilde{\Omega}^t$ as the set of sequences $\tilde{\omega} = (t_n, e_n, a_n)_{n \geq 1}$ contained in $((t, \infty) \times E \times A) \cup \{(\infty, \Delta, \Delta')\}$, where $\Delta \notin E$ (resp. $\Delta' \notin A$) is adjoined to E (resp. to A) as an isolated point. In the sample space $\Omega^t = [0, T] \times E \times A \times \tilde{\Omega}^t$ we define the random variables $T_0(\omega) = t$, $E_0(\omega) = x$, $A_0(\omega) = a$, and, for $n \geq 1$, $T_n^t : \Omega^t \rightarrow (t, \infty]$, $E_n : \Omega^t \rightarrow E \cup \{\Delta\}$, $A_n : \Omega^t \rightarrow A \cup \{\Delta'\}$, as follows: writing $\omega = (t, x, a, \tilde{\omega})$ in the form $\omega = (t, x, a, t_1, e_1, a_1, t_2, e_2, a_2, \dots)$, we set for $n \geq 1$,

$$T_n^t(\omega) = t_n, \quad T_\infty^t(\omega) = \lim_{n \rightarrow \infty} t_n, \quad E_n(\omega) = e_n, \quad A_n(\omega) = a_n.$$

We define the process (X, I) on $(E \times A) \cup \{\Delta, \Delta'\}$ setting

$$(X, I)_s = \begin{cases} (\phi(s - t, x, a), a) & \text{if } s \in [t, T_1^t), \\ (\phi(s - T_n^t, E_n, A_n), A_n) & \text{if } s \in [T_n^t, T_{n+1}^t), \text{ for } n \in \mathbb{N}, \\ (\Delta, \Delta') & \text{if } s \geq T_\infty^t. \end{cases} \quad (3.6)$$

In Ω^t we introduce for all $s \geq t$ the σ -algebras $\mathcal{G}_r^t = \sigma(N(s, G) : s \in (t, r], G \in \mathcal{E} \otimes \mathcal{A})$ generated by the counting processes $N(s, G) = \sum_{n \in \mathbb{N}} \mathbb{1}_{T_n^t \leq s} \mathbb{1}_{(E_n, A_n) \in G}$, and the σ -algebra \mathcal{F}_s^t generated by \mathcal{F}_0 and \mathcal{G}_s^t , where $\mathcal{F}_0 = \mathcal{B}([0, T]) \otimes \mathcal{E} \otimes \mathcal{A} \otimes \{\emptyset, \Omega'\}$. We still denote by $\mathbb{F}^t = (\mathcal{F}_s^t)_{s \geq t}$ and \mathcal{P}^t the corresponding filtration and predictable σ -algebra. The random measure p is now defined on $(t, \infty) \times E \times A$ as

$$p(t; ds dy db) = \sum_{n \in \mathbb{N}} \delta_{(T_n^t, E_n, A_n)}(ds dy db). \quad (3.7)$$

Given any starting point $(t, x, a) \in E \times A$, by Theorem 3.6 in [18], there exists a unique probability measure on $(\Omega^t, \mathcal{F}_\infty^t)$, denoted by $\mathbb{P}^{t, x, a}$, such that its restriction to \mathcal{F}_0 is $\delta_{(x, a)}$ and the \mathbb{F}^t -compensator of the measure $p(t; ds dy db)$ under $\mathbb{P}^{t, x, a}$ is the random measure

$$\tilde{p}(t; ds dy db) = \sum_{n \in \mathbb{N}} \mathbb{1}_{[T_n^t, T_{n+1}^t)}(s) \Lambda(\phi(s - T_n^t, E_n, A_n), A_n, dy db) ds, \quad (3.8)$$

where

$$\Lambda(x, a, dy db) = \lambda(x, a) Q(x, a, dy) \delta_a(db) + \lambda_0(db) \delta_x(dy), \quad \forall (x, a) \in E \times A.$$

We denote by $q = p - \tilde{p}$ the compensated martingale measure associated to p .

The sample path of a process (X, I) with values in $E \times A$, starting from a fixed initial point $(x, a) \in E \times A$ at time t , can be defined iteratively by means of its local characteristics $(\tilde{\phi}, \tilde{\lambda}, \tilde{Q})$ in the following way. Set $F(t, x, a; s) = e^{-\int_t^s (\lambda(\phi(r-t, x, a), a) + \lambda_0(A)) dr}$. For any $B \in \mathcal{E}$, $C \in \mathcal{A}$, we have

$$\mathbb{P}^{t, x, a}(T_1^t > s) = F(t, x, a; s), \quad s \geq t, \quad (3.9)$$

$$\mathbb{P}^{t, x, a}(X_{T_1^t} \in B, I_{T_1^t} \in C | T_1^t) = \tilde{Q}(x, B \times C), \quad (3.10)$$

on $\{T_1^t < \infty\}$, and, for every $n \geq 1$, on $\{T_n^t < \infty\}$,

$$\mathbb{P}^{t, x, a}(T_{n+1}^t > s | \mathcal{F}_{T_n^t}^t) = \exp\left(-\int_{T_n^t}^s (\lambda(\phi(r - T_n^t, X_{T_n^t}^t), I_{T_n^t}^t) + \lambda_0(A)) dr\right), \quad s \geq T_n^t, \quad (3.11)$$

$$\mathbb{P}^{t, x, a}(X_{T_{n+1}^t} \in B, I_{T_{n+1}^t} \in C | \mathcal{F}_{T_n^t}^t, T_{n+1}^t) = \tilde{Q}(\phi(T_{n+1}^t - T_n^t, X_{T_n^t}^t, I_{T_n^t}^t), I_{T_n^t}^t, B \times C). \quad (3.12)$$

We recall the following result, that is a direct consequence of Theorem 4 in [7].

Proposition 3.1. *For any $(t, x, a) \in [0, T] \times E \times A$, let $\phi(t, x, a)$ be the unique mild solution to (3.1), and (X, I) be the process defined in (3.6) with law $\mathbb{P}^{t, x, a}$. Then (X, I) is an homogeneous strong Markov process.*

Moreover, denote by \mathcal{D} the set of all measurable functions $\varphi : E \times A \rightarrow \mathbb{R}$ which are absolutely continuous on \mathbb{R}_+ as maps $s \mapsto \varphi(\phi(s, x, a), a)$, for all $(x, a) \in E \times A$, and such that the map $(x, a, s, \omega) \mapsto \varphi(y, b) - \varphi(X_{s-}, I_{s-})$ is a valid integrand for the random measure (3.8), and set

$$\bar{\mathcal{D}} := \{\varphi \in D(\mathcal{L}), \varphi \in C^{1,0}(E \times A), D\varphi(x, a) \in E \text{ if } x \in E, D\varphi(x, a) \text{ bounded if } x \text{ bounded}\},$$

where $D\varphi$ is the unique element of E such that $\frac{d\varphi}{dx}[x, a](y) = \langle y, D\varphi(x, a) \rangle$, $y \in E$, where $\frac{d\varphi}{dx}[x, a]$ denotes the Fréchet-derivative of φ w.r.t. $x \in E$ evaluated at $(x, a) \in E \times A$. Then the extended generator of (X, I) is given by

$$\begin{aligned} \mathcal{L}\varphi(x, a) := & \langle -Lx + b(x, a), D\varphi(x, a) \rangle + \int_E (\varphi(y, a) - \varphi(x, a)) \lambda(x, a) Q(x, a, dy) \\ & + \int_A (\varphi(x, b) - \varphi(x, a)) \lambda_0(db), \quad \text{for every } \varphi \in \bar{\mathcal{D}}. \end{aligned}$$

3.2 The randomized optimal control problem

We now introduce a randomized optimal control problem associated to the process (X, I) , and formulated in a weak form. For fixed (t, x, a) , we consider a family of probability measures $\{\mathbb{P}_\nu^{t,x,a}, \nu \in \mathcal{V}\}$ in the space $(\Omega^t, \mathcal{F}_\infty^t)$, whose effect is to change the stochastic intensity of the process (X, I) .

Let us proceed with precise definitions. We still assume that **(HbλQ)**, **(Hλ₀)** and **(Hfg)** hold. We recall that $\mathbb{F}^t = (\mathcal{F}_s^t)_{s \geq t}$ is the augmentation of the natural filtration generated by p in (3.7), and that \mathcal{P}^t denotes the σ -field of \mathbb{F}^t -predictable subsets of $[t, \infty) \times \Omega$. We define

$$\mathcal{V} = \{\nu : \Omega \times [0, \infty) \times A \rightarrow (0, \infty) \text{ } \mathcal{P}^0 \otimes \mathcal{A}\text{-measurable and bounded}\}.$$

For every $\nu \in \mathcal{V}$, we consider the predictable random measure

$$\tilde{p}^\nu(t; ds dy db) := \nu_s(b) \lambda_0(db) \delta_{\{X_{s-}\}}(dy) ds + \lambda(X_{s-}, I_{s-}) Q(X_{s-}, I_{s-}, dy) \delta_{\{I_{s-}\}}(db) ds. \quad (3.13)$$

In particular, for any $t \in [0, T]$, by the Radon Nikodym theorem one can find two nonnegative functions d_1, d_2 defined on $\Omega \times [t, \infty) \times E \times A$, $\mathcal{P} \otimes \mathcal{E} \otimes \mathcal{A}$, such that

$$\begin{aligned} \lambda_0(db) \delta_{\{X_{s-}\}}(dy) ds &= d_1(s, y, b) \tilde{p}(ds dy db) \\ \lambda(X_{s-}, I_{s-}, dy) \delta_{\{I_{s-}\}}(db) ds &= d_2(s, y, b) \tilde{p}(ds dy db), \\ d_1(s, y, b) + d_2(s, y, b) &= 1, \quad \tilde{p}(ds dy db)\text{-a.e.} \end{aligned}$$

and we have $d\tilde{p}^\nu = (\nu d_1 + d_2) d\tilde{p}$. For any $t \in [0, T]$, $\nu \in \mathcal{V}$, consider then the Doléans-Dade exponential local martingale $L^{t,\nu}$ defined

$$\begin{aligned} L_s^{t,\nu} &= \exp \left(\int_t^s \int_{E \times A} \log(\nu_r(b) d_1(r, y, b) + d_2(r, y, b)) p(dr dy db) - \int_t^s \int_A (\nu_r(b) - 1) \lambda_0(db) dr \right) \\ &= e^{\int_t^s \int_A (1 - \nu_r(b)) \lambda_0(db) dr} \prod_{n \geq 1: t \leq T_n^t \leq s} (\nu_{T_n^t}(A_n) d_1(T_n^t, E_n, A_n) + d_2(T_n^t, E_n, A_n)), \end{aligned} \quad (3.14)$$

for $s \geq t$. When $(L_s^{t,\nu})_{s \geq t}$ is a true martingale on $[t, T]$, we can define a probability measure $\mathbb{P}_\nu^{t,x,a}$ equivalent to $\mathbb{P}^{t,x,a}$ on $(\Omega^t, \mathcal{F}_T^t)$ by

$$\mathbb{P}_\nu^{t,x,a}(d\omega) = L_T^{t,\nu}(\omega) \mathbb{P}^{t,x,a}(d\omega). \quad (3.15)$$

By the Girsanov theorem for point processes (see Theorem 4.5 in [18]), the restriction of the random measure p to $(t, T] \times E \times A$ admits $\tilde{p}^\nu = (\nu d_1 + d_2) \tilde{p}$ as compensator under $\mathbb{P}_\nu^{t,x,a}$. We set $q^\nu := p - \tilde{p}^\nu$, and we denote by $\mathbb{E}_\nu^{t,x,a}$ the expectation operator under $\mathbb{P}_\nu^{t,x,a}$. Previous considerations are formalized in the following lemma, for a proof see Lemma 3.2 in [5].

Lemma 3.2. *Let Hypotheses **(HbλQ)** and **(Hλ₀)** hold. Then, for every $(t, x, a) \in [0, T] \times E \times A$ and $\nu \in \mathcal{V}$, under the probability $\mathbb{P}^{t,x,a}$, the process $(L_s^{t,\nu})_{s \geq t}$ is a martingale. Moreover, $L_T^{t,\nu}$ is square integrable, and, for every $\mathcal{P}_T^t \otimes \mathcal{E} \otimes \mathcal{A}$ -measurable function $H : \Omega \times [0, T] \times E \times A \rightarrow \mathbb{R}$ such that $\mathbb{E}^{t,x,a} \left[\int_t^T \int_{E \times A} |H_s(y, b)|^2 \tilde{p}(ds dy db) \right] < \infty$, the process $\int_t^\cdot \int_{E \times A} H_r(y, b) q^\nu(dr dy db)$ is a $\mathbb{P}_\nu^{t,x,a}$ -martingale on $[t, T]$.*

Finally, for every $(t, x) \in [0, T] \times E$, $a \in A$ and $\nu \in \mathcal{V}$, we introduce the randomized cost functional

$$J(t, x, a, \nu) := \mathbb{E}_\nu^{t,x,a} \left[\int_t^T f(X_s, I_s) ds + g(X_T) \right], \quad (3.16)$$

and the randomized value function

$$V^*(t, x, a) := \inf_{\nu \in \mathcal{V}} J(t, x, a, \nu). \quad (3.17)$$

4 A constrained BSDEs representation for the value function

In the present section we introduce a BSDE with a sign constraint on its martingale part for which we give existence and uniqueness of a maximal solution in an appropriate sense. This constrained BSDE will provide a probabilistic representation formula for the dual value function introduced in (3.17).

Throughout the section we still assume that $(\mathbf{Hb}\lambda\mathbf{Q})$, $(\mathbf{H}\lambda_0)$ and (\mathbf{Hfg}) hold. For any $(t, x, a) \in [0, T] \times E \times A$, we consider the random measures p , \tilde{p} and q , as well as the dual control setting $\Omega^t, \mathbb{F}^t, (X, I), \mathbb{P}^{t,x,a}$, defined in Section 3.1. We introduce the following notation.

- $\mathbf{L}_{t,x,a}^2(\mathcal{F}_\tau^t)$, the set of \mathcal{F}_τ^t -measurable random variables ξ such that $\mathbb{E}^{t,x,a} [|\xi|^2] < \infty$; here $\tau \geq 0$ is an \mathbb{F}^t -stopping time.
- \mathbf{S}^∞ , the set of real-valued càdlàg adapted processes $Y = (Y_t)_{t \geq 0}$ which are uniformly bounded.
- $\mathbf{L}_{t,x,a}^2(q)$, the set of $\mathcal{P}_T \otimes \mathcal{B}(E) \otimes \mathcal{A}$ -measurable maps $Z : \Omega \times [0, T] \times E \times A \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \|Z\|_{\mathbf{L}_{t,x,a}^2(q)}^2 &:= \mathbb{E}^{t,x,a} \left[\int_t^T \int_E |Z_s(y, I_s)|^2 \lambda(X_s, I_s) Q(X_s, I_s, dy) ds \right] \\ &\quad + \mathbb{E}^{t,x,a} \left[\int_t^T \int_A |Z_s(X_s, b)|^2 \lambda_0(db) ds \right] < \infty. \end{aligned}$$

- $\mathbf{K}_{t,x,a}^2$, the set of nondecreasing càdlàg predictable processes $K = (K_s)_{t \leq s \leq T}$ such that $K_t = 0$ and $\mathbb{E}^{t,x,a} [K_T^2] < \infty$.

We consider the following family of BSDEs with partially nonnegative jumps over a finite horizon T , parametrized by (t, x, a) : $\mathbb{P}^{t,x,a}$ -a.s.,

$$\begin{aligned} Y_s^{t,x,a} &= g(X_T) + \int_s^T f(X_r, I_r) dr - (K_T^{t,x,a} - K_s^{t,x,a}) \\ &\quad - \int_s^T \int_A Z_r^{t,x,a}(X_r, b) \lambda_0(db) dr - \int_s^T \int_{E \times A} Z_r^{t,x,a}(y, b) q(dr dy db), \quad t \leq s \leq T, \end{aligned} \quad (4.1)$$

with

$$Z_s^{t,x,a}(X_{s-}, b) \geq 0, \quad ds \otimes d\mathbb{P}^{t,x,a} \otimes \lambda_0(db) \text{ -a.e. on } [0, T] \times \Omega \times A. \quad (4.2)$$

We are interested in the *maximal solution* $(Y^{t,x,a}, Z^{t,x,a}, K^{t,x,a}) \in \mathbf{S}^\infty \times \mathbf{L}_{t,x,a}^2(q) \times \mathbf{K}_{t,x,a}^2$ to (4.1)-(4.2), in the sense that for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{K}) \in \mathbf{S}^\infty \times \mathbf{L}_{t,x,a}^2(q) \times \mathbf{K}_{t,x,a}^2$ to (4.1)-(4.2), we have $Y_s^{t,x,a} \geq \tilde{Y}_s$, $\mathbb{P}^{t,x,a}$ -a.s., for all $s \geq t$.

Let us introduce the following penalized BSDE, associated to (4.1)-(4.2), parametrized by the integer $n \geq 1$:

$$\begin{aligned} Y_s^{n,t,x,a} &= g(X_T) + \int_s^T f(X_r, I_r) dr - (K_T^{n,t,x,a} - K_s^{n,t,x,a}) \\ &\quad - \int_s^T \int_A Z_r^{n,t,x,a}(X_r, b) \lambda_0(db) dr - \int_s^T \int_{E \times A} Z_r^{n,t,x,a}(y, b) q(dr dy db), \quad t \leq s \leq T, \end{aligned} \quad (4.3)$$

where $K_s^{n,t,x,a} := n \int_0^s \int_A [Z_r^{n,t,x,a}(X_r, b)]^- \lambda_0(db) dr$, $s \in [t, T]$.

Theorem 4.1. *Let Hypotheses **(HbλQ)**, **(Hλ₀)** and **(Hfg)** hold. Then, for every $(t, x, a) \in [0, T] \times E \times A$, there exists a unique maximal solution $(Y^{t,x,a}, Z^{t,x,a}, K^{t,x,a}) \in \mathbf{S}^\infty \times \mathbf{L}_{t,x,a}^2(\mathfrak{q}) \times \mathbf{K}_{t,x,a}^2$ to the BSDE with partially nonnegative jumps (4.1)-(4.2), where $Y^{t,x,a}$ is the nonincreasing limit of $(Y^{n,t,x,a})_n$, $Z^{t,x,a}$ is the weak limit of $(Z^{n,t,x,a})_n$ in $\mathbf{L}_{t,x,a,\text{loc}}^2(\mathfrak{q})$ and $K_s^{t,x,a}$ is the weak limit of $(K_s^{n,t,x,a})_n$ in $\mathbf{L}_{t,x,a}^2(\mathcal{F}_s)$, for any $s \geq 0$. Moreover, $Y^{t,x,a}$ has the explicit representation:*

$$Y_s^{t,x,a} = \text{ess inf}_{\nu \in \mathcal{V}} \mathbb{E}_\nu^{t,x,a} \left[\int_s^T f(X_r, I_r) dr + g(X_T) \middle| \mathcal{F}_s \right], \quad \forall s \in [t, T]. \quad (4.4)$$

In particular, setting $s = t$ in (4.4), we have the following representation formula for the value function of the randomized control problem:

$$V^*(t, x, a) = Y_t^{t,x,a}, \quad (t, x, a) \in [0, T] \times E \times A. \quad (4.5)$$

Proof. The proof of this result is analogous to the one for the BSDE (4.1) with underlying finite-dimensional process X , see Theorem 4.7 in [4], and we do not report it for sake of brevity. \square

Our main purpose is to show how maximal solutions to BSDEs with nonnegative jumps of the form (4.1)-(4.2) provide actually a Feynman-Kac representation to the value function V associated to our optimal control problem for infinite-dimensional PDMPs. Let us introduce a deterministic function $v : [0, T] \times E \times A \rightarrow \mathbb{R}$ as

$$v(t, x, a) := Y_t^{t,x,a}, \quad (t, x, a) \in [0, T] \times E \times A. \quad (4.6)$$

Proposition 4.2. *Assume that Hypotheses **(HL)**, **(HbλQ)**, **(Hλ₀)**, and **(Hfg)** hold. Then the function v in (4.6) does not depend on the variable a :*

$$v(t, x, a) = v(t, x, a'), \quad t \in [0, T], x \in E, a, a' \in A. \quad (4.7)$$

Defining, by abuse of notation, the function v on $[0, T] \times E$ by $v(\cdot, \cdot) = v(\cdot, \cdot, a)$, for any $a \in A$, we get that v admits the representation formula: $\mathbb{P}^{t,x,a}$ -a.s.

$$v(s, X_s) = Y_s^{t,x,a}, \quad s \geq t. \quad (4.8)$$

Proof. By Lemma 5.3 and Remark 5.5 in [4], we have that for any $(t, x, a) \in [0, T] \times E \times A$, $\mathbb{P}^{t,x,a}$ -a.s.,

$$v(s, X_s, I_s) = Y_s^{t,x,a}, \quad s \geq 0. \quad (4.9)$$

Now we recall that, by (4.5) and (4.6), v coincides with the value function V^* of the dual control problem introduced in Section 3.2. Therefore, identity (4.7) corresponds to the fact that $V^*(t, x, a)$ does not depend on a . Proceeding as in the finite-dimensional case (see the proof of Proposition 5.6 in [4]), one can prove that:

for any $t \in [0, T]$, $x \in E$, $a, a' \in A$, $\nu \in \mathcal{V}$, there exists $(\nu^\varepsilon)_\varepsilon \in \mathcal{V}$:

$$\lim_{\varepsilon \rightarrow 0^+} J(t, x, a', \nu^\varepsilon) = J(t, x, a, \nu). \quad (4.10)$$

Property (4.10) implies that $V^*(t, x, a') \leq J(t, x, a, \nu)$ for all $t \in [0, T]$, $x \in E$, $a, a' \in A$, and by the arbitrariness of ν we can conclude that $V^*(t, x, a') \leq V^*(t, x, a)$ for all $t \in [0, T]$, $x \in E$, $a, a' \in A$. In other words $V^*(t, x, a) = v(t, x, a)$ does not depend on a , and (4.7) holds. \square

Theorem 4.3. *Assume that Hypotheses **(HL)**, **(HbλQ)**, **(Hλ₀)**, and **(Hfg)** hold. Then v is bounded and uniformly continuous in the $|\cdot| \times \|\cdot\|_{-1}$ norm. Moreover, v satisfies the so called randomized dynamic programming principle:*

$$v(t, x) = \inf_{\nu \in \mathcal{V}} \mathbb{E}_{\nu}^{t, x, a} \left[\int_t^{T \wedge T_1^t} f(X_r, I_r) dr + v(T \wedge T_1^t, X_{T \wedge T_1^t}) \right]. \quad (4.11)$$

Proof. See Section 7.1. □

We can now give the following important result.

Theorem 4.4. *Assume that Hypotheses **(HL)**, **(HbλQ)**, **(Hλ₀)** and **(Hfg)** hold. Then the function v in (4.6) is a viscosity solution to (2.19)-(2.20).*

Proof. See Section 7.2. □

Finally, we provide a comparison theorem for viscosity sub and supersolutions to the first order IPDE of HJB type (2.19)-(2.20) on Hilbert spaces. To this end, we will need the following additional hypothesis on the transition measure Q :

(HQ') For any $x, x_\varepsilon \in E$, $S_\varepsilon \subset E$, such that $x_\varepsilon \rightarrow x$ and $\cap_\varepsilon S_\varepsilon = \emptyset$,

$$\sup_{a \in A} Q(x_\varepsilon, a, S_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Theorem 4.5. *Let **(HL)**, **(HbλQ)**, **(Hfg)** and **(HQ')** hold. Let $u : [0, T] \times E \rightarrow \mathbb{R}$ (resp. $v : [0, T] \times E \rightarrow \mathbb{R}$) be a bounded and uniformly continuous function in the $|\cdot| \times \|\cdot\|_{-1}$ norm, providing a viscosity subsolution (resp. viscosity supersolution) to (2.19)-(2.20). Suppose that $u(T, x) \leq v(T, x)$ for all $x \in E$. Then $u \leq v$.*

Proof. See Section 7.3. □

By means of Theorems 2.11, 4.3, 4.4, together with the comparison Theorem 4.5, we can finally obtain the following probabilistic representation formula for the value function V .

Theorem 4.6. *Let **(HL)**, **(HbλQ)**, **(Hλ₀)**, **(Hfg)**, and **(HQ')** hold. Then the function v in (4.6) coincides with the value function V , and the following Feynman-Kac representation formula holds:*

$$V(t, x) = Y_t^{t, x, a}, \quad (t, x, a) \in [0, T] \times E \times A. \quad (4.12)$$

5 Application to a Hodgkin-Huxley model of neuronal dynamics

In the present section we apply our theory to an infinite-dimensional stochastic Hodgkin-Huxley model of neuronal dynamics. The deterministic Hodgkin-Huxley system was first introduced in [15], while stochastic versions as Hilbert space valued PDMP have been studied in e.g. [2], [7],[16] and [24], [23].

We focus on the model considered in [23]. The axon is modeled by the interval $[0, 1]$. We consider ion channels of type Na (sodium) or K (potassium), and we assume that they are located along the axon at positions in $I_N = \frac{1}{N}(\mathbb{Z} \cap N(0, 1))$ for some fixed $N \in \mathbb{N}^*$, that we will denote $\frac{i}{N}$ or z_i . The set of possible states of K and Na channels are denoted respectively by D_1 and D_2 , and are given by

$$D_1 := \{n_0, n_1, n_2, n_3, n_4\}, \quad D_2 := \{m_0 h_1, m_1 h_1, m_2 h_1, m_3 h_1, m_0 h_0, m_1 h_0, m_2 h_0, m_3 h_0\}.$$

In the control problem new (rhodopsin) channels that are sensitive to light are inserted in the neuron. Such a rhodopsin channel (denoted by $ChR2$) can have the four possible states O_1, O_2, C_1, C_2 , among which O_1 and O_2 are conductive. Experimentally, the channel is illuminated and the effect of the illumination is to put the channel in one of its conductive states. We set $\bar{D} := D_1 \cup D_2 \cup D_{ChR2}$ with $D_{ChR2} := \{O_1, O_2, C_1, C_2\}$, and $\bar{D}_N := \bar{D}^{I_N}$.

We consider the Hilbert space $E := L^2(0, 1)$ and the operator $L := -\Delta$. The controlled PDMP consists in a set of PDEs written as ODEs in the Hilbert space E indexed by $d \in \bar{D}_N$,

$$\begin{cases} \dot{v}(t) = \frac{1}{C_m} \Delta v(t) + b_d(v(t)), \\ v(0) = v, \\ v(t)(0) = v(t)(1) = 0, \quad \forall t > 0, \end{cases} \quad (5.1)$$

where the membrane capacitance $C_m > 0$ is constant and, for each $(v, d) \in E \times \bar{D}_N$,

$$\begin{aligned} b_d(v) := \frac{1}{N} \sum_{i \in I_N} \left\{ \bar{g}_K \mathbf{1}_{\{d_i = n_4\}} (\bar{V}_K - \Phi_i(v)) + \bar{g}_{Na} \mathbf{1}_{\{d_i = m_3 h_1\}} (\bar{V}_{Na} - \Phi_i(v)) + \bar{g}_l (\bar{V}_l - \Phi_i(v)) \right. \\ \left. + \bar{g}_{ChR2} (\mathbf{1}_{\{d_i = O_1\}} + \rho \mathbf{1}_{\{d_i = O_2\}}) (\bar{V}_{ChR2} - \Phi_i(v)) \right\} \phi_{z_i}, \end{aligned} \quad (5.2)$$

with

$$\Phi_i(v) := \langle v, \phi_{z_i} \rangle, \quad z_i \in I_N, \quad (5.3)$$

where ϕ_{z_i} is a mollifier function supported on a neighborhood of z_i . For a channel of type K , $\bar{g}_K > 0$ is the normalized conductance and $\bar{V}_K \in \mathbb{R}$ is the reversal potential; the same notation holds for $Na, l, ChR2$ (\bar{V}_l and \bar{g}_l denote respectively the leaky reversal potential and conductance). The conductance depends on the number of channels in the conductive state: for K (resp. Na) such a state is unique, and it is n_4 (resp. $m_3 h_1$). The leaky conductance \bar{g}_l remains constant. Formula (5.3) models the mean value of the membrane potential on a neighborhood of z_i .

For any $x = (v, d) \in E \times \bar{D}_N$, we denote by $v_t = \phi_t^d(v)$ the corresponding unique mild solution to the PDE (5.1), that models the membrane potential evolution between two successive changes in the channels configuration. The transitions take place inside the discrete domain \bar{D}_N , and correspond to a continuous-time Markov chain d_t . Denoting by (T_n, d_{T_n}) the jump times and post-jump location, the controlled PDMP starting from $x = (v, d) \in E \times \bar{D}_N$ is

$$X_s = (v_s, d_s) = \begin{cases} (\phi_{s-t}^d(v), d) & \text{if } s \in [t, T_1), \\ (\phi_{s-T_n}^{d_{T_n}}(v), d_{T_n}) & \text{if } s \in [T_n, T_{n+1}), \quad n \in \mathbb{N} \setminus \{0\}. \end{cases}$$

The control process α_t is proportional to the intensity of light (which is necessarily bounded), so that we take as control space $A := [0, a_{\max}]$ with $a_{\max} > 0$. Introducing a family of smooth functions $\sigma_{\zeta, \xi} : \mathbb{R} \rightarrow \mathbb{R}_+^*$ depending on $(\zeta, \xi) \in \bar{D} \times \bar{D}$ for all $x = (v, d) \in E \times \bar{D}_N$, $a \in A$, we define the jump rate function $\lambda : E \times \bar{D}_N \times A \rightarrow \mathbb{R}_+$ by

$$\lambda((v, d), a) := \sum_{i \in I_N} \sum_{\substack{\xi \in \bar{D}, \\ \xi \neq d_i}} \sigma_{d_i, \xi}(\Phi_i(v), a). \quad (5.4)$$

The transition measure $Q : E \times \bar{D}_N \times A \rightarrow \mathcal{P}(\bar{D}_N)$ is such that, for any $x = (v, d) \in E \times \bar{D}_N$, $a \in A$, the measure $Q((v, d), a, \cdot)$ is supported by the set \mathcal{S} of $y = (\tilde{v}, \tilde{d})$ such that $\tilde{v} = v$ (the trajectories

of (v_t) are continuous) and \tilde{d} differs from d only by one component. For $y = (\tilde{v}, \tilde{d}) \in \mathcal{S}$ such that \tilde{d} differs from d only by its component i ,

$$Q((v, d), a, y) := \sum_{\substack{\xi \in \bar{D}, \\ \xi \neq d_i}} \frac{\sigma_{d_i, \xi}(\Phi_i(v), a)}{\lambda((v, d), a)} \delta_v(\tilde{v}) \delta_\xi(\tilde{d}_i), \quad (5.5)$$

if $y \notin \mathcal{S}$, $Q(x, a; dy) := 0$. The transition functions σ from C_1 to O_1 and from C_2 to O_2 are assumed to be proportional to the control α while the other ones are uncontrolled functions. More precisely (see [15], [23]):

$$\begin{aligned} \sigma_{c_1, o_1}(v, a) &= \varepsilon_1 a, & \sigma_{o_1, c_1}(v, a) &= K_{d1}, & \sigma_{o_1, o_2}(v, a) &= e_{12}, & \sigma_{o_2, o_1}(v, a) &= e_{21}, \\ \sigma_{o_2, c_2}(v, a) &= K_{d2}, & \sigma_{c_2, o_2}(v, a) &= \varepsilon_2 a, & \sigma_{c_2, c_1}(v, a) &= K_r, \end{aligned}$$

and

$$\begin{aligned} \sigma_{n_0, n_1}(z) &= 4\alpha_n(z), & \sigma_{n_1, n_2}(z) &= 3\alpha_n(z), & \sigma_{n_2, n_3}(z) &= 2\alpha_n(z), & \sigma_{n_3, n_4}(z) &= \alpha_n(z), \\ \sigma_{n_4, n_3}(z) &= 4\beta_n(z), & \sigma_{n_3, n_2}(z) &= 3\beta_n(z), & \sigma_{n_2, n_1}(z) &= 2\beta_n(z), & \sigma_{n_1, n_0}(z) &= \beta_n(z) \\ \sigma_{m_0 h_1, m_1 h_1}(z) &= \sigma_{m_0 h_0, m_1 h_0}(z) = 3\alpha_m(z), & \sigma_{m_1 h_1, m_2 h_1}(z) &= \sigma_{m_1 h_0, m_2 h_0}(z) = 2\alpha_m(z), \\ \sigma_{m_2 h_1, m_3 h_1}(z) &= \sigma_{m_2 h_0, m_3 h_0}(z) = \alpha_m(z), & \sigma_{m_3 h_1, m_2 h_1}(z) &= \sigma_{m_3 h_0, m_2 h_0}(z) = 3\beta_m(z), \\ \sigma_{m_2 h_1, m_1 h_1}(z) &= \sigma_{m_2 h_0, m_1 h_0}(z) = 2\beta_m(z), & \sigma_{m_1 h_1, m_0 h_1}(z) &= \sigma_{m_1 h_0, m_0 h_0}(z) = \beta_m(z), \end{aligned}$$

where

$$\begin{aligned} \alpha_n(z) &= \frac{0.1 - 0.01z}{e^{1-0.1z} - 1}, & \beta_n(z) &= 0.125e^{-\frac{z}{80}}, & \alpha_m(z) &= \frac{2.5 - 0.1z}{e^{2.5-0.1z} - 1}, & \beta_m(z) &= 4e^{-\frac{z}{18}}, \\ \alpha_h(z) &= 0.07e^{-\frac{z}{20}}, & \beta_h(z) &= \frac{1}{e^{3-0.1z} + 1}. \end{aligned}$$

The optimal control problem consists in mimicking a desired output reference potential V_{ref} , that encodes a given biological behavior while minimizing the intensity of the light applied to the neuron. This corresponds to setting, for any $x = (v, d) \in E \times \bar{D}_N$,

$$f(x, a) = f((v, d), a) = \kappa \|v - V_{ref}\|^2 + a, \quad g(x) = 0, \quad (5.6)$$

so that the cost functional and the value function of the control problem are

$$J(t, x, \alpha) = \mathbb{E}_\alpha^{t, x} \left[\int_t^T (\kappa \|v_s - V_{ref}\|^2 + \alpha(X_s)) ds \right], \quad V(t, x) = \inf_{\alpha \in \mathcal{A}_{ad}^t} J(t, x, \alpha).$$

The reference signal V_{ref} (that we assume not depending on time) may correspond to a healthy behavior that we want the system to recover thanks to the light stimulation. The intensity of the light is modeled by the control $\alpha_s = \alpha(X_s)$. Getting the intensity minimal is crucial for the feasibility of the experiment in relation to the technical characteristics of the devices that are used.

Remark 5.1. *The control of general infinite-dimensional PDMP is considered in [23], [22]. As in the present paper, in [23] the authors deal with piecewise open loop controls (see [29]), and the control may act on the three characteristics of the PDMP; however, the main tools were relaxed controls and the optimal control of Markov Decision Processes, see e.g. [6]. As an application, other types of models can also be considered: the PDEs in (5.1) may depend on the control variable corresponding to the case where b_d depends on the control, ϕ_{z_i} may be replaced by δ_{z_i} or finally the set D_{ChR2} may have three elements, in which case a ChR2 channel has a unique conductive state.*

The rest of this section is devoted to check that the Hodgkin-Huxley stochastic model described above can be put into the framework of the theory developed in the previous sections.

Proposition 5.2. (i) *The operator $L := -\Delta$ is densely defined, maximal monotone and self-adjoint. Moreover, $B := (I - \Delta)^{-1}$ satisfies the strong B -condition with $c_0 = 1$, namely $-\Delta B + B \geq I$ which implies the weak B -condition (2.1).*

(ii) *The semigroup $(S(r))_{r \geq 0} := (e^{-rL})_{r \geq 0} = (e^{r\Delta})_{r \geq 0}$ generated by $L := -\Delta$ is strongly continuous, and for all $r > 0$, and $S(r)$ is a contraction with respect to $\|\cdot\|$ and also with respect to $\|\cdot\|_{-1}$.*

Proof. (i) From [14], example 3.14 at page 155 (see also [22]) $B := (I - \Delta)^{-1}$ satisfies the strong B -condition with $c_0 = 1$ namely $-\Delta B + B \geq I$, which implies in particular the weak B -condition (2.1).

(ii) For any $k \in \mathbb{N}$, let us define

$$f_k = \sqrt{2} \sin(k\pi). \quad (5.7)$$

$(f_k)_{k \geq 1}$ is an orthonormal basis of E , $\Delta f_k = -k^2\pi^2 f_k$, and, for any $v \in E$,

$$\|v\|^2 = \sum_{k \geq 1} (v, f_k)^2, \quad \|v\|_{-1}^2 = ((I - \Delta)^{-1}v, v)_H = \sum_{k \geq 1} \frac{1}{(1 + k^2\pi^2)} (v, f_k)^2.$$

Moreover $S(r) = e^{r\Delta}$ is such that $S(r)v \in \mathcal{D}(\Delta)$ for all $r > 0$, $v \in E$, and satisfies

$$S(r)v = \sum_{k \geq 1} e^{-rk^2\pi^2} (v, f_k) f_k, \quad r \geq 0, \quad v \in E,$$

$$\|S(r)v\|_{-1}^2 = \sum_{k \geq 1} \frac{1}{(1 + k^2\pi^2)} (S(r)v, f_k)^2 = \sum_{k \geq 1} \frac{1}{(1 + k^2\pi^2)} e^{-2rk^2\pi^2} (v, f_k)^2, \quad r \geq 0.$$

We have $\|S(r)v\|^2 \leq e^{-2r\pi^2} \|v\|^2$. Moreover, $\|S(r)v\|_{-1}^2 \leq e^{-2r\pi^2} \|v\|_{-1}^2$. \square

Lemma 5.3. *For any $i \in I_N$, let Φ_i be the function in (5.3). Then there exists a positive constant C_i such that, for all v, v' in E ,*

$$|\Phi_i(v') - \Phi_i(v)| \leq C_i \|v' - v\|_{-1}. \quad (5.8)$$

Proof. We have $\Phi_i(v') - \Phi_i(v) = (v' - v, \phi_{z_i})$, so taking the basis $(f_k)_{k \geq 1}$ in (5.7),

$$(v - v', \phi_{z_i}) = \sum_{k \geq 1} (v - v', f_k) (\phi_{z_i}, f_k) = \sum_{k \geq 1} \frac{1}{\sqrt{1 + k^2\pi^2}} (v - v', f_k) \sqrt{1 + k^2\pi^2} (\phi_{z_i}, f_k).$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} |(v - v', \phi_{z_i})| &\leq \left[\sum_{k \geq 1} \frac{1}{(1 + k^2\pi^2)} (v - v', f_k)^2 \right]^{\frac{1}{2}} \left[\sum_{k \geq 1} (1 + k^2\pi^2) (\phi_{z_i}, f_k)^2 \right]^{\frac{1}{2}} \\ &= \|v' - v\|_{-1} [((I - \Delta)\phi_{z_i}, \phi_{z_i})]^{\frac{1}{2}}. \end{aligned}$$

It remains to prove that $((I - \Delta)\phi_{z_i}, \phi_{z_i}) < +\infty$, so that (5.8) holds with $C_i = [((I - \Delta)\phi_{z_i}, \phi_{z_i})]^{\frac{1}{2}}$. To this end, we take $\phi_{z_i}(z) := \frac{1}{\gamma} M(\frac{z - z_i}{\gamma})$ with $M(z) = \mathbb{1}_{(-1,1)}(z) e^{-\frac{1}{1-z^2}}$. We have $\phi'_{z_i}(z) := \frac{1}{\gamma^2} M'(\frac{z - z_i}{\gamma})$ and $\phi''_{z_i}(z) := \frac{1}{\gamma^3} M''(\frac{z - z_i}{\gamma})$. Moreover

$$M'(\zeta) = -\frac{2\zeta}{(1 - \zeta^2)^2} M(\zeta), \quad M''(\zeta) = M(\zeta) \left[\frac{4\zeta^2}{(1 - \zeta^2)^4} - 2\frac{1 + 3\zeta^2}{(1 - \zeta^2)^3} \right] = M(\zeta) \frac{2(3\zeta^4 - 1)}{(1 - \zeta^2)^4}.$$

Therefore, setting $\zeta = \frac{z-z_i}{\gamma}$,

$$(I - \Delta)\phi_{z_i}(z) = \frac{1}{\gamma}M\left(\frac{z-z_i}{\gamma}\right) - \frac{1}{\gamma^3}M''\left(\frac{z-z_i}{\gamma}\right) = \phi_{z_i}(z)\left(1 - \frac{2}{\gamma^2}\frac{(3\zeta^4-1)}{(1-\zeta^2)^4}\right). \quad (5.9)$$

□

Proposition 5.4. *Let $d \in \bar{D}_N$, $v, v' \in E$. Then*

$$\|b(v, d) - b(v', d)\|_{-1} \leq C\|v - v'\|_{-1}. \quad (5.10)$$

Moreover, for all $R > 0$, there exists a positive constant C_R such that, for all $a \in A$,

$$|\lambda((v, d), a) - \lambda((v', d), a)| \leq C_R\|v - v'\|_{-1}, \quad v, v' \in E \text{ s.t. } \|v\| \vee \|v'\| \leq R. \quad (5.11)$$

Proof. Let $d \in \bar{D}_N$ and $v, v' \in E$. By (5.2) we have

$$b(v, d) = \sum_{i \in I_N} \gamma_i \phi_{z_i} - \sum_{i \in I_N} c_i \Phi_i(v) \phi_{z_i}. \quad (5.12)$$

Therefore

$$\|b(v', d) - b(v, d)\|_{-1} \leq \sum_{i \in I_N} c_i |\Phi_i(v) - \Phi_i(v')| \|\phi_{z_i}\|_{-1}. \quad (5.13)$$

Since $\|\phi_{z_i}\|_{-1} \leq C_i \|\phi_{z_i}\|$ and I_N is a finite set, (5.10) follows from (5.13) and Lemma 5.3.

Let us finally prove (5.11). We assume that $\|v\| \vee \|v'\| \leq R$. By definition (5.4), it is sufficient to check that, for any $i \in I_N$,

$$|\sigma_{d_i, \xi}(\Phi_i(v), a) - \sigma_{d_i, \xi}(\Phi_i(v'), a)| \leq C_R\|v - v'\|_{-1},$$

which in turn corresponds to prove the same property for the functions $\alpha_q(\Phi_i(v))$, $\beta_q(\Phi_i(v))$, $q = n, m, h$. Recalling (5.3) and applying the Cauchy-Schwartz inequality, we see that $\Phi_i(v)$, $\Phi_i(v')$ belong to a bounded interval J_R depending on R . Then, denoting by $K_{q,R}$ the Lipschitz constant of α_q on J_R , from Lemma 5.3

$$|\alpha_q(\Phi_i(v)) - \alpha_q(\Phi_i(v'))| \leq K_{q,R} |\Phi_i(v) - \Phi_i(v')| \leq K_{q,R} C_i \|v - v'\|_{-1},$$

where C_i is the positive constant in (5.8). The conclusion follows recalling that I_N is a finite set.

□

Lemma 5.5. *For any $d \in \bar{D}_N$, and $s, s' \in [t, T]$,*

- (i) $\|\phi_{s-t}^d(v)\| \leq C(1 + \|v\|), \quad v \in E,$
- (ii) $\|\phi_{s-t}^d(v) - \phi_{s'-t}^d(v)\| \leq C \sigma_R(|s - s'|), \quad v \in E : \|v\| \leq R,$
- (iii) $\|\phi_{s-t}^d(v) - \phi_{s-t}^d(v')\| \leq C \omega(\|v - v'\|), \quad v, v' \in E,$
- (iv) $\|\phi_{s-t}^d(v) - \phi_{s'-t}^d(v)\|_{-1} \leq C \sigma_R(|s - s'|), \quad v \in E : \|v\| \leq R,$
- (v) $\|\phi_{s-t}^d(v) - \phi_{s-t}^d(v')\|_{-1} \leq C \omega(\|v - v'\|_{-1}), \quad v, v' \in E.$

Proof. We first prove (i) and (iii). Setting $S(r) = e^{-rL}$, the equation for the mild solution to (5.1) starting from $x = (v, d) \in E \times \bar{D}_N$ reads

$$\phi_{s-t}^d(v) = S(s-t)v + \int_t^s S(s-r)b(\phi_{r-t}^d(v))dr.$$

Concerning (i), using the contraction property of $S(u)$ with respect to $\|\cdot\|$ given in Proposition 5.2-(ii), we obtain

$$\|\phi_{s-t}^d(v)\| \leq \|v\| + \int_t^s \|b(\phi_{r-t}^d(v))\|dr.$$

On the other hand, recalling (5.12),

$$\|\phi_{s-t}^d(v)\| \leq \|v\| + \sum_{i \in I_N} \int_t^s (|\gamma_i| + |c_i| |\Phi_i(\phi_{r-t}^d(v))|) dr \|\phi_{z_i}\|.$$

Using Lemma 5.3 we get

$$\|\phi_{s-t}^d(v)\| \leq (\|v\| + CT) + \Gamma \int_t^s \|\phi_{r-t}^d(v)\|dr,$$

and item (i) follows by Gronwall's Lemma.

Let us now turn to (iii). For any $d \in \bar{D}_N$, $v, v' \in E$, we have

$$\phi_{s-t}^d(v) - \phi_{s-t}^d(v') = S(s-t)(v' - v) + \int_t^s S(s-r)(b_d(\phi_{r-t}^d(v)) - b_d(\phi_{r-t}^d(v'))) dr.$$

Taking the norm $\|\cdot\|$, and applying Proposition 5.2-(ii) together with (5.10), we obtain

$$\|\phi_{s-t}^d(v) - \phi_{s-t}^d(v')\|_{-1} \leq \|v' - v\|_{-1} + C \int_0^s \|\phi_{r-t}^d(v) - \phi_{r-t}^d(v')\|_{-1} dr.$$

The conclusion follows again from the Gronwall Lemma.

Properties (iv) and (v) can be proved analogously, using the contraction property of $S(u)$ with respect to $\|\cdot\|_{-1}$ given in Proposition 5.2-(ii). \square

Additional results on $V = H_0^1(I)$. The space $V = H_0^1(I)$ is continuously embedded in the set of continuous functions on I . For any $k \in \mathbb{N}$, let us set

$$e_k = \frac{\sqrt{2}}{\sqrt{1+k^2\pi^2}} \sin(k\pi). \quad (5.14)$$

Then $(e_k)_{k \geq 1}$ is an orthonormal basis of $V = H_0^1(I)$, and $\Delta e_k = -k^2\pi^2 e_k$. For all $v \in V$, we set $(v, e_k)_V := \int_0^1 v(z)e_k(z)dz + \int_0^1 v'(z)e_k'(z)dz$. We have

$$\begin{aligned} \|v\|_V^2 &= \sum_{k \geq 1} (v, e_k)_V^2, & \|v\|_{-1,V}^2 &= ((I - \Delta)^{-1}v, v)_V = \sum_{k \geq 1} \frac{1}{(1+k^2\pi^2)} (v, e_k)_V^2, \\ \|S(r)v\|_{-1,V}^2 &= \sum_{k \geq 1} \frac{1}{(1+k^2\pi^2)} (S(r)v, e_k)_V^2 = \sum_{k \geq 1} \frac{1}{(1+k^2\pi^2)_V} e^{-2rk^2\pi^2} (v, e_k)_V^2 \quad \forall r \geq 0. \end{aligned}$$

Remark 5.6. Lemma 5.3 and Propositions 5.4. hold true with $V = H_0^1(I)$ in place of $E = L^2(0, 1)$.

The following result for the PDEs (5.1), given in Lemma 4.1 in [23], plays a fundamental role.

Lemma 5.7. *Set $V_- := \min(\bar{V}_{Na}, \bar{V}_K, \bar{V}_L, \bar{V}_{ChR2})$, $V_+ := \max(\bar{V}_{Na}, \bar{V}_K, \bar{V}_L, \bar{V}_{ChR2})$, and let $d \in \bar{D}_N$. If $v \in H_0^1(I)$ is continuous in $I = [0, 1]$, and $v(z) \in [V_-, V_+]$ for all $z \in I$, then, for every $d \in \bar{D}_N$,*

$$\phi_r^d(v)(z) \in [V_-, V_+], \quad r \in [0, T], \quad z \in I. \quad (5.15)$$

Physiologically speaking, we are only interested in the domain $[V_-, V_+]$. Since Lemma 5.7 shows that this domain is invariant for the controlled PDMP, we can modify the local characteristics of the PDMP outside the domain $[V_-, V_+]$ without changing its dynamics inside of $[V_-, V_+]$. We will do so for the rate functions $\sigma_{d_i, \xi}$. From now on, consider a compact set K containing the closed ball of E , centered in 0 with radius $\max\{V_-, V_+\}$. We will rewrite $\sigma_{d_i, \xi}$ outside K such that they all become Lipschitz and bounded functions. We also take V_{ref} taking values in K and \tilde{f} bounded and globally Lipschitz such that

$$\tilde{f}(v) = \|v - V_{\text{ref}}\|^2, \quad \forall v \in K. \quad (5.16)$$

Since the control set $A = [0, a_{\text{max}}]$ is bounded, the corresponding value function and cost

$$\tilde{J}(t, x, \alpha) = \mathbb{E}_\alpha^{t, x} \left[\int_t^T (\kappa \tilde{f}(v_s) + \alpha(X_s)) ds \right], \quad V(t, x) = \inf_{\alpha \in \mathcal{A}_{ad}^t} \tilde{J}(t, x, \alpha),$$

are bounded as well.

The next two results show that the case of the stochastic controlled infinite-dimensional Hodgkin-Huxley model can be actually covered by the theory on controlled infinite-dimensional PDMPs developed in the present paper.

Proposition 5.8. *Let $v, v' \in V$ such that $v(z)$ and $v'(z)$ belong to $[V_-, V_+]$ for all $z \in [0, 1]$. The following hold.*

(i) *There exist a positive constants C_1 such that, for all $d \in \bar{D}_N$, and $a \in A$,*

$$|\lambda((\phi_s^d(v), d), a) - \lambda((\phi_s^d(v'), d), a)| \leq C_1 \|v - v'\|_{-1, V}, \quad r \in [0, T]. \quad (5.17)$$

(ii) *If in addition $\|(I - \Delta)V_{\text{ref}}\| < +\infty$, there exists a positive constant C_2 such that, for all $d \in \bar{D}_N$, $a \in A$, the function f in (5.6) satisfies*

$$|f(\phi_s^d(v), a) - f(\phi_s^d(v'), a)| \leq C_2 \|v - v'\|_{-1, V}, \quad r \in [0, T]. \quad (5.18)$$

Proof. Let us prove item (i). Recalling (5.3) and using the Cauchy-Schwarz inequality we have

$$|\Phi_i(\phi_s^d(v))| \leq \|\phi_s^d(v)\| \|\phi_{z_i}\|. \quad (5.19)$$

Since

$$\|\phi_s^d(v)\| \leq \|\phi_s^d(v)\|_\infty \leq \max\{|V_-|, |V_+|\}, \quad (5.20)$$

and the same inequalities hold for $\phi_s^d(v')$, we have

$$|\alpha_q(\Phi_i(\phi_s^d(v))) - \alpha_q(\Phi_i(\phi_s^d(v')))| \leq K_{q, R} |\Phi_i(\phi_s^d(v)) - \Phi_i(\phi_s^d(v'))|, \quad (5.21)$$

with $R = \max\{|V_-|, |V_+|\}$ and $K_{q, R}$ the Lipschitz constant of α_q depending on R . Taking into account Remark 5.6, we conclude by applying the V -versions of Lemmas 5.3 and 5.4.

Let us now consider item (ii). Using the basis (e_k) introduced in (5.14), and applying the Cauchy-Schwarz inequality,

$$\begin{aligned} |f(\phi_s^d(v), a) - f(\phi_s^d(v'), a)| &= \kappa \left| \sum_{k \geq 1} ((\phi_s^d(v) - V_{ref}, e_k)^2 - (\phi_s^d(v') - V_{ref}, e_k)^2) \right| \\ &\leq \kappa \sum_{k \geq 1} |(\phi_s^d(v_0) - \phi_s^d(v'), e_k)| |(\phi_s^d(v) + \phi_s^d(v') - 2V_{ref}, e_k)| \leq \kappa \|\phi_s^d(v) - \phi_s^d(v')\|_{-1, V} \mathcal{T}, \end{aligned} \quad (5.22)$$

where

$$\mathcal{T} := \sum_{k \geq 1} (1 + k^2 \pi^2) (\phi_s^d(v) + \phi_s^d(v') - 2V_{ref}, e_k)^2 = \|(I - \Delta)(\phi_s^d(v) + \phi_s^d(v') - 2V_{ref})\|_V^2.$$

By Proposition 5.4 and Remark 5.6, it remains to study the boundedness properties of \mathcal{T} . Since by assumption $\|(I - \Delta)V_{ref}\| < +\infty$,

$$\mathcal{T} \leq C(\|(I - \Delta)\phi_s^d(v)\|_V^2 + \|(I - \Delta)\phi_s^d(v')\|_V^2 + \|(I - \Delta)V_{ref}\|_V^2). \quad (5.23)$$

Let us thus consider the term $\|(I - \Delta)\phi_d(s, v)\|_V$. Being $(I - \Delta)$ linear, we can write

$$(I - \Delta)\phi_s^d(v) = (I - \Delta)S(s)v_0 + \int_0^s (I - \Delta)S(s-r)b_d(\phi_r^d(v), a) dr.$$

Moreover, since $(I - \Delta)$ and $S(r)$ commute,

$$\|(I - \Delta)\phi_s^d(v)\|_V \leq \|(I - \Delta)v\|_V + \int_0^s \sum_{i \in I_N} (|\gamma_i| + |c_i| |\Phi_i(\phi_r^d(v))|) \|(I - \Delta)\phi_{z_i}\|_V dr, \quad (5.24)$$

where we have used that (recall formula (5.12))

$$(I - \Delta)b_d(\phi_r^d(v_0), a) = \sum_{i \in I_N} \gamma_i (I - \Delta)\phi_{z_i} - \sum_{i \in I_N} c_i \Phi_i(\phi_r^d(v_0)) (I - \Delta)\phi_{z_i}.$$

Recalling (5.19) and (5.20), (5.24) yields

$$\|(I - \Delta)\phi_s^d(v)\|_V \leq \|(I - \Delta)v\|_V + \int_0^s \sum_{i \in I_N} (|\gamma_i| + |c_i| \max\{|V_-|, |V_+\}|) \|\phi_{z_i}\|_V \|(I - \Delta)\phi_{z_i}\|_V dr.$$

Recalling (5.9) we see that, for any $i \in I_N$, $\|(I - \Delta)\phi_{z_i}\|_V \leq C_i$. Since I_N is finite, we conclude from the above inequality that there exists some constant Γ such that

$$\|(I - \Delta)\phi_d(s, v)\|_V \leq \|(I - \Delta)v\|_V + \Gamma T; \quad (5.25)$$

analogous inequalities holds true for $\phi_s^d(v')$ and v' . Then (5.23), together with (5.25), yields

$$\mathcal{T} \leq 2\|(I - \Delta)v\|_V + 2\Gamma T + 4\|(I - \Delta)V_{ref}\|_V^2. \quad (5.26)$$

and the conclusion follows. \square

Proposition 5.9. *Let $v_0 \in V$ such that $v_0(z) \in [V_-, V_+]$ for all $z \in [0, 1]$. Then there exist two positive constants C_1, C_2 , only depending on $T, N, \max\{|V_-|, |V_+\}|$, such that, for all $d \in D$, $a \in A$,*

$$\|\phi_s^d(v_0)\|_V \leq C_1, \quad s \in [0, T], \quad (5.27)$$

$$|f((\phi_s^d(v_0), d), a)| + |\lambda((\phi_s^d(v_0), d), a)| + \|b(\phi_s^d(v_0), d)\|_V \leq C_2, \quad s \in [0, T]. \quad (5.28)$$

Proof. Estimate (5.27) is obtained arguing as in Lemma 5.5-(i). The boundedness of $f(\phi_s^d(v_0), a)$ follows from (5.27), recalling that

$$|f(\phi_s^d(v_0), a)| = \kappa \left| \sum_{k \geq 1} ((\phi_s^d(v_0) - V_{ref}, e_k)^2) \right| \leq \kappa (\|\phi_s^d(v_0)\|_V^2 + V_{ref}^2 - 2V_{ref} \|\phi_s^d(v_0)\|_V).$$

On the other hand, recalling (5.12) and (5.19),

$$\|b_d(\phi_s^d(v_0))\|_V \leq \sum_{i \in I_N} |\gamma_i| \|\phi_{z_i}\|_V + \sum_{i \in I_N} |c_i| \|\phi_s^d(v_0)\| \|\phi_{z_i}\| \|\phi_{z_i}\|_V,$$

and we obtain the bound from Lemma 5.5-(i) and the fact that $\|\cdot\| \leq \|\cdot\|_V$.

The boundedness of $\lambda_d(\phi_s^d(v_0), a)$ follows from the form of the functions α_q, β_q , together with (5.19) and the fact that $\|\cdot\| \leq \|\cdot\|_V$. \square

6 Proofs of the results in Section 2

6.1 Proof of Proposition 2.5

The boundedness of V directly comes from (2.16) and the boundedness of f and g . Let $C_b([0, T] \times E)$ be the set of bounded functions, continuous on $[0, T] \times E$ with the $\|\cdot\| \times \|\cdot\|_{-1}$ norm. For any bounded Borel-measurable function $\psi : [0, T] \times E \rightarrow \mathbb{R}$ we set

$$\mathcal{T}\psi(t, x) := \inf_{\alpha \in \mathcal{A}_{ad}^t} \mathbb{E}_\alpha^{t, x} \left[\int_t^{T_1^t \wedge T} f(X_s, \alpha_s) ds + g(X_T) \mathbb{1}_{T \leq T_1^t} + \psi(T_1^t, X_{T_1^t}) \mathbb{1}_{T > T_1^t} \right].$$

We aim at proving that

- (i) $\mathcal{T}\psi \in C_b([0, T] \times E)$ for any $\psi \in C_b([0, T] \times E)$;
- (ii) \mathcal{T} is a contracting map in $C_b([0, T] \times E)$ and V is its unique fixed point. In particular, V satisfies the DPP (2.17);
- (iii) V is uniformly continuous in the $\|\cdot\| \times \|\cdot\|_{-1}$ norm.

Proof of item (i). Set $\mathcal{U} = \{u : [0, +\infty) \rightarrow A \text{ measurable}\}$. One can show that

$$\mathcal{T}\psi(t, x) = \inf_{u \in \mathcal{U}} \bar{J}(t, x, u) \tag{6.1}$$

with

$$\bar{J}(t, x, u) = \int_0^{T-t} \chi^u(s, x) (f^u(s, x) + L_\psi^{t, u}(s, x)) ds + \chi^u(T-t, x) g(\phi^u(T-t, x)),$$

where $\chi^u(s, x) = e^{-\int_0^s \lambda(\phi^u(r, x), u_r) dr}$, $f^u(s, x) = f(\phi^u(s, x), u_s)$, and

$$L_\psi^{t, u}(s, x) = \int_E \psi(s+t, y) \lambda(\phi^u(s, x), u_s) Q(\phi^u(s, x), u_s, dy).$$

In the sequel C will denote a generic constant, that may vary from line to line, and that may depend on T . Let $t, t', s \in [0, T]$, $t' \leq t \leq s$, $x, x' \in E$, $u \in \mathcal{U}$. Let $\psi \in C_b([0, T] \times E)$. Recalling

hypotheses **(HbλQ)**-(i), **(Hfg)** and (2.12)-(2.13), we have $|\chi^u(s, x)| \leq 1$, $|f^u(s, x)| \leq C$, and, for any $s' \leq s$,

$$|\chi^u(s', x) - \chi^u(s, x')| \leq (1 - e^{-C\|x-x'\|_{-1}}) + (1 - e^{-C(s-s')}), \quad (6.2)$$

$$|f^u(s, x) - f^u(s, x')| \leq C\|x - x'\|_{-1}, \quad (6.3)$$

$$|g(\phi(T-t, x)) - g(\phi(T-t', x'))| \leq \omega(\|x - x'\|_{-1}). \quad (6.4)$$

On the other hand, by **(HbλQ)**-(i)-(ii), together with the boundedness and continuity of ψ , we have $|L_\psi^{t,u}(s, x)| \leq C$ and, for $s < T-t$,

$$\begin{aligned} |L_\psi^{t,u}(s, x) - L_\psi^{t,u}(s, x')| &\leq |\lambda(\phi^u(s, x), u_s) - \lambda(\phi^u(s, x'), u_s)| \|\psi\|_\infty \\ &+ \|\lambda\|_\infty \left| \int_E \psi(s+t, y) [Q(\phi^u(s, x), u_s, dy) - Q(\phi^u(s, x'), u_s, dy)] \right| \\ &\leq C \sigma(\|\phi^u(s, x) - \phi^u(s, x')\|_{-1}) \leq C\omega(\|x - x'\|_{-1}), \end{aligned} \quad (6.5)$$

where the latter inequality follows from (2.13). Then, for any $t, t' \in [0, T]$, $x, x' \in E$, $u \in \mathcal{U}$,

$$\begin{aligned} &|\bar{J}(t, x, u) - \bar{J}(t', x', u)| \\ &\leq \left| \int_0^{T-t} \chi^u(s, x) f^u(s, x) ds - \int_0^{T-t'} \chi^u(s, x') f^u(s, x') ds \right| \\ &+ \left| \int_0^{T-t} \chi^u(s, x) L_\psi^{t,u}(s, x) ds - \int_0^{T-t'} \chi^u(s, x') L_\psi^{t,u}(s, x') ds \right| \\ &+ |\chi^u(T-t, x)g(\phi^u(T-t, x)) - \chi^u(T-t', x')g(\phi^u(T-t', x'))| \\ &\leq \int_0^{T-t} |\chi^u(s, x) f^u(s, x) - \chi^u(s, x') f^u(s, x')| ds \\ &+ \int_0^{T-t} |\chi^u(s, x) L_\psi^{t,u}(s, x) - \chi^u(s, x') L_\psi^{t,u}(s, x')| ds + C|t - t'| \\ &+ C|g(\phi^u(T-t, x)) - g(\phi^u(T-t', x'))| + C|\chi^u(T-t, x) - \chi^u(T-t', x')| \\ &\leq C \left(\int_0^{T-t} |\chi^u(s, x) - \chi^u(s, x')| ds + \int_0^{T-t} |f^u(s, x) - f^u(s, x')| ds \right. \\ &+ \int_0^{T-t} |L_\psi^{t,u}(s, x) - L_\psi^{t,u}(s, x')| ds + |g(\phi^u(T-t, x)) - g(\phi^u(T-t', x'))| \\ &\left. + |\chi^u(T-t, x) - \chi^u(T-t', x')| + |t - t'| \right) \\ &\leq C(\omega(t-t') + \omega'(\|x - x'\|_{-1})), \end{aligned} \quad (6.6)$$

for some modulus of continuity ω , ω' , where the latter inequality follows from (6.2), (6.3), (6.4), (6.5).

Proof of item (ii). Denote by $\mathcal{P}(A)$ the set of Borel probability measures on A , set $\mathcal{M} = \{u : [0, +\infty) \rightarrow \mathcal{P}(A) \text{ measurable}\}$, and introduce the auxiliary map $\mathcal{T}' : C_b([0, T] \times E) \rightarrow C_b([0, T] \times E)$ defined by

$$\begin{aligned} &\mathcal{T}'\psi(t, x) \\ &:= \inf_{m \in \mathcal{M}} \left\{ \int_0^{T-t} \chi^m(s, x) \int_A (f^m(s, x, u) + L_\psi^{t,m}(s, x, u)) m(s, du) ds + \chi^m(T-t, x)g(\phi^m(T-t, x)) \right\}. \end{aligned}$$

Here $\phi^m(s, x)$ is the unique mild solution to the parabolic partial differential equation

$$\dot{x}(s) = -Lx(s) + \int_A b(x(s), u) m(s, du), \quad x(0) = x \in E.$$

Moreover, $\chi^m(s, x) = e^{-\int_0^s \lambda(\phi^m(r, x), u) m(r, du) dr}$, $f^m(s, x, u) = f(\phi^m(s, x), u)$, and

$$L_\psi^{t, m}(s, x, u) = \int_E \psi(s + t, y) \lambda(\phi^m(s, x), u) Q(\phi^m(s, x), u, dy).$$

Set $\mathcal{A}_{ad}^{t, \mathcal{R}} = \{\gamma : \Omega \times [t, \infty) \rightarrow \mathcal{P}(A) \text{ predictable}\}$. It can be proved that \mathcal{T}' is a contracting map in $C_b([0, T] \times E)$ and that

$$V'(t, x) := \inf_{\gamma \in \mathcal{A}_{ad}^{t, \mathcal{R}}} \mathbb{E}_\gamma^{t, x} \left[\int_t^T f(X_s, u) \gamma(s, du) ds + g(X_T) \right]$$

is its unique fixed point, see e.g. Theorem 3.3 and Lemma 3.4 in [23]. It is clear that $V' = \mathcal{T}'V' \leq \mathcal{T}V'$. The reverse inequality follows from the density of the set \mathcal{U} in \mathcal{M} with respect to the Young topology, see Theorem 3.6 in [8] for more details. Analogously, one proves that $V = V'$, and concludes that V is also the unique fixed point of \mathcal{T} in $C_b([0, T] \times E)$.

Proof of item (iii). It directly follows from item (ii) and estimate (6.6). \square

6.2 Proof of Proposition 2.6

By Proposition 2.5,

$$V(t, x) = \inf_{\alpha \in \mathcal{A}_{ad}^t} \mathbb{E}_\alpha^{t, x} \left[\int_t^{T_1^t \wedge T} f(X_s, \alpha_s) ds + g(X_T) 1_{T \leq T_1^t} + V(T_1^t, X_{T_1^t}) 1_{T > T_1^t} \right]. \quad (6.7)$$

Setting $\mathcal{U} = \{u : [0, +\infty) \rightarrow A \text{ measurable}\}$ and arguing as in the proof of Proposition 2.5, formula (6.7) gives

$$\begin{aligned} V(t, x) = \inf_{u \in \mathcal{U}} \left\{ \int_t^T e^{-\int_t^s \lambda(\tilde{X}_r^{t, x, u}, u_r) dr} f(\tilde{X}_s^{t, x, u}, u_s) ds \right. \\ \left. + \int_t^T e^{-\int_t^s \lambda(\tilde{X}_r^{t, x, u}, u_r) dr} \int_E V(s, y) \lambda(\tilde{X}_s^{t, x, u}, u_s) Q(\tilde{X}_s^{t, x, u}, u_s, dy) ds \right. \\ \left. + e^{-\int_t^T \lambda(\tilde{X}_r^{t, x, u}, u_r) dr} V(T, \tilde{X}_T^{t, x, u}) \right\} \end{aligned}$$

where we have set $\tilde{X}_r^{t, x, u} = \phi^u(r - t, x)$, $r \in [t, T]$. We aim at proving that

$$\begin{aligned} V(t, x) = \Lambda(t, x) := \inf_{u \in \mathcal{U}} \left\{ \int_t^\zeta e^{-\int_t^s \lambda(\tilde{X}_r^{t, x, u}, u_r) dr} f(\tilde{X}_s^{t, x, u}, u_s) ds \right. \\ \left. + \int_t^\zeta e^{-\int_t^s \lambda(\tilde{X}_r^{t, x, u}, u_r) dr} \int_E V(s, y) \lambda(\tilde{X}_s^{t, x, u}, u_s) Q(\tilde{X}_s^{t, x, u}, u_s, dy) ds \right. \\ \left. + e^{-\int_t^\zeta \lambda(\tilde{X}_r^{t, x, u}, u_r) dr} V(\zeta, \tilde{X}_\zeta^{t, x, u}) \right\}, \quad \zeta \in [t, T]. \quad (6.8) \end{aligned}$$

It is easy to see that

$$\Lambda(t, x) = \inf_{\alpha \in \mathcal{A}_{ad}^t} \mathbb{E}_\alpha^{t,x} \left[\int_t^{\zeta \wedge T_1^t} f(X_s, \alpha_s) ds + V(\zeta \wedge T_1^t, X_{\zeta \wedge T_1^t}) \right], \quad \zeta \in [t, T].$$

On the other hand, for any $\alpha \in \mathcal{A}_{ad}^t$,

$$\tau \wedge T_1^t \wedge T = \tau_d \wedge T_1^t \wedge T, \quad \mathbb{P}_\alpha^{t,x}\text{-a.s.}$$

with $\tau_d := \inf \{s \geq t : (s, \phi^{\alpha_0}(s-t, x)) \notin B((t, x); \rho)\}$. Therefore, formula (2.18) would follow from identity (6.8) by choosing $\zeta = T \wedge \tau_d$.

Let us thus prove (6.8). We first prove that $\Lambda(t, x) \leq V(t, x)$. Let us fix $\zeta \in [t, T]$ and $\beta \in \mathcal{U}$. We have

$$\begin{aligned} V(\zeta, \tilde{X}_\zeta^{t,x,\beta}) &= \inf_{u \in \mathcal{U}} \left\{ \int_\zeta^T e^{-\int_\zeta^s \lambda(\tilde{X}_r^{\zeta, \tilde{X}_\zeta^{t,x,u}, u}, u_r) dr} f(\tilde{X}_s^{\zeta, \tilde{X}_\zeta^{t,x,u}, u}, u_s) ds \right. \\ &\quad + \int_\zeta^T e^{-\int_\zeta^s \lambda(\tilde{X}_r^{\zeta, \tilde{X}_\zeta^{t,x,u}, u}, u_r) dr} \int_E V(s, y) \lambda(\tilde{X}_s^{\zeta, \tilde{X}_\zeta^{t,x,u}, u}, u_s) Q(\tilde{X}_s^{\zeta, \tilde{X}_\zeta^{t,x,u}, u}, u_s, dy) ds \\ &\quad \left. + e^{-\int_\zeta^T \lambda(\tilde{X}_r^{\zeta, \tilde{X}_\zeta^{t,x,u}, u}, u_r) dr} V(T, \tilde{X}_T^{\zeta, \tilde{X}_\zeta^{t,x,u}, u}) \right\} \\ &\leq \int_\zeta^T e^{-\int_\zeta^s \lambda(\tilde{X}_r^{\zeta, \tilde{X}_\zeta^{t,x,\beta}, \beta}, \beta_r) dr} f(\tilde{X}_s^{\zeta, \tilde{X}_\zeta^{t,x,\beta}, \beta}, \beta_s) ds \\ &\quad + \int_\zeta^T e^{-\int_\zeta^s \lambda(\tilde{X}_r^{\zeta, \tilde{X}_\zeta^{t,x,\beta}, \beta}, \beta_r) dr} \int_E V(s, y) \lambda(\tilde{X}_s^{\zeta, \tilde{X}_\zeta^{t,x,\beta}, \beta}, \beta_s) Q(\tilde{X}_s^{\zeta, \tilde{X}_\zeta^{t,x,\beta}, \beta}, \beta_s, dy) ds \\ &\quad + e^{-\int_\zeta^T \lambda(\tilde{X}_r^{\zeta, \tilde{X}_\zeta^{t,x,\beta}, \beta}, \beta_r) dr} V(T, \tilde{X}_T^{\zeta, \tilde{X}_\zeta^{t,x,\beta}, \beta}). \end{aligned} \quad (6.9)$$

Recalling the flow property

$$\tilde{X}_r^{\zeta, \tilde{X}_\zeta^{t,x,\beta}, \beta} = \tilde{X}_r^{t,x,\beta}, \quad r \in [\zeta, T],$$

inequality (6.9) yields

$$\begin{aligned} V(\zeta, \tilde{X}_\zeta^{t,x,\beta}) &\leq \int_\zeta^T e^{-\int_\zeta^s \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr} f(\tilde{X}_s^{t,x,\beta}, \beta_s) ds \\ &\quad + \int_\zeta^T e^{-\int_\zeta^s \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr} \int_E V(s, y) \lambda(\tilde{X}_s^{t,x,\beta}, \beta_s) Q(\tilde{X}_s^{t,x,\beta}, \beta_s, dy) ds \\ &\quad + e^{-\int_\zeta^T \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr} V(T, \tilde{X}_T^{t,x,\beta}). \end{aligned} \quad (6.10)$$

Multiplying by $e^{-\int_t^\zeta \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr}$ both sides of (6.10) we get

$$\begin{aligned} e^{-\int_t^\zeta \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr} V(\zeta, \tilde{X}_\zeta^{t,x,\beta}) &\leq \int_\zeta^T e^{-\int_t^s \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr} f(\tilde{X}_s^{t,x,\beta}, \beta_s) ds \\ &\quad + \int_\zeta^T e^{-\int_t^s \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr} \int_E V(s, y) \lambda(\tilde{X}_s^{t,x,\beta}, \beta_s) Q(\tilde{X}_s^{t,x,\beta}, \beta_s, dy) ds \\ &\quad + e^{-\int_t^T \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr} V(T, \tilde{X}_T^{t,x,\beta}). \end{aligned} \quad (6.11)$$

Adding to both sides of (6.11) the quantity

$$\begin{aligned} & \int_t^\zeta e^{-\int_t^s \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr} f(\tilde{X}_s^{t,x,\beta}, \beta_s) ds \\ & + \int_t^\zeta e^{-\int_t^s \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr} \int_E V(s, y) \lambda(\tilde{X}_s^{t,x,\beta}, \beta_s) Q(\tilde{X}_s^{t,x,\beta}, \beta_s, dy) ds \end{aligned}$$

we obtain

$$\begin{aligned} \Lambda(t, x) & \leq \int_t^T e^{-\int_t^s \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr} f(\tilde{X}_s^{t,x,\beta}, \beta_s) ds \\ & + \int_t^T e^{-\int_t^s \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr} \int_E V(s, y) \lambda(\tilde{X}_s^{t,x,\beta}, \beta_s) Q(\tilde{X}_s^{t,x,\beta}, \beta_s, dy) ds \\ & + e^{-\int_t^T \lambda(\tilde{X}_r^{t,x,\beta}, \beta_r) dr} V(T, \tilde{X}_T^{t,x,\beta}). \end{aligned}$$

We conclude by the arbitrariness of $\beta \in \mathcal{U}$.

Let us now prove that $\Lambda(t, x) \geq V(t, x)$. Let us fix $\zeta \in [t, T]$. For any $\varepsilon > 0$, let $u^\varepsilon \in \mathcal{U}$ such that

$$\begin{aligned} \Lambda(t, x) & \geq \int_t^\zeta e^{-\int_t^s \lambda(\tilde{X}_r^{t,x,u^\varepsilon}, u_r^\varepsilon) dr} f(\tilde{X}_s^{t,x,u^\varepsilon}, u_s^\varepsilon) ds \\ & + \int_t^\zeta e^{-\int_t^s \lambda(\tilde{X}_r^{t,x,u^\varepsilon}, u_r^\varepsilon) dr} \int_E V(s, y) \lambda(\tilde{X}_s^{t,x,u^\varepsilon}, u_s^\varepsilon) Q(\tilde{X}_s^{t,x,u^\varepsilon}, u_s^\varepsilon, dy) ds \\ & + e^{-\int_t^\zeta \lambda(\tilde{X}_r^{t,x,u^\varepsilon}, u_r^\varepsilon) dr} V(\zeta, \tilde{X}_\zeta^{t,x,u^\varepsilon}) - \varepsilon. \end{aligned} \tag{6.12}$$

From the definition of $V(\zeta, \tilde{X}_\zeta^{t,x,u^\varepsilon})$, there exists $\beta^\varepsilon \in \mathcal{U}$ such that

$$\begin{aligned} & V(\zeta, \tilde{X}_\zeta^{t,x,u^\varepsilon}) \\ & \geq \int_\zeta^T e^{-\int_\zeta^s \lambda(\tilde{X}_r^{\zeta, \tilde{X}_\zeta^{t,x,u^\varepsilon}, \beta^\varepsilon}, \beta_r^\varepsilon) dr} f(\tilde{X}_s^{\zeta, \tilde{X}_\zeta^{t,x,u^\varepsilon}, \beta^\varepsilon}, \beta_s^\varepsilon) ds \\ & + \int_\zeta^T e^{-\int_\zeta^s \lambda(\tilde{X}_r^{\zeta, \tilde{X}_\zeta^{t,x,u^\varepsilon}, \beta^\varepsilon}, \beta_r^\varepsilon) dr} \int_E V(s, y) \lambda(\tilde{X}_s^{\zeta, \tilde{X}_\zeta^{t,x,u^\varepsilon}, \beta^\varepsilon}, \beta_s^\varepsilon) Q(\tilde{X}_s^{\zeta, \tilde{X}_\zeta^{t,x,u^\varepsilon}, \beta^\varepsilon}, \beta_s^\varepsilon, dy) ds \\ & + e^{-\int_\zeta^T \lambda(\tilde{X}_r^{\zeta, \tilde{X}_\zeta^{t,x,u^\varepsilon}, \beta^\varepsilon}, \beta_r^\varepsilon) dr} V(T, \tilde{X}_T^{\zeta, \tilde{X}_\zeta^{t,x,u^\varepsilon}, \beta^\varepsilon}) - \varepsilon. \end{aligned} \tag{6.13}$$

We set

$$\gamma^\varepsilon = u^\varepsilon \mathbf{1}_{[t, \zeta]} + \beta^\varepsilon \mathbf{1}_{[\zeta, T]}.$$

We have $\gamma^\varepsilon \in \mathcal{U}$. Moreover, the flow property gives

$$\tilde{X}_r^{\zeta, \tilde{X}_\zeta^{t,x,u^\varepsilon}, \beta^\varepsilon} = \tilde{X}_r^{t,x,\gamma^\varepsilon}, \quad r \in [\zeta, T].$$

Therefore (6.13) reads

$$\begin{aligned} V(\zeta, \tilde{X}_\zeta^{t,x,u^\varepsilon}) & \geq \int_\zeta^T e^{-\int_\zeta^s \lambda(\tilde{X}_r^{t,x,\gamma^\varepsilon}, \gamma_r^\varepsilon) dr} f(\tilde{X}_s^{t,x,\gamma^\varepsilon}, \gamma_s^\varepsilon) ds \\ & + \int_\zeta^T e^{-\int_\zeta^s \lambda(\tilde{X}_r^{t,x,\gamma^\varepsilon}, \gamma_r^\varepsilon) dr} \int_E V(s, y) \lambda(\tilde{X}_s^{t,x,\gamma^\varepsilon}, \gamma_s^\varepsilon) Q(\tilde{X}_s^{t,x,\gamma^\varepsilon}, \gamma_s^\varepsilon, dy) ds \end{aligned}$$

$$+ e^{-\int_{\zeta}^T \lambda(\tilde{X}_r^{t,x,\gamma^\varepsilon}, \gamma_r^\varepsilon) dr} V(T, \tilde{X}_T^{t,x,\gamma^\varepsilon}) - \varepsilon,$$

and (6.12) gives

$$\begin{aligned} \Lambda(t, x) &\geq \int_t^T e^{-\int_t^s \lambda(\tilde{X}_r^{t,x,\gamma^\varepsilon}, \gamma_r^\varepsilon) dr} f(\tilde{X}_s^{t,x,\gamma^\varepsilon}, \gamma_s^\varepsilon) ds \\ &\quad + \int_t^T e^{-\int_t^s \lambda(\tilde{X}_r^{t,x,\gamma^\varepsilon}, \gamma_r^\varepsilon) dr} \int_E V(s, y) \lambda(\tilde{X}_s^{t,x,\gamma^\varepsilon}, \gamma_s^\varepsilon) Q(\tilde{X}_s^{t,x,\gamma^\varepsilon}, \gamma_s^\varepsilon, dy) ds \\ &\quad + e^{-\int_t^T \lambda(\tilde{X}_r^{t,x,\gamma^\varepsilon}, \gamma_r^\varepsilon) dr} V(T, \tilde{X}_T^{t,x,\gamma^\varepsilon}) - 2\varepsilon \\ &\geq V(t, x) - 2\varepsilon. \end{aligned}$$

The conclusion follows from the arbitrariness of ε . \square

6.3 Proof of Lemma 2.10

Let us fix $t \in (\varepsilon, T - \varepsilon)$, $\varepsilon > 0$. We first prove that the map

$$\begin{aligned} r \mapsto & -\frac{\partial \psi}{\partial s}(r, \phi^{\alpha_0}(r-t, x)) + \langle \phi^{\alpha_0}(r-t, x), L^* D\varphi(r, \phi^{\alpha_0}(r-t, x)) \rangle \\ & + \langle \phi^{\alpha_0}(r-t, x), h(\|\phi^{\alpha_0}(r-t, x)\|) L^* D\delta(r, \phi^{\alpha_0}(r-t, x)) \rangle \end{aligned}$$

is continuous on $[t, T - \varepsilon)$, uniformly in α_0 , and on $B_R(x) := \{x \in E : \|x\| \leq R\}$, $R > 0$. To this end, let $r, r' \in [t, T - \varepsilon)$. Since ψ satisfies Definition 2.8, in particular $\frac{\partial \psi}{\partial s}$, $L^* D\varphi$, $L^* D\delta$ are bounded on bounded sets of E . In the following C will denote a generic constant that may depend on T and x , and that may vary from line to line. For any $x \in E : \|x\| \leq R$, $R > 0$, we have

$$\begin{aligned} & \left| \frac{\partial \psi}{\partial s}(r, \phi^{\alpha_0}(r-t, x)) - \frac{\partial \psi}{\partial s}(r', \phi^{\alpha_0}(r'-t, x)) \right| \\ & \leq C\omega(|r - r'| + \|\phi^{\alpha_0}(r-t, x) - \phi^{\alpha_0}(r'-t, x)\|) \leq C\sigma_R(|r - r'|), \end{aligned}$$

where in the latter inequality we have used (2.10). Using again the properties of the test functions, together with (2.10)-(2.11), we get

$$\begin{aligned} & |\langle \phi^{\alpha_0}(r-t, x), L^* D\varphi(r, \phi^{\alpha_0}(r-t, x)) \rangle - \langle \phi^{\alpha_0}(r'-t, x), L^* D\varphi(r', \phi^{\alpha_0}(r'-t, x)) \rangle| \\ & \leq |\langle \phi^{\alpha_0}(r-t, x) - \phi^{\alpha_0}(r'-t, x), L^* D\varphi(r', \phi^{\alpha_0}(r'-t, x)) \rangle| \\ & \quad + |\langle \phi^{\alpha_0}(r-t, x), L^* D\varphi(r, \phi^{\alpha_0}(r-t, x)) - L^* D\varphi(r', \phi^{\alpha_0}(r'-t, x)) \rangle|. \\ & \leq C\sigma_R(|r - r'|) + C(1 + \|x\|) \omega(|r - r'| + \|\phi^{\alpha_0}(r-t, x) - \phi^{\alpha_0}(r'-t, x)\|) \\ & \leq C\sigma_R(|r - r'|). \end{aligned}$$

Analogously,

$$\begin{aligned} & |\langle \phi^{\alpha_0}(r-t, x), h(\|\phi^{\alpha_0}(r-t, x)\|) L^* D\delta(r, \phi^{\alpha_0}(r-t, x)) \rangle \\ & \quad - \langle \phi^{\alpha_0}(r'-t, x), h(\|\phi^{\alpha_0}(r'-t, x)\|) L^* D\delta(r', \phi^{\alpha_0}(r'-t, x)) \rangle| \\ & \leq C\sigma_R(|r - r'|). \end{aligned}$$

Moreover, for any $x \in E$, $a \in A$, and any measurable function $\alpha_0 : \mathbb{R}_+ \times E \rightarrow A$, the map

$$r \mapsto f(\phi^{\alpha_0}(r-t, x), a)$$

is continuous on $[t, T - \varepsilon)$, uniformly in α_0 and in a . Indeed, from **(Hfg)** and (2.13), for any $r \in [t, T - \varepsilon)$, $x \in E : \|x\| \leq R$, $R > 0$,

$$\begin{aligned} & |f(\phi^{\alpha_0}(r' - t, x), a) - f(\phi^{\alpha_0}(r - t, x), a)| \\ & \leq C\omega(\|\phi^{\alpha_0}(r - t, x), a) - \phi^{\alpha_0}(r' - t, x), a\|_{-1}) \leq C\sigma_R(|r - r'|). \end{aligned}$$

Let us finally study the continuity of the maps

$$\begin{aligned} r & \mapsto \mathcal{L}^a \psi(r, \phi^{\alpha_0}(r - t, x), a) \\ & = \langle b(\phi^{\alpha_0}(r - t, x), a), D\psi(r, \phi^{\alpha_0}(r - t, x)) \rangle \\ & \quad + \lambda(\phi^{\alpha_0}(r - t, x), a) \int_E (\psi(r, y) - \psi(r, \phi^{\alpha_0}(r - t, x))) Q(\phi^{\alpha_0}(r - t, x), a, dy). \end{aligned}$$

Since by Definition 2.8 $D\psi$ is bounded on bounded sets of E , and using assumption **(Hb λ Q)** for b , for any $x \in E : \|x\| \leq R$, $R > 0$ we get

$$\begin{aligned} & |\langle b(\phi^{\alpha_0}(r - t, x), a), D\psi(t, \phi^{\alpha_0}(r - t, x)) \rangle - \langle b(\phi^{\alpha_0}(r' - t, x), a), D\psi(r, \phi^{\alpha_0}(r' - t, x)) \rangle| \\ & \leq |\langle b(\phi^{\alpha_0}(r - t, x), a) - \langle b(\phi^{\alpha_0}(r' - t, x), a), D\psi(r, \phi^{\alpha_0}(r' - t, x)) \rangle| \\ & \quad + |\langle b(\phi^{\alpha_0}(r - t, x), a), D\psi(t, \phi^{\alpha_0}(r - t, x)) \rangle - D\psi(r, \phi^{\alpha_0}(r' - t, x))| \\ & \leq C\sigma_R(|r - r'|) + C\omega(|r - r'| + \|\phi^{\alpha_0}(r - t, x), a) - \phi^{\alpha_0}(r' - t, x), a\|_{-1}) \\ & \leq C\sigma_R(|r - r'|). \end{aligned}$$

On the other hand, by assumption **(Hb λ Q)** for λ and Q , recalling that ψ is uniformly continuous on $(\varepsilon, T - \varepsilon) \times E$, for any $x \in E : \|x\| \leq R$, $R > 0$ we have

$$\begin{aligned} & \left| \lambda(\phi^{\alpha_0}(r - t, x), a) \int_E (\psi(r, y) - \psi(r, \phi^{\alpha_0}(r - t, x))) Q(\phi^{\alpha_0}(r - t, x), a, dy) \right. \\ & \quad \left. - \lambda(\phi^{\alpha_0}(r' - t, x), a) \int_E (\psi(r', y) - \psi(r', \phi^{\alpha_0}(r' - t, x))) Q(\phi^{\alpha_0}(r' - t, x), a, dy) \right| \\ & \leq |\lambda(\phi^{\alpha_0}(r - t, x), a) - \lambda(\phi^{\alpha_0}(r' - t, x), a)| \left| \int_E (\psi(r', y) - \psi(r', \phi^{\alpha_0}(r' - t, x))) Q(\phi^{\alpha_0}(r' - t, x), a, dy) \right| \\ & \quad + |\lambda(\phi^{\alpha_0}(r - t, x), a)| |\psi(r', \phi^{\alpha_0}(r' - t, x)) - \psi(r, \phi^{\alpha_0}(r - t, x))| \\ & \quad + |\lambda(\phi^{\alpha_0}(r - t, x), a)| \left| \int_E \psi(r, y) Q(\phi^{\alpha_0}(r - t, x), a, dy) - \int_E \psi(r', y) Q(\phi^{\alpha_0}(r' - t, x), a, dy) \right| \\ & \leq C\|\phi^{\alpha_0}(r - t, x) - \phi^{\alpha_0}(r' - t, x)\|_{-1} + C\omega(|r - r'| + \|\phi^{\alpha_0}(r' - t, x) - \phi^{\alpha_0}(r - t, x)\|_{-1}) \\ & \quad + C\sigma_R(|r - r'|) + C\omega(\|\phi^{\alpha_0}(r - t, x), a) - \phi^{\alpha_0}(r' - t, x), a\|_{-1}) \\ & \leq C\sigma_R(|r - r'|), \end{aligned}$$

where the latter inequality follows from (2.10)-(2.11)-(2.12). \square

6.4 Proof of Theorem 2.11

We start by giving the following preliminary result.

Lemma 6.1. *Assume that Hypotheses **(HL)** and **(Hb λ Q)** hold. Let $0 < t < \bar{T} < T$, $\hat{\tau}$ be a stopping time such that $\hat{\tau} \in [t, \bar{T}]$, $x \in E$, $\alpha \in \mathcal{A}_{ad}^t$, and X be the process in (2.6) under $\mathbb{P}_\alpha^{t,x}$. For $R > 0$, let τ_R be the exit time of X from $\{y : \|y\| \leq R\}$, and set $\tau = \hat{\tau} \wedge \tau_R$. Let $\psi = \varphi + h(\|\cdot\|) \delta$ be a test function. Then,*

$$\mathbb{E}_\alpha^{t,x} [\psi(\tau, X_\tau)] \leq \psi(t, x) + \mathbb{E}_\alpha^{t,x} \left[\int_t^\tau \left(\frac{\partial \psi}{\partial t}(r, X_r) + \langle b(X_r, \alpha_r), D\psi(r, X_r) \rangle \right) dr \right]$$

$$\begin{aligned}
& - \mathbb{E}_\alpha^{t,x} \left[\int_t^\tau \langle X_r, L^* D\varphi(r, X_r) + h(\|X_r\|) L^* D\delta(r, X_r) \rangle dr \right] \\
& + \mathbb{E}_\alpha^{t,x} \left[\int_t^\tau \int_E (\psi(r, y) - \psi(r, X_r)) \lambda(X_r, \alpha_r) Q(X_r, \alpha_r, dy) dr \right]. \tag{6.14}
\end{aligned}$$

Proof of Lemma 6.1. The result follows from the Dynkin formula (2.14) and the properties of the test functions ψ in Definition 2.8. In particular, $D\psi(r, X_r) = D\varphi(r, X_r) + h(\|X_r\|) D\delta(r, X_r) + \delta(r, X_r) \frac{h'(\|X_r\|)}{\|X_r\|} X_r$, and $\langle LX_r, \delta(r, X_r) \frac{h'(\|X_r\|)}{\|X_r\|} X_r \rangle \geq 0$, being L monotone. \square

Proof of Theorem 2.11. Viscosity subsolution property. Let $\psi(s, y) = \varphi(s, y) + \delta(s, y) h(\|y\|)$ be a test function of the type introduced in Definition 2.8, such that $V - \psi$ has a global maximum at $(t, x) \in]0, T[\times E$. We also assume that

$$V(t, x) = \psi(t, x), \tag{6.15}$$

and consequently

$$V(s, y) \leq \psi(s, y), \quad \forall (s, y). \tag{6.16}$$

Remember that T_1^t denotes the first jump time of X . We apply the dynamic programming principle (2.18) to $\theta := (t + \eta) \wedge T_1^t$, where $\eta > 0$ is such that $(t + \eta) < T$. By (6.16) we have

$$\psi(t, x) \leq \mathbb{E}_\alpha^{t,x} \left[\psi(\theta, X_\theta) + \int_t^\theta f(X_r, \alpha_r) dr \right], \quad \forall \alpha \in \mathcal{A}_{ad}^t. \tag{6.17}$$

All elements of \mathcal{A}_{ad}^t have the form (2.5). Let us fix $a \in A$, and let us take $\alpha \in \mathcal{A}_{ad}^t$ such that $\alpha_0 \equiv a$. Notice that, $\mathbb{P}_\alpha^{t,x}$ -a.s., $X_r = \phi^a(r - t, x)$ for $r \in [t, \theta]$. In particular, by (2.11),

$$\|X_s\| \leq C(1 + \|x\|) =: R_x.$$

Denoting by τ_R , for any $R > 0$, the exit time of X from $\{y : \|y\| \leq R\}$, it follows that $\theta = (t + h) \wedge T_1^t \wedge \tau_{R_x}$. As a matter of fact, if $(t + h) \leq T_1^t$, then $(t + h) \wedge T_1^t \wedge \tau_{R_x} = t + h$. On the other hand, if $(t + h) > T_1^t$, we have two cases: if $X_{T_1^t} \notin B_{R_x}$, then $(t + h) \wedge T_1^t \wedge \tau_{R_x} = \tau_{R_x} = T_1^t$, if $X_{T_1^t} \in B_{R_x}$, then $(t + h) \wedge T_1^t \wedge \tau_{R_x} = T_1^t$. Then (6.17) for such an α , together with Lemma 6.1, implies that

$$\begin{aligned}
& \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta \left[\frac{\partial \psi}{\partial t}(r, X_r) + \mathcal{L}^a \psi(r, X_r) + f(X_r, a) \right] dr \right] \\
& - \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta \langle X_r, L^* D\psi(r, X_r) + h(\|X_r\|) L^* D\delta(r, X_r) \rangle dr \right] \geq 0, \tag{6.18}
\end{aligned}$$

with $X_r = \phi^a(r - t, x)$. Moreover, by Lemma 2.10, the (deterministic) map

$$\begin{aligned}
r \mapsto & \frac{\partial \psi}{\partial t}(r, \phi^a(r - t, x)) + \mathcal{L}^a \psi(r, \phi^a(r - t, x)) + f(\phi^a(r - t, x), a) \\
& - \langle \phi^a(r - t, x), L^* D\psi(r, \phi^a(r - t, x)) + h(\|\phi^a(r - t, x)\|) L^* D\delta(r, \phi^a(r - t, x)) \rangle
\end{aligned}$$

is continuous at t , uniformly in a . Therefore, for any $\epsilon > 0$ there exists $\eta > 0$ such that (6.18) with θ associated to η becomes

$$\left(\epsilon + \frac{\partial \psi}{\partial t}(t, x) + \mathcal{L}^a \psi(t, x) + f(x, a) - \langle x, L^* D\psi(t, x) + h(\|x\|) L^* D\delta(t, x) \rangle \right) \mathbb{E}_\alpha^{t,x} [\theta - t] \geq 0, \tag{6.19}$$

valid for any $a \in A$. Now we observe that $\mathbb{E}_\alpha^{t,x}[\theta - t] \geq 0$ by definition of θ . Then (6.19) implies

$$\left(\varepsilon + \frac{\partial \psi}{\partial t}(t, x) + \mathcal{L}^a \psi(t, x) + f(x, a) - \langle x, L^* D\psi(t, x) + h(\|x\|) L^* D\delta(t, x) \rangle \right) \geq 0,$$

for any $\varepsilon > 0$ and $a \in A$. The conclusion follows by the arbitrariness of ε and a .

Viscosity supersolution property. Let $\psi(s, y) = \varphi(s, y) + \delta(s, y) h(\|y\|)$ be a test function of the type introduced in Definition 2.8, such that $V + \psi$ has a global minimum at $(t, x) \in]0, T[\times E$. We also assume that

$$V(t, x) + \psi(t, x) = 0, \quad (6.20)$$

so we have

$$V(s, y) \geq -\psi(s, y), \quad \forall (s, y). \quad (6.21)$$

We will show that V is a viscosity supersolution by contradiction. Let us use the notations of Lemma 2.10. Assume that

$$\inf_{a \in A} G_a^\psi(t, x) = \mu > 0. \quad (6.22)$$

By Lemma 2.10, there exists $\eta > 0$, independent from α_0 , such that

$$\mathcal{G}^{\alpha_0}(r) \geq \frac{\mu}{2} > 0, \quad \forall r \in [t, t + \eta), \quad (6.23)$$

Let us now set $\theta := (t + \eta) \wedge T_1^t$ where η satisfies $(t + \eta) < T$. By the dynamic programming principle (2.18), for all $\gamma > 0$ there exists $\alpha \in \mathcal{A}_{ad}^t$ such that

$$V(t, x) + \gamma \geq \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta f(X_r, \alpha_r) dr + V(\theta, X_\theta) \right],$$

and therefore, recalling (6.20) and (6.21),

$$-\psi(t, x) + \gamma \geq \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta f(X_r, \alpha_r) dr - \psi(\theta, X_\theta) \right]. \quad (6.24)$$

As in the proof of the viscosity subsolution property, we set R_x to the the bound in (2.11), and we notice that $\theta = (t + h) \wedge T_1^t \wedge \tau_{R_x}$, where τ_R denotes the exit time of X from $\{y : \|y\| \leq R\}$. Applying Lemma 6.1 to ψ between t and θ , we get

$$\begin{aligned} \gamma &\geq \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta f(X_r, \alpha_r) dr \right] - \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta \left[\frac{\partial \psi}{\partial t}(r, X_r) + \langle b(X_r, \alpha_r), D\psi(X_r, \alpha_r) \rangle \right] dr \right] \\ &\quad - \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta \int_E (\psi(r, y) - \psi(r, X_r)) \lambda(X_r, \alpha_r) Q(X_r, \alpha_r, dy) dr \right] \\ &\quad + \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta \langle X_r, L^* D\varphi(r, X_r) + h(\|X_r\|) L^* D\delta(r, X_r) \rangle dr \right]. \end{aligned}$$

Then

$$\begin{aligned} \gamma &\geq \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta \left(-\frac{\partial \psi}{\partial t}(r, X_r) + \langle X_r, L^* D\varphi(r, X_r) + h(\|X_r\|) L^* D\delta(r, X_r) \rangle \right) dr \right] \\ &\quad + \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta \inf_{a \in A} (-\mathcal{L}^a \psi(r, X_r) + f(X_r, a)) dr \right] \end{aligned}$$

$$= \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta \inf_{a \in A} G_a^\psi(r, X_r) dr \right]. \quad (6.25)$$

By the definition of θ , together with (2.5) and (2.6), for all $r \in [t, \theta)$, $\alpha_r = \alpha_0(r - t, x)$, with α_0 as in (2.5) and $X_r = \phi^{\alpha_0}(r - t, x)$. Thus (6.23) yields

$$\gamma \geq \mathbb{E}_\alpha^{t,x} \left[\int_t^\theta \mathcal{G}^{\alpha_0}(r) dr \right] \geq \frac{\mu}{2} \mathbb{E}_\alpha^{t,x} [(\theta - t)]. \quad (6.26)$$

Now we notice that

$$\begin{aligned} \mathbb{E}_\alpha^{t,x}(\theta - t) &= \eta \mathbb{E}_\alpha^{t,x}(\mathbb{1}_{T_1 > t + \eta}) + \mathbb{E}_\alpha^{t,x}((T_1 - t)\mathbb{1}_{T_1 \leq t + \eta}) \\ &\geq \eta \mathbb{P}_\alpha^{t,x}(\mathbb{1}_{T_1 > t + \eta}) \\ &= \eta e^{-\int_t^{t+\eta} \lambda(\phi^{\alpha_0}(s,x), \alpha_0(s,x)) ds} \\ &\geq \eta e^{-\eta M}, \end{aligned}$$

where in the latter inequality we have used that by assumption λ is bounded by some constant M . By letting γ go to zero we obtain the contradiction.

7 Proofs of the results in Section 4

7.1 Proof of Theorem 4.3

The boundedness of v follows from (4.9), (4.5), together with the definition of V^* in (3.17) and the assumption **(Hfg)**.

Let us now turn to the continuity properties. We argue as in the proof of Proposition 2.5. For any $a \in A$, for any bounded Borel-measurable functions ψ on $[0, T] \times E$, we define the map

$$\mathcal{T}_a \psi(t, x) := \inf_{\nu \in \mathcal{V}} \mathbb{E}_\nu^{t,x,a} \left[\int_t^{T_1^t \wedge T} f(X_s, I_s) ds + g(X_T) \mathbb{1}_{T \leq T_1^t} + \psi(T_1^t, X_{T_1^t}) \mathbb{1}_{T > T_1^t} \right].$$

We have $\mathcal{T}_a \psi(t, x) = \inf_{\nu \in \bar{\mathcal{V}}} \bar{J}(t, x, a, \nu)$, with

$$\bar{J}(t, x, a, \nu) = \int_0^{T-t} \chi^\nu(s, x, a) (f(s, x, a) + L_\psi^t(s, x, a)) ds + \chi^\nu(T - t, x, a) g(\phi(T - t, x, a)),$$

where $\chi^\nu(s, x, a) = e^{-\int_0^s (\lambda(\phi(r,x,a), a) + \int_A \nu_s(b) \lambda_0(db)) dr}$, $f(s, x, a) = f(\phi(s, x, a), a)$, and

$$L_\psi^t(s, x, a) = \int_E \psi(s + t, y) \lambda(\phi(s, x, a), a) Q(\phi(s, x, a), a, dy),$$

and $\bar{\mathcal{V}} = \{\nu : [0, \infty) \times A \rightarrow (0, \infty) \text{ measurable and bounded}\}$. Let us denote by $C_b([0, T] \times E)$ the set of bounded functions, continuous on $[0, T] \times E$ with the $\|\cdot\|_{-1}$ norm.

We aim at proving that, for any $\psi \in C_b([0, T] \times E)$, for any $a \in A$ one has $\mathcal{T}_a \psi \in C_b([0, T] \times E)$. Once this is achieved, proceeding along the same lines of the proof of Proposition 2.5, item (ii), one can get that, for any $a \in A$, \mathcal{T}_a is a contracting map in $C_b([0, T] \times E)$ and v is its unique fixed point. In particular, v satisfies the randomized dynamic programming principle (4.11).

Let thus $\psi \in C_b([0, T] \times E)$. In what follows C will denote a generic constant, that may vary from line to line, and that may depend on T . Let $t, t', s \in [0, T]$, $t' \leq t \leq s$, $x, x' \in E$, $a \in A$, $\nu \in \mathcal{V}$.

Recalling hypotheses **(HbλQ)**-(i), **(Hfg)** and (3.2), we have $|\chi^\nu(s, x, a)| \leq 1$, $|f(s, x, a)| \leq C$, and, for any $s' \leq s$,

$$|\chi^\nu(s', x, a) - \chi^\nu(s, x', a)| \leq (1 - e^{-C\|x-x'\|_{-1}}) + (1 - e^{-C(s-s')}), \quad (7.1)$$

$$|f(s, x, a) - f(s, x', a)| \leq C\|x - x'\|_{-1}, \quad (7.2)$$

$$|g(\phi(T-t, x, a)) - g(\phi(T-t', x', a))| \leq \omega(\|x - x'\|_{-1}). \quad (7.3)$$

On the other hand, by **(HbλQ)**-(i)-(ii), together with the boundedness and continuity of ψ , we have $|L_\psi(s, x, a)| \leq C$ and, for $s < T-t$,

$$\begin{aligned} & |L_\psi^t(s, x, a) - L_\psi^t(s, x', a)| \leq |\lambda(\phi(s, x, a), a) - \lambda(\phi(s, x', a), a)| \|\psi\|_\infty \\ & + \|\lambda\|_\infty \left| \int_E \psi(s+t, y) [Q(\phi(s, x, a), a, dy) - Q(\phi(s, x', a), a, dy)] \right| \\ & \leq C \sigma(\|\phi(s, x, a) - \phi(s, x', a)\|_{-1}) \leq C\omega(\|x - x'\|_{-1}), \end{aligned} \quad (7.4)$$

where the latter inequality follows from (3.2). Then, for any $t, t' \in [0, T]$, $x, x' \in E$, $a \in A$, $\nu \in \mathcal{V}$,

$$\begin{aligned} & |\bar{J}(t, x, a, \nu) - \bar{J}(t', x', a, \nu)| \\ & \leq \left| \int_0^{T-t} \chi^\nu(s, x, a) f(s, x, a) ds - \int_0^{T-t'} \chi^\nu(s, x', a) f(s, x', a) ds \right| \\ & + \left| \int_0^{T-t} \chi^\nu(s, x, a) L_\psi^t(s, x, a) ds - \int_0^{T-t'} \chi^\nu(s, x', a) L_\psi^t(s, x', a) ds \right| \\ & + |\chi^\nu(T-t, x, a)g(\phi(T-t, x, a)) - \chi^\nu(T-t', x', a)g(\phi(T-t', x', a))| \\ & \leq \int_0^{T-t} |\chi^\nu(s, x, a) f(s, x, a) - \chi^\nu(s, x', a) f(s, x', a)| ds \\ & + \int_0^{T-t} |\chi^\nu(s, x, a) L_\psi^t(s, x, a) - \chi^\nu(s, x', a) L_\psi^t(s, x', a)| ds + C|t - t'| \\ & + C|g(\phi(T-t, x, a)) - g(\phi(T-t', x', a))| + C|\chi^\nu(T-t, x, a) - \chi^\nu(T-t', x', a)| \\ & \leq C \left(\int_0^{T-t} |\chi^\nu(s, x, a) - \chi^\nu(s, x', a)| ds + \int_0^{T-t} |f(s, x, a) - f(s, x', a)| ds \right. \\ & + \int_0^{T-t} |L_\psi^t(s, x, a) - L_\psi^t(s, x', a)| ds + |g(\phi(T-t, x, a)) - g(\phi(T-t', x', a))| \\ & \left. + |\chi^\nu(T-t, x, a) - \chi^\nu(T-t', x', a)| + |t - t'| \right) \\ & \leq C(\omega(t-t') + \omega'(\|x - x'\|_{-1})) \end{aligned}$$

for some modulus of continuity ω, ω' , where the latter inequality follows from (7.1), (7.2), (7.3), (7.4). This also allows to conclude that v is uniformly continuous in the $|\cdot| \times \|\cdot\|_{-1}$ norm.

7.2 Proof of Theorem 4.4

We first give the following preliminary result.

Lemma 7.1. *Let $0 < t < \bar{T} < T$, $\hat{\tau}$ be a stopping time such that $\hat{\tau} \in [t, \bar{T}]$, $x \in E$, $a \in A$, $\nu \in \mathcal{V}$, and (X, I) be the PDMP constructed in Section 3.1 under the probability $\mathbb{P}_\nu^{t, x, a}$. For $R > 0$, let τ_R*

be the exit time of X from $\{y : \|y\| \leq R\}$, and set $\tau = \hat{\tau} \wedge \tau_R$. Let $\psi = \varphi + h(\|\cdot\|) \delta$ be a test function. Then,

$$\begin{aligned} \mathbb{E}_\nu^{t,x,a} [\psi(\tau, X_\tau)] &\leq \psi(t, x) + \mathbb{E}_\nu^{t,x,a} \left[\int_t^\tau \left(\frac{\partial \psi}{\partial t}(r, X_r) + \langle b(X_r, I_r), D\psi(r, X_r) \rangle \right) dr \right] \\ &\quad - \mathbb{E}_\nu^{t,x,a} \left[\int_t^\tau \langle X_r, L^* D\varphi(r, X_r) + h(\|X_r\|) L^* D\delta(r, X_r) \rangle dr \right] \\ &\quad + \mathbb{E}_\nu^{t,x,a} \left[\int_t^\tau \int_E (\psi(r, y) - \psi(r, X_r)) \lambda(X_r, I_r) Q(X_r, I_r, dy) dr \right]. \end{aligned} \quad (7.5)$$

Proof of Lemma 7.1. By Proposition 3.1, applying the Dynkin formula to $\psi(s, X_s)$ between t and τ and taking the expectation under $\mathbb{P}_\nu^{t,x,a}$, we get

$$\begin{aligned} \mathbb{E}_\nu^{t,x,a} [\psi(\tau, X_\tau)] &= \psi(t, x) + \mathbb{E}_\nu^{t,x,a} \left[\int_t^\tau \left(\frac{\partial \psi}{\partial t}(r, X_r) + \langle b(X_r, I_r), D\psi(r, X_r) \rangle \right) dr \right] \\ &\quad + \mathbb{E}_\nu^{t,x,a} \left[\int_t^\tau \left(-\langle LX_r, D\psi(r, X_r) \rangle + \int_E (\psi(r, y) - \psi(r, X_r)) \lambda(X_r, I_r) Q(X_r, I_r, dy) \right) dr \right]. \end{aligned}$$

We conclude noticing that $D\psi(r, X_r) = D\varphi(r, X_r) + h(\|X_r\|) D\delta(r, X_r) + \delta(r, X_r) \frac{h'(\|X_r\|)}{\|X_r\|} X_r$, and that $\langle LX_r, \delta(r, X_r) \frac{h'(\|X_r\|)}{\|X_r\|} X_r \rangle \geq 0$, being L is monotone.

Proof of Theorem 4.4. Viscosity subsolution property. Let $\psi(s, y) = \varphi(s, y) + \delta(s, y) h(\|x\|)$ be a test function of the type introduced in Definition 2.8, such that $v - \psi$ has a global maximum at $(t, x) \in [0, T] \times E$. We also assume that

$$v(t, x) = \psi(t, x), \quad (7.6)$$

so we have

$$v(s, y) \leq \psi(s, y), \quad \forall (s, y). \quad (7.7)$$

Fix (t, x, a) and $\nu \in \mathcal{V}$. Let $\eta > 0$ and define $\theta = (t + \eta) \wedge T_1^t$, where T_1^t denotes the first jump time of (X, I) . Using the identification property (4.9), from the randomized dynamic programming principle (4.11), together with (7.7), we get

$$\psi(t, x) \leq \mathbb{E}_\nu^{t,x,a} \left[\psi(\theta, X_\theta) + \int_t^\theta f(X_r, I_r) dr \right].$$

Applying Lemma 7.1, we obtain

$$\begin{aligned} \mathbb{E}_\nu^{t,x,a} \left[\int_t^\theta \left[\frac{\partial \psi}{\partial t}(r, X_r) + \mathcal{L}^{I_r} \psi(r, X_r) + f(X_r, I_r) \right] dr \right] \\ - \mathbb{E}_\nu^{t,x,a} \left[\int_t^\theta \langle X_r, L^* D\psi(r, X_r) + h(\|X_r\|) L^* D\delta(r, X_r) \rangle dr \right] \geq 0, \end{aligned} \quad (7.8)$$

where

$$\mathcal{L}^{I_r} \psi(r, X_r) = \langle b(X_r, I_r), D\psi(r, X_r) \rangle + \int_E (\psi(r, y) - \psi(r, X_r)) \lambda(X_r, I_r) Q(X_r, I_r, dy). \quad (7.9)$$

Now we notice that $\mathbb{P}^{t,x,a}$ -a.s., for all $r \in (t, \theta)$, $(X_r, I_r) = (\phi(r - t, x, a), a)$. Moreover, by Lemma 2.10, the map

$$r \mapsto \frac{\partial \psi}{\partial t}(r, \phi(r - t, x, a)) + \mathcal{L}^a \psi(r, \phi(r - t, x, a)) + f(\phi(r - t, x, a), a)$$

$$- \langle \phi(r-t, x, a) L^* D\psi(r, \phi(r-t, x, a)) + h(\|\phi(r-t, x, a)\|) L^* D\delta(r, \phi(r-t, x, a)) \rangle$$

is continuous, uniformly with respect to $a \in A$. We can proceed as in the proof of Theorem 2.11. By the latter continuity property, for any $\epsilon > 0$, we can find $\eta > 0$ independent of a such that (7.8) holds true for θ corresponding to η . Since $\mathbb{E}_\alpha^{t,x}[\theta - t] \geq 0$ by definition of θ , then identity (7.8) implies

$$\left(\epsilon + \frac{\partial \psi}{\partial t}(t, x) + \mathcal{L}^a \psi(t, x) + f(x, a) - \langle x, L^* D\psi(t, x) + h(\|x\|) L^* D\delta(t, x) \rangle \right) \geq 0,$$

for any $\epsilon > 0$ and $a \in A$. As in the proof of Theorem 2.11, we conclude by the arbitrariness of ϵ and a .

Viscosity supersolution property. Let $\psi(s, y) = \varphi(s, y) + \delta(s, y) h(\|y\|)$ be a test function of the type introduced in Definition 2.8, such that $v + \psi$ has a global minimum at $(t, x) \in [0, T] \times E$. We also assume that

$$v(t, x) + \psi(t, x) = 0, \quad (7.10)$$

so we have

$$v(s, y) \geq -\psi(s, y), \quad \forall (s, y). \quad (7.11)$$

We will show that v is a viscosity supersolution by contradiction. Let us use the notations of Lemma 2.10. Assume that

$$G^\psi(t, x, \psi, D\varphi, D\delta) = \mu > 0. \quad (7.12)$$

By Lemma 2.10 that we apply for $\alpha_0 \equiv a$, $a \in A$ arbitrary, there exists $\eta > 0$, independent from a , such that

$$\mathcal{G}^a(r) \geq \frac{\mu}{2} > 0 \quad \forall r \in [t, t + \eta]. \quad (7.13)$$

Let us set $\theta = (t + \eta) \wedge T_1^t$ and fix $a \in A$. By the dynamic programming principle (4.11) together with the identification property (4.9), we see that, for all $\gamma > 0$, it exists a strictly positive, predictable and bounded function ν such that

$$v(t, x) + \gamma \geq \mathbb{E}_\nu^{t,x,a} \left[\int_t^\theta f(X_r, I_r) dr + v(\theta, X_\theta) \right].$$

Recalling (7.10) and (7.11), we get

$$-\psi(t, x) + \gamma \geq \mathbb{E}_\nu^{t,x,a} \left[\int_t^\theta f(X_r, I_r) dr - \psi(\theta, X_\theta) + \beta(\eta) \mathbb{1}_{\tau \wedge T_1 \leq T} \right]. \quad (7.14)$$

Applying Lemma 7.1, inequality (7.14) yields

$$\begin{aligned} \gamma &\geq \mathbb{E}_\nu^{t,x,a} \left[\int_t^\theta f(X_r, I_r) dr - \int_t^\theta \left(\frac{\partial \psi}{\partial t}(r, X_r) + \langle b(X_r, I_r), D\psi(X_r, I_r) \rangle \right) dr \right] \\ &- \mathbb{E}_\nu^{t,x,a} \left[\int_t^\theta \int_E (\psi(r, y) - \psi(r, X_r)) \lambda(X_r, I_r) Q(X_r, I_r, dy) dr \right] \\ &+ \mathbb{E}_\nu^{t,x,a} \left[\int_t^\theta \langle X_r, L^* D\psi(r, X_r) + h(\|X_r\|) L^* D\delta(r, X_r) \rangle dr \right]. \end{aligned} \quad (7.15)$$

Noticing that

$$-\mathcal{L}^{I_r} \psi(r, X_r) + f(X_r, I_r) = U^\psi(r, X_r, I_r, D\psi) \geq \inf_{a \in A} U^\psi(r, X_r, a, D\psi),$$

with \mathcal{L}^I is the operator in (7.9), previous inequality gives

$$\begin{aligned}
\gamma &\geq \mathbb{E}_\nu^{t,x,a} \left[\int_t^\theta \left(-\frac{\partial\psi}{\partial t}(r, X_r) + \langle X_r, L^* D\varphi(r, X_r) + h(\|X_r\|) L^* D\delta(r, X_r) \rangle \right) dr \right] \\
&+ \mathbb{E}_\nu^{t,x,a} \left[\int_t^\theta \inf_{a \in A} U^\psi(r, X_r, a, D\psi) dr \right] \\
&= \mathbb{E}_\nu^{t,x,a} \left[\int_t^\theta G^\psi(r, X_r, \psi, D\psi, D\varphi, D\delta) dr \right].
\end{aligned} \tag{7.16}$$

By the definition of θ , together with (3.6), for all $r \in [t, \theta]$, $X_r = \phi(r-t, x, a)$. Thus, (7.16) together with (7.13) yields

$$\gamma \geq \mathbb{E}_\nu^{t,x,a} \left[\int_t^\theta G^\psi(r, \phi^{\alpha_0}(r-t, x), \psi, D\psi, D\varphi, D\delta) dr \right] \geq \frac{\mu}{2} \mathbb{E}_\nu^{t,x,a} [(\theta - t)]. \tag{7.17}$$

We conclude as in the proof of Theorem 2.11 using that

$$\begin{aligned}
\mathbb{E}_\alpha^{t,x}(\theta - t) &= \eta \mathbb{E}_\alpha^{t,x}(\mathbb{1}_{T_1 > t + \eta}) + \mathbb{E}_\alpha^{t,x}((T_1 - t)\mathbb{1}_{T_1 \leq t + \eta}) \\
&\geq \eta \mathbb{P}_\alpha^{t,x}(\mathbb{1}_{T_1 > t + \eta}) \\
&= \eta e^{-\int_t^{t+\eta} \lambda(\phi^\alpha(s,x), \alpha_0(s,x)) ds} \\
&\geq \eta e^{-\eta M},
\end{aligned}$$

where M is an upper bound of λ . We obtain the contradiction by letting γ go to zero. \square

7.3 Proof of the comparison Theorem 4.5

We begin recalling the following result concerning an equivalent definition of viscosity super and subsolution to (2.19)-(2.20).

Definition 7.2. *Let assumptions (HL), (Hb λ Q) and (Hfg) be satisfied. We will say that a function ψ is a test function in the sense of Definition 7.2 if $\psi(s, y) = \varphi(s, y) + h(\|y\|)$, where φ, h are as in Definition 2.8 without being bounded, however φ is bounded on every set $(\varepsilon, T - \varepsilon) \times \{x \in E : \|x\| \leq R\}$, $\varepsilon \in (0, T)$, $R > 0$.*

- (i) *A bounded B-upper-semicontinuous function $u : (0, T] \times E \rightarrow \mathbb{R}$ is a viscosity subsolution in the sense of Definition 7.2 of (2.19)-(2.20) if $u(T, x) \leq g(x)$ on E , and, whenever $u - \psi$ has a global maximum at a point (t, x) for a test function $\psi(s, y) = \varphi(s, y) + h(\|y\|)$, then*

$$\begin{aligned}
&\frac{\partial\psi}{\partial t}(t, x) - \langle x, L^* D\varphi(t, x) \rangle \\
&+ \inf_{a \in A} \left\{ \langle b(x, a), D\psi(t, x) \rangle + \int_E (u(t, y) - u(t, x)) \lambda(x, a) Q(x, a, dy) + f(x, a) \right\} \geq 0.
\end{aligned}$$

- (ii) *A bounded B-lower-semicontinuous function $w : (0, T] \times E \rightarrow \mathbb{R}$ is a viscosity supersolution in the sense of Definition 7.2 of (2.19)-(2.20) if $w(T, x) \geq g(x)$ on E , and, whenever $w + \psi$ has a global minimum at a point (t, x) for a test function $\psi(s, y) = \varphi(s, y) + h(\|y\|)$, then*

$$\begin{aligned}
&-\frac{\partial\psi}{\partial t}(t, x) + \langle x, L^* D\varphi(t, x) \rangle \\
&+ \inf_{a \in A} \left\{ \langle b(x, a), -D\psi(t, x) \rangle + \int_E (w(t, y) - w(t, x)) \lambda(x, a) Q(x, a, dy) + f(x, a) \right\} \leq 0.
\end{aligned}$$

(iii) A viscosity solution of (2.19)-(2.20) in the sense of Definition 7.2 is a function which is both a viscosity subsolution and a viscosity supersolution.

Lemma 7.3. *Let assumptions (HL), (HbλQ) and (Hfg) be satisfied. If a function $u : (0, T) \times E \rightarrow \mathbb{R}$ (resp. $w : (0, T) \times E \rightarrow \mathbb{R}$) is bounded and uniformly continuous in the $|\cdot| \times \|\cdot\|_{-1}$ norm, and is a viscosity subsolution (resp. supersolution) of equation (2.19)-(2.20), then it is a viscosity subsolution (resp. supersolution) of equation (2.19)-(2.20) in the sense of Definition 7.2.*

Proof of Lemma 7.3. This lemma extends to the infinite-dimensional framework a well known result in the finite-dimensional case, see e.g. Lemma 2.1 in [26].

We consider the subsolution case, the supersolution case can be proved analogously. Let thus $u : (0, T) \times E \rightarrow \mathbb{R}$ be bounded and uniformly continuous function in the $|\cdot| \times \|\cdot\|_{-1}$ norm, providing a viscosity subsolution to (2.19)-(2.20). Let $u - \psi$ has a global maximum at (t, x) for a test function $\psi(s, y) = \varphi(s, y) + h(\|y\|)$, where without loss of generality we can assume that φ and $h(\|\cdot\|)$ are bounded and that $u(t, x) = \psi(t, x)$. By assumption, it exists a modulus σ_u such that

$$|u(s, y) - u(s, z)| \leq \sigma_u(\|y - z\|_{-1}) \quad s \in (0, T), \quad y, z \in E. \quad (7.18)$$

For any $\varepsilon > 0$, let \bar{u}^ε be the sup-inf convolution of u (see e.g. Definition D.24 in [14]), namely

$$\bar{u}^\varepsilon(s, x) = \inf_{z \in E} \sup_{w \in E} \left(u(w) - \frac{\|z - w\|_{-1}^2}{2\varepsilon} + \frac{\|z - x\|_{-1}^2}{\varepsilon} \right).$$

Then, according to Proposition D.26 in [14], \bar{u}^ε , $\frac{\partial \bar{u}^\varepsilon}{\partial t}$, $D\bar{u}^\varepsilon$ are uniformly continuous in the $|\cdot| \times \|\cdot\|_{-1}$ norm and bounded, and for any $s \in [0, T]$, $y \in E$,

$$u(s, y) \leq \bar{u}^\varepsilon(s, y), \quad (7.19)$$

$$|u(s, y) - \bar{u}^\varepsilon(s, y)| \leq \sigma_u(t_\varepsilon), \quad (7.20)$$

where $\frac{t_\varepsilon}{\sqrt{\varepsilon}} \rightarrow 0$ as $\varepsilon \rightarrow 0$. This implies in particular that \bar{u}^ε , $\frac{\partial \bar{u}^\varepsilon}{\partial t}$, $D\bar{u}^\varepsilon$ and $A^*D\bar{u}^\varepsilon$ are uniformly continuous with respect in the $|\cdot| \times \|\cdot\|$ norm.

Let η be a smooth function, such that $\eta(\tau) = 1$ for $\tau < 1$, $\eta(\tau) = 0$ for $\tau > 2$, and which is strictly decreasing on $[1, 2]$. We define

$$\psi^\varepsilon(s, y) := \psi(s, y) \eta\left(\frac{\|y - x\|_{-1}}{\varepsilon}\right) + \bar{u}^\varepsilon(s, y) \left[1 - \eta\left(\frac{\|y - x\|_{-1}}{\varepsilon}\right)\right].$$

By definition $u(t, x) - \psi^\varepsilon(t, x) = 0$. Moreover

$$\begin{aligned} u(s, y) - \psi^\varepsilon(s, y) &= u(s, y) \eta\left(\frac{\|y - x\|_{-1}}{\varepsilon}\right) + u(s, y) \left[1 - \eta\left(\frac{\|y - x\|_{-1}}{\varepsilon}\right)\right] \\ &\quad - \psi(s, y) \eta\left(\frac{\|y - x\|_{-1}}{\varepsilon}\right) - \bar{u}^\varepsilon(s, y) \left[1 - \eta\left(\frac{\|y - x\|_{-1}}{\varepsilon}\right)\right] \\ &= (u(s, y) - \psi(s, y)) \eta\left(\frac{\|y - x\|_{-1}}{\varepsilon}\right) + (u(s, y) - \bar{u}^\varepsilon(s, y)) \left[1 - \eta\left(\frac{\|y - x\|_{-1}}{\varepsilon}\right)\right]. \end{aligned}$$

For all $(s, y) \in [0, T] \times E$, $u(s, y) \leq \bar{u}^\varepsilon(s, y)$ by (7.19), and $u(s, y) - \psi(s, y) \leq 0$ by assumption.

It follows that $u - \psi^\varepsilon$ has a global maximum at (t, x) . Therefore, we apply Definition 2.9 with $\psi^\varepsilon(s, y) = \varphi^\varepsilon(s, y) + h(\|y\|) \delta^\varepsilon(s, y)$, where

$$\varphi^\varepsilon(s, y) = \varphi(s, y) \eta\left(\frac{\|y - x\|_{-1}}{\varepsilon}\right) + \bar{u}^\varepsilon(s, y) \left[1 - \eta\left(\frac{\|y - x\|_{-1}}{\varepsilon}\right)\right],$$

$$\delta^\varepsilon(s, y) = \eta \left(\frac{\|y - x\|_{-1}}{\varepsilon} \right).$$

Notice that $\psi^\varepsilon(t, x) = \psi(t, x) = u(t, x)$, $\frac{\partial \psi^\varepsilon}{\partial t}(t, x) = \frac{\partial \psi}{\partial t}(t, x)$, $D\psi^\varepsilon(t, x) = D\psi(t, x)$. We get

$$\begin{aligned} 0 &\leq \frac{\partial \psi}{\partial t}(t, x) - \langle x, L^* D\varphi(t, x) \rangle \\ &+ \inf_{a \in A} \left\{ f(x, a) + \langle b(x, a), D\psi(t, x) \rangle + \int_E (\psi^\varepsilon(t, y) - u(t, x)) \lambda(x, a) Q(x, a, dy) \right\} \\ &= \frac{\partial \psi}{\partial t}(t, x) - \langle x, L^* D\varphi(t, x) \rangle \\ &+ \inf_{a \in A} \left\{ f(x, a) + \langle b(x, a), D\psi(t, x) \rangle + \int_E (u(t, y) - u(t, x)) \lambda(x, a) Q(x, a, dy) \right. \\ &\quad \left. + \int_E (\psi^\varepsilon(t, y) - u(t, y)) \lambda(x, a) Q(x, a, dy) \right\}. \end{aligned}$$

At this point we notice that

$$\begin{aligned} |\psi^\varepsilon(t, y) - u(t, y)| &= \left| (\psi(t, y) - \bar{u}^\varepsilon(t, y)) \eta \left(\frac{\|y - x\|_{-1}}{\varepsilon} \right) + \bar{u}^\varepsilon(t, y) - u(t, y) \right| \\ &\leq |\psi(t, y) - \bar{u}^\varepsilon(t, y)| \eta \left(\frac{\|y - x\|_{-1}}{\varepsilon} \right) + |\bar{u}^\varepsilon(t, y) - u(t, y)| \\ &\leq |\psi(t, y) - u(t, y)| \eta \left(\frac{\|y - x\|_{-1}}{\varepsilon} \right) + |\bar{u}^\varepsilon(t, y) - u(t, y)| \left[1 + \eta \left(\frac{\|y - x\|_{-1}}{\varepsilon} \right) \right] \\ &\leq |\psi(t, y) - u(t, y)| \eta \left(\frac{\|y - x\|_{-1}}{\varepsilon} \right) + \sigma_u(t_\varepsilon) \left[1 + \eta \left(\frac{\|y - x\|_{-1}}{\varepsilon} \right) \right], \end{aligned}$$

where in the latter inequality we have used (7.20). The conclusion follows by the Lebesgue dominated convergence theorem.

Proof of Theorem 4.5. We will show the result by contradiction. Assume therefore that $u \not\leq v$.

Step 1. Set $u^\eta(t, x) = u(t, x) - \frac{\eta}{t}$, $v^\eta(s, y) = v(s, y) + \frac{\eta}{s}$, $\eta > 0$, and, for $\varepsilon, \delta, \beta > 0$, define the function

$$\Phi^{\varepsilon, \delta, \beta}(t, s, x, y) := u^\eta(t, x) - v^\eta(s, y) - \frac{\|x - y\|_{-1}^2}{2\varepsilon} - \delta(\|x\|^2 + \|y\|^2) - \frac{(t - s)^2}{2\beta}.$$

By perturbed optimization (see, e.g. Corollary 3.26 in [14]) there exist sequences $a_n, b_n \in \mathbb{R}$, $p_n, q_n \in E$ such that

$$|a_n| + |b_n| + |q_n| + |p_n| \leq \frac{1}{n}, \quad n\delta \rightarrow \infty, \quad (7.21)$$

and

$$\Phi^{\varepsilon, \delta, \beta}(t, s, x, y) + a_n t + b_n s + \langle Bp_n, x \rangle + \langle Bq_n, y \rangle$$

attains a strict maximum at some point $(\bar{t}, \bar{s}, \bar{x}, \bar{y}) \in (0, T] \times (0, T] \times E \times E$. Standard considerations yield (see e.g. [14], page 209)

$$\lim_{\beta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{|\bar{t} - \bar{s}|^2}{2\beta} = 0, \quad \forall \delta, \varepsilon > 0, \quad (7.22)$$

$$\lim_{\delta \rightarrow 0} \limsup_{\beta \rightarrow 0} \limsup_{n \rightarrow \infty} \delta(\|\bar{x}\|^2 + \|\bar{y}\|^2) = 0, \quad \forall \varepsilon > 0, \quad (7.23)$$

$$\lim_{\varepsilon \rightarrow 0} \limsup_{\delta \rightarrow 0} \limsup_{\beta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{2\varepsilon} \|\bar{x} - \bar{y}\|_{-1}^2 = 0. \quad (7.24)$$

Then, recalling that by assumption $u \not\leq v$, it follows from (7.22)-(7.23)-(7.24) and the uniform continuity of u, v , that for sufficiently small $\varepsilon, \eta, \delta, \beta > 0$ and n large enough, $\bar{t}, \bar{s} < T$.

Step 2. From Step 1 we deduce that

$$\begin{aligned} u(t, x) - (\varphi(t, x) + h(\|x\|)) &\text{ has a global maximum over } (0, T) \times E \text{ at } (\bar{t}, \bar{x}), \\ v(s, y) + (\psi(s, y) + h(\|y\|)) &\text{ has a global minimum over } (0, T) \times E \text{ at } (\bar{s}, \bar{y}), \end{aligned}$$

where $h(\|z\|) := \delta\|z\|^2$, and

$$\begin{aligned} \varphi(t, x) &:= \frac{\eta}{t} - a_n t - \langle Bp_n, x \rangle + \frac{\|x - \bar{y}\|_{-1}^2}{2\varepsilon} + \frac{(t - \bar{s})^2}{2\beta}, \\ \psi(s, y) &:= \frac{\eta}{s} - b_n s - \langle Bq_n, y \rangle + \frac{\|\bar{x} - y\|_{-1}^2}{2\varepsilon} + \frac{(\bar{t} - s)^2}{2\beta}. \end{aligned}$$

In particular, $\nabla h(\|z\|) = 2\delta z$, and

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}) &= -\frac{\eta}{\bar{t}^2} - a_n + \frac{\bar{t} - \bar{s}}{\beta}, & \frac{\partial \psi}{\partial t}(\bar{s}, \bar{y}) &= -\frac{\eta}{\bar{s}^2} - b_n - \frac{\bar{t} - \bar{s}}{\beta}, \\ B^{-1}D\varphi(\bar{t}, \bar{x}) &= -p_n + \frac{\bar{x} - \bar{y}}{\varepsilon}, & B^{-1}D\psi(\bar{s}, \bar{y}) &= -q_n - \frac{\bar{x} - \bar{y}}{\varepsilon}. \end{aligned}$$

Step 3. Viscosity inequalities. By Lemma 7.3, u is a viscosity subsolution of equation (2.19)-(2.20) in the sense of Definition 7.2. Therefore, using Step 2, we have

$$\begin{aligned} &\frac{\bar{t} - \bar{s}}{\beta} - \frac{\eta}{T^2} - a_n - \left\langle \bar{x}, L^* \left(\frac{B(\bar{x} - \bar{y})}{\varepsilon} - Bp_n \right) \right\rangle + \inf_{a \in A} \left\{ \left\langle b(\bar{x}, a), \frac{B(\bar{x} - \bar{y})}{\varepsilon} - Bp_n + 2\delta\bar{x} \right\rangle \right. \\ &\left. + \int_E (u(\bar{t}, y) - u(\bar{t}, \bar{x})) \lambda(\bar{x}, a) Q(\bar{x}, a, dy) + f(\bar{x}, a) \right\} \geq 0. \end{aligned} \quad (7.25)$$

Similarly, being v is a viscosity supersolution of equation (2.19)-(2.20) in the sense of Definition 7.2 by Lemma 7.3, proceeding as before one gets

$$\begin{aligned} &\frac{\bar{t} - \bar{s}}{\beta} + \frac{\eta}{T^2} + b_n - \left\langle \bar{y}, L^* \left(\frac{B(\bar{x} - \bar{y})}{\varepsilon} + Bq_n \right) \right\rangle + \inf_{a \in A} \left\{ \left\langle b(\bar{y}, a), \frac{B(\bar{x} - \bar{y})}{\varepsilon} + Bq_n - 2\delta\bar{y} \right\rangle \right. \\ &\left. + \int_E (v(\bar{s}, y) - v(\bar{s}, \bar{y})) \lambda(\bar{y}, a) Q(\bar{y}, a, dy) + f(\bar{y}, a) \right\} \leq 0. \end{aligned} \quad (7.26)$$

Subtracting (7.26) from (7.25) we obtain

$$\begin{aligned} \frac{2\eta}{T^2} &\leq -(a_n + b_n) - \frac{1}{\varepsilon} \langle (\bar{x} - \bar{y}), L^* (B(\bar{x} - \bar{y})) \rangle + \langle \bar{x}, L^* Bp_n \rangle + \langle \bar{y}, L^* Bq_n \rangle \\ &+ \sup_{a \in A} \left\{ \left\langle b(\bar{x}, a), \frac{B(\bar{x} - \bar{y})}{\varepsilon} - Bp_n + 2\delta\bar{x} \right\rangle - \left\langle b(\bar{y}, a), \frac{B(\bar{x} - \bar{y})}{\varepsilon} + Bq_n - 2\delta\bar{y} \right\rangle \right. \\ &\left. + \int_E (u(\bar{t}, y) - u(\bar{t}, \bar{x})) \lambda(\bar{x}, a) Q(\bar{x}, a, dy) \right\} \end{aligned} \quad (7.27)$$

$$- \int_E (v(\bar{s}, y) - v(\bar{s}, \bar{y})) \lambda(\bar{y}, a) Q(\bar{y}, a, dy) + f(\bar{x}, a) - f(\bar{y}, a) \Big\},$$

where we have used that $\inf A_1 - \inf A_2 \leq \sup(A_1 - A_2)$. Using condition (2.1), together with the assumptions on the functions b and f , (7.27) yields

$$\begin{aligned} & \frac{2\eta}{T^2} + a_n + b_n \leq \langle \bar{x}, L^* Bp_n \rangle + \langle \bar{y}, L^* Bq_n \rangle \\ & + \sup_{a \in A} \left\{ \frac{1}{\varepsilon} \langle b(\bar{x}, a) - b(\bar{y}, a), B(\bar{x} - \bar{y}) \rangle - \langle b(\bar{x}, a), Bp_n \rangle + \langle b(\bar{x}, a), 2\delta\bar{x} \rangle \right. \\ & \quad - \langle b(\bar{y}, a), Bq_n \rangle + \langle b(\bar{y}, a), 2\delta\bar{y} \rangle + \int_E (u(\bar{t}, y) - u(\bar{t}, \bar{x})) \lambda(\bar{x}, a) Q(\bar{x}, a, dy) \\ & \quad \left. - \int_E (v(\bar{s}, y) - v(\bar{s}, \bar{y})) \lambda(\bar{y}, a) Q(\bar{y}, a, dy) + f(\bar{x}, a) - f(\bar{y}, a) \right\} \\ & \leq \langle \bar{x}, L^* Bp_n \rangle + \langle \bar{y}, L^* Bq_n \rangle \\ & + C \left(\frac{\|\bar{x} - \bar{y}\|_{-1}^2}{2\varepsilon} + (|Bp_n| + |Bq_n|) + \omega(\|\bar{x} - \bar{y}\|_{-1}) + \delta(1 + \|\bar{x}\|^2 + \|\bar{y}\|^2) \right) \\ & + \sup_{a \in A} \left\{ \int_E u(\bar{t}, y) \lambda(\bar{x}, a) Q(\bar{x}, a, dy) - \int_E u(\bar{t}, y) \lambda(\bar{y}, a) Q(\bar{y}, a, dy) \right\} \\ & + \sup_{a \in A} \left\{ \int_E (u(\bar{t}, y) - u(\bar{t}, \bar{x}) - v(\bar{s}, y) + v(\bar{s}, \bar{y})) \lambda(\bar{y}, a) Q(\bar{y}, a, dy) \right\}. \end{aligned} \quad (7.28)$$

At this point, by Hypothesis **(HbλQ)**-(i)-(ii), we get

$$\sup_{a \in A} \left\{ \int_E u(\bar{t}, y) \lambda(\bar{x}, a) Q(\bar{x}, a, dy) - \int_E u(\bar{t}, y) \lambda(\bar{y}, a) Q(\bar{y}, a, dy) \right\} \leq C\omega(\|\bar{x} - \bar{y}\|_{-1}).$$

Therefore it remains to prove that

$$\sup_{a \in A} \left\{ \int_E (u(\bar{t}, y) - u(\bar{t}, \bar{x}) - v(\bar{s}, y) + v(\bar{s}, \bar{y})) \lambda(\bar{y}, a) Q(\bar{y}, a, dy) \right\} \quad (7.29)$$

converges to 0 when the parameters go to their respective limits.

Step 4. Proof of the convergence of (7.29) to 0. Set $m := 2(\|u\|_\infty \vee \|v\|_\infty)$ and

$$M := \Phi^{\varepsilon, \delta, \beta}(\bar{t}, \bar{s}, \bar{x}, \bar{y}) + a_n \bar{t} + b_n \bar{s} + \langle Bp_n, \bar{x} \rangle + \langle Bq_n, \bar{y} \rangle.$$

By Step 1, we know that M is a strict maximum on $(0, T] \times (0, T] \times E \times E$ of the function

$$\Phi^{\varepsilon, \delta, \beta}(t, s, x, y) + a_n t + b_n s + \langle Bp_n, x \rangle + \langle Bq_n, y \rangle.$$

The definition of $\Phi^{\varepsilon, \delta, \beta}$ implies that

$$\begin{aligned} M &= u(\bar{t}, \bar{x}) - \frac{\eta}{\bar{t}} - v(\bar{s}, \bar{y}) - \frac{\eta}{\bar{s}} - \frac{\|\bar{x} - \bar{y}\|_{-1}^2}{2\varepsilon} - \delta(\|\bar{x}\|^2 + \|\bar{y}\|^2) - \frac{(\bar{t} - \bar{s})^2}{2\beta} + a_n \bar{t} + b_n \bar{s} + \langle Bp_n, \bar{x} \rangle + \langle Bq_n, \bar{y} \rangle \\ &= u(\bar{t}, \bar{x}) - \frac{\eta}{\bar{t}} - v(\bar{s}, \bar{y}) - \frac{\eta}{\bar{s}} - \frac{(\bar{t} - \bar{s})^2}{2\beta} - \frac{\|\bar{x} - \bar{y}\|_{-1}^2}{2\varepsilon} - \delta\|\bar{x} - \frac{Bp_n}{2\delta}\|^2 - \delta\|\bar{y} - \frac{Bq_n}{2\delta}\|^2 \end{aligned}$$

$$+ \frac{\|Bp_n\|^2}{4\delta} + \frac{\|Bq_n\|^2}{4\delta} + a_n\bar{t} + b_n\bar{s}, \quad (7.30)$$

which in turn implies that

$$\begin{aligned} & \frac{\eta}{\bar{t}} + \frac{\eta}{\bar{s}} + \frac{(\bar{t} - \bar{s})^2}{2\beta} + \frac{\|\bar{x} - \bar{y}\|_{-1}^2}{2\varepsilon} + \delta\|\bar{x} - \frac{Bp_n}{2\delta}\|^2 + \delta\|\bar{y} - \frac{Bq_n}{2\delta}\|^2 \\ & = u(\bar{t}, \bar{x}) - v(\bar{s}, \bar{y}) - M + \frac{\|Bp_n\|^2}{4\delta} + \frac{\|Bq_n\|^2}{4\delta} + a_n\bar{t} + b_n\bar{s}. \end{aligned}$$

Moreover $u(\bar{t}, \bar{x}) - v(\bar{s}, \bar{y}) \leq m$ and $a_n\bar{t} + b_n\bar{s} \leq T$ for all $n \geq 2$, since $|a_n| + |b_n| \leq \frac{1}{2}$ for $n \geq 2$. Therefore

$$\delta\|\bar{x} - \frac{Bp_n}{2\delta}\|^2 + \delta\|\bar{y} - \frac{Bq_n}{2\delta}\|^2 \leq m - M + \frac{\|Bp_n\|^2}{4\delta} + \frac{\|Bq_n\|^2}{4\delta} + T, \quad (7.31)$$

$$M \leq m + \frac{\|Bp_n\|^2}{4\delta} + \frac{\|Bq_n\|^2}{4\delta} + T. \quad (7.32)$$

Let us take $K \in \mathbb{N}$ satisfying

$$2KM > m + T + \frac{\|Bp_n\|^2}{4\delta} + \frac{\|Bq_n\|^2}{4\delta} - M, \quad (7.33)$$

and define the set

$$\Gamma_{1,d} := \left\{ (x, y) \in E \times E; \|x - \frac{Bp_n}{2\delta}\|^2 + \|y - \frac{Bq_n}{2\delta}\|^2 \leq \frac{2KM}{\delta} \right\}. \quad (7.34)$$

Notice that from (7.31) we have $(\bar{x}, \bar{y}) \in \Gamma_{1,d}$. Let also $\alpha > 0$ be such that

$$m + T + \frac{\|Bp_n\|^2}{4\delta} + \frac{\|Bq_n\|^2}{4\delta} - 2KM + \alpha < M \quad (7.35)$$

and D be a smooth function on $E \times E$ satisfying

$$\begin{aligned} D(x, y) &= -\delta \left(\|x - \frac{Bp_n}{2\delta}\|^2 + \|y - \frac{Bq_n}{2\delta}\|^2 \right), \quad \forall (x, y) \in \Gamma_{1,d}, \\ -2KM &\leq D(x, y) \leq -2KM + \alpha, \quad \forall (x, y) \in \Gamma_{1,d}^c. \end{aligned} \quad (7.36)$$

Then the function

$$u^\eta(t, x) - v^\eta(s, y) - \frac{\|x - y\|_{-1}^2}{2\varepsilon} - \frac{(t - s)^2}{2\beta} + a_nt + b_ns + D(x, y) + \frac{\|Bp_n\|^2}{4\delta} + \frac{\|Bq_n\|^2}{4\delta} \quad (7.37)$$

admits a strict maximum at $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$. Indeed, if $(x, y) \in \Gamma_{1,d}$ the expression (7.37) coincides with $\Phi^{\varepsilon, \delta, \beta}(t, s, x, y) + a_nt + b_ns + \langle Bp_n, x \rangle + \langle Bq_n, y \rangle$, and if $(x, y) \notin \Gamma_{1,d}$, by the definition of $D(x, y)$ the expression (7.37) is smaller or equal to

$$\begin{aligned} & u^\eta(t, x) - v^\eta(s, y) - \frac{\|x - y\|_{-1}^2}{2\varepsilon} - \frac{(t - s)^2}{2\beta} + a_nt + b_ns - 2KM + \alpha + \frac{\|Bp_n\|^2}{4\delta} + \frac{\|Bq_n\|^2}{4\delta} \\ & \leq u^\eta(t, x) - v^\eta(s, y) + a_nt + b_ns - 2KM + \alpha + \frac{\|Bp_n\|^2}{4\delta} + \frac{\|Bq_n\|^2}{4\delta} \\ & \leq m + T - 2KM + \alpha + \frac{\|Bp_n\|^2}{4\delta} + \frac{\|Bq_n\|^2}{4\delta}, \end{aligned}$$

the latter being strictly smaller than M by the choice of α (cf. (7.35)). Using Step 1 with $x = y$ we obtain that, for all $y \in E$,

$$\Phi^{\varepsilon, \delta, \beta}(\bar{t}, \bar{s}, y, y) + a_n \bar{t} + b_n \bar{s} + \langle Bp_n, y \rangle + \langle Bq_n, y \rangle \leq \Phi^{\varepsilon, \delta, \beta}(\bar{t}, \bar{s}, \bar{x}, \bar{y}) + a_n \bar{t} + b_n \bar{s} + \langle Bp_n, \bar{x} \rangle + \langle Bq_n, \bar{y} \rangle,$$

which implies

$$\begin{aligned} & u(\bar{t}, y) - u(\bar{t}, \bar{x}) - v(\bar{s}, y) + v(\bar{s}, \bar{y}) \\ & \leq -\frac{\|\bar{x} - \bar{y}\|_{-1}^2}{2\varepsilon} + \delta \|y - \frac{Bp_n}{2\delta}\|^2 + \delta \|y - \frac{Bq_n}{2\delta}\|^2 - \delta \|\bar{x} - \frac{Bp_n}{2\delta}\|^2 - \delta \|\bar{y} - \frac{Bq_n}{2\delta}\|^2 \\ & \leq \delta \left(\|y - \frac{Bp_n}{2\delta}\|^2 + \|y - \frac{Bq_n}{2\delta}\|^2 \right). \end{aligned} \quad (7.38)$$

Let us set

$$\Sigma_1 := \left\{ y \in E : \|y - \frac{Bp_n}{2\delta}\|^2 + \|y - \frac{Bq_n}{2\delta}\|^2 \leq \frac{2KM}{\sqrt{\delta}} \right\}.$$

For any $y \in \Sigma_1$ we obtain by (7.38) that

$$u(\bar{t}, y) - u(\bar{t}, \bar{x}) - v(\bar{s}, y) + v(\bar{s}, \bar{y}) \leq 2KM \sqrt{\delta}. \quad (7.39)$$

Let us now set (since we are interested in $\delta \in (0, 1)$ we have $\frac{2KM}{\sqrt{\delta}} < \frac{2KM}{\delta}$)

$$\Sigma_2 := \left\{ y \in E : \frac{2KM}{\sqrt{\delta}} < \|y - \frac{Bp_n}{2\delta}\|^2 + \|y - \frac{Bq_n}{2\delta}\|^2 \leq \frac{2KM}{\delta} \right\}.$$

Inequality (7.38) yields

$$u(\bar{t}, y) - u(\bar{t}, \bar{x}) - v(\bar{s}, y) + v(\bar{s}, \bar{y}) \leq 2KM, \quad \forall y \in \Sigma_2. \quad (7.40)$$

Finally set

$$\Sigma_3 := \left\{ y \in E : \|y - \frac{Bp_n}{2\delta}\|^2 + \|y - \frac{Bq_n}{2\delta}\|^2 > \frac{2KM}{\delta} \right\}.$$

Let us now take $y \in \Sigma_3$. Then (y, y) belongs to $\Gamma_{1,d}^c$. From the previous arguments

$$u(\bar{t}, y) - v(\bar{s}, y) - \frac{\eta}{\bar{t}} - \frac{\eta}{\bar{s}} - \frac{(\bar{t} - \bar{s})^2}{2\beta} + a_n \bar{t} + b_n \bar{s} + D(y, y) + \frac{\|Bp_n\|^2}{4\delta} + \frac{\|Bq_n\|^2}{4\delta} \leq M,$$

which together with (7.30) implies

$$u(\bar{t}, y) - u(\bar{t}, \bar{x}) - v(\bar{s}, y) + v(\bar{s}, \bar{y}) \leq -\frac{\|\bar{x} - \bar{y}\|_{-1}^2}{2\varepsilon} - \delta \|\bar{x} - \frac{Bp_n}{2\delta}\|^2 - \delta \|\bar{y} - \frac{Bq_n}{2\delta}\|^2 - D(y, y).$$

Thus we obtain

$$u(\bar{t}, y) - u(\bar{t}, \bar{x}) - v(\bar{s}, y) + v(\bar{s}, \bar{y}) \leq 2KM, \quad \forall y \in \Sigma_3. \quad (7.41)$$

At this point, let us go back to (7.29). Using the partitioning $E = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, in (7.29) we split the integral on E in the integrals over the sets Σ_i . From (7.39) together with $(\mathbf{Hb}\lambda\mathbf{Q})$, we get

$$\sup_{a \in A} \int_{\Sigma_1} (u(\bar{t}, y) - u(\bar{t}, \bar{x}) - v(\bar{s}, y) + v(\bar{s}, \bar{y})) \lambda(\bar{y}, a) Q(\bar{y}, a, dy) \leq \|\lambda\|_{\infty} 2KM \sqrt{\delta},$$

which obviously converges to zero. On the other hand, by (7.40) and (7.41), we obtain

$$\begin{aligned} & \sup_{a \in A} \left\{ \int_{\Sigma_2} (u(\bar{t}, y) - u(\bar{t}, \bar{x}) - v(\bar{s}, y) + v(\bar{s}, \bar{y})) \lambda(\bar{y}, a) Q(\bar{y}, a, dy) \right. \\ & \quad \left. + \int_{\Sigma_3} (u(\bar{t}, y) - u(\bar{t}, \bar{x}) - v(\bar{s}, y) + v(\bar{s}, \bar{y})) \lambda(\bar{y}, a) Q(\bar{y}, a, dy) \right\} \\ & \leq \|\lambda\|_\infty 2KM \sup_{a \in A} (Q(\bar{x}, a, \Sigma_2) + Q(\bar{x}, a, \Sigma_3)). \end{aligned}$$

Since we have chosen the parameters according to (7.21), in particular $\frac{\|B_{p_n}\|}{\delta} \leq \frac{1}{n\delta}$ converges to 0. This completes the proof recalling assumption **(HQ')** (see Section 4) and the respective definitions of Σ_2, Σ_3 . \square

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