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The identification problem for BSDEs driven by possibly non quasi-left-continuous random measures

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Abstract

In this paper we focus on the so called *identification problem* for a backward SDE driven by a continuous local martingale and a possibly non quasi-left-continuous random measure. Supposing that a solution (Y, Z, U) of a backward SDE is such that $Y_t = v(t, X_t)$, where X is an underlying process and v is a deterministic function, solving the identification problem consists in determining Z and U in term of v. We study the over-mentioned identification problem under various sets of assumptions and we provide a family of examples including the case when X is a non-semimartingale jump process solution of an SDE with singular coefficients.

Key words: Backward SDEs; identification problem; non quasi-left-continuous random measure; weak Dirichlet processes; piecewise deterministic Markov processes; martingale problem with jumps and distributional drift.

MSC 2010: 60J75; 60G57; 60H30

1 Introduction

This paper considers a BSDE driven by a compensated random measure $\mu - \nu$, of the form

$$
\begin{aligned}
Y_t = \xi + \int_{]t,\,T]\times\mathbb{R}} \tilde{f}(s,\, e,\, Y_{s-},\, Z_s,\, U_s(e))\, d\zeta_s \\
- \int_{]t,\,T]} Z_s\, dM_s - \int_{]t,\,T]\times\mathbb{R}} U_s(e)\,(\mu - \nu)(ds\, de),
\end{aligned}
\tag{1.1}
$$

whose solution is a triplet of processes (Y, Z, U), with Y a càdlàg adapted process, Z a predictable process and $U(\cdot)$ a predictable random field. Besides μ and ν appear two driving random elements, namely a continuous martingale M and a non-decreasing adapted càdlàg process ζ, while ξ is a square integrable random variable, and $\tilde{f}$ is a random function. Often Y turns out to be of the type $v(t, X_t)$ where v is a deterministic function, and X is a càdlàg adapted process. The *identification problem* consists in determining Z and U in terms of v.

BSDEs have been deeply studied since the seminal paper [28], where the Brownian context appears as a particular case of (1.1), setting $\mu = 0$, $\zeta_s \equiv s$. There, M is a standard Brownian motion and ξ is measurable with respect to the Brownian σ-field at terminal time. In that case the unknown can be reduced to (Y, Z), since U can be arbitrarily chosen. BSDEs with a discontinuous

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driving term of the form (1.1) have been studied as well; in almost all cases, the random measure μ is quasi-left-continuous, i.e. $\mu(\{S\} \times \mathbb{R}) = 0$ on $\{S < \infty\}$ for every predictable time S, see, e.g., [34], [12], [33], [5], [4]. Existence and uniqueness for BSDEs driven by a random measure which is not necessarily quasi-left-continuous are very recent, and were discussed in [1] in the purely discontinuous case, and in [27] in the jump-diffusion case.

When the random dependence of $\tilde{f}$ is provided by a Markov solution X of a forward SDE, and ξ is a real function of X at the terminal time T, then the BSDE (1.1) is called forward BSDE. In the Brownian context, when X is the solution of a classical SDE with diffusion coefficient σ, forward BSDEs generally constitute stochastic representations of a partial differential equation. If $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a classical (smooth) solution of the mentioned PDE, then $Y_s = v(s, X_s)$, $Z_s = \sigma(s, X_s)\, \partial_x v(s, X_s)$, generate a solution to the forward BSDE, see e.g. [30], [29], [31], which provide the solution to the identification problem in that particular case. Conversely, solutions of forward BSDEs generate solutions of PDEs in the viscosity sense, or in other generalized sense, see e.g. [11]. More precisely, for each given couple $(t, x) \in [0, T] \times \mathbb{R}$, consider an underlying process X given by the solution $X^{t,x}$ of an SDE starting at x at time t; if $(Y^{t,x}, Z^{t,x})$ is a family of solutions of the forward BSDE, under reasonable general assumptions, the function $v(t, x) := Y_t^{t,x}$ is a viscosity solution of the related PDE. In the Brownian context, the identification problem of Z has been faced even if $v \in C^{0,1}$, including the infinite dimensional case, see for instance [20] under slightly more general conditions.

In the general case when the forward BSDEs are also driven by random measures, these equations generally constitute stochastic representations of a partial integro-differential equation (PIDE). When v is a classical solution of the PIDE and X is a solution to a Markov jump-diffusion equation, the identification problem was solved in [9]. Analogous results can be obtained when X is a purely discontinuous Markov process, see [14]. In both cases, the BSDE is driven by a compensated random measure $\mu - \nu$ with μ quasi-left-continuous. In the context of alternative BSDEs with jumps, i.e. the one of martingale driven forward BSDEs of [13], the identification problem was discussed for instance in [26] and [10].

In [8] we extended the above-mentioned identification results in two directions. Firstly, we generalized [14] to the case of BSDEs driven by non-quasi-left-continuous random measures, related to a special class of piecewise deterministic Markov process (PDMPs). Secondly, in the non purely discontinuous case, we extended the study to the case when $Y_t = v(t, X_t)$, with X a special weak Dirichlet process of finite quadratic variation and v of class $C^{0,1}$. A similar technique was used earlier in the different context of verification theorems for control problems, see [21].

Besides the survey aspects, the present paper extends the results of [8] along three lines.

1. One investigates the identification problems, going beyond the forward BSDEs formalism, even though following the same lines of [8].

2. We generalize the results in [8] by alleviating some important hypotheses. As a matter of fact, Proposition 3.7 improves the achievements of Proposition 2.17 in [8], since the condition

$$\nu(\{S\}, de) = \mu(\{S\}, de) \text{ a.s. for every predictable time } S \text{ such that } [[S]] \subset K \tag{1.2}$$

   is no longer needed here. This allows to formulate Theorems 4.2 and 4.5 under more general assumptions, and extends the applicability of our results. Among others, we are able to solve the identification result for more general jump-diffusion processes and piecewise deterministic Markov processes, see respectively Corollaries 5.5 and 5.9.

3. We apply our results to the completely new case when X is the solution to a martingale problem with general jumps and distributional drift, related thus to an operator of the form

$\beta'(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}$, with β only continuous, and to some predictable random measure ν possibly discontinuous. Martingale problems of this type have been studied in the companion paper [6]. We solve the identification problem in this context, see Corollary 5.15.

2 Notation and preliminaries

We fix a positive horizon T. Given a topological space E, in the sequel $\mathcal{B}(E)$ will denote the Borel σ-field associated with E. We will indicate by $C^{0,1}$ the space of all functions $u : [0, T] \times \mathbb{R} \to \mathbb{R}$, $(t, x) \mapsto u(t, x)$, that are continuous together their derivative $\partial_x u$.

For a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, we will always suppose that $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions, with $\mathcal{F} = \mathcal{F}_T$. Related to it, $\mathcal{P}$ (resp. $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R})$) will denote the predictable σ-field on $\Omega \times [0, T]$ (resp. on $\tilde{\Omega} = \Omega \times [0, T] \times \mathbb{R}$). Analogously, we set $\mathcal{O}$ (resp. $\tilde{\mathcal{O}} = \mathcal{O} \otimes \mathcal{B}(\mathbb{R})$) as the optional σ-field on $\Omega \times [0, T]$ (resp. on $\tilde{\Omega}$). Moreover, $\tilde{\mathcal{F}}$ will be σ-field $\mathcal{F} \otimes \mathcal{B}([0, T] \times \mathbb{R})$, and we will indicate by $\mathcal{F}^{\mathbb{P}}$ the completion of $\mathcal{F}$ with the $\mathbb{P}$-null sets. We set $\tilde{\mathcal{F}}^{\mathbb{P}} = \mathcal{F}^{\mathbb{P}} \otimes \mathcal{B}([0, T] \times \mathbb{R})$. By default, all the stochastic processes will be considered with parameter $t \in [0, T]$. By convention, any càdlàg process defined on $[0, T]$ is extended to $\mathbb{R}_+$ by continuity. A random set $A \subset \tilde{\Omega}$ is called evanescent if the set $\{\omega : \exists\, t \in \mathbb{R}_+ \text{ with } (\omega, t) \in A\}$ is $\mathbb{P}$-null. Generically, all the equalities of random sets will be intended up to an evanescent set.

For a measurable process X we denote by ${}^p(X)$ its predictable projection, see e.g. Theorem 5.2 in [23]. A bounded variation process X on $[0, T]$ will be said to be with integrable variation if the expectation of its total variation is finite. $\mathcal{A}$ (resp. $\mathcal{A}_{\mathrm{loc}}$) will denote the collection of all adapted processes with integrable variation (resp. with locally integrable variation), and $\mathcal{A}^+$ (resp $\mathcal{A}^+_{\mathrm{loc}}$) the collection of all adapted integrable increasing (resp. adapted locally integrable) processes. In general, these notions refer to the underlying probability $\mathbb{P}$; when this is not the case, we will mention the specific probability. The significance of locally is the usual one which refers to localization by stopping times, see e.g. (0.39) of [24].

We also recall that a random kernel $\phi(a, db)$ of a measurable space $(A, \mathcal{A})$ into another measurable space $(B, \mathcal{B})$ is a family $\{\phi(a, \cdot),\ a \in A\}$ of positive measures on $(B, \mathcal{B})$, such that $\phi(\cdot, C)$ is $\mathcal{A}$-measurable for any $C \in \mathcal{B}$. Finally, the concept of random measure is extensively used throughout the paper: for a detailed discussion on this topic and the unexplained notations see Chapter I and Chapter II, Section 1, in [25], Chapter III in [24], and Chapter XI, Section 1, in [23]. In particular, if μ is a random measure on $[0, T] \times \mathbb{R}$, for any measurable real function H defined on $\tilde{\Omega}$, one denotes $H \star \mu_t := \int_{]0,\,t]\times\mathbb{R}} H(\cdot, s, e)\, \mu(\cdot, ds\, de)$.

2.1 Stochastic integration with respect to integer-valued random measures

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. In the sequel of Section 2, μ will be an integer-valued random measure on $[0, T] \times \mathbb{R}$, and ν will be its compensator, for which we will choose the "good" version as constructed in point (c) of Proposition 1.17, Chapter II, in [25]. Set

$$D = \{(\omega, t) : \mu(\omega, \{t\} \times \mathbb{R}) > 0\}, \tag{2.1}$$

$$J = \{(\omega, t) : \nu(\omega, \{t\} \times \mathbb{R}) > 0\}, \tag{2.2}$$

$$K = \{(\omega, t) : \nu(\omega, \{t\} \times \mathbb{R}) = 1\}. \tag{2.3}$$

We define $\nu^d := \nu\, \mathbb{1}_J$ and $\nu^c := \nu\, \mathbb{1}_{J^c}$. Similar conventions will be used for the other integer-valued random measures appearing in the paper. We will sometimes use the form of μ given in Proposition 1.14, Chapter II, in [25], i.e.

$$\mu(dt\, de) = \sum_{s\geq 0} \mathbb{1}_D(s, \omega)\, \delta_{(s, \beta_s(\omega))}(dt\, de), \tag{2.4}$$

where β is a real-valued optional process. In what follows $[[\tau, \tau']]$ will denote the stochastic interval $\{(\omega, t) : t \in \mathbb{R}_+,\, \tau(\omega) \leq t \leq \tau'(\omega)\}$ associated to two stopping times τ, τ'.

Remark 2.1. (i) D is a thin set, namely $D = \cup_n [[T_n]]$ with $(T_n)_n$ random times, see Theorem 11.13 in [23].

(ii) J is the predictable support of D, namely $J = \{^p(\mathbb{1}_D) > 0\}$, see Theorem 5.39 in [23]. This is equivalent to $\mathbb{1}_J = {}^p(\mathbb{1}_D)$.

(iii) There exists a sequence of predictable times $(R_n)_n$ with disjoint graphs, such that $J = \cup_n [[R_n]]$, see Proposition 2.23, Chapter I, in [24].

(iv) K is the largest predictable subset of D, see Theorems 11.14 in [23]. Since K is predictable, we have ${}^p(\mathbb{1}_K) = \mathbb{1}_K$.

(v) A progressive set B contained in a thin set is also thin set, see Theorem 3.19 in [23]. In particular, K is a thin set.

Remark 2.2. ν admits a disintegration of the type

$$\nu(\omega, ds\, de) = dA_s(\omega)\, \phi(\omega, s, de), \tag{2.5}$$

where ϕ is a random kernel from $(\Omega \times [0, T], \mathcal{P})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and A is a right-continuous nondecreasing predictable process, such that $A_0 = 0$, see for instance Remark 4.4 in [1].

We recall an important notion of measure associated with μ, given in formula (3.10) in [24].

Definition 2.3. *Let $(\tilde{\Omega}_n)$ be a partition of $\tilde{\Omega}$ constituted by elements of $\tilde{\mathcal{O}}$, such that $\mathbb{1}_{\tilde{\Omega}_n} \star \mu \in \mathcal{A}$. $M^{\mathbb{P}}_\mu$ denotes the σ-finite measure on $(\tilde{\Omega}, \tilde{\mathcal{F}}^{\mathbb{P}})$, such that for every $W : \tilde{\Omega} \to \mathbb{R}$ positive, bounded, $\tilde{\mathcal{F}}^{\mathbb{P}}$-measurable function,*

$$M^{\mathbb{P}}_\mu(W\, \mathbb{1}_{\tilde{\Omega}_n}) = \mathbb{E}\big[W\, \mathbb{1}_{\tilde{\Omega}_n} \star \mu_T\big]. \tag{2.6}$$

Let us set $\hat{\nu}_t(de) := \nu(\{t\}, de)$ for all $t \in [0, T]$. For any $W \in \tilde{\mathcal{O}}$, we define

$$\hat{W}_t = \int_{\mathbb{R}} W_t(e)\, \hat{\nu}_t(de), \quad \tilde{W}_t = \int_{\mathbb{R}} W_t(e)\, \mu(\{t\}, de) - \hat{W}_t, \quad t \geq 0,$$

with the convention that $\tilde{W}_t = +\infty$ if $\hat{W}_t$ is not defined. For every $q \in [1, \infty[$, we introduce the linear spaces

$$\mathcal{G}^q(\mu) = \Big\{W \in \tilde{\mathcal{P}} :\ \Big[\sum_{s \leq \cdot} |\tilde{W}_s|^2\Big]^{q/2} \in \mathcal{A}^+\Big\}, \quad \mathcal{G}^q_{\mathrm{loc}}(\mu) = \Big\{W \in \tilde{\mathcal{P}} :\ \Big[\sum_{s \leq \cdot} |\tilde{W}_s|^2\Big]^{q/2} \in \mathcal{A}^+_{\mathrm{loc}}\Big\}.$$

Given $W \in \tilde{\mathcal{P}}$, we define the increasing (possibly infinite) predictable process

$$C(W) := |W - \hat{W}\, \mathbb{1}_J|^2 \star \nu + \sum_{s \leq \cdot} (1 - \hat{\nu}_s(\mathbb{R}))\, |\hat{W}_s|^2\, \mathbb{1}_{J \setminus K}(s), \tag{2.7}$$

provided the right-hand side is well-defined. By Theorem 11.21, point 3) in [23], if $W \in \mathcal{G}^2(\mu)$, then $\langle W \star (\mu - \nu) \rangle$ is well defined and

$$C(W) = \langle W \star (\mu - \nu) \rangle. \tag{2.8}$$

We set

$$||W||^2_{\mathcal{G}^2(\mu)} := \mathbb{E}\left[C(W)_T\right]. \tag{2.9}$$

We also introduce the space

$$\mathcal{L}^2(\mu) := \left\{ W \in \tilde{\mathcal{P}} : \; ||W||_{\mathcal{L}^2(\mu)} := \mathbb{E}\Big[\int_{]0,T]\times\mathbb{R}} |W_s(e)|^2 \, \nu(ds\, de) \Big] < \infty \right\}.$$

Remark 2.4 (Lemma 2.4 in [8]). If $W \in \mathcal{L}^2(\mu)$ then $W \in \mathcal{G}^2(\mu)$ and $||W||^2_{\mathcal{G}^2(\mu)} \leq ||W||^2_{\mathcal{L}^2(\mu)}$. Moreover $\mathcal{L}^2_{\text{loc}}(\mu) \subset \mathcal{G}^2_{\text{loc}}(\mu)$.

The following result is the object of Proposition 2.8 in [8].

Proposition 2.5. *If $C(W)_T = 0$ a.s., then $||W - \hat{W}\, \mathbb{1}_K||_{\mathcal{L}^2(\mu)} = 0$, or, equivalently, there exists a predictable process (l_s) such that*

$$W_s(e) = l_s \, \mathbb{1}_K(s), \quad d\mathbb{P}\, \nu(ds\, de)\text{-a.e.} \tag{2.10}$$

In particular, $W_s(e) = 0$, $d\mathbb{P}\, \nu^c(ds\, de)$-a.e., and there is a predictable process (l_s) such that $W_s(e) = l_s \, \mathbb{1}_K(s)$, $d\mathbb{P}\, \nu^d(ds\, de)$-a.e.

We recall the following definition, fundamental for the sequel of the paper.

Definition 2.6. *Given a càdlàg process X, we introduce the associated jump measure, namely the integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}$ defined as*

$$\mu^X(\omega; dt\, dx) := \sum_{s \in]0,\, T]} \mathbb{1}_{\{\Delta X_s(\omega) \neq 0\}} \, \delta_{(s,\, \Delta X_s(\omega))}(dt\, dx). \tag{2.11}$$

The compensator of $\mu^X(ds\, dx)$ will be denoted by $\nu^X(ds\, dx)$.

We will consider the following condition for a couple $(\chi, \mathbb{Q})$ where χ is a random measure and $\mathbb{Q}$ is a given probability (of simply for χ when $\mathbb{Q}$ is the self-explanatory probability $\mathbb{P}$):

$$\int_{]0,\cdot]\times\mathbb{R}} (|x| \wedge |x|^2) \, \chi(ds\, dx) \in \mathcal{A}^+_{\text{loc}}. \tag{2.12}$$

A more restrictive condition will be also considered, namely

$$\int_{]0,\cdot]\times\mathbb{R}} (|x| \wedge |x|^{1+\alpha}) \, \chi(ds\, dx) \in \mathcal{A}^+_{\text{loc}} \quad \text{for some } \alpha \in [0,\, 1]. \tag{2.13}$$

By abuse of notations, when $\chi = \mu^X$ for a given càdlàg process X, we will say that X verifies condition (2.12) or condition (2.13) (under $\mathbb{Q}$).

Remark 2.7. Condition (2.13) implies in particular condition (2.12).

We will be also interested in functions $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ of class $C^{0,1}$ fulfilling the integrability property (with respect to $(\chi, \mathbb{Q})$ or simply for χ when $\mathbb{P}$ is self-explanatory)

$$\int_{]0,\cdot]\times\mathbb{R}} |v(s, X_{s-} + x) - v(s, X_{s-}) - x\, \partial_x v(s, X_{s-})| \, \mathbb{1}_{\{|x|>1\}} \, \chi(ds\, dx) \in \mathcal{A}^+_{\text{loc}}. \tag{2.14}$$

Also in this case, by abuse of notations, when $\chi = \mu^X$ for a given càdlàg process X, we will say that v and X verify condition (2.14) (under $\mathbb{Q}$).

2.2 Chain rules for special weak Dirichlet processes

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Special weak Dirichlet processes constitute a further development of weak Dirichlet processes, which were introduced by [17], [22] in the continuous case and by [15] in the jump case.

O is an $(\mathcal{F}_t)$-orthogonal process if $[O, N] = 0$ for every N continuous local $(\mathcal{F}_t)$-martingale. We recall that $[\cdot, \cdot]$ is the covariation extending the classical covariation of semimartingales, see [32] and Definition 2.4 in [7]. An $(\mathcal{F}_t)$-local martingale M is said to be purely discontinuous if it is $(\mathcal{F}_t)$-orthogonal. A special weak Dirichlet process is a process of the type $X = M + A$, where M is an $(\mathcal{F}_t)$-local martingale and A is an $(\mathcal{F}_t)$-predictable orthogonal process, see Definition 5.6 in [7]. When A has bounded variation, then X is a special $(\mathcal{F}_t)$-semimartingale.

Remark 2.8 (Proposition 5.9 in [7]). Any $(\mathcal{F}_t)$-special weak Dirichlet process X admits a unique decomposition of the type

$$X = X^c + M^d + A, \tag{2.15}$$

where X^c is a continuous local martingale, M^d is a purely discontinuous local martingale, and A is an $(\mathcal{F}_t)$-predictable and orthogonal process, with $A_0 = 0$. (2.15) is called the canonical decomposition of X.

In the sequel we will consider the following assumptions on a couple (X, Y) of adapted processes.

Hypothesis 2.9. *X is an $(\mathcal{F}_t)$-special weak Dirichlet process of finite quadratic variation with its canonical decomposition $X = X^c + M^d + A$, satisfying condition (2.12). $Y_t = v(t, X_t)$ for some (deterministic) function $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ of class $C^{0,1}$ such that v and X verify condition (2.14).*

Remark 2.10.

(i) [Proposition 4.5 in [7]]. If X is a càdlàg process such that $\sum_{s\leq T} |\Delta X_s|^2 < \infty$ a.s., and $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a function of class $C^{0,1}$, then

$$|v(s, X_{s-} + x) - v(s, X_{s-})|^2 \, \mathbb{1}_{\{|x|\leq 1\}} \star \mu^X \in \mathcal{A}^+_{\mathrm{loc}}. \tag{2.16}$$

(ii) [Lemma 5.29 in [7]]. If X is a càdlàg process satisfying condition (2.12), and $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a function of class $C^{0,1}$ fulfilling (2.14), then

$$|v(s, X_{s-} + x) - v(s, X_{s-})| \, \mathbb{1}_{\{|x|>1\}} \star \mu^X \in \mathcal{A}^+_{\mathrm{loc}}. \tag{2.17}$$

(iii) [Remark 5.30 in [7]]. Condition (2.14) is automatically verified if X is a càdlàg process satisfying (2.12) and $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a function of class $C^{0,1}$ with $\partial_x v$ bounded.

Theorem 2.11 (Theorem 5.31 in [7]). *Let (X, Y) be a couple of $(\mathcal{F}_t)$-adapted processes satisfying Hypothesis 2.9 with corresponding function v. Then we have*

$$\begin{aligned} v(t, X_t) = v(0, X_0) &+ \int_0^t \partial_x v(s, X_s) \, dX^c_s \\ &+ \int_{]0,\, t]\times\mathbb{R}} (v(s, X_{s-} + x) - v(s, X_{s-})) \, (\mu^X - \nu^X)(ds \, dx) + A^v(t), \end{aligned} \tag{2.18}$$

where A^v is a predictable $(\mathcal{F}_t)$-orthogonal process.

We now need to formulate a technical assumption under which Proposition 2.14 below holds, see item (iii) of Hypothesis 2.12. Let E be a closed subset of $\mathbb{R}$ on which X takes values. Given a càdlàg function $\varphi : [0, T] \to \mathbb{R}$, we denote by $\mathcal{C}_\varphi$ the set of times $t \in [0, T]$ for which there is a left (resp. right) neighborhood $I_{t-} =]t - \varepsilon, t[$ (resp. $I_{t+} = [t, t + \varepsilon[$) such that φ is constant on I_{t-} and I_{t+}. Let us then consider the following assumptions on a couple (X, Y) of adapted processes.

Hypothesis 2.12.

(i) Y is an $(\mathcal{F}_t)$-orthogonal process such that $\sum_{s\leq T}|\Delta Y_s| < \infty$, a.s.

(ii) X is a càdlàg process and $Y_t = v(t, X_t)$ for some deterministic function $v : [0,T]\times\mathbb{R} \to \mathbb{R}$, satisfying the integrability condition

$$\int_{]0,\,\cdot]\times\mathbb{R}} |v(t, X_{t-}+x) - v(t, X_{t-})|\,\mu^X(dt\,dx) \ \in \mathcal{A}^+_{\rm loc}. \tag{2.19}$$

(iii) There exists $\mathcal{C} \in [0,\,T]$ such that for ω a.s. $\mathcal{C} \supset \mathcal{C}_X(\omega)$, and

(i) $\forall t \in \mathcal{C}$, $t \mapsto v(t,x)$ is continuous for all $x \in E$;

(ii) $\forall t \in \mathcal{C}^c$, $x \in E$, (t,x) is a continuity point of v.

Remark 2.13. Item (iii) of Hypothesis 2.12 is fulfilled in two typical situations.

1. $\mathcal{C} = [0,\,T]$. Almost surely X admits a finite number of jumps and $t \mapsto v(t,x)$ is continuous for all $x \in E$.

2. $\mathcal{C} = \emptyset$ and $v|_{[0,\,T]\times E}$ is continuous.

Proposition 2.14 (Proposition 5.37 in [7])**.** *Let (X,Y) be a couple of $(\mathcal{F}_t)$-adapted processes satisfying Hypothesis 2.12 with corresponding function v. Then $v(t,X_t)$ is an $(\mathcal{F}_t)$-special weak Dirichlet càdlàg process with decomposition*

$$v(t,X_t) = v(0,X_0) + \int_{]0,\,t]\times\mathbb{R}} (v(s,X_{s-}+x) - v(s,X_{s-}))\,(\mu^X - \nu^X)(ds\,dx) + A^v(t), \tag{2.20}$$

where A^v is a predictable $(\mathcal{F}_t)$-orthogonal process.

3 A class of stochastic processes X related in a specific way to an integer-valued random measure μ

Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ be a filtered probability space. We will make use of the following assumption relating a càdlàg process X and an integer-valued random measure $\mu(ds\,de)$ on $[0,T]\times\mathbb{R}$, with compensator $\nu(ds\,de)$ (the random sets D, J and K are defined in (2.1)-(2.2)-(2.3)).

Hypothesis 3.1. *We suppose that X is an adapted càdlàg process with decomposition $X = X^i + X^p$, where the conditions below hold.*

1.a) $Y := X^i$ is a càdlàg quasi-left-continuous adapted process satisfying $\{\Delta Y \neq 0\} \subset D$.

1.b) There exists a $\tilde{\mathcal{P}}$-measurable map $\tilde\gamma : \Omega\times]0,\,T]\times\mathbb{R} \to \mathbb{R}$ such that

$$\Delta Y_t(\omega)\,\mathbb{1}_{]0,\,T]}(t) = \tilde\gamma(\omega,t,\cdot) \quad dM^{\mathbb{P}}_\mu\text{-a.e.} \tag{3.1}$$

2. X^p is a càdlàg predictable process satisfying $\{\Delta X^p \neq 0\} \subset J$.

Remark 3.2. We recall that a random time T is totally inaccessible if $\mathbb{1}_{[[T]]}(\omega,S(\omega))\,\mathbb{1}_{\{S<\infty\}} = 0$ for every predictable random time S see Definition 2.20, Chapter I, in [25], while a process Y is quasi-left-continuous if $\Delta Y_S \mathbb{1}_{S<\infty} = 0$ for all predictable random time S, see Definition 2.25, Chapter I, in [24]. Y is quasi-left-continuous if and only if there is a sequence of totally inaccessible times (T_n), with $[[T_n]]\cap[[T_m]] = \emptyset$, $n \neq m$, such that $\{\Delta Y \neq 0\} = \cup_n [[T_n]]$, see Proposition 2.26, Chapter I, in [25].

In the sequel we will also need the following assumption on μ.

Hypothesis 3.3. *$J = K$ (up to an evanescent set).*

Remark 3.4. Hypothesis 3.3, is equivalent to ask that

$$\begin{aligned} D \text{ is the disjoint union of } K \text{ and } \cup_n [[T_n^i]] \text{ (up to an evanescent set)} \\ \text{with } (T_n^i)_n \text{ disjoint totally inaccessible times.} \end{aligned} \tag{3.2}$$

Indeed, if (3.2) holds, then Hypothesis 3.3 holds true, see Proposition 2.7 in [8]. On the other hand, assume that Hypothesis 3.3 is satisfied. Then, recalling Remark 2.1-(ii)-(iv) and taking into account the additivity of the predictable projection operator, we have

$$\mathbb{1}_K = {}^p(\mathbb{1}_D) = {}^p(\mathbb{1}_K) + {}^p\left(\mathbb{1}_{D\setminus K}\right) = \mathbb{1}_K + {}^p\left(\mathbb{1}_{D\setminus K}\right)$$

so that ${}^p\left(\mathbb{1}_{D\setminus K}\right) = 0$. It follows that the predictable support of $D \setminus K$ is an evanescent set. By Remark 2.1-(v) $D\setminus K$ a thin set, therefore $D\setminus K = \cup_n [[T_n^i]]$, with $(T_n^i)_n$ disjoint totally inaccessible times, see Corollary 5.43 in [23] and Remark at page 122 in [23].

Remark 3.5. If $\nu(\{t\} \times \mathbb{R}) = 0$ for every $t \geq 0$, then $J = K = \emptyset$ and Hypothesis 3.3 trivially holds.

Proposition 3.6 (Proposition 2.12 in [7])**.** *Let μ be an integer-valued random measure and X be an adapted càdlàg process, such that (X, μ) verifies Hypothesis 3.1. Then, there exists a null set $\mathcal{N}$ such that, for every Borel function $\varphi : [0,\, T] \times \mathbb{R} \to \mathbb{R}_+$ satisfying $\varphi(s, 0) = 0$, $s \in [0,\, T]$, we have, for every $\omega \notin \mathcal{N}$,*

$$\int_{]0,\,\cdot] \times \mathbb{R}} \varphi(s, x)\, \mu^X(\omega, ds\, dx) = \int_{]0,\,\cdot] \times \mathbb{R}} \varphi(s, \tilde{\gamma}(\omega, s, e))\, \mu(\omega, ds\, de) + \sum_{0 < s \leq \cdot} \varphi(s, \Delta X_s^p(\omega)).$$

Proposition 3.7. *Let μ be an integer-valued random measure with compensator ν satisfying Hypothesis 3.3, and let X be an adapted càdlàg process such that and (X, μ) verifies Hypothesis 3.1. Let $\varphi : \Omega \times [0,\, T] \times \mathbb{R} \to \mathbb{R}_+$ be a $\tilde{\mathcal{P}}$-measurable function such that $\varphi(\omega, s, 0) = 0$ for every $s \in [0,\, T]$, up to indistinguishability, and assume that there exists a $\tilde{\mathcal{P}}$-measurable subset A of $\Omega \times [0,\, T] \times \mathbb{R}$ satisfying*

$$|\varphi|\, \mathbb{1}_A \star \mu^X \in \mathcal{A}^+_{\mathrm{loc}}, \quad |\varphi|^2\, \mathbb{1}_{A^c} \star \mu^X \in \mathcal{A}^+_{\mathrm{loc}}. \tag{3.3}$$

Then

$$\int_{]0,\,\cdot] \times \mathbb{R}} \varphi(s, x)\ (\mu^X - \nu^X)(ds\, dx) = \int_{]0,\,\cdot] \times \mathbb{R}} \varphi(s, \tilde{\gamma}(s, e))\, (\mu - \nu)(ds\, de). \tag{3.4}$$

Remark 3.8. The result above consistently improves the achievements of Proposition 2.17 in [8]. As a matter of fact, condition (1.2) is no longer asked here. This allows to solve the identification problems under more general assumptions, see Theorems 4.2 and 4.5, and therefore it extends the applicability of our results, see e.g. Section 5.1.

Proof. Clearly the result holds if we show that φ verifies (3.4) under one of the two following assumptions: (i) $|\varphi| \star \mu^X \in \mathcal{A}^+_{\mathrm{loc}}$, (ii) $|\varphi|^2 \star \mu^X \in \mathcal{A}^+_{\mathrm{loc}}$. By localization arguments, it is enough to show it when $|\varphi| \star \mu^X \in \mathcal{A}^+$, $|\varphi|^2 \star \mu^X \in \mathcal{A}^+$.

Case $|\varphi| \star \mu^X \in \mathcal{A}^+$. We will separate the proof into the following steps.

1. Assume that $\theta : \Omega \times [0,\, T] \times \mathbb{R} \to \mathbb{R}_+$ be a $\tilde{\mathcal{P}}$-measurable function such that $\theta(\omega, s, 0) = 0$ for every $s \in [0,\, T]$, satifying

$$\int_{]0,\,\cdot] \times \mathbb{R}} \theta(s,x)\; (\mu^X - \nu^X)(ds\, dx) = \int_{]0,\,\cdot] \times \mathbb{R}} \theta(s, \tilde{\gamma}(s,e))\, (\mu - \nu)(ds\, de) + \Gamma^\theta_\cdot \tag{3.5}$$

with Γ^θ a predictable process. Then, $\Gamma^\theta = 0$. As a matter of fact, Γ^θ is a predictable local martingale, and therefore continuous, see Remark 4 pag 194, Chapter VII, in [23]. On the other hand, being also a purely discontinuous martingale, it follows that it is the null process.

2. Let $\phi : \Omega \times [0,\, T] \times \mathbb{R} \to \mathbb{R}_+$ be a $\tilde{\mathcal{P}}$-measurable function such that $\phi(\omega, s, 0) = 0$ for every $s \in [0,\, T]$, and $|\phi| \star \mu^X \in \mathcal{A}^+$. Assume that $\phi = \phi \mathbb{1}_K$. Then (3.4) holds true for $\varphi = \phi$.

3. Let $\psi : \Omega \times [0,\, T] \times \mathbb{R} \to \mathbb{R}_+$ be a $\tilde{\mathcal{P}}$-measurable function such that $\psi(\omega, s, 0) = 0$ for every $s \in [0,\, T]$, and $|\psi| \star \mu^X \in \mathcal{A}^+$. Assume that $\psi \mathbb{1}_K = 0$. Then (3.4) holds true for $\varphi = \psi$.

4. Let $\varphi : \Omega \times [0,\, T] \times \mathbb{R} \to \mathbb{R}_+$ be a $\tilde{\mathcal{P}}$-measurable function such that $\varphi(\omega, s, 0) = 0$ for every $s \in [0,\, T]$, and $|\varphi| \star \mu^X \in \mathcal{A}^+$. Then $\varphi = \phi + \psi$, with $\phi := \varphi \mathbb{1}_K$, $\psi = \varphi - \varphi \mathbb{1}_K$. In particular, $\phi = \phi \mathbb{1}_K$ and $\psi \mathbb{1}_K = 0$, and $|\phi| \star \mu^X \in \mathcal{A}^+$, $|\psi| \star \mu^X \in \mathcal{A}^+$. By Steps 2. and 3. and the additivity property of the stochastic integral, it follows that (3.4) holds true for φ.

It remains to prove Steps 2. and 3.
Step 2. We have

$$\begin{aligned}\int_{]0,\,\cdot] \times \mathbb{R}} \phi(s,x)\, \mu^X(ds\, dx) &= \int_{]0,\,\cdot] \times \mathbb{R}} \phi(s,x)\, \mathbb{1}_K(s)\, \mu^X(ds\, dx) \\ &= \sum_{s \leq \cdot} \phi(s, \Delta X_s)\, \mathbb{1}_K(s) = \sum_{s \leq \cdot} \phi(s, \Delta X^i_s + \Delta X^p_s)\, \mathbb{1}_K(s),\end{aligned}$$

where in the latter equality we have used that (X, μ) satisfies Hypotheses 3.1 with $X = X^i + X^p$. Since X^i is a càdlàg quasi-left-continuous process, $\Delta X^i_S = 0$ for all predictable random time S. Recalling Remark 2.1-(iii) and Hypothesis 3.3, we have

$$\begin{aligned}\sum_{s \leq \cdot} \phi(s, \Delta X^i_s + \Delta X^p_s)\, \mathbb{1}_K(s) &= \sum_{s \leq \cdot} \phi(s, \Delta X^i_s + \Delta X^p_s)\, \mathbb{1}_{\cup_n [[R_n]]}(s) \\ &= \sum_{n:\, R_n \leq \cdot} \phi(R_n, \Delta X^i_{R_n} + \Delta X^p_{R_n})\, \mathbb{1}_{[[R_n]]}(s) \\ &= \sum_{n:\, R_n \leq \cdot} \phi(R_n, \Delta X^p_{R_n})\, \mathbb{1}_{[[R_n]]}(s) = \sum_{s \leq \cdot} \phi(s, \Delta X^p_s)\, \mathbb{1}_K(s),\end{aligned}$$

so that (3.7) yields

$$\int_{]0,\,\cdot] \times \mathbb{R}} \phi(s,x)\, \mu^X(ds\, dx) = \sum_{0 < s \leq \cdot} \phi(s, \Delta X^p_s). \tag{3.6}$$

By Proposition 3.6 together with (3.6), $\int_{]0,\,\cdot] \times \mathbb{R}} \phi(s, \tilde{\gamma}(s,e))\, \mu(ds\, de) = 0$. Therefore, being ϕ a non negative function, $\int_{]0,\,\cdot] \times \mathbb{R}} \phi(s, \tilde{\gamma}(s,e))\, \nu(ds\, de) = 0$. It follows that

$$\int_{]0,\,\cdot] \times \mathbb{R}} \phi(s, \tilde{\gamma}(s,e))\, (\mu - \nu)(ds\, de) = 0.$$

Adding and subtracting $\int_{]0,\,\cdot]\times\mathbb{R}} \phi(s,x)\,\nu^X(ds\,dx)$ in (3.6) we get that (3.5) holds for ϕ with

$$\Gamma^\phi := \sum_{0<s\leq\cdot} \phi(s,\Delta X^p_s) - \int_{]0,\,\cdot]\times\mathbb{R}} \phi(s,x)\,\nu^X(ds\,dx).$$

Being Γ^ϕ predictable, by Step 1 it follows that $\Gamma^\phi = 0$, so that

$$\int_{]0,\,\cdot]\times\mathbb{R}} \phi(s,x)\ (\mu^X - \nu^X)(ds\,dx) = 0.$$

It follows that (3.4) holds true for ϕ, and reads $0=0$.
Step 3. Since (X,μ) satisfies Hypotheses 3.1 and $J=K$ by Hypothesis 3.3, $\{\Delta X^p \neq 0\} \subset K$, so that $\mathbb{1}_{\{\Delta X^p\neq 0\}} \leq \mathbb{1}_K$. We have

$$\sum_{0<s\leq\cdot} \psi(s,\Delta X^p_s) = \sum_{0<s\leq\cdot} \psi(s,\Delta X^p_s)\mathbb{1}_{\{\Delta X^p\neq 0\}}(s) \leq \sum_{0<s\leq\cdot} \psi(s,\Delta X^p_s)\mathbb{1}_K(s) = 0.$$

Therefore, $\sum_{0<s\leq\cdot} \psi(s,\Delta X^p_s) = 0$, and the equality in Proposition 3.6 reads

$$\int_{]0,\,\cdot]\times\mathbb{R}} \psi(s,x)\,\mu^X(ds\,dx) = \int_{]0,\,\cdot]\times\mathbb{R}} \psi(s,\tilde\gamma(s,e))\,\mu(ds\,de). \tag{3.7}$$

Adding and subtracting $\int_{]0,\,\cdot]\times\mathbb{R}} \psi(s,x)\,\nu^X(ds\,dx)$ (resp. $\int_{]0,\,\cdot]\times\mathbb{R}} \psi(s,\tilde\gamma(s,e)\,\nu(ds\,de))$ in the left-hand side of (3.7) (resp. in the right-hand side of (3.7)), we get that (3.5) holds for ψ with

$$\Gamma^\psi := \int_{]0,\,\cdot]\times\mathbb{R}} \psi(s,\tilde\gamma(s,e)\,\nu(ds\,de) - \int_{]0,\,\cdot]\times\mathbb{R}} \psi(s,x)\,\nu^X(ds\,dx).$$

Being Γ^ψ predictable, by Step 1 it follows that $\Gamma^\psi = 0$, so that (3.4) holds true for ψ. This concludes the proof in the case $|\varphi| \star \mu^X \in \mathcal{A}^+$.
Case $|\varphi|^2 \star \mu^X \in \mathcal{A}^+$. This will follow from the previous one by approaching in $\mathcal{L}^2(\mu^X)$ the function φ with $\varphi_\varepsilon(s,x) := \varphi(s,x)\,\mathbb{1}_{\varepsilon<|x|\leq 1/\varepsilon}\,\mathbb{1}_{s\in[0,T]}$. Indeed, $\varphi_\varepsilon(s,x) \star \mu^X \in \mathcal{A}^+$, by Cauchy-Schwarz inequality, taking into account the fact that μ^X, restricted to $\varepsilon \leq |x| \leq 1/\varepsilon$, is finite, since μ^X is σ-finite on $[0,\infty)\times\mathbb{R}$. The proof is done along the same steps as above, with the following slight modifications.
Step 2'. Set $\phi_\varepsilon(s,x) := \phi(s,x)\,\mathbb{1}_{\varepsilon<|x|\leq 1/\varepsilon}\,\mathbb{1}_{s\in[0,T]}$, and notice that $\phi_\varepsilon(s,x) = \phi_\varepsilon(s,x)\mathbb{1}_K$. Applying Step 2 with $\phi = \phi_\varepsilon$, we get that

$$\int_{]0,\,\cdot]\times\mathbb{R}} \phi_\varepsilon(s,\tilde\gamma(s,e))\,(\mu-\nu)(ds\,de) = \int_{]0,\,\cdot]\times\mathbb{R}} \phi_\varepsilon(s,x)\ (\mu^X - \nu^X)(ds\,dx) = 0. \tag{3.8}$$

We remind that, by (2.7)-(2.8)-(2.9), if $||\phi_\varepsilon(s,x) - \phi(s,x)||^2_{\mathcal{G}^2(\mu^X)}$ and $||\phi_\varepsilon(s,\tilde\gamma(s,e)) - \phi(s,\tilde\gamma(s,e))||^2_{\mathcal{G}^2(\mu)}$ converges to zero as ε goes to zero, then (3.4) holds for φ replaced by ϕ, and reads $0=0$. Recalling Remark 2.4, we have $||\phi_\varepsilon(s,x) - \phi(s,x)||^2_{\mathcal{G}^2(\mu^X)} \leq ||\phi_\varepsilon(s,x) - \phi(s,x)||^2_{\mathcal{L}^2(\mu^X)}$ and $||\phi_\varepsilon(s,\tilde\gamma(s,e)) - \phi(s,\tilde\gamma(s,e))||^2_{\mathcal{G}^2(\mu)} \leq ||\phi_\varepsilon(s,\tilde\gamma(s,e)) - \phi(s,\tilde\gamma(s,e)))||^2_{\mathcal{L}^2(\mu)}$. By the Lebesgue theorem, and the fact that ϕ_ε converges pointwise to ϕ, we have that $||\phi_\varepsilon(s,x) - \phi(s,x)||^2_{\mathcal{L}^2(\mu^X)} \to 0$ and $||\phi_\varepsilon(s,\tilde\gamma(s,e)) - \phi(s,\tilde\gamma(s,e)))||^2_{\mathcal{L}^2(\mu)} \to 0$, and the conclusion follows.
Step 3'. Set $\psi_\varepsilon(s,x) := \psi(s,x)\,\mathbb{1}_{\varepsilon<|x|\leq 1/\varepsilon}\,\mathbb{1}_{s\in[0,T]}$, and notice that $\psi_\varepsilon(s,x) = \psi_\varepsilon(s,x)\mathbb{1}_K$. Arguing as in Step 3, we get that (3.4) holds true for ψ_ε, namely

$$\int_{]0,\,\cdot]\times\mathbb{R}} \psi_\varepsilon(s,x)\ (\mu^X - \nu^X)(ds\,dx) = \int_{]0,\,\cdot]\times\mathbb{R}} \psi_\varepsilon(s,\tilde\gamma(s,e))\,(\mu-\nu)(ds\,de).$$

The conclusion follows arguing as in Step 2'. □

4 The identification problem

In the present section we address the identification problem in two cases, the first one consisting in Theorem 4.2 and the second one consisting in Theorem 4.5.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space. Let μ be an integer-valued random measure defined on $[0, T] \times \mathbb{R}$, with compensator ν, and let M be a local martingale, with $M_0 = 0$. Let ζ be a non-decreasing adapted càdlàg process. We will focus on BSDEs driven by a compensated random measure $\mu - \nu$ of the form (1.1). Here ξ is an $\mathcal{F}_T$-measurable square integrable random variable, $\tilde{f} : \Omega \times [0,\, T] \times \mathbb{R}^4 \to \mathbb{R}$ is a measurable function, whose domain is equipped with the σ-field $\mathcal{F} \otimes \mathcal{B}([0,\, T] \times \mathbb{R}^4)$. A solution of BSDE (1.1) is a triple of processes (Y, Z, U) such that the first two integrals in (1.1) exist and are finite in the Lebesgue sense, Y is adapted and càdlàg, Z is progressively measurable with $Z \in L^2([0,\, T], d\langle M \rangle_t)$ a.s., and $U \in \mathcal{G}^2_{\mathrm{loc}}(\mu)$.

Remark 4.1. Uniqueness means the following: if (Y, Z, U), (Y', Z', U') are solutions of the BSDE (1.1), then $Y = Y'$ in the sense of indistinguishability, $Z = Z'$ $d\mathbb{P}\, d\langle M \rangle_t$ a.e., and $U_t(e) - U'_t(e)$ in the sense of $\mathcal{G}^2_{\mathrm{loc}}(\mu)$, namely, there is a predictable process (l_t) such that $U_t(e) - U'_t(e) = l_t\, \mathbb{1}_K(t)$, $d\mathbb{P}\, \nu(dt\, de)$-a.e. The latter fact is a direct consequence of Proposition 2.5. In particular, if $K = \emptyset$, then the third component of the BSDE solution is uniquely characterized in $\mathcal{L}^2(\mu)$.

Theorem 4.2. *Let μ be a random measure with compensator ν satisfying Hypothesis 3.3, and assume that X is a càdlàg process such that (X, μ) verifies Hypothesis 3.1. Let (Y, Z, U) be a solution to the BSDE (1.1) such that the pair (X, Y) satisfies Hypothesis 2.9 with corresponding function v. Let X^c denote the continuous local martingale of X given in the canonical decomposition (2.15). Then, the pair (Z, U) fulfills*

$$Z_t = \partial_x v(t, X_t)\, \frac{d\langle X^c, M \rangle_t}{d\langle M \rangle_t} \quad d\mathbb{P}\, d\langle M \rangle_t \text{-a.e.}, \tag{4.1}$$

$$\int_{]0,\, t] \times \mathbb{R}} H_s(e)\, (\mu - \nu)(ds\, de) = 0, \quad \forall\ t \in]0,\, T],\ \text{a.s.} \tag{4.2}$$

with $H_s(e) := U_s(e) - (v(s, X_{s-} + \tilde{\gamma}(s, e)) - v(s, X_{s-}))$.
If, in addition, $H \in \mathcal{G}^2_{\mathrm{loc}}(\mu)$, then there exists a predictable process (l_s) such that

$$H_s(e) = l_s\, \mathbb{1}_K(s), \quad d\mathbb{P}\, \nu(ds\, de)\text{-a.e.} \tag{4.3}$$

Remark 4.3. In particular, it follows from (4.3) and Hypothesis 3.3 for μ, that $H_s(e) = 0$, $d\mathbb{P}\, \nu^c(ds\, de)$-a.e. and $H_s(e) = l_s$, $d\mathbb{P}\, \nu^d(ds\, de)$-a.e.

Proof. By assumption, the couple (X, Y) satisfies Hypothesis 2.9 with corresponding function v. We are thus in the condition to apply Theorem 2.11 to $v(t,\, X_t)$. We set $\varphi(s, x) := v(s, X_{s-} + x) - v(s, X_{s-})$. Since X is of finite quadratic variation and verifies (2.12), and X and v satisfy (2.14), by (2.16) and (2.17) we see that the process φ verifies condition (3.3) with $A = \{|x| > 1\}$. Moreover $\varphi(s, 0) = 0$. Since μ verifies Hypothesis 3.3 and (X, μ) verifies Hypothesis 3.1, we can apply Proposition 3.7 to $\varphi(s, x)$. Identity (2.18) in Theorem 2.11 becomes

$$\begin{aligned} v(t,\, X_t) = v(0, X_0) &+ \int_{]0,\, t] \times \mathbb{R}} (v(s, X_{s-} + \tilde{\gamma}(s, e)) - v(s, X_{s-}))\, (\mu - \nu)(ds\, de) \\ &+ \int_{]0,\, t]} \partial_x v(s, X_s)\, dX^c_s + A^v(t), \end{aligned} \tag{4.4}$$

where A^v is a predictable $(\mathcal{F}_t)$-orthogonal process. In particular, $v(t, X_t)$ is a special weak Dirichlet process. On the other hand, the process $Y_t = v(t, X_t)$ fulfills the BSDE (1.1). In particular it is a special semimartingale, and therefore a special weak Dirichlet process. By Remark 2.8, which states the uniqueness of the decomposition of a special weak Dirichlet process, we get (4.2) and

$$\int_{]0,\,t]} Z_s\, dM_s = \int_{]0,\,t]} \partial_x v(s, X_s)\, dX_s^c. \tag{4.5}$$

In particular, from (4.5) we get

$$\begin{aligned} 0 &= \langle \int_{]0,\,t]} Z_s dM_s - \int_{]0,\,t]} \partial_x v(s, X_s)\, dX_s^c,\, M_t \rangle \\ &= \int_{]0,\,t]} Z_s d\langle M \rangle_s - \int_{]0,\,t]} \partial_x v(s, X_s)\, \frac{d\langle X^c,\, M\rangle_s}{d\langle M \rangle_s}\, d\langle M \rangle_s \\ &= \int_{]0,\,t]} \left(Z_s - \partial_x v(s, X_s)\, \frac{d\langle X^c,\, M\rangle_s}{d\langle M \rangle_s} \right) d\langle M \rangle_s, \end{aligned}$$

that gives identification (4.1). If in addition $H \in \mathcal{G}^2_{\text{loc}}(\mu)$, the predictable bracket at time t of the purely discontinuous martingale in identity (4.2) is well-defined, and by (2.8) equals $C(H)$ given in (2.7). Since $C(H)_T = 0$ a.s., the conclusion follows from Proposition 2.5. □

If $\mu = \mu^X$, Theorem 4.2 simplifies in the following way.

Theorem 4.4. *Let X be a càdlàg process, whose jump measure μ^X with compensator ν^X satisfies Hypothesis 3.3. Let (Y, Z, U) be a solution to the BSDE (1.1) with $\mu = \mu^X$, such that the pair (X, Y) satisfies Hypothesis 2.9 with corresponding function v. Let X^c denote the continuous local martingale of X given in the canonical decomposition (2.15). Then, the pair (Z, U) fulfills*

$$Z_t = \partial_x v(t, X_t)\, \frac{d\langle X^c, M\rangle_t}{d\langle M \rangle_t} \quad d\mathbb{P}\, d\langle M \rangle_t\text{-a.e.}, \tag{4.6}$$

$$\int_{]0,\,t]\times\mathbb{R}} H_s(x)\, (\mu^X - \nu^X)(ds\, dx) = 0, \quad \forall\, t \in]0,\, T],\ \text{a.s.} \tag{4.7}$$

with $H_s(x) := U_s(x) - (v(s, X_{s-} + x) - v(s, X_{s-}))$.
If, in addition, $H \in \mathcal{G}^2_{\text{loc}}(\mu^X)$, then there exists a predictable process (l_s) such that

$$H_s(x) = l_s\, \mathbb{1}_K(s), \quad d\mathbb{P}\, \nu^X(ds\, dx)\text{-a.e.} \tag{4.8}$$

Proof. The proof goes along the same lines of the one of Theorem 4.2, the only difference being that we replace (4.4) directly with identity (2.18) in Theorem 2.11. □

Let us now consider a BSDE driven only by a purely discontinuous martingale, of the form

$$Y_t = \xi + \int_{]t,\,T]\times\mathbb{R}} \tilde{f}(s,\, e,\, Y_{s-},\, U_s(e))\, d\zeta_s - \int_{]t,\,T]\times\mathbb{R}} U_s(e)\, (\mu - \nu)(ds\, de). \tag{4.9}$$

Theorem 4.5. *Let μ be a random measure with compensator ν satisfying Hypothesis 3.3, and assume that X is a process such that (X, μ) verifies Hypothesis 3.1. Let (Y, U) be a solution to the BSDE (4.9), such that (X, Y) satisfies Hypothesis 2.12 with corresponding function v. Then, the random field U satisfies*

$$\int_{]0,\,t]\times\mathbb{R}} H_s(e)\, (\mu - \nu)(ds\, de) = 0 \quad \forall t \in]0,\, T],\ \text{a.s.} \tag{4.10}$$

with $H_s(e) := U_s(e) - (v(s, X_{s-} + \tilde\gamma(s,e)) - v(s, X_{s-}))$.
If, in addition, $H \in \mathcal{G}^2_{\rm loc}(\mu)$, *then there exists a predictable process* (l_s) *such that*

$$H_s(e) = l_s\, \mathbb{1}_K(s), \quad d\mathbb{P}\, \nu(ds\, de)\text{-a.e.} \tag{4.11}$$

Proof. Set $\varphi(s,x) := v(s, X_{s-}+x) - v(s, X_{s-})$. By condition (ii) in Hypothesis 2.12, the process φ verifies condition (3.3) with $A = \Omega \times [0,\,T] \times \mathbb{R}$. Moreover $\varphi(s,0) = 0$. Since μ verifies Hypothesis 3.3, and (X, μ) verifies Hypothesis 3.1, we can apply Proposition 3.7 to $\varphi(s,x)$. Identity (2.20) in Proposition 2.14 becomes

$$v(t,\, X_t) = v(0, X_0) + \int_{]0,\,t]\times\mathbb{R}} (v(s, X_{s-} + \tilde\gamma(s,e)) - v(s, X_{s-}))\,(\mu-\nu)(ds\,de) + A^v(t), \tag{4.12}$$

where A^v is a predictable $(\mathcal{F}_t)$-orthogonal process. At this point we recall that the process $Y_t = v(t, X_t)$ fulfills BSDE (4.9). Again, the uniqueness of a special weak Dirichlet process (see Remark 2.8) yields identity (4.10). If in addition we assume that $H \in \mathcal{G}^2_{\rm loc}(\mu)$, the predictable bracket at time t of the purely discontinuous martingale in identity (4.10) is well-defined, and by (2.8) equals $C(H)$ given in (2.7). Since $C(H)_T = 0$ a.s., the conclusion follows from Proposition 2.5. □

Also in this case, the result simplifies when $\mu = \mu^X$.

Theorem 4.6. *Let* X *be a càdlàg process, whose jump measure* μ^X *with compensator* ν^X *satisfies Hypothesis 3.3. Let* (Y, U) *be a solution to the BSDE (4.9) with* $\mu = \mu^X$, *such that* (X, Y) *satisfies Hypothesis 2.12 with corresponding function* v. *Then, the random field* U *satisfies*

$$\int_{]0,\,t]\times\mathbb{R}} H_s(x)\,(\mu^X - \nu^X)(ds\,dx) = 0 \quad \forall t \in]0,\,T], \text{ a.s.} \tag{4.13}$$

with $H_s(x) := U_s(x) - (v(s, X_{s-} + x) - v(s, X_{s-}))$.
If, in addition, $H \in \mathcal{G}^2_{\rm loc}(\mu^X)$, *then there exists a predictable process* (l_s) *such that*

$$H_s(x) = l_s\, \mathbb{1}_K(s), \quad d\mathbb{P}\, \nu^X(ds\, dx)\text{-a.e.} \tag{4.14}$$

Proof. The proof goes along the same lines of the one of Theorem 4.5, the only difference being that we replace (4.12) directly with identity (2.18) in Theorem 2.11. □

5 Applications

5.1 The jump diffusion case

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. In the present section we consider a random measure μ and a process X satisfying the following.

Hypothesis 5.1. $\mu(ds\,de)$ *is an integer-valued random measure with compensator* $\nu(ds\,de) = dA_s\,\phi_s(de)$ *satisfying Hypothesis 3.3.* X *is a solution of the equation*

$$X_t = X_0 + \int_0^t b(s, X_{s-})\,dC_s + \int_0^t \sigma(s, X_s)\,dN_s + \int_{]0,\,t]\times\mathbb{R}} \gamma(s, X_{s-}, e)\,(\mu-\nu)(ds\,de). \tag{5.1}$$

Here N *is a continuous martingale,* C *is an increasing predictable càdlàg process, with* $C_0 = 0$, *such that* $\{\Delta C \neq 0\} \subset J$. *Moreover* $b, \sigma : \Omega \times [0,\,T] \times \mathbb{R} \to \mathbb{R}$, $\gamma : \Omega \times [0,\,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are* $\tilde{\mathcal{P}}$*-measurable maps such that* $(\omega, s, e) \mapsto \gamma(\omega, s, X_{s-}(\omega), e) \in \mathcal{G}^1_{\rm loc}(\mu)$ *and*

$$\int_0^t |b(s, X_{s-})|\,dC_s < \infty \text{ a.s.,} \tag{5.2}$$

$$\int_0^t |\sigma(s, X_s)|^2 \, d\langle N\rangle_s < \infty \text{ a.s.}, \tag{5.3}$$

$$\gamma(s, X_{s-}, e)\mathbb{1}_K(s) \equiv 0. \tag{5.4}$$

We have the following results.

Lemma 5.2. *Let X be a càdlàg process and μ be a random measure such that (X, μ) satisfies Hypothesis 5.1. Then (X, μ) satisfies Hypothesis 3.1 with decomposition $X = X^i + X^p$, where*

$$X_t^i = \int_{]0,\, t]\times\mathbb{R}} \tilde{\gamma}(s, e)\, (\mu - \nu)(ds\, de), \tag{5.5}$$

$$X_t^p = X_0 + \int_0^t b(s, X_{s-})\, dC_s + \int_0^t \sigma(s, X_s)\, dN_s, \tag{5.6}$$

with $\tilde{\gamma}(\omega, s, e) = \gamma(\omega, s, X_{s-}(\omega), e)$.

Remark 5.3. This result extends Lemma 2.19 in [8] in two ways. Firstly, we allow the coefficients b, γ and σ to be random. Secondly, we no longer ask condition (1.2) on μ, and instead we ask condition (5.4) on the coefficient γ in (5.1). This allows for instance to consider the case when μ does not fulfills condition (1.2) and γ satisfies (5.4), which was not included in Lemma 2.19 in [8].

Proof. By (5.1)-(5.5)-(5.6), together with (5.4), it straightly follows that $X = X^i + X^p$. Let us now show that X^i and X^p in (5.5)-(5.6) are respectively a càdlàg quasi-left-continuous adapted process and a càdlàg predictable process. The fact that X^p is predictable straight follow from (5.6). Concerning X^i, it is enough to prove that $\Delta X_S^i \, \mathbb{1}_{\{S<\infty\}} = 0$ a.s., for any S predictable time, see Remark 3.2. Recalling that by Hypothesis 3.3 we have $J = K$, and that γ fulfills (5.4), we get

$$\Delta X_s^i = \int_{\mathbb{R}} \gamma(s, X_{s-}, e)\, \mathbb{1}_{D\setminus K}(s)\, \mu(\{s\}, de). \tag{5.7}$$

Recalling (2.4), (5.7) can be rewritten as

$$\Delta X_s^i(\omega) = \gamma(\omega, s, X_{s-}(\omega), \beta_s(\omega))\; \mathbb{1}_{D\setminus K}(\omega, s), \tag{5.8}$$

$$= \gamma(\omega, s, X_{s-}(\omega), \beta_s(\omega))\; \mathbb{1}_{\cup_n [[T_n^i]]}(\omega, s), \tag{5.9}$$

where the second line follows by the fact that $D \setminus K = \cup_n [[T_n^i]]$ up to an evanescent set, $(T_n^i)_n$ being a sequence of totally inaccessible times with disjoint graphs, see Remark 3.4. Identity (5.9) gives, for any S finite predictable time,

$$\Delta X_S^i(\omega)\, \mathbb{1}_{\{S<\infty\}} = \gamma(\omega, S, X_{S-}(\omega), \beta_S(\omega)) \sum_n \mathbb{1}_{[[T_n^i]]}(\omega, S(\omega))\, \mathbb{1}_{\{S<\infty\}}$$

which is zero being $(T_n^i)_n$ a sequence of totally inaccessible times. The fact that $\{\Delta X^i \neq 0\} \subset D$ also directly follows from (5.7). To prove that $\Delta X_s^i(\omega) = \tilde{\gamma}(\omega, s, \cdot)$, $dM_\mu^{\mathbb{P}}(\omega, s)$-a.e., it is enough to show that

$$\mathbb{E}\left[\int_{]0,\, T]\times\mathbb{R}} \mu(\omega, ds\, de)\, |\tilde{\gamma}(\omega, s, e) - \Delta X_s^i(\omega)|\, \mathbb{1}_{\tilde{\Omega}_n}(\omega, s)\right] = 0.$$

By the structure of μ it follows that, for every $n \in \mathbb{N}$,

$$\mathbb{E}\left[\int_{]0,\, T]\times\mathbb{R}} \mu(\omega, ds\, de)\, |\tilde{\gamma}(\omega, s, e) - \Delta X_s^i(\omega)|\mathbb{1}_{\tilde{\Omega}_n}(\omega, s)\right] \leqslant \sum_{s\in]0,\, T]} \mathbb{E}\left[\mathbb{1}_D(\cdot, s)\, |\tilde{\gamma}(\cdot, s, \beta_s(\cdot)) - \Delta X_s^i(\cdot)|\right],$$

which vanishes taking into account (5.8) and that γ fulfills (5.4). Finally, since N is continuous, it follows from (5.6) that

$$\Delta X^p_s = b(s, X_{s-})\,\Delta C_s, \tag{5.10}$$

so that $\{\Delta X^p \neq 0\} \subset \{\Delta C \neq 0\} \subset J$. □

Lemma 5.4. *Let X be a càdlàg process and μ be a random measure such that (X, μ) satisfies Hypothesis 5.1. Assume that*

$$\sum_{s \in]0,\,\cdot]} |b(s, X_{s-})|^2 |\Delta C_s|^2 + \int_{]0,\cdot] \times \mathbb{R}} |\gamma(X_{s-}, e)|^2\, \nu(ds\, de) \in \mathcal{A}^+_{\rm loc}. \tag{5.11}$$

Then the following holds.

(i) X is a special weak Dirichlet process with finite quadratic variation such that (X, μ) verifies condition (2.12);

(ii) if $v : [0,\,T] \times \mathbb{R} \to \mathbb{R}$ is a function of $C^{0,1}$ class such that $x \mapsto \partial_x v(s, x)$ has linear growth, uniformly in s, condition (2.14) holds for X and v.

Proof. (i) By (5.1)-(5.2)-(5.3), X is a special semimartingale. In particular condition (2.12) holds, see Corollary 11.26 in [23]. Moreover, obviously X has finite quadratic variation. (ii) For some constant c we have

$$\begin{aligned}
&\int_{]0,\cdot] \times \mathbb{R}} |v(s, X_{s-} + x) - v(s, X_{s-}) - x\, \partial_x v(s, X_{s-})|\, \mathbb{1}_{\{|x|>1\}}\, \mu^X(ds\, dx)\\
&= \sum_{0<s\leq \cdot} |v(s, X_s) - v(s, X_{s-}) - \partial_x v(s, X_{s-})\, \Delta X_s|\, \mathbb{1}_{\{|\Delta X_s|>1\}}\\
&\leq \sum_{0<s\leq \cdot} |\Delta X_s|\, \mathbb{1}_{\{|\Delta X_s|>1\}} \left(\int_0^1 |\partial_x v(s, X_{s-} + a\, \Delta X_s)|\, da + \int_0^1 |\partial_x v(s, X_{s-})|\, da \right)\\
&\leq 2\, c \int_{]0,\cdot] \times \mathbb{R}} |X_{s-}|\, |x|\, \mathbb{1}_{\{|x|>1\}}\, \mu^X(ds\, dx) + \sum_{s \leq \cdot} |\Delta X_s|^2\, \mathbb{1}_{\{|\Delta X_s|>1\}}.
\end{aligned} \tag{5.12}$$

The first term in the right-hand side of (5.12) belongs to $\mathcal{A}^+_{\rm loc}$, taking into account (2.12) and the fact that X_{s-} is locally bounded being càglàd.

On the other hand, since X is of finite quadratic variation, by Lemma 2.10-(ii) in [7] we have that $\sum_{s \in]0,\,T]} |\Delta X_s|^2 < \infty$, a.s. Consequently, the second term in the right-hand side of (5.12) belongs to $\mathcal{A}^+_{\rm loc}$ if we prove that

$$\sum_{s \in]0,\,\cdot]} |\Delta X_s|^2 \in \mathcal{A}^+_{\rm loc}. \tag{5.13}$$

By (5.9)-(5.10), $\Delta X_s = b(s, X_{s-}) \Delta C_s + \int_{\mathbb{R}} \gamma(X_{s-}, e)\, \mu(\{s\},\, de)$, so that

$$\sum_{s \in]0,\,\cdot]} |\Delta X_s|^2 \leq \sum_{s \in]0,\,\cdot]} |b(s, X_{s-})|^2 |\Delta C_s|^2 + \int_{]0,\cdot] \times \mathbb{R}} |\gamma(X_{s-}, e)|^2\, \mu(ds\, de),$$

which belongs to $\mathcal{A}^+_{\rm loc}$ because of (5.11). □

Let W be a Brownian motion and $\mu(ds\,de)$ be a random measure with compensator $\nu(ds\,de) = \phi_s(de)dA_t$. We will focus on the BSDE

$$Y_t = g(X_T) + \int_{]t,\,T]} f(s,\, X_s,\, Y_s,\, Z_s,\, U_s(\cdot))\, dA_s - \int_{]t,\,T]} Z_s\, dW_s - \int_{]t,\,T]\times\mathbb{R}} U_s(e)\,(\mu-\nu)(ds\,de), \tag{5.14}$$

which constitutes a particular case of the BSDE (1.1). The process X appearing in (5.14) is a solution to (5.1) satisfying (5.2)- (5.3)- (5.4) -(5.11). BSDEs of the type (5.14) are considered in [27]; under suitable assumptions, the existence and uniqueness of a solution $(Y, Z, U) \in \mathcal{S}^2 \times \mathcal{L}^2 \times \mathcal{G}^2(\mu)$ is established.

We are ready to give the identification result in the present framework.

Corollary 5.5. *Let $(Y, Z, U) \in \mathcal{S}^2 \times \mathcal{L}^2 \times \mathcal{G}^2(\mu)$ be a solution to the BSDE (5.14). Then the pair (Z, U) satisfies*

$$Z_t = \sigma(X_t)\,\partial_x u(t, X_t) \quad d\mathbb{P}\,dt\text{-a.e.}, \tag{5.15}$$

$$\int_{]0,\,t]\times\mathbb{R}} H_s(e)\,(\mu-\nu)(ds\,de) = 0, \quad \forall t \in]0,\,T],\ \text{a.s.}, \tag{5.16}$$

where $H_s(e) := U_s(e) - (v(s, X_{s-} + \gamma(s, X_{s-}, e)) - v(s, X_{s-}))$.
If in addition $H \in \mathcal{G}^2_{\mathrm{loc}}(\mu)$,

$$U_s(e) = v(s, X_{s-} + \gamma(s, X_{s-}, e)) - v(s, X_{s-}), \quad d\mathbb{P}\,\nu^c(ds\,de)\text{-a.e.} \tag{5.17}$$

and there exists a predictable process (l_s) such that

$$U_s(e) = l_s, \quad d\mathbb{P}\,\nu^d(ds\,de)\text{-a.e.} \tag{5.18}$$

Proof. By Lemma 5.4, the pair (X, Y) verifies Hypothesis 2.9. On the other hand, by Lemma 5.2, (X, μ) verifies Hypothesis 3.1 with X^i and X^p given respectively by (5.5) and (5.6). We can then apply Theorem 4.2: since $X^c_\cdot = \int_0^\cdot \sigma(X_t)\,dW_t$ and $M = W$, (4.1) gives (5.15), while (4.2) with $\tilde\gamma(s, e) = \gamma(s, X_{s-}(e))$ yields (5.16). If in addition $H \in \mathcal{G}^2(\mu)$, (5.17)-(5.18) follows by (4.3) and Remark 4.3. □

Remark 5.6. This result significantly extends Corollary 4.3 in [8], where $\mu(ds\,de)$ where a Poisson random measure with deterministic compensator $\nu(de)ds$ (and in particular condition (1.2) held being $K = \emptyset$). Here we insist on the fact that we deal with a general random measure μ not necessarily verifying condition (1.2).

5.2 The PDMPs case

We assume that X is a piecewise deterministic Markov process (PDMP) generated by a marked point process (T_n, ζ_n), where $(T_n)_n$ are increasing random times such that $T_n \in]0, \infty[$, where either there is a finite number of times $(T_n)_n$ or $\lim_{n\to\infty} T_n = +\infty$, and ζ_n are random variables in $[0, 1]$. We will follow the notations in [16], Chapter 2, Sections 24 and 26. The behavior of the PDMP X is described by a triplet of local characteristics (h, λ, Q): $h :]0,\,1[\to \mathbb{R}$ is a Lipschitz continuous function, $\lambda :]0, 1[\to \mathbb{R}$ is a measurable function such that $\sup_{x\in]0,1[} |\lambda(x)| < \infty$, and Q is a transition probability measure on $[0, 1] \times \mathcal{B}(]-1, 1[)$. Some other technical assumptions are specified in the over-mentioned reference, that we do not recall here. Let us denote by $\Phi(s, x)$ the unique solution of $g'(s) = h(g(s))$, $g(0) = x$. The process X can be defined as

$$X(t) = \begin{cases} \Phi(t, x), & t \in [0,\, T_1[\\ \Phi(t - T_n, \zeta_n), & t \in [T_n,\, T_{n+1}[, \end{cases} \tag{5.19}$$

and verifies the equation

$$X_t = X_0 + \int_0^t h(X_s)\, ds + \int_{]0,\, t] \times \mathbb{R}} x\, \mu^X(ds\, dx) \tag{5.20}$$

with

$$\mu^X(ds\, dx) = \sum_{n \geq 1} \mathbb{1}_{\{\zeta_n \in]0,1[\}} \delta_{(T_n,\, \zeta_n - \zeta_{n-1})}(ds\, dx). \tag{5.21}$$

The knowledge of $(h,\, \lambda,\, Q)$ completely specifies the law of X, see Section 24 in [16], and also Proposition 2.1 in [2]. In particular, let $\mathbb{P}$ be the unique probability measure under which the compensator of μ^X has the form

$$\nu^X(ds\, dx) = (\lambda(X_{s-})\, ds + dp^*_s)\, Q(X_{s-},\, dx), \tag{5.22}$$

where λ has been trivially extended to $[0, 1]$ by the zero value, and

$$p^*_t = \sum_{n=1}^{\infty} \mathbb{1}_{\{t \geq T_n\}}\, \mathbb{1}_{\{X_{T_n -} \in \{0,1\}\}} \tag{5.23}$$

is the predictable process counting the number of jumps of X from the boundary of its domain.

From (5.22), we can write decomposition $\nu^X(ds\, dx) = \phi_s(dx) dA_s$ with $dA_s = \lambda(X_{s-})\, ds + dp^*_s$ and $\phi_s(dx) = Q(X_{s-}, dx)$. In particular, A is predictable (not deterministic) and discontinuous, with jumps $\Delta A_s(\omega) = \Delta p^*_s(\omega) = \mathbb{1}_{\{X_{s-}(\omega) \in \{0,1\}\}}$. Consequently,

$$J = K = \{(\omega, t) : X_{t-}(\omega) \in \{0, 1\}\}. \tag{5.24}$$

Remark 5.7. In [8] we asked the measure μ^X to satisfy condition (1.2), which entails the existence of a function $\beta : \{0, 1\} \to] - 1, 1[$ such that

$$Q(y,\, dx) = \delta_{\beta(y)}(dx) \quad \text{a.s.,}$$

see Lemma 4.11 in [8]. In the present paper we can avoid this assumption and work with the whole class of PDMPs.

Let us consider a BSDE driven by the compensated random measure $\mu^X - \nu^X$, where μ^X is the integer-valued random measure in (5.21) associated to a piecewise deterministic Markov process X with values in the interval $[0, 1]$, of the form

$$Y_t = g(X_T) + \int_{]t,\, T]} f(s,\, X_{s-},\, Y_{s-},\, U_s(\cdot))\, dA_s - \int_{]t,\, T] \times \mathbb{R}} U_s(e)\, (\mu^X - \nu^X)(ds\, de). \tag{5.25}$$

Existence and uniqueness results for solutions $(Y, U) \in \mathcal{L}^2 \times \mathcal{G}^2(\mu)$ to BSDEs driven by purely discontinuous martingales (that include (5.25) as a special case) were established under suitable assumptions in [1] and in the recent work [3].

Lemma 5.8. *We set $E = [0,\, 1]$. Let Y be a special semimartingale such that its martingale component is purely discontinuous. Let X be a càdlàg process with values in E, with a finite number of jumps on each compact interval. Assume that $Y_t = v(t, X_t)$ for some function $v : [0,\, T] \times \mathbb{R} \to \mathbb{R}$ such that its restriction to $[0,\, T] \times E$ is continuous. Then (X, Y) satisfies Hypothesis 2.12 with corresponding function v.*

Proof. The proof is the same as the one of Lemma 4.13 in [8]. □

Corollary 5.9. *Let X be a PDMP with jump measure μ^X with compensator ν^X given by (5.22), and let $(Y, U) \in \mathcal{L}^2 \times \mathcal{G}^2(\mu^X)$ be a solution to the BSDE (5.25). Assume that $Y_t = v(t, X_t)$ for some continuous function v. Then the random field U satisfies*

$$\int_{]0,\,t]\times\mathbb{R}} H_s(x)\,(\mu^X - \nu^X)(ds\,dx) = 0, \quad \forall\, t \in]0,\,T], \text{ a.s.} \tag{5.26}$$

with $H_s(x) := U_s(x) - (v(s, X_{s-} + x) - v(s, X_{s-}))$. If in addition $H_s(x) \in \mathcal{G}^2_{\mathrm{loc}}(\mu^X)$,

$$U_s(x) = v(s, X_{s-} + x) - v(s, X_{s-}) \quad d\mathbb{P}\,\lambda(X_{s-})\,Q(X_{s-},\,dx)\,\mathbb{1}_{X_{s-}\in]0,\,1[}ds\text{-a.e.} \tag{5.27}$$

and there exists a predictable process (l_s) such that

$$U_s(x) = v(s, X_{s-} + x) - v(s, X_{s-}) + l_s, \quad d\mathbb{P}\,Q(X_{s-},\,dx)\mathbb{1}_{X_{s-}\in\{0,\,1\}}\,dp^*_s\text{-a.e.} \tag{5.28}$$

Proof. Hypothesis 2.12 holds for (X, Y) by Lemma 5.8. We are in condition to apply Theorem 4.6, which gives (5.26). If, in addition, $H \in \mathcal{G}^2_{\mathrm{loc}}(\mu^X)$, by (4.14) together with Remark 4.3 with $\mu = \mu^X$,

$$H_s(x) = 0, \quad d\mathbb{P}\,\nu^{X,c}(ds\,de)\text{-a.e.},$$

and there exists a predictable process (l_s) such that

$$H_s(x) = l_s, \quad d\mathbb{P}\,\nu^{X,d}(ds\,de)\text{-a.e.}$$

At this point, recalling (5.22), and being $J = K$, we see that

$$\begin{aligned}
\nu^{X,c}(ds\,dx) &= \nu^X(ds\,dx)\,\mathbb{1}_{K^c}(s) = \lambda(X_{s-})\,Q(X_{s-}, dx)\,\mathbb{1}_{K^c}(s)\,ds,\\
\nu^{X,d}(ds\,dx) &= \nu^X(ds\,dx)\,\mathbb{1}_K(s) = Q(X_{s-},\,dx)\,\mathbb{1}_K(s)\,dp^*_s.
\end{aligned}$$

Then, since by (5.24) we have $K = \{(\omega, s) : X_{s-}(\omega) \in \{0, 1\}\}$, and (5.27)-(5.28) follow. □

5.3 The jump-diffusion case with distributional drift

Let $\sigma, \beta \in C^0$ such that $\sigma > 0$. We consider formally a PDE operator, obtained by mollification (see e.g. [18], [19]), of the type

$$L\psi = \frac{1}{2}\sigma^2\psi'' + \beta'\psi'. \tag{5.29}$$

Hypothesis 5.10. *We assume the existence of a function $\Sigma(x) := \lim_{n\to\infty} 2\int_0^x \frac{\beta_n'}{\sigma_n^2}(y)dy$ in C^0, independently from the mollifier. Moreover $\Sigma \in C^\alpha$ for some $\alpha \in [0, 1]$, the function Σ is lower bounded, and*

$$\int_{-\infty}^0 e^{-\Sigma(x)}dx = \int_0^{+\infty} e^{-\Sigma(x)}dx = +\infty.$$

Definition 5.11. *We will denote by $\mathcal{D}_L$ the set of all $f \in C^1$ such that there exists some $\dot{l} \in C^0$ with $Lf = \dot{l}$ in the sense of [18]. This defines without ambiguity $L : \mathcal{D}_L \to C^0$.*

We introduce the following definition of martingale problem. For an increasing process A, we will use the notation $A_t = A^c_t + \sum_{s\leq t}\Delta A_s$. We also denote by $\mathcal{C}^+(\mathbb{R})$ the set of bounded Borel functions of $\mathbb{R}$, vanishing inside a neighborhood of 0. In particular, if any two positive measures η, η' on $\mathbb{R}$ with $\eta(\{0\}) = \eta'(\{0\}) = 0$, and $\eta(x : |x| > \varepsilon) < \infty$, $\eta'(x : |x| > \varepsilon) < \infty$ are such that $\eta(f) = \eta'(f)$ for all $f \in \mathcal{C}^+(\mathbb{R})$, then $\eta = \eta'$.

Definition 5.12. *A couple $(X, \mathbb{P})$ is said to solve the martingale problem related to a given operator L of the form (5.29), a random measure $\nu(ds\,dx) = \phi_s(dx)\,dA_s$, an initial condition $X_0 = x_0 \in \mathbb{R}$, and a domain $\mathcal{D} \subset \mathcal{D}_L$, if the following holds.*

(i) for any $f \in \mathcal{D}$,

$$\int_0^t \int_{\mathbb{R}} [f(X_{s-} + x) - f(X_{s-}) - x\, f'(X_{s-})]\, \nu(ds\,dx) \in \mathcal{A}^+_{\mathrm{loc}}, \tag{5.30}$$

with respect to $\mathbb{P}$;

(ii) for any $f \in \mathcal{D}$, the process

$$Z^f := f(X_\cdot) - f(X_0) - \int_0^\cdot Lf(X_s) dA^c_s - \int_{]0,\cdot]} \int_{\mathbb{R}} [f(X_{s-}+x) - f(X_{s-}) - x\, f'(X_{s-}) \mathbb{1}_{J^c}]\, \nu(ds\,dx)$$

is a local martingale under $\mathbb{P}$;

(iii) for every $g \in \mathcal{C}^+(\mathbb{R})$, $G^g := g * \mu^X - g * \nu$ is a local martingale under $\mathbb{P}$.

Remark 5.13. Assume that $(\nu, \mathbb{P})$ satisfies condition (2.13) for som $\alpha \in [0, 1]$. Then condition (5.30) holds with respect to $\mathbb{P}$ for any $f \in \mathcal{D}$ with

$$\mathcal{D} := \{f \in \mathcal{D}_L \text{ with } f \in C^{1+\alpha}_{\mathrm{loc}}, f' \text{ bounded}\}. \tag{5.31}$$

Let L be an operator of the form (5.29), for which Hypotheses 5.10 holds, and let $\nu(ds\,dx) = \phi_s(dx)\,dA_s$ be a predictable random measure. Let $(X, \mathbb{P})$ be a solution to the martingale problem in Definition 5.12 related to X_0, L, $\nu(ds\,dx)$, and $\mathcal{D}$ given in (5.31), and such that $(\nu, \mathbb{P})$ satisfies condition (2.13).

The following result is given in [6], where we study under suitable assumptions the well-posedness of the martingale problem in Definition 5.12.

Proposition 5.14. *X is a special weak Dirichlet process (with respect to its canonical filtration) of finite quadratic variation with canonical decomposition $X = X_0 + M^X + \Gamma$, with Γ a predictable and $\mathcal{F}^X_t$-orthogonal process, and $M^X = M^{X,d} + X^c$, satisfying condition (2.12) under $\mathbb{P}$. In particular, ν is the $\mathbb{P}$-compensator of μ^X, and*

$$M^{X,d} = \int_{]0,\,\cdot]} \int_{\mathbb{R}} x\, (\mu^X - \nu)(ds\,dx), \quad \langle X^c \rangle = \int_0^\cdot \sigma^2(X_s)\, dA^c_s.$$

We are interested in BSDEs under $\mathbb{P}$ driven by the compensated random measure $\mu^X - \nu$ and the continuous martingale X^c, of the form

$$\begin{aligned} Y_t = g(X_T) &+ \int_{]t,\,T]} f(s,\, X_s, Y_s,\, Z_s,\, U_s(\cdot))\, dA_s \\ &- \int_{]t,\,T]} Z_s \frac{1}{\sigma(X_s)} dX^c_s - \int_{]t,\,T] \times \mathbb{R}} U_s(x)\, (\mu^X - \nu)(ds\,dx). \end{aligned} \tag{5.32}$$

A consequence of our identification Theorem 4.4 is the following.

Corollary 5.15. *Assume that $\mu^X(ds\,dx)$ satisfies Hypothesis 3.3. Let $(Y, Z, U) \in \mathcal{S}^2 \times \mathcal{L}^2 \times \mathcal{G}^2(\mu^X)$ be a solution to the BSDE (5.32) Assume that $Y_t = v(t, X_t)$ for some deterministic function $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ of class $C^{0,1}$ such that v and X verify condition (2.14) under $\mathbb{P}$. Then the pair (Z, U) satisfies*

$$Z_t = \sigma(X_t)\, \partial_x v(t, X_t) \quad d\mathbb{P}\, dA^c_t\text{-a.e.}, \tag{5.33}$$

$$\int_{]0,\,t]\times\mathbb{R}} H_s(x)\,(\mu^X-\nu)(ds\,dx)=0, \quad \forall\, t\in]0,\,T], \text{ a.s.} \tag{5.34}$$

with $H_s(x):=U_s(x)-(v(s,X_{s-}+x)-v(s,X_{s-}))$*. If, in addition,* $H\in\mathcal{G}^2_{\rm loc}(\mu^X)$*, then there exists a predictable process* (l_s) *such that*

$$H_s(x)=l_s\,\mathbb{1}_K(s), \quad d\mathbb{P}\,\nu(ds\,dx)\text{-a.e.} \tag{5.35}$$

Proof. We aim at applying Theorem 4.4. By assumption μ^X satisfies Hypothesis 3.3. On the other hand, by Proposition 5.14, X is a special weak Dirichlet process of finite quadratic variation with its canonical decomposition $X=X^c+M^d+\Gamma$. In addition, by assumption X satisfies condition (2.12) under $\mathbb{P}$, and condition (2.14) holds for X and v under $\mathbb{P}$, see Remark 5.13. This implies the validity of Hypothesis 2.9 for (X,Y). We can then apply Theorem 4.4: since $\langle X^c_\cdot\rangle=\int_0^\cdot\sigma^2(X_t)\,dA^c_t$ and $M=\int_0^\cdot\frac{1}{\sigma(X_s)}dX^c_s$, formula (4.6) gives (5.33), while (4.7) yields (5.34). If in addition $H\in\mathcal{G}^2(\mu^X)$, then (5.35) follows by (4.8), recalling that ν is the $\mathbb{P}$ compensator of μ^X. □

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References

[1] Bandini, E. Existence and uniqueness for backward stochastic differential equations driven by a random measure. *Electronic Communications in Probability*, 20(71):1–13, 2015.

[2] Bandini, E. Optimal control of Piecewise-Deterministic Markov Processes: a BSDE representation of the value function. *ESAIM: Control, Optimization and Calculus of Variations*, 4(1):311–354, 2018.

[3] Bandini, E. Constrained BSDEs driven by a non quasi-left-continuous random measure and optimal control of PDMPs on bounded domains. *SIAM Journal on Control and Optimization*, 57(6):3767–3798, 2019.

[4] Bandini, E. & Confortola, F. Optimal control of semi-Markov processes with a backward stochastic differential equations approach. *Mathematics of Control, Signals and Systems*, 29(1):1–35, 2017.

[5] Bandini, E. & Fuhrman, M. Constrained BSDEs representation of the value function in optimal control of pure jump Markov processes. *Stochastic Process. Appl.*, 127(5):1441–1474, 2017.

[6] Bandini, E. & Russo, F. The martingale problem related to an SDE with jumps and distributional drift. *Work in progress*.

[7] Bandini, E. & Russo, F. Weak Dirichlet processes with jumps. *Stochastic Processes and their Applications*, 127(12):4139–4189, 2017.

[8] Bandini, E. & Russo, F. Special weak Dirichlet processes and BSDEs driven by a random measure. *Bernoulli*, 24(4A):2569–2609, 2018.

[9] Barles, G. & Buckdahn, R. & Pardoux, E. Backward stochastic differential equations and integral-partial differential equations. *Stochastic Rep.*, 60(1-2):57–83, 1997.

[10] Barrasso, A. & Russo, F. Backward Stochastic Differential Equations with no driving martingale, Markov processes and associated Pseudo Partial Differential Equations. 2017. Preprint, hal-01431559, v1.

[11] Barrasso, A. & Russo, F. Martingale driven BSDEs, PDEs and other related deterministic problems. 2017. Preprint, hal-01566883.

[12] Buckdahn, R. & Pardoux, E. BSDE's with jumps and associated integral-stochastic differential equations. *Preprint*, 1994.

[13] Carbone, R. & Ferrario, B. & Santacroce, M. Backward stochastic differential equations driven by càdlàg martingales. *Theory Probab. Appl.*, 52:304–314, 2008.

[14] Confortola, F. & Fuhrman, M. Backward stochastic differential equations associated to Markov jump processes and applications. *Stochastic Processes and their Applications*, 124:289–316, 2014.

[15] Coquet, F. & Jakubowski, A. & Mémin, J. & Słomiński, L. Natural decomposition of processes and weak Dirichlet processes. In *In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX*, volume 1874 of *Lecture Notes in Math.*, pages 81–116. Springer, Berlin, 2006.

[16] Davis, M. H. A. *Markov models and optimization.*, volume 49 of *Monographs on Statistics and Applied Probability*. Chapman & Hall., 1993.

[17] Errami, M. & Russo, F. n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stochastic Process. Appl.*, 104(2):259–299, 2003.

[18] Flandoli, F. & Russo, F. & Wolf, J. Some SDEs with distributional drift. I. General calculus. *Osaka J. Math.*, 40(2):493–542, 2003.

[19] Flandoli, F. & Russo, F. & Wolf, J. Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô's formula and semimartingale characterization. *Random Oper. Stochastic Equations*, 12(2):145–184, 2004.

[20] Fuhrman, M. & Tessitore, G. Generalized directional gradients, backward stochastic differential equations and mild solutions of semilinear parabolic equations. *Appl Math Optim*, 108:263–298, 2003.

[21] Gozzi, F. & Russo, F. Verification theorems for stochastic optimal control problems via a time dependent Fukushima-Dirichlet decomposition. *Stochastic Process. Appl.*, 116(11):1530–1562, 2006.

[22] Gozzi, F. & Russo, F. Weak Dirichlet processes with a stochastic control perspective. *Stochastic Process. Appl.*, 116(11):1563–1583, 2006.

[23] He, S. & Wang, J. & Yan, J. *Semimartingale theory and stochastic calculus*. Science Press Bejiing New York, 1992.

[24] Jacod, J. *Calcul Stochastique et Problèmes de martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[25] Jacod, J. & Shiryaev, A. N. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.

[26] Laachir, I. & Russo, F. BSDEs, càdlàg martingale problems, and orthogonalization under basis risk. *SIAM J. Financial Math.*, 7(1):308–356, 2016.

[27] Papapantoleon A. & Possamai D. & Saplaouras A. Existence and uniqueness results for BSDE with jumps: the whole nine yards. *Electron. J. Probab.*, 23(121):1–68, 2018.

[28] Pardoux, É. & Peng, S. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 14(1):55–61, 1990.

[29] Pardoux, É. & Peng, S. Backward Stochastic Differential Equations and Quasilinear Parabolic Partial Differential Equations. *Lecture Notes in CIS*, 176:200–217, 1992.

[30] Peng, S. Probabilistic Interpretation for Systems of Quasilinear Parabolic Partial Differential Equations. *Stochastics*, 37:61–74, 1991.

[31] Peng, S. A generalized dynamic programming principle and Hamilton-Jacobi-Bellman Equation. *Stochastics*, 38:119–134, 1992.

[32] Russo, F. & Vallois, P. The generalized covariation process and Itô formula. *Stochastic Process. Appl.*, 59(1):81–104, 1995.

[33] Tang, S. J. & Li, X. J. Necessary conditions for optimal control of stochastic systems with random jumps. *SIAM J. Control Optim.*, 32:1447–1475, 1994.

[34] Xia, J. Backward stochastic differential equations with random measures. *Acta Mathematicae Applicatae Sinica*, 16(3):225–234, 2000.