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SYMMETRY AND MONOTONICITY OF SINGULAR SOLUTIONS OF DOUBLE PHASE PROBLEMS

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ABSTRACT. We consider positive singular solutions of PDEs arising from double phase functionals. Exploiting a rather new version of the moving plane method originally developed by Sciunzi, we prove symmetry and monotonicity properties of such solutions.

1. INTRODUCTION

Integral functionals of the form

$$u \mapsto \int_\Omega f(\nabla u)\,\mathrm{d}x \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ and f has *non-standard growth*, have been object of intensive study in the Calculus of Variations since the seminal papers of Marcellini [32, 33]. Subsequently, the interest has moved towards the study of non-autonomous functionals, whose prototype can be given by

$$u \mapsto \int_\Omega |\nabla u|^p + a(x)|\nabla u|^q\,\mathrm{d}x, \tag{1.2}$$

where, in general, $1 < p < q < N$ and $a(\cdot) \geq 0$. Without any aim of completeness, we refer e.g. [2, 10, 11, 22]. Functionals of the type (1.2) are called *double phase functionals* and have been introduced in homogenization theory by Zhikov [47, 48] with the aim of modeling strongly anisotropic materials. The common trait between the aforementioned papers is the study of regularity properties of minimizers, a pretty technical topic which has brought to light first the deep interplay between the two exponents p and q, and second the great influence of the term $a(\cdot)$ when dealing with non-autonomous functionals. In both cases, the upper bounds on q are strictly related to the so called *Lavrentiev phenomenon*, see e.g. [17].

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In this paper we are more interested in considering the natural PDE counterpart of the functional

$$u \mapsto \int_\Omega |\nabla u|^p + a(x)|\nabla u|^q \,\mathrm{d}x - F(u), \tag{1.3}$$

whose Euler-Lagrange equations is given by

$$\begin{cases} -\mathrm{div}\,(p|\nabla u|^{p-2}\nabla u + qa(x)|\nabla u|^{q-2}\nabla u) = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Here F is a primitive of f, which is assumed to be a locally Lipschitz and positive function. We stress that some fundamental analytical tools like weak comparison principle and summability estimates for the second order derivatives of the solutions of (1.3) have been recently established in [39]. The literature concerning existence and multiplicity results of double phase problems is rapidly growing, see e.g. [31, 36, 28] up to the very recent [27] where double phase Kirchhoff problems have been object of study by means of variational techniques.

The aim of this work is different from the above mentioned papers and it is of classical flavour. Indeed, we will prove purely qualitative properties of solutions of double phase problems of the form (1.4). In particular, we are interested in generalizing some very recent results contained in [20] in order to get some symmetry and monotonicity results for nontrivial solutions $u \in C^1(\overline{\Omega} \setminus \Gamma)$ to the following quasilinear elliptic boundary value problem

$$\begin{cases} -\mathrm{div}\,(p|\nabla u|^{p-2}\nabla u + qa(x)|\nabla u|^{q-2}\nabla u) = f(u) & \text{in } \Omega \setminus \Gamma, \\ u > 0 & \text{in } \Omega \setminus \Gamma, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain, with $N \geq 2$ and $1 < p < q < N$, while $\Gamma \subset \Omega$ is a closed set. See below for the details. The solution u has a possible singularity on the critical set Γ and in fact we shall only assume that u is of class C^1 far from the critical set. Before stating our main result, we need to properly describe what a solution to equation (1.5) is.

Definition 1.1. *We say that a function $u \in C^1(\overline{\Omega} \setminus \Gamma)$ is a solution to problem (1.5) if it satisfies the following two properties:*

(1) *$u > 0$ in $\Omega \setminus \Gamma$ and $u = 0$ on $\partial\Omega$;*
(2) *for every $\varphi \in C^1_c(\Omega \setminus \Gamma)$ one has*

$$\int_\Omega \big(p|\nabla u|^{p-2} + qa(x)|\nabla u|^{q-2}\big)\langle \nabla u, \nabla\varphi\rangle \,\mathrm{d}x = \int_\Omega f(u)\varphi \,\mathrm{d}x, \tag{1.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^N$.

Now we state our main result.

Theorem 1.2. *Let $q > p \geq 2$ and let $\Omega \subseteq \mathbb{R}^N$ be a convex open set, symmetric with respect to the x_1-direction. Moreover, let $\Gamma \subseteq \Omega \cap \{x_1 = 0\}$ be a closed set such that*

$$\mathrm{Cap}_q(\Gamma) = 0.$$

Finally, we assume that the following 'structural' assumptions are satisfied:

(1) $a \in L^\infty(\Omega) \cap C^1(\Omega)$ *is* non-negative and independent of x_1
(2) $f : \mathbb{R} \to \mathbb{R}$ *is a locally Lipschitz continuous function with $f(s) > 0$ for $s > 0$.*

Then, any solution $u \in C^1(\overline{\Omega} \setminus \Gamma)$ to (1.5) is symmetric wrt the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$. Furthermore,

$$\frac{\partial u}{\partial x_1} > 0 \quad \text{in } \Omega \cap \{x_1 < 0\}.$$

We stress that for $a \equiv p/q$, the problem (1.3) reduces to

$$\begin{cases} -\Delta_p u - \Delta_q u = f(u) & \text{in } \Omega \setminus \Gamma \\ u > 0 & \text{in } \Omega \setminus \Gamma \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

where the operator appearing on the left hand-side is called $(p, q) -$ *Laplacian*. Obviously, by taking $a \equiv 0$ our result boils down to the p–Laplacian case considered in [20].

We notice that in the planar case $N = 2$, Γ reduces to a point: in this case, Theorem 1.2 can be seen as a generalization of [43, 7] to the double phase setting. The case of *point singularity* for cooperative elliptic systems has been considered in [5].

Let us now spend a few comments on the Theorem 1.2. The technique that we will develop to prove Theorem 1.2 is a quite recent version of the moving plane method introduced by Sciunzi in [40] in order to deal with singular solutions of semilinear elliptic problems driven by the classical Laplacian operator. The technique is so powerful and flexible that has been recently extended to cover the case of unbounded sets [19], the p–Laplacian operator [34], cooperative elliptic systems [6, 18], the fractional Laplacian [34] and mixed local–nonlocal elliptic operators [4]. We want to stress that the technique we will use to prove Theorem 1.2 actually works for any $q > p \geq 2$. Nevertheless, the result is *more meaningful* if stated for $2 \leq p < q \leq N$ because there are no sets of zero q-capacity when $q > N$, see e.g. [30]. We also want to highlight that our result holds for $q > p \geq 2$; this lower bound for p is somehow necessary for a quite technical reason. Indeed, if $1 < p < 2$ the operator may become *very singular* near the set Γ and hence the inverse of the weight $\rho := (|\nabla u| + |\nabla u_\lambda|)^{p-2}$ may not have the right summability properties (see Remark 4.1 for more details). This issue already occurs when dealing with the p–Laplacian, see [20]. In that case however, the authors made an accurate analysis of the behaviour of the gradient of the solution near the set Γ based on previous results contained in [38]: whether a similar approach could be fruitful in our setting is currently an open problem and will be the aim of future projects.

It is clear from our previous considerations that the lower bound on p (i.e. $p \geq 2$) may be avoided if $\Gamma = \emptyset$. To the best of our knowledge, Theorem 1.2 is new even in this simpler setting. In this case, symmetry and monotocity properties of the solutions hold true for every $q > p > 1$, and the proof can be performed by using the classical moving plane method introduced by Alexandrov [1] and Serrin [42], and subsequently improved in the celebrated papers [29] and [3] in the context of semilinear elliptic equations. Since then, the literature relative to generalizations of these result to more and more general situations has become so huge that we do not even attempt at recalling all the contributions. We limit ourselves to [44, 16, 15, 8, 9] for the case of cooperative elliptic systems (also on unbounded/non smooth domains) and for the case of the composite plate problem. Finally, we refer to [12, 13, 14, 21, 24, 25, 26, 35] for the quasilinear case, which is very close to our present needs. For completeness we state the following:

Theorem 1.3. *Let $p, q > 1$ and let $\Omega \subseteq \mathbb{R}^N$ be a convex open set, symmetric with respect to the x_1-direction. Moreover, let us assume that the following 'structural' assumptions are satisfied:*

(1) $a \in L^\infty(\Omega) \cap C^1(\Omega)$ *is* non-negative and independent of x_1
(2) $f : \mathbb{R} \to \mathbb{R}$ *is a locally Lipschitz continuous function with $f(s) > 0$ for $s > 0$.*

Then, any solution $u \in C^1(\overline{\Omega})$ to (1.5) with $\Gamma = \emptyset$ is symmetric wrt the hyperplane $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$. Furthermore,

$$\frac{\partial u}{\partial x_1} > 0 \quad \text{in } \Omega \cap \{x_1 < 0\}.$$

Before closing the Introduction, we must comment on the assumption made on the term $a(x)$. We believe that the requirement of being independent of x_1 is a merely technical issue strictly related to the moving plane method. Indeed, our assumption on a appears also in [37] where the authors prove symmetry results for nonnegative solutions of fully nonlinear operators. We plan to come back to the possibility of removing such an assumption in a future paper.

The plan of the paper is the following:

- In Section 2 we fix the notations used in all the paper. Moreover, we recall the notion of r-capacity and some related theorems. Finally, we prove Lemma 2.3 and Lemma 2.4 that are two key ingredients in order to apply the moving plane procedure.
- In Section 3 we prove Theorem 1.2 performing the moving plane technique in the x_1-direction and using the results stated in Section 2.
- In Section 4 we prove Theorem 1.3 using some results contained in [20, 39] and performing the moving plane method in a standard way (since $\Gamma = \emptyset$).
- In the Appendix we state two essential results: a strong comparison principle and a Hopf-type lemma that apply specifically to our context.

2. Notations and auxiliary results

The aim of this section is twofold: on the one hand, we fix once and for all the relevant notations used throughout the paper; on the other hand, we present some auxiliary results which shall be key ingredients for the proof of Theorem 1.2.

2.1. **A review of r-capacity.** Let $1 \leq r \leq N$ be fixed, and let $K \subseteq \mathbb{R}^N$ be a *compact set.* We remind that the r-capacity of K is defined as

$$\mathrm{Cap}_r(K) := \inf\left\{\int_{\mathbb{R}^N} |\nabla\varphi|^r\,\mathrm{d}x \,:\, \varphi \in C_c^\infty(\mathbb{R}^N) \text{ and } \varphi \geq 1 \text{ on } K\right\}. \tag{2.1}$$

Moreover, if $D \subseteq \mathbb{R}^N$ is any bounded set containing K, it is possible to define the r-capacity of *the condenser* (K, D) in the following way

$$\mathrm{Cap}_r^D(K) := \inf\left\{\int_{\mathbb{R}^N} |\nabla\varphi|^r\,\mathrm{d}x \,:\, \varphi \in C_c^\infty(D) \text{ and } \varphi \geq 1 \text{ on } K\right\}. \tag{2.2}$$

As already described in the Introduction, the main aim of this paper is to investigate symmetry/monotonicity properties of the solutions to (1.5), which may present singularities on the (compact) set Γ. Since the key assumption on Γ is that

$$\mathrm{Cap}_q(\Gamma) = 0$$

(broadly put, Γ has to be 'small enough'), it is worth reviewing some basic facts about compact sets with vanishing capacity. In what follows, we denote by $\mathcal{H}^d(\cdot)$ the standard d-dimensional Hausdorff measure on $\mathbb{R}^N$, as defined, e.g., in [23].

Theorem 2.1. *The following assertions hold true.*

1. *If* $\mathrm{Cap}_r(K) = 0$*, then* $\mathrm{Cap}_r^D(K) = 0$ *for any bounded set* $D \supseteq K$*.*
2. *If* $\mathrm{Cap}_r(K) = 0$*, then* $\mathcal{H}^s(K) = 0$ *for every* $s > N - r$*.*
3. *If* $\mathcal{H}^{N-r}(K) < \infty$*, then* $\mathrm{Cap}_r(K) = 0$*.*

For a complete proof of Theorem 2.1, we refer the Reader to [30, Sec. 2.24].

Corollary 2.2. *Let* $1 \leq p < q \leq N$ *and let* $K \subseteq \mathbb{R}^N$ *be compact. Then,*

$$\mathrm{Cap}_q(K) = 0 \implies \mathrm{Cap}_p(K) = 0.$$

Proof. Since, by assumption, $\mathrm{Cap}_q(K) = 0$, by Theorem 2.1-(2) we have $\mathcal{H}^s(K) = 0$ for every $s > N - q$; in particular, as $p < q$, we derive that

$$\mathcal{H}^{N-p}(K) = 0.$$

Using this fact and Theorem 2.1-(3), we then conclude that $\mathrm{Cap}_p(K) = 0$. □

On account of Corollary 2.2, if $\Gamma \subseteq \mathbb{R}^N$ is as in Theorem 1.2 we have

$$\mathrm{Cap}_p(\Gamma) = 0.$$

2.2. **Notations for the moving plane method.** Let $\Gamma \subseteq \Omega \subseteq \mathbb{R}^N$ be as in the statement of Theorem 1.2, and let $u \in C^1(\overline{\Omega} \setminus \Gamma)$ be a solution of (1.5). For any fixed $\lambda \in \mathbb{R}$, we indicate by R_λ the reflection trough the hyperplane $\Pi_\lambda := \{x_1 = \lambda\}$, that is,

$$R_\lambda(x) = x_\lambda := (2\lambda - x_1, x_2, \ldots, x_N) \qquad (\text{for all } x \in \mathbb{R}^N); \tag{2.3}$$

accordingly, we define the function

$$u_\lambda(x) := u(x_\lambda), \qquad \text{for all } x \in R_\lambda\big(\overline{\Omega} \setminus \Gamma\big). \tag{2.4}$$

We point out that, since u solves (1.5) and a is *independent of* x_1, one has

1. $u_\lambda \in C^1(R_\lambda(\overline{\Omega} \setminus \Gamma))$;
2. $u_\lambda > 0$ in $R_\lambda(\Omega \setminus \Gamma)$ and $u_\lambda \equiv 0$ on $R_\lambda(\partial\Omega \setminus \Gamma)$;
3. for every test function $\varphi \in C^1_c(R_\lambda(\Omega \setminus \Gamma))$ one has

$$\int_{R_\lambda(\Omega)} \big(p|\nabla u_\lambda|^{p-2} + qa(x)|\nabla u_\lambda|^{q-2}\big)\langle \nabla u_\lambda, \nabla\varphi\rangle \,\mathrm{d}x \;=\; \int_{R_\lambda(\Omega)} f(u_\lambda)\varphi \,\mathrm{d}x. \tag{2.5}$$

To proceed further, we let

$$\mathbf{a} = \mathbf{a}_\Omega := \inf_{x\in\Omega} x_1 \tag{2.6}$$

and we observe that, since Ω is (bounded and) symmetric with respect to the x_1-direction, we certainly have $-\infty < \mathbf{a} < 0$. Hence, for every $\lambda \in (\mathbf{a}, 0)$ we can set

$$\Omega_\lambda := \{x \in \Omega :\, x_1 < \lambda\}. \tag{2.7}$$

Notice that the convexity of Ω in the x_1-direction ensures that

$$\Omega_\lambda \subseteq R_\lambda(\Omega) \cap \Omega. \tag{2.8}$$

Finally, for every $\lambda \in (\mathbf{a}, 0)$ we define the function

$$w_\lambda(x) := (u - u_\lambda)(x), \qquad \text{for } x \in (\overline{\Omega} \setminus \Gamma) \cap R_\lambda(\overline{\Omega} \setminus \Gamma).$$

On account of (2.8), w_λ is surely well-posed on $\overline{\Omega}_\lambda \setminus R_\lambda(\Gamma)$.

2.3. **Auxiliary results.** From now on, we assume that all the hypotheses of Theorem 1.2 are satisfied. Moreover, we tacitly inherit all the notations introduced so far.

To begin with, we remind some identities between vectors in $\mathbb{R}^N$ which are very useful in dealing with quasilinear operators: *for every $s > 1$ there exist constants $C_1, C_2, C_3 > 0$, only depending on s, such that, for every $\eta, \eta' \in \mathbb{R}^N$, one has*

$$\begin{aligned} &\langle |\eta|^{s-2}\eta - |\eta'|^{s-2}\eta', \eta - \eta'\rangle \geq C_1(|\eta| + |\eta'|)^{s-2}|\eta - \eta'|^2, \\ &\big||\eta|^{s-2}\eta - |\eta'|^{s-2}\eta'| \leq C_2(|\eta| + |\eta'|)^{s-2}|\eta - \eta'|, \\ &\langle |\eta|^{s-2}\eta - |\eta'|^{s-2}\eta', \eta - \eta'\rangle \geq C_3|\eta - \eta'|^s \qquad (\text{if } s \geq 2), \\ &\big||\eta|^{s-2}\eta - |\eta'|^{s-2}\eta'| \leq C_4|\eta - \eta'|^{s-1} \qquad (\text{if } 1 < s < 2). \end{aligned} \tag{2.9}$$

We refer, e.g., to [12] for a proof of (2.9).

Next, we need to define an *ad-hoc* family of Sobolev functions in Ω allowing us to 'cut off' of the singular set Γ. To this end, let $\varepsilon > 0$ be small enough and let

$$\mathcal{B}_\epsilon^\lambda := \big\{x \in \mathbb{R}^N : \mathrm{dist}(x, R_\lambda(\Gamma)) < \varepsilon\big\}.$$

Since R_λ is an affine map, it is easy to see that

$$\mathrm{Cap}_q\big(R_\lambda(\Gamma)\big) = 0;$$

as a consequence, by Theorem 2.1-(1) there exists $\varphi_\varepsilon \in C_c^\infty(\mathcal{B}_\varepsilon^\lambda)$ such that

$$\varphi_\varepsilon \geq 1 \text{ on } R_\lambda(\Gamma) \qquad \text{and} \qquad \int_{\mathcal{B}_\varepsilon^\lambda} |\nabla \varphi_\varepsilon|^q \,\mathrm{d}x < \varepsilon. \tag{2.10}$$

We then consider the Lipschitz functions

- $T(s) := \max\{0; \min\{s; 1\}\}$ (for $s \in \mathbb{R}$),
- $g(t) := \max\{0; -2s + 1\}$ (for $t \geq 0$)

and we define, for $x \in \mathbb{R}^N$,

$$\psi_\varepsilon(x) := g\big(T(\varphi_\varepsilon(x))\big). \tag{2.11}$$

In view of (2.10), and taking into account the very definitions of T and g, it is not difficult to recognize that ψ_ε satisfy the following properties:

(1) $\psi_\varepsilon \equiv 1$ on $\mathbb{R}^N \setminus \mathcal{B}_\varepsilon^\lambda$ and $\psi_\varepsilon \equiv 0$ on some neighborhood of $R_\lambda(\Gamma)$, say $\mathcal{V}_\varepsilon^\lambda$;
(2) $0 \leq \psi_\varepsilon \leq 1$ on $\mathbb{R}^N$;
(3) ψ_ε is Lipschitz-continuous in $\mathbb{R}^N$, so that $\psi_\varepsilon \in W^{1,\infty}(\mathbb{R}^N)$;
(4) there exists a constant $C > 0$, independent of ε, such that

$$\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^q \,\mathrm{d}x \leq C\varepsilon. \tag{2.12}$$

In particular, by combining (1), (2.12) and Hölder's inequality we get

$$\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^r \,\mathrm{d}x = \int_{\mathcal{B}_\varepsilon^\lambda} |\nabla \psi_\varepsilon|^r \,\mathrm{d}x \leq C' \varepsilon^{r/q} \qquad \text{for every } 1 \leq r < q, \tag{2.13}$$

where $C' > 0$ is a constant which can be chosen independently of ε.

With the family $\{\psi_\varepsilon\}_\varepsilon$ at hand, we can prove the following key lemma.

Lemma 2.3. *For any fixed $\lambda \in (\mathbf{a}, 0)$ we have*

$$\int_{\Omega_\lambda} \big(p(|\nabla u| + |\nabla u_\lambda|)^{p-2} + q a(x)(|\nabla u| + |\nabla u_\lambda|)^{q-2}\big) \cdot |\nabla w_\lambda^+|^2 \,\mathrm{d}x \leq \mathbf{c}_0, \tag{2.14}$$

where $\mathbf{c}_0 > 0$ is a constant only depending on p, q, λ and $\|u\|_{L^\infty(\Omega_\lambda)}$.

Proof. For every fixed $\varepsilon > 0$, we consider the function

$$\varphi_\varepsilon(x) := \begin{cases} w_\lambda^+(x)\, \psi_\varepsilon^{p+q}(x) = (u - u_\lambda)^+(x)\, \psi_\varepsilon^{p+q}(x), & \text{if } x \in \Omega_\lambda, \\ 0, & \text{otherwise.} \end{cases}$$

We claim that the following assertions hold:

(i) $\varphi_\varepsilon \in \mathrm{Lip}(\mathbb{R}^N)$;

(ii) $\mathrm{supp}(\varphi_\varepsilon) \subseteq \Omega_\lambda$ and $\varphi_\varepsilon \equiv 0$ near $R_\lambda(\Gamma)$.

In fact, since $u \in C^1(\overline{\Omega}_\lambda)$ and $u_\lambda \in C^1(\overline{\Omega}_\lambda \setminus R_\lambda(\Omega))$, we have $w_\lambda^+ \in \mathrm{Lip}(\overline{\Omega}_\lambda \setminus V)$ *for every open set* $V \supseteq R_\lambda(\Gamma)$; as a consequence, reminding that $\psi_\varepsilon \in \mathrm{Lip}(\mathbb{R}^N)$ and $\psi_\varepsilon \equiv 0$ on a neighborhood of $R_\lambda(\Gamma)$, we get $\varphi_\varepsilon \in \mathrm{Lip}(\overline{\Omega}_\lambda)$. On the other hand, since $\varphi_\varepsilon \equiv 0$ on $\partial\Omega_\lambda$, we easily conclude that $\varphi_\varepsilon \in \mathrm{Lip}(\mathbb{R}^N)$, as claimed. As for assertion (ii), it is a direct consequence of the very definition of φ_ε and of the fact that

$$\psi_\varepsilon \equiv 0 \text{ on } \mathcal{V}_\varepsilon^\lambda \supseteq R_\lambda(\Gamma).$$

On account of properties (i)-(ii) of φ_ε, a standard density argument allows us to use φ_ε as a test function *both in* (1.6) *and* (2.5); reminding that a is independent of x_1, this gives

$$\begin{aligned} p\int_{\Omega_\lambda} &\langle |\nabla u|^{p-2}\nabla u - |\nabla u_\lambda|^{p-2}\nabla u_\lambda, \nabla\varphi_\varepsilon\rangle \,\mathrm{d}x \\ &\qquad + q\int_{\Omega_\lambda} a(x)\,\langle |\nabla u|^{q-2}\nabla u - |\nabla u_\lambda|^{q-2}\nabla u_\lambda, \nabla\varphi_\varepsilon\rangle \,\mathrm{d}x \\ &= \int_{\Omega_\lambda} (f(u)-f(u_\lambda))\varphi_\varepsilon \,\mathrm{d}x. \end{aligned}$$

By unraveling the very definition of φ_ε, we then obtain

$$\begin{aligned} p\int_{\Omega_\lambda} &\psi_\varepsilon^{p+q}\cdot \langle |\nabla u|^{p-2}\nabla u - |\nabla u_\lambda|^{p-2}\nabla u_\lambda, \nabla w_\lambda^+\rangle \,\mathrm{d}x \\ &+ q\int_{\Omega_\lambda} \psi_\varepsilon^{p+q}\cdot a(x)\,\langle |\nabla u|^{q-2}\nabla u - |\nabla u_\lambda|^{q-2}\nabla u_\lambda, \nabla w_\lambda^+\rangle \,\mathrm{d}x \\ &+ p(p+q)\int_{\Omega_\lambda} w_\lambda^+\cdot\psi_\varepsilon^{p+q-1}\cdot\langle |\nabla u|^{p-2}\nabla u - |\nabla u_\lambda|^{p-2}\nabla u_\lambda, \nabla\psi_\varepsilon\rangle \,\mathrm{d}x \\ &+ q(p+q)\int_{\Omega_\lambda} w_\lambda^+\cdot\psi_\varepsilon^{p+q-1}\cdot a(x)\,\langle |\nabla u|^{q-2}\nabla u - |\nabla u_\lambda|^{q-2}\nabla u_\lambda, \nabla w_\lambda^+\rangle \,\mathrm{d}x \\ = \int_{\Omega_\lambda} &(f(u)-f(u_\lambda))\,w_\lambda^+\,\psi_\varepsilon^{p+q}\,\mathrm{d}x. \end{aligned} \tag{2.15}$$

We now observe that the integral in the left-hand side of (2.15) is actually performed on the set $\mathcal{O}_\lambda := \{x\in\Omega_\lambda:\ u\geq u_\lambda\}\setminus R_\lambda(\Gamma)$; moreover, for every $x\in\mathcal{O}_\lambda$ we have

$$0\leq u_\lambda(x)\leq u(x)\leq \|u\|_{L^\infty(\Omega_\lambda)}.$$

As a consequence, since f is locally Lipschitz-continuous on $\mathbb{R}$, we have

$$\begin{aligned} \int_{\Omega_\lambda} (f(u)-f(u_\lambda))\,w_\lambda^+\,\psi_\varepsilon^{p+q}\,\mathrm{d}x &= \int_{\Omega_\lambda} \frac{f(u)-f(u_\lambda)}{u-u_\lambda}\,(w_\lambda^+)^2\,\psi_\varepsilon^{p+q}\,\mathrm{d}x \\ &\leq L\int_{\Omega_\lambda} (w_\lambda^+)^2\,\psi_\varepsilon^{p+q}\,\mathrm{d}x, \end{aligned} \tag{2.16}$$

where $L = L(f, u, \lambda) > 0$ is the Lipschitz constant of f on the interval $[0, \|u\|_{L^\infty(\Omega_\lambda)}] \subseteq \mathbb{R}$. Using (2.16) and the estimates in (2.9), from (2.15) we then obtain

$$
\begin{aligned}
& C_1 \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} \left\{ p(|\nabla u| + |\nabla u_\lambda|)^{p-2} + q a(x) \left(|\nabla u| + |\nabla u_\lambda|\right)^{q-2} \right\} \cdot |\nabla w_\lambda^+|^2 \, \mathrm{d}x \\
& \quad \leq p \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} \, \langle |\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla w_\lambda^+ \rangle \, \mathrm{d}x \\
& \qquad + q \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} \, a(x) \, \langle |\nabla u|^{q-2} \nabla u - |\nabla u_\lambda|^{q-2} \nabla u_\lambda, \nabla w_\lambda^+ \rangle \, \mathrm{d}x \\
& \quad \leq p(p+q) \int_{\Omega_\lambda} w_\lambda^+ \, \psi_\varepsilon^{p+q-1} \cdot \big| |\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda \big| \, |\nabla \psi_\varepsilon| \, \mathrm{d}x \\
& \qquad + q(p+q) \int_{\Omega_\lambda} w_\lambda^+ \, \psi_\varepsilon^{p+q-1} \cdot a(x) \, \big| |\nabla u|^{q-2} \nabla u - |\nabla u_\lambda|^{q-2} \nabla u_\lambda \big| \, |\nabla w_\lambda^+| \, \mathrm{d}x \\
& \qquad + L_f \int_{\Omega_\lambda} (w_\lambda^+)^2 \, \psi_\varepsilon^{p+q} \, \mathrm{d}x \\
& \quad \leq C_0 \left(I_p + I_q + \int_{\Omega_\lambda} (w_\lambda^+)^2 \, \psi_\varepsilon^{p+q} \, \mathrm{d}x \right),
\end{aligned} \tag{2.17}
$$

where $C_0 = C_0(p, q, \lambda, \|u\|_{L^\infty(\Omega)}, f) > 0$ is a suitable constant and

$$
\begin{aligned}
I_p &:= \int_{\Omega_\lambda} w_\lambda^+ \, \psi_\varepsilon^{p+q-1} \cdot (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla w_\lambda^+| \, |\nabla \psi_\varepsilon| \, \mathrm{d}x, \\
I_q &:= \int_{\Omega_\lambda} w_\lambda^+ \, \psi_\varepsilon^{p+q-1} \cdot (|\nabla u| + |\nabla u_\lambda|)^{q-2} |\nabla w_\lambda^+| \, |\nabla \psi_\varepsilon| \, \mathrm{d}x.
\end{aligned} \tag{2.18}
$$

In order to complete the proof, we start from (2.17) and we provide an estimate of both I_p and I_q. Actually, we limit ourselves to consider I_p, since I_q can be treated analogously.

To begin with, we split the set Ω_λ as $\Omega_\lambda = \Omega_\lambda^{(1)} \cup \Omega_\lambda^{(2)}$, where

$$
\begin{gathered}
\Omega_\lambda^{(1)} = \{x \in \Omega_\lambda \setminus R_\lambda(\Gamma) : |\nabla u_\lambda(x)| < 2|\nabla u|\} \qquad \text{and} \\
\Omega_\lambda^{(2)} = \{x \in \Omega_\lambda \setminus R_\lambda(\Gamma) : |\nabla u_\lambda(x)| \geq 2|\nabla u|\};
\end{gathered}
$$

accordingly, since Theorem 2.1-(2) ensures that $\mathcal{H}^N(R_\lambda(\Gamma)) = 0$, we write

$$
I_p = I_{p,1} + I_{p,2}, \qquad \text{with } I_{p,i} = \int_{\Omega_\lambda^{(i)}} \{\cdots\} \, \mathrm{d}x \quad (i = 1, 2).
$$

We then proceed by estimating $I_{p,1}$, $I_{p,2}$ separately.

Step I: Estimate of $I_{1,p}$. By definition, for every $x \in \Omega_\lambda^{(1)}$ we have

$$
|\nabla u_\lambda(x)| + |\nabla u(x)| < 3|\nabla u(x)|; \tag{2.19}
$$

Using the weighted Young inequality and (2.19), for every $\rho > 0$ we get

$$\begin{aligned} I_{p,1} &\le \frac{\rho}{2}\int_{\Omega_\lambda^{(1)}} (|\nabla u| + |\nabla u_\lambda|)^{p-2}|\nabla w_\lambda^+|^2\,\psi_\varepsilon^{p+q}\,\mathrm{d}x \\ &\qquad + \frac{1}{2\rho}\int_{\Omega_\lambda^{(1)}} (|\nabla u| + |\nabla u_\lambda|)^{p-2}|\nabla \psi_\varepsilon|^2\,(w_\lambda^+)^2\,\psi_\varepsilon^{p+q-2}\,\mathrm{d}x \\ &\le \frac{\rho}{2}\int_{\Omega_\lambda^{(1)}} \psi_\varepsilon^{p+q}\,(|\nabla u| + |\nabla u_\lambda|)^{p-2}|\nabla w_\lambda^+|^2\,\mathrm{d}x \\ &\qquad + \frac{3^{p-2}}{2\rho}\int_{\Omega_\lambda^{(1)}} |\nabla u|^{p-2}|\nabla \psi_\varepsilon|^2\,(w_\lambda^+)^2\,\psi_\varepsilon^{p+q-2}\,\mathrm{d}x = (\bigstar); \end{aligned}$$

from this, reminding that $0 \le \psi_\varepsilon \le 1$ and using Hölder's inequality, we have

$$\begin{aligned} (\bigstar) &\le \frac{\rho}{2}\int_{\Omega_\lambda^{(1)}} \psi_\varepsilon^{p+q}\,(|\nabla u| + |\nabla u_\lambda|)^{p-2}|\nabla w_\lambda^+|^2\,\mathrm{d}x \\ &\qquad + \frac{3^{p-2}}{2\rho}\int_{\Omega_\lambda^{(1)}} |\nabla u|^{p-2}|\nabla \psi_\varepsilon|^2\,(w_\lambda^+)^2\,\psi_\varepsilon^{p-2}\,\mathrm{d}x \\ &\le \frac{\rho}{2}\int_{\Omega_\lambda^{(1)}} \psi_\varepsilon^{p+q}\,(|\nabla u| + |\nabla u_\lambda|)^{p-2}|\nabla w_\lambda^+|^2\,\mathrm{d}x \\ &\qquad + \frac{3^{p-2}}{2\rho}\bigg(\int_{\Omega_\lambda^{(1)}} |\nabla u|^p\psi_\varepsilon^p\,\mathrm{d}x\bigg)^{\frac{p-2}{p}}\bigg(\int_{\Omega_\lambda^{(1)}} |\nabla \psi_\varepsilon|^p\,(w_\lambda^+)^p\bigg)^{\frac{2}{p}} \\ &\quad (\text{since } 0 \le \psi_\varepsilon \le 1 \text{ and } 0 \le w_\lambda^+ \le u \le \|u\|_{L^\infty(\Omega_\lambda)}) \\ &\le \frac{\rho}{2}\int_{\Omega_\lambda^{(1)}} \psi_\varepsilon^{p+q}\,(|\nabla u| + |\nabla u_\lambda|)^{p-2}|\nabla w_\lambda^+|^2\,\mathrm{d}x \\ &\qquad + \frac{\mathbf{c}}{\rho}\bigg(\int_{\Omega_\lambda^{(1)}} |\nabla u|^p\,\mathrm{d}x\bigg)^{\frac{p-2}{p}}\bigg(\int_{\Omega_\lambda^{(1)}} |\nabla \psi_\varepsilon|^p\bigg)^{\frac{2}{p}} \\ &\quad (\text{reminding that } a \ge 0 \text{ and } p, q \ge 2) \\ &\le \frac{\rho}{2}\int_{\Omega_\lambda} \psi_\varepsilon^{p+q}\,\big\{p(|\nabla u| + |\nabla u_\lambda|)^{p-2} + qa(x)\,(|\nabla u| + |\nabla u_\lambda|)^{q-2}\big\}\cdot|\nabla w_\lambda^+|^2\,\mathrm{d}x \\ &\qquad + \frac{\mathbf{c}}{\rho}\bigg(\int_{\Omega_\lambda} |\nabla u|^p\,\mathrm{d}x\bigg)^{\frac{p-2}{p}}\bigg(\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p\bigg)^{\frac{2}{p}}, \end{aligned}$$

where $\mathbf{c} = \mathbf{c}(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}) > 0$. Summing up, we have obtained the estimate

$$\begin{aligned} I_{p,1} &\le \frac{\rho}{2}\int_{\Omega_\lambda} \psi_\varepsilon^{p+q}\,\big\{p(|\nabla u| + |\nabla u_\lambda|)^{p-2} + qa(x)\,(|\nabla u| + |\nabla u_\lambda|)^{q-2}\big\}|\nabla w_\lambda^+|^2\,\mathrm{d}x \\ &\qquad + \frac{\mathbf{c}}{\rho}\bigg(\int_{\Omega_\lambda} |\nabla u|^p\,\mathrm{d}x\bigg)^{\frac{p-2}{p}}\bigg(\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p\bigg)^{\frac{2}{p}}, \end{aligned} \tag{2.20}$$

holding true for every choice of $\rho > 0$.

STEP II: ESTIMATE OF $I_{2,p}$. By definition, for every $x \in \Omega_\lambda^{(2)}$ we have

$$\frac{1}{2}|\nabla u_\lambda| \le |\nabla u_\lambda| - |\nabla u| \le |\nabla w_\lambda| \le |\nabla u_\lambda| + |\nabla u| \le \frac{3}{2}|\nabla u_\lambda|. \tag{2.21}$$

Using again the weighted Young inequality and (2.21), for every $\rho > 0$ we get

$$\begin{aligned}
I_{p,2} &\le \Big(1 - \frac{1}{p}\Big)\rho^{\frac{p}{p-1}} \int_{\Omega_\lambda^{(2)}} (|\nabla u| + |\nabla u_\lambda|)^{\frac{p(p-2)}{p-1}} |\nabla w_\lambda^+|^{\frac{p}{p-1}} \psi_\varepsilon^{\frac{p(p+q-1)}{p-1}} \,\mathrm{d}x \\
&\qquad + \frac{1}{p\rho^p} \int_{\Omega_\lambda^{(2)}} |\nabla \psi_\varepsilon|^p \,(w_\lambda^+)^p \,\mathrm{d}x \\
&= \Big(1 - \frac{1}{p}\Big)\rho^{\frac{p}{p-1}} \int_{\Omega_\lambda^{(2)}} (|\nabla u| + |\nabla u_\lambda|)^{\frac{p(p-2)}{p-1}} |\nabla w_\lambda^+|^{\frac{p}{p-1}-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^{p+q+\frac{q}{p-1}} \,\mathrm{d}x \\
&\qquad + \frac{1}{p\rho^p} \int_{\Omega_\lambda^{(2)}} |\nabla \psi_\varepsilon|^p \,(w_\lambda^+)^p \,\mathrm{d}x \\
&\quad \text{(remind that } p \ge 2, \text{ so that } p/(p-1) - 2 \le 0) \\
&\quad \le c_p\, \rho^{\frac{p}{p-1}} \int_{\Omega_\lambda^{(2)}} |\nabla u_\lambda|^{\frac{p(p-2)}{p-1}} |\nabla u_\lambda|^{\frac{p}{p-1}-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^{p+q+\frac{q}{p-1}} \,\mathrm{d}x \\
&\qquad + \frac{1}{p\rho^p} \int_{\Omega_\lambda^{(2)}} |\nabla \psi_\varepsilon|^p \,(w_\lambda^+)^p \,\mathrm{d}x = (\bullet),
\end{aligned}$$

where we have used the notation

$$c_p := \Big(1 - \frac{1}{p}\Big)\Big(\frac{3}{2}\Big)^{\frac{p(p-2)}{p-1}} \Big(\frac{1}{2}\Big)^{\frac{p}{p-1}-2};$$

from this, reminding that $0 \le \psi_\varepsilon \le 1$ and $0 \le w_\lambda^+ \le \|u\|_{L^\infty(\Omega_\lambda)}$, we have

$$\begin{aligned}
(\bullet) &= c_p\, \rho^{\frac{p}{p-1}} \int_{\Omega_\lambda^{(2)}} |\nabla u_\lambda|^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^{p+q} \cdot \psi_\varepsilon^{\frac{q}{p-1}} \,\mathrm{d}x \\
&\qquad + \frac{1}{p\rho^p} \int_{\Omega_\lambda^{(2)}} |\nabla \psi_\varepsilon|^p \,(w_\lambda^+)^p \,\mathrm{d}x \\
&\le \mathbf{c}'\, \rho^{\frac{p}{p-1}} \int_{\Omega_\lambda^{(2)}} |\nabla u_\lambda|^{p-2} |\nabla w_\lambda^+|^2 \psi_\varepsilon^{p+q} \,\mathrm{d}x + \frac{\mathbf{c}'}{\rho^p} \int_{\Omega_\lambda^{(2)}} |\nabla \psi_\varepsilon|^p \,\mathrm{d}x \\
&\quad \text{(since } p \ge 2 \text{ and the function } a \text{ is non-negative)} \\
&\le \mathbf{c}'\, \rho^{\frac{p}{p-1}} \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} \left\{ p(|\nabla u| + |\nabla u_\lambda|)^{p-2} + q a(x) \,(|\nabla u| + |\nabla u_\lambda|)^{q-2} \right\} \cdot |\nabla w_\lambda^+|^2 \,\mathrm{d}x \\
&\qquad + \frac{\mathbf{c}'}{\rho^p} \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p \,\mathrm{d}x,
\end{aligned}$$

where $\mathbf{c}' = \mathbf{c}'(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}) > 0$. Summing up, we have obtained the estimate

$$
\begin{aligned}
I_{p,2} \leq \mathbf{c}'\, \rho^{\frac{p}{p-1}} \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} \big\{p(|\nabla u| + |\nabla u_\lambda|)^{p-2} + q a(x)\, (|\nabla u| + |\nabla u_\lambda|)^{q-2}\big\} |\nabla w_\lambda^+|^2 \, \mathrm{d}x \\
+ \frac{\mathbf{c}'}{\rho^p} \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p \, \mathrm{d}x,
\end{aligned} \tag{2.22}
$$

Gathering together (2.20) and (2.22), we finally derive

$$
\begin{aligned}
I_p &\leq \Big(\frac{\rho}{2} + \kappa \rho^{\frac{p}{p-1}}\Big) \times \\
&\quad \times \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} \big\{p(|\nabla u| + |\nabla u_\lambda|)^{p-2} + q a(x)\, (|\nabla u| + |\nabla u_\lambda|)^{q-2}\big\} |\nabla w_\lambda^+|^2 \, \mathrm{d}x \\
&\quad + \frac{\kappa}{\rho} \Big(\int_{\Omega_\lambda} |\nabla u|^p \, \mathrm{d}x \Big)^{\frac{p-2}{p}} \Big(\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p \Big)^{\frac{2}{p}} + \frac{\kappa}{\rho^p} \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^p \, \mathrm{d}x,
\end{aligned} \tag{2.23}
$$

for a suitable constant κ only depending on p, λ and $\|u\|_{L^\infty(\Omega_\lambda)}$. Furthermore, by arguing exactly in the same way, we obtain the following analogous estimate for I_q,

$$
\begin{aligned}
I_q &\leq \Big(\frac{\rho}{2} + \kappa' \rho^{\frac{q}{q-1}}\Big) \times \\
&\quad \times \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} \big\{p(|\nabla u| + |\nabla u_\lambda|)^{p-2} + q a(x)\, (|\nabla u| + |\nabla u_\lambda|)^{q-2}\big\} |\nabla w_\lambda^+|^2 \, \mathrm{d}x \\
&\quad + \frac{\kappa'}{\rho} \Big(\int_{\Omega_\lambda} |\nabla u|^q \, \mathrm{d}x \Big)^{\frac{q-2}{q}} \Big(\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^q \Big)^{\frac{2}{q}} + \frac{\kappa'}{\rho^q} \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^q \, \mathrm{d}x,
\end{aligned} \tag{2.24}
$$

where κ' is another constant only depending on q, λ and $\|u\|_{L^\infty(\Omega_\lambda)}$.

With the above estimates for I_p and I_q at hand, we are finally ready to conclude the proof: in fact, by combining (2.17), (2.23) and (2.24), we get

$$
\begin{aligned}
&\Big(C_1 - C_0\rho - C_0\kappa\,\rho^{\frac{p}{p-1}} - C_0\kappa'\rho^{\frac{q}{q-1}}\Big)\times \\
&\quad\times \int_{\Omega_\lambda} \psi_\varepsilon^{p+q}\,\big\{p(|\nabla u|+|\nabla u_\lambda|)^{p-2} + qa(x)\,(|\nabla u|+|\nabla u_\lambda|)^{q-2}\big\}|\nabla w_\lambda^+|^2\,\mathrm{d}x \\
&\qquad \leq C_0\bigg\{\frac{\kappa}{\rho}\Big(\int_{\Omega_\lambda}|\nabla u|^p\,\mathrm{d}x\Big)^{\frac{p-2}{p}}\Big(\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^p\Big)^{\frac{2}{p}} + \frac{\kappa}{\rho^p}\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^p\,\mathrm{d}x \\
&\qquad\qquad + \frac{\kappa'}{\rho}\Big(\int_{\Omega_\lambda}|\nabla u|^q\,\mathrm{d}x\Big)^{\frac{q-2}{q}}\Big(\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^q\Big)^{\frac{2}{q}} + \frac{\kappa}{\rho^q}\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^q\,\mathrm{d}x \\
&\qquad\qquad + \int_{\Omega_\lambda}(w_\lambda^+)^2\,\psi_\varepsilon^{p+q}\,\mathrm{d}x\bigg\} \qquad (2.25)\\
&\qquad \text{(since } 0\leq\psi_\varepsilon\leq 1 \text{ and } 0\leq w_\lambda^+\leq\|u\|_{L^\infty(\Omega_\lambda)}) \\
&\qquad \leq C_0\bigg\{\frac{\kappa}{\rho}\Big(\int_{\Omega_\lambda}|\nabla u|^p\,\mathrm{d}x\Big)^{\frac{p-2}{p}}\Big(\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^p\Big)^{\frac{2}{p}} + \frac{\kappa}{\rho^p}\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^p\,\mathrm{d}x \\
&\qquad\qquad + \frac{\kappa'}{\rho}\Big(\int_{\Omega_\lambda}|\nabla u|^q\,\mathrm{d}x\Big)^{\frac{q-2}{q}}\Big(\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^q\Big)^{\frac{2}{q}} + \frac{\kappa}{\rho^q}\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^q\,\mathrm{d}x \\
&\qquad\qquad + \|u\|^2_{L^\infty(\Omega_\lambda)}\cdot\mathcal{H}^N(\Omega_\lambda)\bigg\};
\end{aligned}
$$

from this, by choosing $\rho>0$ in such a way that

$$C_1 - C_0\rho - C_0\kappa\,\rho^{\frac{p}{p-1}} - C_0\kappa'\rho^{\frac{q}{q-1}} < \frac{1}{2},$$

and by letting $\varepsilon\to 0$ with the aid of Fatou's lemma (remind the properties (1)-to-(4) of the function ψ_ε and that $u\in C^1(\overline{\Omega}_\lambda)$ if $\lambda<0$), we obtain

$$\int_{\Omega_\lambda}\big[p(|\nabla u|+|\nabla u_\lambda|)^{p-2} + qa(x)\,(|\nabla u|+|\nabla u_\lambda|)^{q-2}\big]|\nabla w_\lambda^+|^2\,\mathrm{d}x \leq \mathbf{c}_0,$$

where $\mathbf{c}_0 = 2\,C_0\,\|u\|^2_{L^\infty(\Omega_\lambda)}\cdot\mathcal{H}^N(\Omega_\lambda)$. This is ends the proof. □

Another key tool for the proof of Theorem 1.2 is Lemma 2.4 below. Before stating this result, we first introduce a notation: for every fixed $\lambda\in(\mathbf{a},0)$, we define

$$\mathcal{Z}_\lambda := \big\{x\in\Omega_\lambda\setminus R_\lambda(\Gamma) :\, \nabla u(x) = \nabla u_\lambda(x) = 0\big\}. \tag{2.26}$$

We also notice that, since $u,u_\lambda\in C^1(\overline{\Omega}_\lambda\setminus R_\lambda(\Gamma))$, the set $\mathcal{Z}_\lambda$ is closed (in Ω_λ).

Lemma 2.4. *Let $\lambda\in(\mathbf{a},0)$ and let $\mathcal{C}_\lambda\subseteq\Omega_\lambda\setminus(R_\lambda(\Gamma)\cup\mathcal{Z}_\lambda)$ be a* connected component *of (the open set) $\Omega_\lambda\setminus(R_\lambda(\Gamma)\cup\mathcal{Z}_\lambda)$. If $u\equiv u_\lambda$ in $\mathcal{C}_\lambda$, then*

$$\mathcal{C}_\lambda = \emptyset.$$

Proof. We first notice that, since it is a connected component, the set $\mathcal{C}_\lambda$ is surely open. In order to prove the lemma, we then consider the Π_λ-symmetric set

$$\mathcal{C} := \mathcal{C}_\lambda \cup R_\lambda(\mathcal{C}_\lambda),$$

and we show that $\mathcal{C} = \varnothing$. To this end, we argue by contradiction and we assume that

$$\mathcal{C} \neq \varnothing.$$

Since $\mathcal{C}_\lambda$ is open and R_λ is a bijective linear map, also the set $\mathcal{C}$ is open; as a consequence, since $u > 0$ on $\Omega \setminus \Gamma$ and f is (continuous and) positive on $(0, \infty)$, we have

$$\int_{\mathcal{C}} f(u)\,\mathrm{d}x > 0. \tag{2.27}$$

On the other hand, since u is a solution of (1.5), one has

$$0 \leq \int_\Omega f(u)\varphi\,\mathrm{d}x = \int_\Omega \big(p|\nabla u|^{p-2} + qa(x)|\nabla u|^{q-2}\big)\langle \nabla u, \nabla\varphi\rangle\,\mathrm{d}x \tag{2.28}$$

for every function $\varphi \in C^1_c(\Omega \setminus \Gamma)$ such that $\varphi \geq 0$ on $\Omega \setminus \Gamma$. We now aim at choosing an *ad-hoc* test function in (2.28) allowing us to contradict (2.27).

To begin with, since $\Gamma_0 := \Gamma \cup R_\lambda(\Gamma)$ has vanishing q-capacity (as the same is true of both Γ and $R_\lambda(\Gamma)$), we can imitate the construction of the family $\{\psi_\varepsilon\}$ in (2.11): this leads to another family of functions, say $\{\gamma_\varepsilon\}$, satisfying the following properties:

(1) γ_ε is Lipschitz-continuous in $\mathbb{R}^N$, so that $\gamma_\varepsilon \in W^{1,\infty}(\mathbb{R}^N)$;
(2) $0 \leq \gamma_\varepsilon \leq 1$ on $\mathbb{R}^N$;
(3) $\gamma_\varepsilon \equiv 1$ on $\mathbb{R}^N \setminus \mathcal{O}^\lambda_\varepsilon$ and $\gamma_\varepsilon \equiv 0$ on $\mathcal{W}^\lambda_\varepsilon$, where

$$\mathcal{O}^\lambda_\varepsilon := \big\{x \in \mathbb{R}^N : \operatorname{dist}(x, \Gamma_0) < \varepsilon\big\},$$

and $\mathcal{W}^\lambda_\varepsilon \subseteq \mathcal{O}^\lambda_\varepsilon$ is a suitable neighborhood of Γ_0;
(4) there exists a constant $C > 0$, independent of ε, such that

$$\int_{\mathbb{R}^N} |\nabla\gamma_\varepsilon|^q\,\mathrm{d}x \leq C\varepsilon. \tag{2.29}$$

Moreover, for every fixed $\varepsilon > 0$ we consider the maps $G_\varepsilon, h_\varepsilon : [0, \infty) \to \mathbb{R}$ defined as

$$G_\varepsilon(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq \varepsilon, \\ 2t - 2\varepsilon, & \text{if } \varepsilon < t \leq 2\varepsilon, \\ t, & \text{if } t > 2\varepsilon. \end{cases} \qquad \text{and} \qquad h_\varepsilon(t) := \frac{G_\varepsilon(t)}{t}.$$

Using the family $\{\gamma_\varepsilon\}$ and the function h_ε just introduced, we set

$$\varphi_\varepsilon : \mathbb{R}^N \to \mathbb{R}, \qquad \varphi_\varepsilon(x) := \begin{cases} h_\varepsilon(|\nabla u(x)|) \cdot \gamma^2_\varepsilon(x), & \text{if } x \in \mathcal{C}, \\ 0, & \text{if } x \notin \mathcal{C}, \end{cases}$$

and we claim that φ_ε can be chosen as a test function in (2.28). In fact, since both γ_ε and h_ε are Lipschitz-continuous on the whole of $\mathbb{R}^N$, we clearly have $\varphi_\varepsilon \in \mathrm{Lip}(\mathcal{C})$; moreover, since $u \equiv u_\lambda$ in $\mathcal{C}_\lambda$ and $u \equiv 0$ on $\partial\Omega$, one has

$$u \equiv 0 \text{ on } K := R_\lambda(\partial\mathcal{C}_\lambda \cap \partial\Omega). \tag{2.30}$$

Now, reminding that $u > 0$ in $\Omega \setminus \Gamma$ and $u \in C^1(\overline{\Omega} \setminus \Gamma)$, from (2.30) we infer that

$$\nabla u = \nabla u_\lambda = 0 \text{ on } \partial\mathcal{C} \setminus \Gamma_0;$$

as a consequence, taking into account the very definition of h_ε, it is not difficult to recognize that $\varphi_\varepsilon \in \mathrm{Lip}(\mathbb{R}^N)$ and that $\mathrm{supp}(\varphi_\varepsilon) \subseteq \mathcal{C} \setminus \Gamma_0$. By a standard density argument we are then entitled to use φ_ε as a test function in (2.28), obtaining

$$\begin{aligned} 0 \leq \int_{\mathcal{C}} f(u) h_\varepsilon(|\nabla u|)\gamma_\varepsilon^2 \,\mathrm{d}x &= \int_{\mathcal{C}} \gamma_\varepsilon^2 \cdot \left(p|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}\right) \langle \nabla u, \nabla(h_\varepsilon \circ |\nabla u|)\rangle \,\mathrm{d}x \\ &+ 2\int_{\mathcal{C}} h_\varepsilon(|\nabla u|)\,\gamma_\varepsilon \cdot \left(p|\nabla u|^{p-2} + qa(x)|\nabla u|^{q-2}\right) \langle \nabla u, \nabla \gamma_\varepsilon\rangle \,\mathrm{d}x. \end{aligned}$$

To proceed further we observe that, since $u \in C^1(\overline{\Omega}_\lambda)$ and $u \equiv u_\lambda$ on $\mathcal{C}_\lambda \subseteq \Omega_\lambda$, we have $u \in C^1(\overline{\mathcal{C}})$. We can then invoke the regularity results proved by Riey [39], ensuring that

$$u \in W^{2,s}(\mathcal{C}) \quad \text{for a suitable } s = s_p \in (1, 2]. \tag{2.31}$$

As a consequence, we can write

$$\begin{aligned} &\int_{\mathcal{C}} \gamma_\varepsilon^2 \cdot \left(p|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}\right) \langle \nabla u, \nabla(h_\varepsilon \circ |\nabla u|)\rangle \,\mathrm{d}x \\ &\qquad = \int_{\mathcal{C}} h'_\varepsilon(|\nabla u|)\,\gamma_\varepsilon^2 \cdot \left(p|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}\right) \langle \nabla u, \nabla|\nabla u|\rangle \,\mathrm{d}x. \end{aligned} \tag{2.32}$$

From this, since by definition one has

$$h_\varepsilon(t) \leq 1 \quad \text{and} \quad h'_\varepsilon(t) \leq 2/\varepsilon, \tag{2.33}$$

using Schwartz's inequality, (2.33) and reminding that $0 \leq \gamma_\varepsilon \leq 1$, we then get

$$
\begin{aligned}
0 &\leq \int_{\mathcal{C}} f(u) h_\varepsilon(|\nabla u|)\gamma_\varepsilon^2 \,\mathrm{d}x \\
&\leq 2\int_{\mathcal{C}\cap\{\varepsilon<|\nabla u|<2\varepsilon\}} \frac{|\nabla u|}{\varepsilon}\cdot\left(p|\nabla u|^{p-2}+qa(x)|\nabla u|^{q-2}\right)\|D^2u\|\,\gamma_\varepsilon^2\,\mathrm{d}x \\
&\qquad + 2\int_{\mathcal{C}} \psi_\varepsilon\cdot\left(p|\nabla u|^{p-1}+qa(x)|\nabla u|^{q-1}\right)|\nabla\gamma_\varepsilon|\,\mathrm{d}x \\
&\leq 4p\int_{\mathcal{C}\cap\{\varepsilon<|\nabla u|<2\varepsilon\}} |\nabla u|^{p-2}\|D^2u\|\gamma_\varepsilon^2\,\mathrm{d}x + 4q\int_{\mathcal{C}\cap\{\varepsilon<|\nabla u|<2\varepsilon\}} a(x)|\nabla u|^{q-2}\|D^2u\|\gamma_\varepsilon^2\,\mathrm{d}x \\
&\qquad + 2\int_{\mathcal{C}} \psi_\varepsilon\cdot\left(p|\nabla u|^{p-1}+qa(x)|\nabla u|^{q-1}\right)|\nabla\gamma_\varepsilon|\,\mathrm{d}x \\
&\leq 4p\int_{\mathcal{C}} |\nabla u|^{p-2}\|D^2u\|\gamma_\varepsilon^2\cdot\mathbf{1}_{\mathcal{D}_\varepsilon}\,dx + 4q\int_{\mathcal{C}} a(x)|\nabla u|^{q-2}\|D^2u\|\gamma_\varepsilon^2\cdot\mathbf{1}_{\mathcal{D}_\varepsilon}\,\mathrm{d}x \\
&\qquad + 2p\bigg(\int_{\mathcal{C}}|\nabla u|^p\,\mathrm{d}x\bigg)^{\frac{p-1}{p}}\bigg(\int_{\mathcal{C}}|\nabla\gamma_\varepsilon|^p\,\mathrm{d}x\bigg)^{\frac{1}{p}} \\
&\qquad\qquad + 2q\,\|a\|_{L^\infty(\Omega)}\bigg(\int_{\mathcal{C}}|\nabla u|^q\,\mathrm{d}x\bigg)^{\frac{q-1}{q}}\bigg(\int_{\mathcal{C}}|\nabla\gamma_\varepsilon|^q\,\mathrm{d}x\bigg)^{\frac{1}{q}},
\end{aligned}
\tag{2.34}
$$

where $\mathcal{D}_\varepsilon := \mathcal{C}\cap\{\varepsilon<|\nabla u|<2\varepsilon\}$ and $\mathbf{1}_{\mathcal{D}_\varepsilon}$ is the indicator function of $\mathcal{D}_\varepsilon$.

We now aim to apply a dominated-convergence argument to let $\varepsilon\to 0^+$ in (2.34). To this end we first notice that, by definition of $\mathcal{D}_\varepsilon$, we have (a.e. on $\mathcal{C}$)

$$
\lim_{\varepsilon\to 0^+}\left(|\nabla u|^{p-2}\|D^2u\|\gamma_\varepsilon^2\cdot\mathbf{1}_{\mathcal{D}_\varepsilon}\right) = \lim_{\varepsilon\to 0^+}\left(a(\cdot)|\nabla u|^{q-2}\|D^2u\|\gamma_\varepsilon^2\cdot\mathbf{1}_{\mathcal{D}_\varepsilon}\right) = 0;
$$

moreover, reminding that $u\in C^1(\overline{\mathcal{C}})$ and $q>p$, one has

$$
\begin{gathered}
\left||\nabla u|^{p-2}\|D^2u\|\gamma_\varepsilon^2\cdot\mathbf{1}_{\mathcal{D}_\varepsilon}\right| \leq |\nabla u|^{p-2}\|D^2u\| \qquad \text{and} \\
\left|a(x)|\nabla u|^{q-2}\|D^2u\|\gamma_\varepsilon^2\cdot\mathbf{1}_{\mathcal{D}_\varepsilon}\right| \leq C\,|\nabla u|^{p-2}\|D^2u\|,
\end{gathered}
$$

where $C := \|a\|_{L^\infty(\Omega)}\cdot\|\nabla u\|^{q-p-2}_{L^\infty(\overline{\mathcal{C}})}$. Appealing once again to some results by Riey [39] (see, precisely, Corollary 1 with $\beta=\gamma=0$), we know that

$$
|\nabla u|^{p-2}\|D^2u\|\in L^1(\mathcal{C});
$$

as a consequence, by taking into account (2.33) and the properties of γ_ε, we can pass to the limit as $\varepsilon\to 0^+$ in (2.34) with the aid of Lebesgue's theorem, obtaining

$$
\int_{\mathcal{C}} f(u)\,\mathrm{d}x = 0.
$$

This is clearly in contradiction with (2.27), and the proof is complete. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2. By assumptions, the singular set Γ is contained in the hyperplane $\{x_1 = 0\}$, then the moving plane procedure can be started in the standard way, see e.g [20] for the p-laplacian case, by using the weak comparison principle in small domains, see [39, Theorem 4.3]. Indeed, for $\mathbf{a} < \lambda < \mathbf{a} + \tau$ with $\tau > 0$ small enough, the singularity does not play any role. Therefore, recalling that w_λ has a singularity at Γ and at $R_\lambda(\Gamma)$, we have that $w_\lambda \leq 0$ in Ω_λ. To proceed further we define

$$\Lambda_0 = \{\mathbf{a} < \lambda < 0 : u \leq u_t \ \text{ in } \ \Omega_t \setminus R_t(\Gamma) \ \text{ for all } t \in (\mathbf{a}, \lambda]\}$$

and $\lambda_0 = \sup \Lambda_0$, since we proved above that Λ_0 is not empty. To prove our result we have to show that $\lambda_0 = 0$. To do this we assume that $\lambda_0 < 0$ and we reach a contradiction by proving that $u \leq u_{\lambda_0+\tau}$ in $\Omega_{\lambda_0+\tau} \setminus R_{\lambda_0+\tau}(\Gamma)$ for any $0 < \tau < \bar{\tau}$ for some small $\bar{\tau} > 0$. We remark that $|\mathcal{Z}_{\lambda_0}| = 0$, see [14, 39]. Let us take $\mathcal{H}_{\lambda_0} \subset \Omega_{\lambda_0}$ be an open set such that $\mathcal{Z}_{\lambda_0} \cap \Omega_{\lambda_0} \subset \mathcal{H}_{\lambda_0} \subset\subset \Omega$. We note that the existence of such a set is guaranteed by Theorem A.2. Moreover note that, since $|\mathcal{Z}_{\lambda_0}| = 0$, we can take $\mathcal{H}_{\lambda_0}$ of arbitrarily small measure. By continuity we know that $u \leq u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$. We can exploit Theorem A.1 to get that, in any connected component of $\Omega_{\lambda_0} \setminus \mathcal{Z}_{\lambda_0}$, we have

$$u < u_{\lambda_0} \qquad \text{or} \qquad u \equiv u_{\lambda_0}.$$

The case $u \equiv u_{\lambda_0}$ in some connected component $\mathcal{C}_{\lambda_0}$ of $\Omega_{\lambda_0} \setminus \mathcal{Z}_{\lambda_0}$ is not possible, since by symmetry, it would imply the existence of a local symmetry phenomenon and consequently that $\Omega \setminus \mathcal{Z}_{\lambda_0}$ would be not connected, in spite of what we proved in Lemma 2.4. Hence we deduce that $u < u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$. Therefore, given a compact set $\mathcal{K} \subset \Omega_{\lambda_0} \setminus (R_{\lambda_0}(\Gamma) \cup \mathcal{H}_{\lambda_0})$, by uniform continuity we can ensure that $u < u_{\lambda_0+\tau}$ in $\mathcal{K}$ for any $0 < \tau < \bar{\tau}$ for some small $\bar{\tau} > 0$. Note that to do this we implicitly assume, with no loss of generality, that $R_{\lambda_0}(\Gamma)$ remains bounded away from $\mathcal{K}$.

Arguing in a similar fashion as in Lemma 2.3, we consider

$$\varphi_\varepsilon := w_{\lambda_0+\tau}^+ \psi_\varepsilon^{p+q} \cdot \mathbf{1}_{\Omega_{\lambda_0+\tau}} = \begin{cases} w_{\lambda_0+\tau}^+ \psi_\varepsilon^{p+q}, & \text{in } \Omega_{\lambda_0+\tau}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

By density arguments as above, we plug φ_ε as test function in (1.6) and (2.5) so that, subtracting, we get

$$
\begin{aligned}
& p \int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} \langle |\nabla u|^{p-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{p-2}\nabla u_{\lambda_0+\tau}, \nabla w^+_{\lambda_0+\tau}\rangle \, \psi_\varepsilon^{p+q} \, \mathrm{d}x \\
& + q \int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} a(x) \langle |\nabla u|^{q-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{q-2}\nabla u_{\lambda_0+\tau}, \nabla w^+_{\lambda_0+\tau}\rangle \, \psi_\varepsilon^{p+q} \, \mathrm{d}x \\
& + p(p+q) \int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} \langle |\nabla u|^{p-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{p-2}\nabla u_{\lambda_0+\tau}, \nabla \psi_\varepsilon\rangle \, \psi_\varepsilon^{p+q-1} w^+_{\lambda_0+\tau} \, \mathrm{d}x \\
& + q(p+q) \int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} a(x) \langle |\nabla u|^{q-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{q-2}\nabla u_{\lambda_0+\tau}, \nabla \psi_\varepsilon\rangle \, \psi_\varepsilon^{p+q-1} w^+_{\lambda_0+\tau} \, \mathrm{d}x \\
& = \int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} (f(u) - f(u_\lambda)) w^+_{\lambda_0+\tau} \psi_\varepsilon^{p+q} \, \mathrm{d}x.
\end{aligned} \tag{3.2}
$$

Now we split the set $\Omega_{\lambda_0+\tau} \setminus \mathcal{K}$ as the union of two disjoint subsets $\Omega^{(1)}_{\lambda_0+\tau}$ and $\Omega^{(2)}_{\lambda_0+\tau}$ such that $\Omega_{\lambda_0+\tau} \setminus \mathcal{K} = \Omega^{(1)}_{\lambda_0+\tau} \cup \Omega^{(2)}_{\lambda_0+\tau}$. In particular, we set

$$
\Omega^{(1)}_{\lambda_0+\tau} := \{x \in \Omega_{\lambda_0+\tau} \setminus \mathcal{K} \ : \ |\nabla u_{\lambda_0+\tau}(x)| < 2|\nabla u(x)|\} \quad \text{and}
$$

$$
\Omega^{(2)}_{\lambda_0+\tau} := \{x \in \Omega_{\lambda_0+\tau} \setminus \mathcal{K} \ : \ |\nabla u_{\lambda_0+\tau}(x)| \geq 2|\nabla u(x)|\}.
$$

From (3.2) and using (2.9), repeating verbatim arguments along the proof of Lemma 2.3, we have

$$
\begin{aligned}
& C_1 \int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} \psi_\varepsilon^{p+q} \left\{ p(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} + q a(x) \left(|\nabla u| + |\nabla u_{\lambda_0+\tau}|\right)^{q-2} \right\} |\nabla w^+_{\lambda_0+\tau}|^2 \, \mathrm{d}x \\
\leq & C_0 \Big(\rho + \kappa \rho^{\frac{p}{p-1}} + \kappa' \rho^{\frac{q}{q-1}} \Big) \times \\
& \quad \times \int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} \psi_\varepsilon^{p+q} \left\{ p(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} + q a(x) \left(|\nabla u| + |\nabla u_{\lambda_0+\tau}|\right)^{q-2} \right\} |\nabla w^+_{\lambda_0+\tau}|^2 \, \mathrm{d}x \\
& + C_0 \left\{ \frac{\kappa}{\rho} \Big(\int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} |\nabla u|^p \, \mathrm{d}x \Big)^{\frac{p-2}{p}} \Big(\int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} |\nabla \psi_\varepsilon|^p \Big)^{\frac{2}{p}} + \frac{\kappa}{\rho^p} \int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} |\nabla \psi_\varepsilon|^p \, \mathrm{d}x \right. \\
& \qquad + \frac{\kappa'}{\rho} \Big(\int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} |\nabla u|^q \, \mathrm{d}x \Big)^{\frac{q-2}{q}} \Big(\int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} |\nabla \psi_\varepsilon|^q \Big)^{\frac{2}{q}} + \frac{\kappa}{\rho^q} \int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} |\nabla \psi_\varepsilon|^q \, \mathrm{d}x \\
& \qquad \left. + L_f \int_{\Omega_{\lambda_0+\tau} \setminus \mathcal{K}} (w^+_{\lambda_0+\tau})^2 \psi_\varepsilon^{p+q} \, \mathrm{d}x. \right\},
\end{aligned}
$$

where $C_0 = C_0(p, q, \lambda_0, \tau, \|u\|_{L^\infty(\Omega)})$. Taking $\rho > 0$ sufficiently small in such a way that

$$
C_1 - C_0\rho - C_0\kappa\rho^{\frac{p}{p-1}} - C_0\kappa'\rho^{\frac{q}{q-1}} < \frac{1}{2},
$$

as we did above passing to the limit for $\varepsilon \to 0$ (thanks to Fatou's lemma) we obtain

$$\begin{aligned}
&\int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} \left[p(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} + qa(x)(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{q-2}\right] |\nabla w^+_{\lambda_0+\tau}|^2 \psi_\varepsilon^{p+q}\,\mathrm{d}x \\
&\leq 2C_0 L_f \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} (w^+_{\lambda_0+\tau})^2\,\mathrm{d}x.
\end{aligned} \tag{3.3}$$

We now observe that, since $p > 2$, we have

$$|\nabla u|^{p-2} \leq p(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} \leq p(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} + qa(x)(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{q-2}.$$

Setting $\rho := |\nabla u|^{p-2}$, we see that ρ is bounded in $\Omega_{\lambda_0+\tau}$, hence $\rho \in L^1(\Omega_{\lambda_0+\tau})$. By applying the weighted Poincaré inequality to (3.3), see [39, Theorem 4.2], we deduce that

$$\begin{aligned}
&\int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} \rho|\nabla w^+_{\lambda_0+\tau}|^2\,\mathrm{d}x \\
&\qquad\leq \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} \left[p(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} + qa(x)(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{q-2}\right] |\nabla w^+_{\lambda_0+\tau}|^2\,\mathrm{d}x \\
&\qquad\leq 2C_0 L_f \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} (w^+_{\lambda_0+\tau})^2\,\mathrm{d}x \\
&\qquad\leq 2C_0 L_f C_p(|\Omega_{\lambda_0+\tau}\setminus\mathcal{K}|) \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} \rho|\nabla w^+_{\lambda_0+\tau}|^2\,\mathrm{d}x
\end{aligned} \tag{3.4}$$

where $C_p(\cdot)$ tends to zero if the measure of the domain tends to zero. For $\bar\tau$ small and $\mathcal{K}$ large, we may assume that

$$2C_0 L_f C_p(|\Omega_{\lambda_0+\tau}\setminus\mathcal{K}|) < \frac{1}{2};$$

as a consequence by (3.4) and Lemma 2.3 we deduce that

$$\int_{\Omega_{\lambda_0+\tau}} \rho|\nabla w^+_{\lambda_0+\tau}|^2\,\mathrm{d}x = \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} \rho|\nabla w^+_{\lambda_0+\tau}|^2\,\mathrm{d}x = 0,$$

and this proves that $u \leq u_{\lambda_0+\tau}$ in $\Omega_{\lambda_0+\tau}\setminus R_{\lambda_0+\tau}(\Gamma)$ for any $0 < \tau < \bar\tau$ (provided $\bar\tau > 0$ is small enough). Such a contradiction shows that

$$\lambda_0 = 0.$$

Since the moving plane procedure can be performed in the same way but in the opposite direction, then this proves the desired symmetry result. The fact that the solution is increasing in the x_1-direction in $\{x_1 < 0\}$ is implicit in the moving plane procedure. $\square$

4. Proof of Theorem 1.3

Proof of Theorem 1.3. First of all we observe that, in the case $q > p \geq 2$, Theorem 1.3 immediately follows from Theorem 1.2 by choosing $\Gamma = \emptyset$. As a consequence, we assume from now on that $1 < p < 2 \leq q$ (the case $1 < p < q < 2$ is very similar).

By proceeding exactly as in the proof of Theorem 1.2, we easily recognize that

$$\Lambda_0 = \{\mathbf{a} < \lambda < 0 : u \leq u_t \text{ in } \Omega_t \text{ for all } t \in (\mathbf{a}, \lambda]\} \neq \emptyset,$$

and thus $\lambda_0 := \sup(\Lambda_0) \in (\mathbf{a}, 0]$. Arguing by contradiction, we suppose that

$$\lambda_0 < 0$$

and we proceed again as in the proof of Theorem 1.2: choosing an open neighborhood $\mathcal{H}_{\lambda_0}$ of $\mathcal{Z}_{\lambda_0}$ with arbitrary small Lebesgue measure, for every compact set $\mathcal{K} \subseteq \Omega_{\lambda_0} \setminus \mathcal{H}_{\lambda_0}$ we are able to find a suitable $\bar{\tau} > 0$ such that

$$u < u_{\lambda_0+\tau} \text{ in } \mathcal{K} \text{ for any } 0 < \tau < \bar{\tau}. \tag{4.1}$$

We now fix $\tau \in (0, \bar{\tau})$ and we consider the function $\varphi : \mathbb{R}^N \to \mathbb{R}$ defined as follows:

$$\varphi := w^+_{\lambda_0+\tau} \cdot \mathbf{1}_{\Omega_{\lambda_0+\tau}}.$$

Since $u \in C^1(\overline{\Omega})$, we clearly have that $\varphi \in \mathrm{Lip}(\mathbb{R}^N)$ and $\mathrm{supp}(\varphi) \subseteq \Omega_{\lambda_0+\tau}$. By a standard density argument we can use φ as a test function in (1.6) and (2.5), thus obtaining

$$\begin{aligned} & p \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} \langle |\nabla u|^{p-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{p-2}\nabla u_{\lambda_0+\tau}, \nabla w^+_{\lambda_0+\tau}\rangle \,\mathrm{d}x \\ & \qquad + q \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} a(x)\langle |\nabla u|^{q-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{q-2}\nabla u_{\lambda_0+\tau}, \nabla w^+_{\lambda_0+\tau}\rangle \,\mathrm{d}x \\ & = \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} (f(u) - f(u_\lambda)) w^+_{\lambda_0+\tau} \,\mathrm{d}x. \end{aligned} \tag{4.2}$$

Starting from (4.2), we closely follow the proof of Lemma 2.3 up to formula (2.17): since in our case we formally have $\psi_\varepsilon \equiv 1$ (and thus $\nabla\psi_\varepsilon \equiv 0$), we get

$$\begin{aligned} & \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} \left\{ p(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} + q a(x)(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{q-2} \right\} \cdot |\nabla w^+_{\lambda_0+\tau}|^2 \,\mathrm{d}x \\ & \qquad \leq C \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} (w^+_{\lambda_0+\tau})^2 \,\mathrm{d}x, \end{aligned} \tag{4.3}$$

where $C = C(p, q, \|u\|_{L^\infty(\Omega)}, f) > 0$ is a suitable constant. To proceed further, we set

$$\varrho := \left(1 + |\nabla u|^2 + |\nabla u_{\lambda_0+\tau}|^2\right)^{\frac{p-2}{2}}$$

in order to exploit the weighted Sobolev inequality from [46]. We remind that the results of [46] do apply if $\varrho \in L^1(\Omega_\lambda)$ and if there exists some $t > N/2$ such that

$$1/\varrho \in L^t(\Omega_\lambda).$$

In particular, if $\mathcal{O} \subseteq \mathbb{R}^N$ is any (non-void) open set, the space $H^1_{0,\varrho}(\mathcal{O})$ (see [14, 46]) coincides with the closure of $C_c^\infty(\mathcal{O})$ with respect to the norm

$$\|w\|_\varrho := \||\nabla w|\|_{L^2(\mathcal{O},\varrho)} := \left(\int_{\mathcal{O}} \rho\,|\nabla w|^2\,\mathrm{d}x\right)^{\frac{1}{2}},$$

and there exists a suitable constant $C_S > 0$ such that

$$\|w\|_{L^{2^*_\varrho}(\mathcal{O})} \leq C_S\,\||\nabla w|\|_{L^2(\mathcal{O},\varrho)} \qquad \text{for any } w \in H^1_{0,\varrho}(\mathcal{O}), \tag{4.4}$$

where the exponent 2^{*_ϱ} is defined via the relation

$$\frac{1}{2^*_\varrho} := \frac{1}{2}\left(1 + \frac{1}{t}\right) - \frac{1}{N}.$$

We now observe that, since $u \in C^1(\overline{\Omega})$ (and $1 < p < 2$), it is possible to find two constants $K_1, K_2 > 0$, only depending on p and on $\|u\|_{C^1(\overline{\Omega})}$, such that

$$\left(1 + |\nabla u|^2 + |\nabla u_{\lambda_0+\tau}|^2\right)^{\frac{2-p}{2}} \leq K_1 + K_2|\nabla u_{\lambda_0+\tau}|^{2-p} \qquad \text{in } \Omega_{\lambda_0+\tau}. \tag{4.5}$$

Using (4.5) and the fact both u and $u_{\lambda_0+\tau}$ are of class C^1 on $\overline{\Omega}_{\lambda_0+\tau}$ (see (2.7) and remind that, since $\Gamma = \emptyset$, one actually has $u \in C^1(\overline{\Omega})$), we deduce that

$$1/\varrho := \left(1 + |\nabla u|^2 + |\nabla u_{\lambda_0+\tau}|^2\right)^{\frac{2-p}{2}} \in L^\infty(\Omega_{\lambda_0+\tau}).$$

We are then entitled to use the Sobolev inequality (4.4) in (4.3): observing that

$$\begin{aligned} \left(|\nabla u| + |\nabla u_{\lambda_0+\tau}|\right)^{2-p} &\leq 2^{\frac{2-p}{2}}\left(|\nabla u|^2 + |\nabla u_{\lambda_0+\tau}|^2\right)^{\frac{2-p}{2}} \\ &\leq 2^{\frac{2-p}{2}}\left(1 + |\nabla u|^2 + |\nabla u_{\lambda_0+\tau}|^2\right)^{\frac{2-p}{2}}, \end{aligned} \tag{4.6}$$

by exploiting (4.6) and Hölder's inequality we obtain

$$\begin{aligned} &\int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} \varrho|\nabla w^+_{\lambda_0+\tau}|^2\,\mathrm{d}x \\ &\quad\leq 2^{\frac{2-p}{2}} \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} \left\{p(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} + qa(x)(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{q-2}\right\} \cdot |\nabla w^+_{\lambda_0+\tau}|^2\,\mathrm{d}x \\ &\quad\leq C_p \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} (w^+_{\lambda_0+\tau})^2\,\mathrm{d}x \\ &\quad\leq C_p \cdot \mathcal{H}^N(\Omega_{\lambda_0+\tau}\setminus\mathcal{K})^{\frac{1}{(\frac{2^*_\varrho}{2})'}} \left(\int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} (w^+_{\lambda_0+\tau})^{2^*_\varrho}\,\mathrm{d}x\right)^{\frac{2}{2^*_\varrho}} \\ &\quad\leq \Theta_p(|\Omega_{\lambda_0+\tau}\setminus\mathcal{K}|) \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} \varrho|\nabla w^+_{\lambda_0+\tau}|^2\,\mathrm{d}x, \end{aligned} \tag{4.7}$$

where $\Theta_p(\cdot)$ tends to zero if the measure of the domain tends to zero. For $\bar\tau$ sufficiently small and $\mathcal{K}$ sufficiently large, we may assume that

$$\Theta_p(|\Omega_{\lambda_0+\tau} \setminus \mathcal{K}|) < \frac{1}{2};$$

as a consequence, from (4.7) and (4.1) we deduce that

$$\int_{\Omega_{\lambda_0+\tau}} \rho |\nabla w^+_{\lambda_0+\tau}|^2 \,\mathrm{d}x = \int_{\Omega_{\lambda_0+\tau}\setminus\mathcal{K}} \rho |\nabla w^+_{\lambda_0+\tau}|^2 \,\mathrm{d}x = 0,$$

and this proves that $u \leq u_{\lambda_0+\tau}$ in $\Omega_{\lambda_0+\tau}$ for any $0 < \tau < \bar\tau$ (provided $\bar\tau > 0$ is sufficiently small). Such a contradiction shows that

$$\lambda_0 = 0.$$

Since the moving plane procedure can be performed in the same way but in the opposite direction, then this proves the desired symmetry result. The fact that the solution is increasing in the x_1-direction in $\{x_1 < 0\}$ is implicit in the moving plane procedure. $\square$

Remark 4.1. *As already mentioned in the Introduction, the approach adopted in the proof of Theorem 1.3* cannot be reproduced *if $\Gamma \neq \emptyset$ and $p \in (1,2)$.*

In fact, when $\Gamma \neq \emptyset$ the reflected function $u_{\lambda_0+\tau}$ is not *of class C^1 on $\overline{\Omega}_{\lambda_0+\tau}$; as a consequence, even if (4.5) remains valid (at least out of $R_{\lambda_0+\tau}(\Gamma)$, which has zero Lebesgue measure),* we cannot deduce *from this estimate that*

$$1/\varrho = \left(1 + |\nabla u|^2 + |\nabla u_{\lambda_0+\tau}|^2\right)^{\frac{2-p}{2}} \in L^\infty(\Omega_{\lambda_0+\tau}),$$

nor that $1/\varrho \in L^t(\Omega_{\lambda_0+\tau})$ for a sufficiently large t. This lack of information on the summability of $1/\varrho$ prevents us to apply the weighted Sobolev inequality (4.4) in (4.3).

On the other hand, when $p \in (1,2)$ we cannot use $\rho = |\nabla u|^{p-2}$ as a weight for the Sobolev inequality: this is due to the fact that, in general, we cannot expect that

$$\rho \leq p(|\nabla u| + |\nabla u_{\lambda_0+\tau})^{p-2} + qa(x)(|\nabla u| + |\nabla u_{\lambda_0+\tau})^{q-2}.$$

Appendix A. The strong comparison principle and the Hopf lemma.

We provide here a strong comparison principle and a Hopf-type lemma which apply to our context. Though these results seems to be very well-known as consequences of rather general results by Serrin [41], we explicitly write them here for future reference.

In what follows, if $\mathcal{V} \subseteq \mathbb{R}^N$ is open and $v \in C^1(\mathcal{V})$, we say that v satisfies

$$-\mathrm{div}\left(p|\nabla v|^{p-2}\nabla v + qa(x)|\nabla v|^{q-2}\nabla v\right) \geq [\leq]\ 0 \quad \text{in } \mathcal{V}$$

if, for every *non-negative* test function $\varphi \in C^1_c(\mathcal{V})$, one has

$$\int_{\mathcal{V}} \left(p|\nabla v|^{p-2} + qa(x)|\nabla v|^{q-2}\right)\langle \nabla v, \nabla\varphi\rangle \,\mathrm{d}x \geq [\leq]\ 0.$$

The function a is assumed to be non-negative, bounded and of class C^1 on $\mathcal{V}$.

Theorem A.1. *Let $\mathcal{V} \subseteq \mathbb{R}^N$ be a connected open set, and let $v_1, v_2 \in C^1(\mathcal{V})$ satisfy*

$$-\operatorname{div}\big(p|\nabla v_1|^{p-2}\nabla v_1 + qa(x)|\nabla v_1|^{q-2}\nabla v_1\big) \leq 0 \quad \textit{in } \mathcal{V},$$
$$-\operatorname{div}\big(p|\nabla v_2|^{p-2}\nabla v_2 + qa(x)|\nabla v_2|^{q-2}\nabla v_2\big) \geq 0 \quad \textit{in } \mathcal{V},$$

respectively. We assume that

$$|\nabla v_1| \neq 0 \ \textit{or} \ |\nabla v_2| \neq 0 \ \textit{on the whole of } \mathcal{V}. \tag{A.1}$$

Then, either $v_1 \equiv v_2$ or $v_1 < v_2$ throughout $\mathcal{V}$.

Proof. If $p \geq 2$, this result is a particular case of [41, Theorem 1]: in fact, following the notation of this cited theorem, our setting corresponds to the choices

(1) $A(x, z, \xi) = p|\xi|^{p-2}\xi + qa(x)|\xi|^{q-2}\xi$ (for $x \in \mathcal{V}, z \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$);
(2) $B(x, z, \xi) \equiv 0$.

We explicitly notice that, since $q > p \geq 2$ and $a \in C^1(\mathcal{V})$, the function $A(x, z, \xi)$ is of class C^1 on $\mathcal{V} \times \mathbb{R} \times \mathbb{R}^N$; moreover, since for every $(x, z, \xi) \in \mathcal{V} \times \mathbb{R} \times \mathbb{R}^N$ we have

$$\partial_\xi A(x, z, \xi) = p|\xi|^{p-2}\Big[\mathrm{I}_N + \frac{p-2}{|\xi|^2}(\xi_i \cdot \xi_j)_{i,j=1}^N\Big] + qa(x)|\xi|^{q-2}\Big[\mathrm{I}_N + \frac{q-2}{|\xi|^2}(\xi_i \cdot \xi_j)_{i,j=1}^N\Big],$$

it follows from assumption (A.1) that *at least one of the two matrices* $\partial_\xi A(x, v_1, \nabla v_1)$ and $\partial_\xi A(x, v_2, \nabla v_2)$ is positive definite for every $x \in \mathcal{V}$.

If, instead, $p < 2$, the function A is *no longer* differentiable at $\xi = 0$; however, we claim that estimates (8) and (10) in [41] are still satisfied in our case: more precisely, if $K \subseteq \mathcal{V}$ is compact, there exist constants $\mathbf{c}_1, \mathbf{c}_2 > 0$ such that, for any $x \in K$, one has

$$\big|A(x, v_2, \nabla v_2) - A(x, v_1, \nabla v_1)\big| \leq \mathbf{c}_1|\nabla v_2 - \nabla v_1|; \tag{A.2}$$

$$\langle A(x, v_2, \nabla v_2) - A(x, v_1, \nabla v_1), \nabla v_2 - \nabla v_1\rangle \geq \mathbf{c}_2|\nabla v_2 - \nabla v_1|^2. \tag{A.3}$$

In fact, let us assume (to fix ideas) that $q \geq 2$; the case $1 < q < 2$ can be faced analogously. Using the explicit expression of A and the second estimate in (2.9), we have

$$\begin{aligned} &\big|A(x, v_2, \nabla v_2) - A(x, v_1, \nabla v_1)\big| \\ &\qquad \text{(since } a \text{ is bounded on } \mathcal{V}) \\ &\qquad \leq C_2\Big(p(|\nabla v_1| + |\nabla v_2|)^{p-2} + q\|a\|_{L^\infty(\mathcal{V})}(|\nabla v_1| + |\nabla v_2|)^{q-2}\Big)|\nabla v_2 - \nabla v_1|. \end{aligned} \tag{A.4}$$

Moreover, since $v_1, v_2 \in C^1(\mathcal{V})$, $K \subseteq \mathcal{V}$ is compact and $q \geq 2$, one has

$$(|\nabla v_1| + |\nabla v_2|)^{q-2} \leq \big(\max_K |\nabla v_1| + \max_K |\nabla v_2|\big)^{q-2} =: \kappa^{(1)}_{v_1, v_2}. \tag{A.5}$$

Finally, since $p \in (1, 2)$, by crucially exploiting assumption (A.1) we get

$$(|\nabla v_1| + |\nabla v_2|)^{p-2} \leq \big(\inf_K |\nabla v_1| + \inf_K |\nabla v_2|\big)^{p-2} =: \kappa^{(2)}_{v_1, v_2} < \infty. \tag{A.6}$$

Gathering together (A.4), (A.5) and (A.6), we then obtain (A.2).

As regards (A.3), we proceed essentially in the same way: using the explicit expression of the function A and the first estimate in (2.9), we have

$$\begin{aligned}&\langle A(x,v_2,\nabla v_2)-A(x,v_1,\nabla v_1),\nabla v_2-\nabla v_1\rangle\\ &\quad\geq C_1\Big(p(|\nabla v_1|+|\nabla v_2|)^{p-2}+qa(x)(|\nabla v_1|+|\nabla v_2|)^{q-2}|\Big)|\nabla v_2-\nabla v_1|^2\\ &\quad\text{(reminding that } a\geq 0 \text{ on } \mathcal{V})\\ &\quad\geq p\,C_1(|\nabla v_1|+|\nabla v_2|)^{p-2}|\nabla v_2-\nabla v_1|^2.\end{aligned}\tag{A.7}$$

On the other hand, since $p\in(1,2)$, again by exploiting assumption (A.1) we get

$$(|\nabla v_1|+|\nabla v_2|)^{p-2}\geq\big(\max_K|\nabla v_1|+\max_K|\nabla v_2|)^{p-2}=\kappa^{(1)}_{v_1,v_2}\in(0,\infty).\tag{A.8}$$

Gathering together (A.7) and (A.8), we then obtain (A.3).

With (A.2)-(A.3) at hand, we can exploit the celebrated Harnack inequality established by Trudinger [45] and conclude exactly as in the proof of [41, Theorem 1]. □

Theorem A.2. *Let $\mathcal{V}\subseteq\mathbb{R}^N$ be a fixed open set, and let $v\in C^1(\overline{\mathcal{V}})$ satisfy*

$$-\mathrm{div}\,\big(p|\nabla v|^{p-2}\nabla v+qa(x)|\nabla u|^{q-2}\nabla v\big)\geq 0\quad\text{in } \mathcal{V}.$$

We assume that $v<0$ on $\mathcal{V}$ and that there exists a point $y\in\partial\mathcal{V}$ such that $v(y)=0$. Then, if $\mathcal{V}$ satisfies the interior cone condition *at y and $|\nabla v|\neq 0$ on $\mathcal{V}$, one has*

$$\nabla v(y)\neq 0.$$

Proof. First of all, by [41, Theorem 3] (applied here with $u\equiv 0$) we know that the zero of v ad y is of finite order, say $m\in\mathbb{N}\cup\{0\}$. In fact, following the notations in this cited theorem, our context corresponds to the choices

(1) $A(x,z,\xi)=p|\xi|^{p-2}\xi+qa(x)|\xi|^{q-2}\xi$ (for $x\in\mathcal{V}$, $z\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$);
(2) $B(x,z,\xi)\equiv 0$.

We explicitly notice that, when $p\geq 2$, the function A is continuously differentiable on the whole of $\mathcal{V}\times\mathbb{R}\times\mathbb{R}^N$; when $p\in(1,2)$, instead, the function A is no longer differentiable ad $\xi=0$ but we have at our disposal the estimates (A.2)-(A.3).

To conclude the demonstration it suffices to observe that, since in our case we have $B(x,z,\xi)\equiv 0$, a closer inspection to the proof of [41, Theorem 3] shows that

$$m=0;$$

from this, we immediately deduce that $\nabla v(y)\neq 0$, as desired. □

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