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TOWARDS A BREZIS-OSWALD-TYPE RESULT FOR FRACTIONAL PROBLEMS WITH ROBIN BOUNDARY CONDITIONS

DIMITRI MUGNAI, ANDREA PINAMONTI, AND EUGENIO VECCHI

Abstract. We consider a boundary value problem driven by the $p-$fractional Laplacian with nonlocal Robin boundary conditions and we provide necessary and sufficient conditions which ensure the existence of a unique positive (weak) solution. The results proved in this paper can be considered a first step towards a complete generalization of the classical result by Brezis and Oswald [6] to the nonlocal setting.

1. Introduction

The celebrated result by Brezis and Oswald [6] states that, given $\Omega \subset \mathbb{R}^n$ open and bounded with smooth boundary and $f : \Omega \times [0, \infty) \to \mathbb{R}$ a "nice function" (see below for the precise definition) such that the map $u \mapsto \frac{f(x,u)}{u}$ is decreasing in $(0, \infty)$, the problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u \geq 0, u \neq 0, & \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

has at most one solution, and such a solution exists if and only if

$$\lambda_1(-\Delta - \tilde{a}_0(x)) < 0 \tag{1.2}$$

and

$$\lambda_1(-\Delta - \tilde{a}_\infty(x)) > 0, \tag{1.3}$$

where $\lambda_1(-\Delta - a(x))$ denotes the first eigenvalue of $-\Delta - a(x)$ with zero Dirichlet condition and

$$\tilde{a}_0(x) := \lim_{u \downarrow 0} \frac{f(x,u)}{u} \quad \text{and} \quad \tilde{a}_\infty(x) := \lim_{u \uparrow \infty} \frac{f(x,u)}{u}. \tag{1.4}$$

After the paper of Brezis and Oswald [6] there has been a wide interest in providing necessary and/or sufficient conditions for the solvability of more general problems, aiming at considering different operators and boundary conditions. In particular, we refer to [3, 9, 15–17, 21, 26] for the case of semilinear Dirichlet problems, and to [2, 5, 7, 13, 20, 22] for an extension to quasilinear problems with Dirichlet boundary conditions, even in the nonlocal case. Finally, we want to mention the paper [18] where the authors considered the case of Robin boundary conditions for a quite general class of quasilinear operators. Having in mind the last mentioned paper, we wish now to extend the Brezis-Oswald result to a fractional setting in presence of nonlocal Robin boundary conditions. To the best of our knowledge, this is one of the first contributions aiming at considering such kind of boundary conditions in the nonlocal framework.

Let us now properly introduce the problem we are interested in. Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with smooth boundary. We consider

$$\begin{cases} (-\Delta)_p^s u = f(x,u) & \text{in } \Omega \\ u \geq 0, u \not\equiv 0, \\ \mathcal{N}_{s,p}u + \beta(x)|u|^{p-2}u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases} \tag{1.5}$$

where

$$(-\Delta)_p^s u(x) = c_{n,s,p} \text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\, dy$$

denotes the standard fractional $p-$Laplacian, while

$$\mathcal{N}_{s,p}u(x) = c_{n,s,p} \int_{\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\, dy$$

is the nonlocal normal derivative associated to $(-\Delta)_p^s$, see [1, 4, 8, 10, 24, 25] and [14] for its introduction in the case $p = 2$ and $c_{n,s,p}$ is a suitable positive normalization constant only depending on n, s and p. Finally, β is a nonnegative given function. We would like to point out that the Neumann operator $\mathcal{N}_{s,2}u$ recovers the classical Neumann condition as a limit case, and has a clear probabilistic and variational interpretation as well, see [14] for the details.

In order to extend the previous result to our setting, we encounter several difficulties, which make our studies not trivial. First, the fact that we consider a nonlinear nonlocal operator in presence of nonlinear and nonlocal boundary conditions of Robin type requires the introduction of an appropriate function space to work with. Second, and probably more complicated, we have to prove a maximum principle following a rather uncommon path to prove positivity of weak solutions of our problem. We must mention that this technique has been recently used in [10] in a different context, and it is heavily based on previous results contained in [11]. Third, we completely miss a regularity theory for weak solutions. For this reason, we can prove the uniqueness and the existence part, the necessity of the analogue of (1.2), but not the validity of the necessity of the analogue of (1.3). However, if we knew that any solution to problem (1.5) were bounded, we would have the complete analogue of the classical Brezis-Oswald result, see Theorem 1.2 for the precise statement of our main results.

Let us start making precise our framework. Let $\Omega \subset \mathbb{R}^n$ be an open, connected and bounded set with smooth boundary $\partial\Omega$. We further assume that $\mathbb{R}^n \setminus \partial\Omega$ is made of a finite number of connected components. In other words, there exists a positive integer M such that

$$\mathbb{R}^n \setminus \partial\Omega = \bigcup_{i=1}^{M} C_i,$$

with $C_i \subset \mathbb{R}^n$ open and connected for every $i = 1, \ldots, M$. We notice that there exists $i_0 \in \{1, \ldots, M\}$ such that $\Omega = C_{i_0}$. We now fix the standing assumptions on the reaction term f and on β:

- (f1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.
- (f2) For all $t \geq 0$, $f(\cdot, t) \in L^\infty(\Omega)$ and there exists $c_1 > 0$ such that
$$|f(x,t)| \leq c_1(1 + t^{p-1}) \qquad \text{for a.e. } x \in \Omega, \text{and all } t \geq 0.$$
- (f3) For a.e. $x \in \Omega$, the function $t \mapsto \frac{f(x,t)}{t^{p-1}}$ is strictly decreasing on $(0, \infty)$.
- (β1) $\beta \in L^\infty(\mathbb{R}^n \setminus \Omega)$ and $\beta \geq 0$.

Aiming at providing necessary and sufficient conditions which ensure the existence of a unique positive (weak) solution of (1.5), we need to consider the eigenvalue problem for $(-\Delta)^s_p$ plus an indefinite potential $\xi \in L^\infty(\Omega)$ in presence of *fractional* Robin boundary conditions, namely

$$\begin{cases}(-\Delta)^s_p u + \xi(x)|u|^{p-2}u = \lambda |u|^{p-2}u, & \text{in } \Omega,\\ \mathcal{N}_{s,p}u + \beta(x)|u|^{p-2}u = 0, & \text{in } \mathbb{R}^n \setminus \overline{\Omega}.\end{cases} \tag{1.6}$$

The necessity of considering indefinite (and possibly unbounded!) weights is a consequence of the fact that, by (1.9), we have

$$-\infty < a_0(x) \le +\infty \quad \text{and} \quad -\infty \le a_\infty(x) < +\infty \text{ for a.e. } x \in \Omega,$$

where the nonlocal quasilinear analogous of the functions defined in (1.4) are

$$a_0(x) := \lim_{u \downarrow 0} \frac{f(x,u)}{u^{p-1}} \quad \text{and} \quad a_\infty(x) := \lim_{u \uparrow \infty} \frac{f(x,u)}{u^{p-1}}. \tag{1.7}$$

Of course, in the extremal cases, concerning problem (1.1), we would have

$$\lambda_1(-\Delta - \tilde{a}_0(x)) = -\infty \quad \text{and} \quad \lambda_1(-\Delta - \tilde{a}_\infty(x)) = +\infty.$$

We stress the following immediate consequences of (f1), (f2) and (f3), (see also [18]):

$$\frac{f(x,t)}{t^{p-1}} \ge f(x,1) \ge -\|f(\cdot,1)\|_{L^\infty(\Omega)} =: -c_f, \tag{1.8}$$

for a.e. $x \in \Omega$ and for every $t \in (0,1]$ and

$$a_0(x) \ge \frac{f(x,t)}{t^{p-1}} \ge a_\infty(x) \tag{1.9}$$

for a.e. $x \in \Omega$ and for every $t > 0$ and

$$f(x,0) \ge 0 \tag{1.10}$$

for a.e. $x \in \Omega$. To simplify the notation, for any couple of functions u, v, we set

$$\mathcal{H}_{s,p}(u,v) := \frac{c_{n,s,p}}{2} \iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp}}\, dx\, dy.$$

With this at hand, we define weak solutions of (1.5) and (1.6) (see [14] and [25]): we say that $u \in X^{s,p}_\beta$ is a weak solution of (1.5) if

$$\mathcal{H}_{s,p}(u,v) = \int_\Omega f(x,u(x))v(x)\, dx - \int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u(x)|^{p-2}u(x)v(x)\, dx, \tag{1.11}$$

for every $v \in X^{s,p}_\beta$. Analogously, we say that $u \in X^{s,p}_\beta$ is a weak solution of (1.6) if

(1.12)

$$\mathcal{H}_{s,p}(u,v) + \int_\Omega \xi(x)|u(x)|^{p-2}u(x)v(x)\, dx = \lambda \int_\Omega |u(x)|^{p-2}u(x)v(x)\, dx - \int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u(x)|^{p-2}u(x)\, dx,$$

for every $v \in X^{s,p}_\beta$.

Remark 1.1. *Using (1.12), (f3) and the positivity of β, it is easy to prove that if $f(x,s) \le 0$ for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$, then (1.6) admits only the trivial solution. Hence, it is easy to deduce that $f(x,s) > 0$ for a.e. $x \in \Omega$ and all $s \in (0,\kappa)$ for some $\kappa \in (0,+\infty]$.*

We refer to Section 2 for the definition and main properties of the function space $X^{s,p}_\beta$. The following is our main result

Theorem 1.2. *Let (f1), (f2), (f3) and* (β_1) *hold. Then problem* (1.5) *admits at most one solution. Moreover, a solution exists if*

$$\lambda_1(-a_0, \beta, s, p) < 0 < \lambda_1(-a_\infty, \beta, s, p), \tag{1.13}$$

and if a solution exists, then $\lambda_1(-a_0, \beta, s, p) < 0$*. If* $p = 2$ *and any solution is bounded, then a solution exists if and only if* (1.13) *holds.*

As already pointed out, the lack of a suitable regularity result for solutions of problem (1.5) does not allow to provide the precise analogue of the Brezis-Oswald result.

The paper is organized as follows: in Section 2 we introduce the appropriate functional space where we look for solution of (1.5) and we prove a few important properties needed later on, like a sort of strong maximum principle for weak solutions. In Section 3 we provide a few results concerning the eigenvalue problem (1.6). Finally in Section 4 we prove the existence of solutions of (1.5).

2. Functional setting and Maximum principle

From now on, we will work in the following *fractional Sobolev space*, suitably modeled to deal with fractional Robin boundary conditions. Precisely, given $\beta \in L^\infty(\mathbb{R}^n \setminus \Omega)$, we define the function space

$$X^{s,p}_\beta := \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ measurable } : \|u\|_{X^{s,p}_\beta} < +\infty \right\},$$

where

$$\begin{aligned} \|u\|^p_{X^{s,p}_\beta} &:= \int_\Omega |u|^p dx + \iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy + \int_{\mathbb{R}^n\setminus\Omega} |\beta||u|^p dx \\ &=: \|u\|_{L^p(\Omega)} + [u]^p_{s,p} + \|u\|^p_{L^p(\beta;\mathbb{R}^n\setminus\Omega)}. \end{aligned}$$

We observe that

$$[u]_{s,p} := \left(\int_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p},$$

is strictly related to the Gagliardo seminorm

$$[u] = \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p}.$$

We stress that $X^{s,p}_\beta$ is a real vector space, and we further note that for $\beta \equiv 0$, this space coincides with the one introduced in [10, 14] for $g = 0$.

We want now to collect several technical results needed in the upcoming sections.

Lemma 2.1. *The space* $X^{s,p}_\beta$ *is a reflexive Banach space for every* $1 < p < \infty$.

Proof. We set

$$Y := L^p(\Omega) \times L^p(\mathbb{R}^n \setminus \Omega) \times L^p(\mathbb{R}^{2n} \setminus (\mathbf{C}\Omega)^2).$$

We endow Y with the norm

$$\begin{aligned} \|v\|^p_Y &:= \|(v_1(x), v_2(x), v_3(x,y))\|^p_Y \\ &= \int_\Omega |v_1(x)|^p\, dx + \int_{\mathbb{R}^n\setminus\Omega} |v_2(x)|^p\, dx + \iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} |v_3(x,y)|^p\, dx\, dy. \end{aligned}$$

We notice that $(Y, \|\cdot\|_Y)$ is a reflexive Banach space. We now consider the map $T : X^{s,p}_\beta \to Y$ defined as

$$T(u) := \left(u, \beta^{1/p} u, \frac{|u(x) - u(y)|}{|x-y|^{s+n/p}} \right).$$

By construction, we have that

$$\|T(u)\|^p_Y = \|u\|^p_{X^{s,p}_\beta},$$

and hence T is an isometry from $X^{s,p}_\beta$ to the reflexive space Y. This concludes the proof. □

Lemma 2.2. *The embedding $X^{s,p}_\beta \hookrightarrow L^q(\Omega)$ is compact for every $q \in [1, p^\sharp_s)$, where*

$$p^\sharp_s := \begin{cases} \dfrac{pn}{n-ps} & \text{if } n < ps, \\ \infty & \text{if } n \geq ps. \end{cases}$$

Proof. From the definition of $\|\cdot\|_{X^{s,p}_\beta}$, we easily get that

$$\|u\|^p_{L^p(\Omega)} + [u]^p_{s,p} \leq \|u\|^p_{X^{s,p}_\beta},$$

for every $u \in X^{s,p}_\beta$. Now, since $\Omega \times \Omega \subset \mathbb{R}^{2n} \setminus (\mathbf{C}\Omega)^2$, we also have that

$$[u]^p \leq [u]^p_{s,p}.$$

Since the classical fractional Sobolev space $W^{s,p}(\Omega)$ compactly embeds in $L^q(\Omega)$ (see [12, Theorem 7.1]), the proof is complete. □

For any $u \in X^{s,p}_\beta$ and $\varepsilon > 0$, define the truncation

$$u_\varepsilon := \min\left\{u, \tfrac{1}{\varepsilon}\right\}. \tag{2.1}$$

We prove a technical Lemma which will be very useful hereinafter.

Lemma 2.3. *Let $u_1, u_2 \in X^{s,p}_\beta$, $u_1, u_2 \geq 0$ and set*

$$v := \frac{u^p_{2,\varepsilon}}{(u_1 + \varepsilon)^{p-1}} - u_{1,\varepsilon},$$

where $u_{1,\varepsilon}, u_{2,\varepsilon}$ are as in (2.1). Then

$$v \in X^{s,p}_\beta.$$

Proof. Since the function $\mathbb{R} \ni t \mapsto \min\{|t|, 1/\varepsilon\}$ is $1-$Lipschtz we have

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq |u(x) - u(y)|, \tag{2.2}$$

which immediately implies that $u_{1,\varepsilon} \in X^{s,p}_\beta$. By the Lagrange theorem, it is readily seen that for any $a, b \geq 0$ and for every $r \geq 0$, we have

$$|a^r - b^r| \leq r|a-b| \max\{a^{r-1}, b^{r-1}\}. \tag{2.3}$$

Then, since $\varepsilon^{p-1} \leq (u_{1,\varepsilon}+\varepsilon)^{p-1}$ and $u_{2,\varepsilon} \leq \frac{1}{\varepsilon}$, by (2.3) we have

$$
\begin{aligned}
&\left|\frac{u_{2,\varepsilon}^p(x)}{(u_1(x)+\varepsilon)^{p-1}} - \frac{u_{2,\varepsilon}^p(y)}{(u_1(y)+\varepsilon)^{p-1}}\right| \\
&= \left|\frac{u_{2,\varepsilon}^p(x)-u_{2,\varepsilon}^p(y)}{(u_1(x)+\varepsilon)^{p-1}} + u_{2,\varepsilon}^p(y)\frac{(u_1(y)+\varepsilon)^{p-1}-(u_1(x)+\varepsilon)^{p-1}}{(u_1(x)+\varepsilon)^{p-1}(u_1(y)+\varepsilon)^{p-1}}\right| \\
&\leq \frac{p}{\varepsilon^{2p-2}}|u_{2,\varepsilon}(x)-u_{2,\varepsilon}(y)| + \frac{1}{\varepsilon^p}\left|\frac{(u_1(y)+\varepsilon)^{p-1}-(u_1(x)+\varepsilon)^{p-1}}{(u_1(x)+\varepsilon)^{p-1}(u_1(y)+\varepsilon)^{p-1}}\right| \\
&\leq \frac{p}{\varepsilon^{2p-2}}|u_{2,\varepsilon}(x)-u_{2,\varepsilon}(y)| \\
&\quad + \frac{p-1}{\varepsilon^p}\max\left\{(u_1(x)+\varepsilon)^{p-2},(u_1(y)+\varepsilon)^{p-2}\right\}\frac{|u_1(x)-u_1(y)|}{(u_1(x)+\varepsilon)^{p-1}(u_1(y)+\varepsilon)^{p-1}}.
\end{aligned}
$$

By (2.2), for every $p>1$ we can estimate the last quantity by

$$\frac{p}{\varepsilon^{2p-2}}|u_2(x)-u_2(y)| + \frac{p-1}{\varepsilon^{2p}}|u_1(x)-u_1(y)|.$$

Thus the Gagliardo seminorm of v is finite. Moreover,

$$\frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}} = \frac{u_{2,\varepsilon}^{p-1}}{(u_1+\varepsilon)^{p-1}}u_{2,\varepsilon} \leq \frac{1}{\varepsilon^{2p-2}}u_2,$$

hence,

$$\int_\Omega |v|^p \leq 2^{p-1}\left(\int_\Omega \left|\frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}}\right|^p + \int_\Omega |u_{1,\varepsilon}|^p\right) \leq C(p,\varepsilon)(\|u_2\|_{L^p(\Omega)} + \|u_1\|_{L^p(\Omega)}) < +\infty,$$

where $C(p,\varepsilon)>0$. A similar argument works for $\|v\|_{L^p(\beta;\mathbb{R}^n\setminus\Omega)}$. This finally gives that $v \in X_\beta^s$. $\square$

The rest of the section is devoted to the proof of a strong maximum principle for weak solutions of either (1.5) or (1.6). The main idea is taken from [10, Lemma 2.2], which in turn heavily relies on [11, Lemma 1.3].

Lemma 2.4. *Let $u \in X_\beta^{s,p}$ be a weak solution of (1.5) or (1.6). Assume that $u \geq 0$ in $\mathbb{R}^n$. If $B_R(x_0) \subset \mathbb{R}^n \setminus \partial\Omega$, then for every $B_r(x_0) \subset B_{R/2}(x_0)$ and for every $\delta \in (0,1)$, there exists a positive constant $C>0$ independent of δ, such that*

$$\int_{B_r(x_0)}\int_A \frac{1}{|x-y|^{n+sp}}\left|\ln\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p dxdy \leq Cr^{n-sp}(1+r^{sp}),$$

where

$$A := \begin{cases} B_r(x_0) & \text{if } B_R(x_0)\subset\Omega, \\ \Omega & \text{if } B_R(x_0)\subset\mathbb{R}^n\setminus\overline{\Omega}. \end{cases}$$

Proof. We start with u solution of (1.5), the remaining case can be treated in a similar way and we omit the proof. Let $r \in \left(0,\frac{R}{2}\right)$ and let $\delta\in(0,1)$. We further consider a smooth function $\phi \in C_0^\infty(B_{3r/2})$ such that

(i) $0 \leq \phi \leq 1$;
(ii) $\phi \equiv 1$ in $B_r(x_0)$;
(iii) $|D\phi| < Cr^{-1}$ in $B_{3r/2}(x_0) \subset B_R(x_0)$.

Let us now define the function

$$v := (u+\delta)^{1-p}\phi^p.$$

We stress that $v \in X^{s,p}_\beta$, hence it can be used as a test function in (1.11) finding

$$\begin{aligned}
\mathcal{H}_{s,p}(u,v) &= \int_\Omega f(x,u(x))v(x)\,dx - \int_{\mathbb{R}^n\setminus\Omega} \beta(x)u(x)^{p-1}v(x)\,dx \\
&= \int_\Omega f(x,u(x))\frac{\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx - \int_{\mathbb{R}^n\setminus\Omega} \beta(x)\frac{u(x)^{p-1}\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx.
\end{aligned} \tag{2.4}$$

On the other hand, in the proof of [11, Lemma 1.3] it has been showed that there exists a positive constants $C>0$, independent of δ, such that for every u the following inequality holds:

$$\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+ps}} \le -C\left(\frac{1}{|x-y|^{n+sp}}\left|\ln\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p\phi(y)^p\right)+\frac{1}{C}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+ps}}. \tag{2.5}$$

Now, in order to complete the proof, we need to consider a couple of different cases:

- **Case 1**: $B_R(x_0)\subset\Omega$;
- **Case 2**: $B_R(x_0)\subset\mathbb{R}^n\setminus\overline{\Omega}$.

First, notice that for any $r>0$ we have

$$\iint_{B_r(x_0)\times B_r(x_0)} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+ps}}dxdy \le cr^{n-sp}$$

for some $c>0$, see the end of the proof of [11, Lemma 1.3].

Let us start with **Case 1**. From (2.5), by integrating in $\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2$ and by recalling the properties of ϕ and (2.4), we get that

$$\begin{aligned}
\iint_{B_r(x_0)\times B_r(x_0)} \frac{1}{|x-y|^{n+sp}}\left|\ln\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p\,dxdy &\le Cr^{n-sp} - \mathcal{H}_{s,p}\left(u,\tfrac{\phi^p}{(u+\delta)^{p-1}}\right) \\
&= Cr^{n-sp} - \int_\Omega f(x,u(x))\frac{\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx + \int_{\mathbb{R}^n\setminus\Omega}\beta(x)\frac{u(x)^{p-1}\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx \\
&=: Cr^{n-sp} + (I) + (II)
\end{aligned}$$

for a, possibly different, $C>0$. We proceed by estimating (I) and (II) with quantities independent of δ. We start noticing that, since $B_{3r/2}(x_0)\subset B_R(x_0)\subset\Omega$, then

$$(II)=0.$$

We now focus on (I). By (1.10), $f(x,0) \geq 0$. Hence

$$\begin{aligned}
(I) &= \int_{\Omega\cap\{u=0\}} \frac{-f(x,0)\phi(x)^p}{\delta^{p-1}}\,dx + \int_{\Omega\cap\{0<u<1\}} \frac{-f(x,u)\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx + \int_{\Omega\cap\{u\geq 1\}} \frac{-f(x,u)\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx \\
&\leq \int_{\Omega\cap\{0<u<1\}} \frac{-f(x,u)\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx + \int_{\Omega\cap\{u\geq 1\}} \frac{|f(x,u)|\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx \\
&\leq \int_{\Omega\cap\{0<u<1\}} c_f \frac{u(x)^{p-1}}{(u(x)+\delta)^{p-1}}\phi(x)^p\,dx + c_1 \int_{\Omega\cap\{u\geq 1\}} \frac{(1+u(x)^{p-1})\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx \\
&\leq c_f|\Omega\cap\{0<u<1\}\cap \operatorname{supp}(\phi)| + 2c_1 \int_{\Omega\cap\{u\geq 1\}} \frac{u(x)^{p-1}\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx \\
&\leq c_f|\Omega\cap\{0<u<1\}\cap \operatorname{supp}(\phi)| + 2c_1|\Omega\cap\{u\geq 1\}\cap\operatorname{supp}(\phi)| \\
&\leq 2\max\{c_f, 2c_1\}|\Omega\cap\operatorname{supp}(\phi)| \\
&\leq 2\max\{c_f, 2c_1\}|B_{3r/2}(x_0)|,
\end{aligned}$$

where the constant $c_f > 0$ has been defined in (1.8). We then get

$$\begin{aligned}
\iint_{B_r(x_0)\times B_r(x_0)} &\frac{1}{|x-y|^{n+sp}}\left|\ln\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p dxdy \\
&\leq Cr^{n-sp} + 2\max\{c_f, 2c_1\}|B_{3r/2}(x_0)|.
\end{aligned}$$

This completes the proof when **Case 1** holds.

Let us now consider **Case 2**. Due to the assumptions, we have that $B_{3r/2}(x_0)\cap\Omega = \emptyset$. Therefore,

$$\begin{aligned}
\mathcal{H}_{s,p}\left(u, \frac{\phi^p}{(u+\delta)^{p-1}}\right) &= -\int_{\mathbb{R}^n\setminus\Omega} \beta(x)\frac{u(x)^{p-1}\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx \\
&= -\int_{(\mathbb{R}^n\setminus\Omega)\cap B_{3r/2}(x_0)} \beta(x)\frac{u(x)^{p-1}\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx
\end{aligned}$$

Therefore, from (2.5), since $\phi \equiv 1$ on $B_r(x_0)$, we have

$$\begin{aligned}
&\iint_{B_r(x_0)\times\Omega} \frac{1}{|x-y|^{n+sp}}\left|\ln\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p dxdy \\
&\qquad\leq Cr^{n-sp} + \int_{(\mathbb{R}^n\setminus\Omega)\cap B_{3r/2}(x_0)} \beta(x)\frac{u(x)^{p-1}\phi(x)^p}{(u(x)+\delta)^{p-1}}\,dx + c\iint_{B_{3r/2}(x_0)\times\Omega} \frac{|\phi(x)|^p}{|x-y|^{n+sp}}\,dxdy \\
&\qquad\leq Cr^{n-sp} + \|\beta\|_{L^\infty(\mathbb{R}^n\setminus\Omega)}|B_{3r/2}(x_0)| + C\frac{r^n}{\operatorname{dist}(B_R(x_0),\Omega)^{sp}}.
\end{aligned}$$

This concludes the proof.

□

Lemma 2.5. *Let $u \in X^{s,p}_\beta$ be a weak solution of either (1.5) or (1.6). Assume further that $u \geq 0$ in $\mathbb{R}^n$ and that $u \not\equiv 0$ in C_i for every $i = 1,\dots,M$. Then*

$$u > 0, \quad \textit{for a.e. } x \textit{ in } \mathbb{R}^n.$$

Proof. Let us suppose that u vanishes on a set of full measure, namely

$$|Z| = |\{x\in\mathbb{R}^n : u(x) = 0\}| > 0.$$

By assumption there exists a point $x_0 \in \mathbb{R}^n$ for which the following holds:

- there exists a radius $R > 0$ such that $B_R(x_0) \subset \mathbb{R}^n \setminus \partial\Omega$;
- there exists $r \in (0, R/2)$ such that $|B_r(x_0) \cap Z| > 0$ with

$$u \not\equiv 0 \quad \text{in } B_r(x_0). \tag{2.6}$$

Now, we follow the proof of [10, Theorem 2.3]. To this aim, let us define the function

$$F_\delta(x) := \ln\left(1 + \frac{u(x)}{\delta}\right), \quad \text{for any } \delta > 0,\ x \in \mathbb{R}^n.$$

Recalling that, by definition of the set Z, $F_\delta(y) = 0$ whenever $y \in Z$, we have that

$$|F_\delta(x)|^p = |F_\delta(x) - F_\delta(y)|^p \leq \frac{(2r)^{n+sp}}{|x-y|^{n+sp}} \left|\ln\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p,$$

for every $x \in B_r(x_0)$ and $y \in B_r(x_0) \cap Z$, while

$$|F_\delta(x)|^p = |F_\delta(x) - F_\delta(y)|^p \leq \frac{(r + |x-x_0|)^{n+sp}}{|x-y|^{n+sp}} \left|\ln\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p,$$

for every $x \in \mathbb{R}^n \setminus B_r(x_0)$ and $y \in B_r(x_0) \cap Z$.

Then, integrating the previous inequalities (in y) over $B_r(x_0) \cap Z$, we get

$$|F_\delta(x)|^p \leq \frac{(2r)^{n+sp}}{|Z \cap B_r(x_0)|} \int_{B_r(x_0)} \frac{1}{|x-y|^{n+sp}} \left|\ln\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p dy,$$

if $x \in B_r(x_0)$ and

$$|F_\delta(x)|^p \leq \frac{(r + |x-x_0|)^{n+sp}}{|Z \cap B_r(x_0)|} \int_{B_r(x_0)} \frac{1}{|x-y|^{n+sp}} \left|\ln\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p dy$$

if $x \in \mathbb{R}^n \setminus B_r(x_0)$. Therefore, integrating now (in x) over the set

$$A := \begin{cases} B_r(x_0) & \text{if } B_R(x_0) \subset \Omega \\ \Omega & \text{if } B_R(x_0) \subset \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

yields

$$\int_A |F_\delta(x)|^p \, dx \leq \frac{(2r)^{n+sp}}{|Z \cap B_r(x_0)|} \iint_{B_r(x_0) \times A} \frac{1}{|x-y|^{n+sp}} \left|\ln\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p dydx$$

in the first case, and

$$\int_A |F_\delta(x)|^p \, dx \leq \frac{(2r + dist(\Omega, B_r(x_0)) + diam(\Omega))^{n+sp}}{|Z \cap B_r(x_0)|} \iint_{B_r(x_0) \times A} \frac{1}{|x-y|^{n+sp}} \left|\ln\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p dydx$$

in the latter. In any case, we can now apply Lemma 2.4 which provides the existence of a positive constant $C_r > 0$ (independent of δ but depending on r) such that

$$\int_A |F_\delta(x)|^p \, dx \leq \frac{C_r}{|Z \cap B_r(x_0)|}.$$

Letting $\delta \to 0^+$ we finally obtain that

$$u \equiv 0 \quad \text{a.e. in } A.$$

If $A = B_r(x_0)$, this is in contradiction with (2.6), while the case $A = \Omega$ cannot take place, since Ω is one of the C_i's. Therefore $u > 0$ a.e. in $\mathbb{R}^n$. □

We are now ready to state a strong maximum principle for weak solutions of (1.5) or (1.6).

Theorem 2.6. *Assume* ($\beta 1$) *and let* $u \in X^{s,p}_\beta$ *be a weak solution of either* (1.5) *or* (1.6). *Assume further that* $u \geq 0$ *in* $\mathbb{R}^n$. *Then either*

$$u \equiv 0 \quad \text{or } u > 0 \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Proof. If $u \equiv 0$, there is nothing to prove. Thus, let us assume that $u \not\equiv 0$. Due to Lemma 2.5, it is enough to show that $u \not\equiv 0$ on *every* C_i. To this aim, let us assume by contradiction that there exists $\bar{i} \in \mathbb{N}$ such that $u \equiv 0$ on $C_{\bar{i}}$. Since $C_{\bar{i}}$ is open (and connected), we can take $\psi \in C^\infty_0(C_{\bar{i}})$. In particular, we notice that $\psi \in X^{s,p}_\beta$ and hence we can use it as a test function in (1.11) (or analogously in (1.12)). By doing so, we get

$$\mathcal{H}_{s,p}(u,\psi) = \int_\Omega f(x,u(x))\psi(x)\,dx - \int_{\mathbb{R}^n\setminus\Omega} \beta(x)u(x)^{p-1}\psi(x)\,dx, \quad \text{for all } \psi \in C^\infty_0(C_{\bar{i}}).$$

Therefore, if $\bar{i} \neq i_0$, we get

$$\mathcal{H}_{s,p}(u,\psi) = -c_{n,s,p}\int_\Omega u(x)^{p-1}\int_{C_{\bar{i}}} \frac{\psi(y)}{|x-y|^{n+sp}}\,dydx = -\int_{C_{\bar{i}}} \beta u^{p-1}\psi dy, \quad \text{for all } \psi \in C^\infty_0(C_{\bar{i}}).$$

Thus

$$c_{n,s,p}\int_\Omega \frac{u(x)^{p-1}}{|x-y|^{n+sp}}\,dx = \beta(y)u(y)^{p-1} \text{ for a.e. } y \in C_{\bar{i}}.$$

Since $u = 0$ in $C_{\bar{i}}$, we get that $u = 0$ in Ω.

Thus, we are reduced to consider the case $\bar{i} = i_0$. By testing (1.11) with u itself, we get

$$\begin{aligned}\mathcal{H}_{s,p}(u,u) &= \int_\Omega f(x,u(x))u(x)\,dx - \int_{\mathbb{R}^n\setminus\Omega}\beta(x)u(x)^p \\ &= -\int_{\mathbb{R}^n\setminus\Omega}\beta(x)u(x)^p \leq 0,\end{aligned}$$

where the last inequality follows from the fact that $\beta \geq 0$ and $u \geq 0$ (in $\mathbb{R}^n \setminus \Omega$). On the other hand,

$$\mathcal{H}_{s,p}(u,u) = \frac{1}{2}c_{n,s,p}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy \geq 0,$$

which yields that

$$\mathcal{H}_{s,p}(u,u) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n\setminus\Omega}\beta(x)u(x)^p\,dx = 0.$$

This implies that $\|u\|_{X^{s,p}_\beta} = 0$, so that

$$u = 0, \quad \text{for a.e. } x \in \mathbb{R}^n.$$

This yields a contradiction and the theorem holds. □

Remark 2.7. *We stress that the proofs of Lemma* 2.4, *Lemma* 2.5 *and Theorem* 2.6 *in the case of the eigenvalue problem* (1.6) *follow precisely the same scheme and are even easier than those in the case of solutions to* (1.5), *and for this reason we omit them.*

3. The first eigenvalue

In this section we provide a few results concerning the auxiliary eigenvalue problem (1.6).

Proposition 3.1. *Let* $\xi \in L^\infty(\Omega)$ *and assume* ($\beta 1$). *Then problem* (1.6) *admits a smallest eigenvalue* $\lambda_1(\xi,\beta,s,p) \in \mathbb{R}$ *which is simple and the associated eigenfunction do not change sign in* $\mathbb{R}^n$. *Moreover, every eigenfunction associated to an eigenvalue* $\lambda > \lambda_1(\xi,\beta,s,p)$ *is nodal, i.e. sign changing.*

Proof. Let $\gamma : X^{s,p}_\beta \to \mathbb{R}$ be the C^1-functional defined by

$$\gamma(u) = \frac{c_{n,s,p}}{2}[u]^p_{s,p} + \int_\Omega \xi(x)|u|^p dx + \int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u|^p dx$$

for all $u \in X^{s,p}_\beta$, define $M \subseteq X^{s,p}_\beta$ as the C^1- Banach manifold defined by

$$M := \left\{ u \in X^{s,p}_\beta : \int_\Omega |u|^p dx = 1 \right\},$$

and set

$$\lambda_1(\xi, \beta, s, p) := \inf \{\gamma(u) : u \in M\}. \tag{3.1}$$

Of course, since $\xi \in L^\infty(\Omega)$ and $\beta \geq 0$, we immediately have

$$\lambda_1(\xi, \beta, s, p) \geq -\|\xi\|_{L^\infty(\Omega)}.$$

Let $\{u_n\}_{n\geq 1} \subseteq M$ be a minimizing sequence for (3.1), that is $\gamma(u_n) \downarrow \lambda_1(\xi, \beta, s, p)$. Being $\xi \in L^\infty(\Omega)$, we immediately get that $\{u_n\}_{n\geq 1} \subseteq X^{s,p}_\beta$ is bounded and so we may assume that there exists $e_1 \in M$ such that

$$u_n \rightharpoonup e_1 \ \text{ in } X^{s,p}_\beta \tag{3.2}$$

as $n \to \infty$. In particular, by Lemma 2.2,

$$u_n \to e_1 \text{ in } L^q(\Omega) \text{ for every } q \in [1, p^\sharp_s). \tag{3.3}$$

By (3.2), we have

$$\frac{c_{n,s,p}}{2}[e_1]^p_{s,p} + \int_{\mathbb{R}^n\setminus\Omega} \beta(x)|e_1|^p dx \leq \liminf_{n\to\infty} \left(\frac{c_{n,s,p}}{2}[u_n]^p + \int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u_n|^p dx \right), \tag{3.4}$$

and, by (3.3),

$$\int_\Omega \xi(x)|u_n|^p dx \to \int_\Omega \xi(x)|e_1|^p dx. \tag{3.5}$$

Then, (3.4) and (3.5) imply $\gamma(e_1) \leq \lambda_1(\xi, \beta, s, p)$. Finally, since $e_1 \in M$ by (3.1), we get

$$\gamma(e_1) = \lambda_1(\xi, \beta, s, p).$$

By the Lagrange multiplier rule we immediately get that $\lambda_1(\xi, \beta, s, p)$ is an eigenvalue for problem (1.6), and precisely the smallest one, with associated eigenfunction $e_1 \in X^{s,p}_\beta$. Finally, notice that

$$\gamma(|u|) \leq \gamma(u) \ \text{ for all } u \in X^s_\beta,$$

and so we may assume that $e_1 \geq 0$ in $\mathbb{R}^n$. Since $\|e_1\|_{L^p(\Omega)} = 1$ by construction, we can then apply Theorem 2.6 to conclude that

$$e_1(x) > 0, \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Now, we prove that e_1 is simple. Thus, let $e_2 \in X^{s,p}_\beta$ be another eigenfunction associated to $\lambda_1(\xi, \beta, s, p)$. Now, take $\varepsilon > 0$, define

$$e_{2,\varepsilon} := \min \left\{ e_2, \frac{1}{\varepsilon} \right\},$$

and take $v = \dfrac{e^p_{2,\varepsilon}}{(e_1 + \varepsilon)^{p-1}}$. By Lemma 2.3 we know that $v \in X^s_\beta$. Thus, we can use v as test function in the problem solved by e_1. Setting

$$J_p(t) := |t|^{p-2}t, \quad t \in \mathbb{R},$$

we get

(3.6)

$$\frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{J_p((e_1+\varepsilon)(x)-(e_1+\varepsilon)(y))\left(\frac{e_{2,\varepsilon}^p}{(e_1+\varepsilon)^{p-1}}(x)-\frac{e_{2,\varepsilon}^p}{(e_1+\varepsilon)^{p-1}}(y)\right)}{|x-y|^{n+2s}}dxdy$$
$$=\lambda_1(\xi,\beta,s,p)\int_\Omega e_1^{p-1}\frac{e_{2,\varepsilon}^p}{(e_1+\varepsilon)^{p-1}}dx-\int_\Omega \xi e_1^{p-1}\frac{e_{2,\varepsilon}^p}{(e_1+\varepsilon)^{p-1}}dx+\int_{\mathbb{R}^n\setminus\Omega}\beta e_1^{p-1}\frac{e_{2,\varepsilon}^p}{(e_1+\varepsilon)^{p-1}}dx.$$

By Picone's inequality (see e.g. [5, Proposition 2.2]), we get

$$J_p((e_1+\varepsilon)(x)-(e_1+\varepsilon)(y))\left(\frac{e_{2,\varepsilon}^p}{(e_1+\varepsilon)^{p-1}}(x)-\frac{e_{2,\varepsilon}^p}{(e_1+\varepsilon)^{p-1}}(y)\right)\leq |e_{2,\varepsilon}(x)-e_{2,\varepsilon}(y)|^p.$$

Moreover, since the map $t\mapsto \min\{|t|,1/\varepsilon\}$ is 1−Lipschitz, we obtain

$$J_p((e_1+\varepsilon)(x)-(e_1+\varepsilon)(y))\left(\frac{e_{2,\varepsilon}^p}{(e_1+\varepsilon)^{p-1}}(x)-\frac{e_{2,\varepsilon}^p}{(e_1+\varepsilon)^{p-1}}(y)\right)\leq |e_2(x)-e_2(y)|^p.$$

Hence, passing to the limit as $\varepsilon\to 0$ and by applying the Fatou Lemma in the left had side of (3.6) and the Lebesgue Theorem in the right hand side, we find

$$\frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{J_p(e_1(x)-e_1(y))\left(\frac{e_2^p(x)}{e_1^{p-1}(x)}-\frac{e_2^p(y)}{e_1^{p-1}(y)}\right)}{|x-y|^{n+2s}}dxdy$$
$$\geq \lambda_1(\xi,\beta,s,p)\int_\Omega e_2^p dx-\int_\Omega \xi e_2^p dx+\int_{\mathbb{R}^n\setminus\Omega}\beta e_2^p dx=\frac{c_{n,s,p}}{2}[e_2]_{s,p}^p. \tag{3.7}$$

By the Picone inequality we can estimate the left hand side of (3.7) obtaining

$$\frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{J_p(e_1(x)-e_1(y))\left(\frac{e_2^p(x)}{e_1^{p-1}(x)}-\frac{e_2^p(y)}{e_1^{p-1}(y)}\right)}{|x-y|^{n+2s}}dxdy\leq \frac{c_{n,s,p}}{2}[e_2]_{s,p}^p \tag{3.8}$$

Hence, all previous inequalities are actually equalities. In particular, the Picone inequality implies

$$\frac{e_1(x)}{e_1(y)}=\frac{e_2(x)}{e_2(y)}\ \text{in}\ \mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2,$$

and so, by choosing alternately $x,y\in\Omega$ or $x,y\in\mathbb{R}^n\setminus\Omega$, we can conclude that there exists $\alpha\in\mathbb{R}$ such that $e_1=\alpha e_2$ in $\mathbb{R}^n$, as claimed.

Now, suppose that $\lambda>\lambda_1(\xi,\beta,s,p)$ is another eigenvalue of (1.6) with associated L^p−normalized eigenfunction $u\in X$, and assume assume by contradiction that u has constant sign, say $u\geq 0$.

Then, starting from the equation solved by u and using

$$\frac{(e_{1,\varepsilon})^p}{(u+\varepsilon)^{p-1}}$$

as test function, acting similarly as for reaching the equality after (3.7) and (3.8), we get

$$\frac{c_{n,s,p}}{2}[e_1]_{s,p}^p=\lambda\int_\Omega e_1^p dx-\int_\Omega \xi e_1^p dx+\int_{\mathbb{R}^n\setminus\Omega}\beta e_1^p dx=\frac{c_{n,s,p}}{2}[e_1]_{s,p}^p+(\lambda-\lambda_1(\xi,\beta,s,p)),$$

which is absurd, since $\lambda>\lambda_1(\xi,\beta,s,p)$, and thus u must change sign. □

Remark 3.2. *Uniqueness of positive eigenvalues is not strange, even in our general situation, and it parallels analogous results for fractional problems, e.g. see [23].*

Before going on with our result, we need to recall the following *Locality Theorem*, proved in [25, Theorem 2.8]:

Theorem 3.3. *Let u be a weak solution of (1.5). Then, $\mathscr{N}_{s,p}u + \beta|u|^{p-2}u = 0$ a.e. in $\mathbb{R}^N \setminus \overline{\Omega}$.*

Proposition 3.4. *Let $(u, \lambda) \in X^{s,p}_\beta \times \mathbb{R}$ be an eigen-pair. Then $u \in L^\infty(\mathbb{R}^n)$ with $\|u\|_{L^\infty(\mathbb{R}^N)} = \|u\|_{L^\infty(\Omega)}$.*

Proof. The first part of the proof follows as the corresponding ones for Dirichlet problems given in [19] and the one for Neumann problems in [25], and for this reason we'll be sketchy.

Being both $\pm u$ are solutions to the eigenvalue problem, it is enough to prove that $u_+ = \max\{0, u\}$ is bounded in Ω, and to prove this, it is enough to show that

$$\|u_+\|_{L^\infty(\Omega)} \leq 1 \text{ provided that } \|u_+\|_{L^p(\Omega)} \leq \delta$$

for some $\delta > 0$; but this assumption can be done without loss of generality, due to the homogeneity of the problem.

Then, for any integer $k \geq 0$ set

$$w_k := (u - (1 - 2^{-k}))_+.$$

We notice that $w_k \in X^{s,p}_\beta$ for any $k \geq 0$.

Moreover, the following inequalities can be easily proved:

$$\begin{aligned} &w_{k+1} \leq w_k \text{ in } \mathbb{R}^n, \\ &u(x) < (2^{k+1} - 1)w_k(x) \text{ for a.e. } x \in \{w_{k+1} > 0\}. \end{aligned}$$

Moreover, $\{w_{k+1} > 0\} \subseteq \{w_k > 2^{-(k+1)}\}$ and

$$|v(x) - v(y)|^{p-2}\big(v_+(x) - v_+(y)\big)\big(v(x) - v(y)\big) \geq |v_+(x) - v_+(y)|^p \tag{3.9}$$

for all $x, y \in \mathbb{R}^n$ and every function v, see [18].

Now, by using (3.9) with $v = u - (1 - 2^{-k-1})$, we get

$$[w_{k+1}]^p_{s,p} \leq \iint_{\mathbb{R}^{2n} \setminus (\mathbf{C}\Omega)^2} \frac{J_p(u(x) - u(y))\,(w_{k+1}(x) - w_{k+1}(y))}{|x - y|^{n+sp}} dxdy,$$

and taking w_{k+1} as test function in (1.6), it implies

$$\frac{c_{n,s,p}}{2}[w_{k+1}]^p_{s,p} \leq |\lambda| \int_{\{w_{k+1} > 0\}} |u|^{p-2}uw_{k+1}dx.$$

By adding the inequality

$$\int_\Omega w^p_{k+1}dx \leq \int_{\{w_{k=1} > 0\}} u^{p-1}w_{k+1}dx,$$

obtained by using that $0 < w_{k+1} \leq u$ in the set $\{w_{k=1} > 0\}$, we obtain as in [11] and [25]

$$\|w_{k+1}\|^p_{W^{s,p}(\Omega)} \leq (|\lambda| + 1) \int_{\{w_{k+1} > 0\}} u^{p-1}w_{k+1}dx.$$

Now proceed as in [11] to get that $u \in L^\infty(\Omega)$.

In order to conclude, from Theorem 3.3, we have that for $x \in \mathbb{R}^N \setminus \overline{\Omega}$ with $u(x) \geq 0$ and non constant (otherwise the conclusion is trivial), we have

$$\beta(x)u(x)^{p-1} + u(x) \int_\Omega \frac{|u(x) - u(y)|^{p-2}}{|x - y|^{N+ps}} dy = \int_\Omega \frac{|u(x) - u(y)|^{p-2}u(y)}{|x - y|^{N+ps}} dy.$$

Being β nonnegative, we have

$$0 \leq u(x) \leq \frac{\displaystyle\int_\Omega \frac{|u(x)-u(y)|^{p-2}u(y)}{|x-y|^{N+ps}}dy}{\displaystyle\int_\Omega \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+ps}}dy} \leq \|u\|_{L^\infty(\Omega)}.$$

On the other hand, if $x \in \mathbb{R}^N \setminus \overline{\Omega}$ with $u(x) \leq 0$ we have

$$u(x)\left(\beta(x)|u(x)|^{p-2} + \int_\Omega \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+ps}}dy\right) = \int_\Omega \frac{|u(x)-u(y)|^{p-2}u(y)}{|x-y|^{N+ps}}dy,$$

and so

$$u(x) \geq -\frac{\|u\|_{L^\infty(\Omega)} \displaystyle\int_\Omega \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+ps}}dy}{\beta(x)|u(x)|^{p-2} + \displaystyle\int_\Omega \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+ps}}dy} \geq -\|u\|_{L^\infty(\Omega)}.$$

□

4. Existence and uniqueness of positive weak solutions

In this section we prove the existence of positive weak solution to problem (1.5), and for this reason from now on we set $f(x,u) = f(x,0)$ for all $u \leq 0$.

Formally, weak solutions of (1.5) coincide with the critical points of the the functional $I : X^{s,p}_\beta \to \mathbb{R} \cup \{\infty\}$ defined as

$$I(u) := \frac{c_{n,s,p}}{2} \iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy + \frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega} \beta|u|^p - \int_\Omega F(x,u)\,dx, \tag{4.1}$$

where

$$F(x,u) := \int_0^u f(x,s)\,ds.$$

In order for I to be differentiable, we need to prove that I is well defined and that the Nemytszkii operator associated to f along $u \in X^{s,p}_\beta$, $\mathfrak{f}_u$, defined pointwise as $\mathfrak{f}_u(x) := f(x,u(x))$, is such that $\mathfrak{f}_u \in (L^{p^\sharp_s}(\Omega))'$. For this purpose, we need the next two lemmas.

Lemma 4.1. *If (f1), (f2) and (β_1) hold, then the functional defined in (4.1) is well defined on the space $X^{s,p}_\beta$.*

Proof. First, we notice that

$$\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy \leq c\|u\|^p_{X^{s,p}_\beta}.$$

Moreover, we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^n\setminus\Omega} \beta|u|^p\right| &\leq \frac{1}{p}\|\beta\|_{L^\infty(\mathbb{R}^n\setminus\Omega)}\|u\|^p_{\mathbb{R}^n\setminus\Omega} \\ &\leq C(p, \|\beta\|_{L^\infty(\mathbb{R}^n\setminus\Omega)})\|u\|^p_{X^{s,p}_\beta}.\end{aligned}$$

Finally, recalling $(f2)$,

$$\left|\int_\Omega F(x,u)\,dx\right| \leq c\int_\Omega (|u| + |u|^p)\,dx \leq c\left(\|u\|_{X^{s,p}_\beta} + \|u\|^p_{X^{s,p}_\beta}\right).$$

□

Lemma 4.2. *If (f1), (f2) and (β_1) hold, then every critical point of I is a weak solution of* (1.5).

Proof. Let us take $\varepsilon \in \mathbb{R}$ such that $|\varepsilon| < 1$ and $u, v \in X^{s,p}_\beta$. Certainly $u + \varepsilon v \in X^{s,p}_\beta$. Since,

$$\begin{aligned} I(u+\varepsilon v) = \frac{c_{n,s,p}}{2} \iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} \frac{|(u+\varepsilon v)(x) - (u+\varepsilon v)(y)|^p}{|x-y|^{n+sp}} dxdy \\ - \int_\Omega F(x, u+\varepsilon v)\, dx + \frac{1}{p} \int_{\mathbb{R}^n\setminus\Omega} \beta |u+\varepsilon v|^p, \end{aligned}$$

we have that

$$\lim_{\varepsilon\to 0} \frac{I(u+\varepsilon v) - I(u)}{\varepsilon} =: A_1 + A_2 + A_3.$$

Now,

(4.2)

$$\begin{aligned} A_1 &:= \lim_{\varepsilon\to 0} \frac{c_{n,s,p}}{2\varepsilon} \left\{ \iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} \frac{|(u+\varepsilon v)(x) - (u+\varepsilon v)(y)|^p}{|x-y|^{n+sp}} dxdy - \iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right\} \\ &= \frac{c_{n,s,p}}{2} \iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} \lim_{\varepsilon\to 0} \frac{|(u+\varepsilon v)(x) - (u+\varepsilon v)(y)|^p - |u(x)-u(y)|^p}{\varepsilon|x-y|^{n+sp}} dxdy \\ &= \frac{c_{n,s,p}}{2} \iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} \frac{|u(x)-u(y)|^{p-1}(v(x)-v(y))}{|x-y|^{n+sp}} dxdy. \end{aligned}$$

Similarly,

$$A_2 := \frac{1}{p} \int_{\mathbb{R}^n\setminus\Omega} \beta \lim_{\varepsilon\to 0} \frac{|u+\varepsilon v|^p - |u|^p}{\varepsilon}\, dx = \int_{\mathbb{R}^n\setminus\Omega} \beta |u|^{p-1} v\, dx \tag{4.3}$$

and

$$\begin{aligned} A_3 &:= \lim_{\varepsilon\to 0} \int_\Omega \frac{F(x, u+\varepsilon v) - F(x,u)}{\varepsilon} \varepsilon\, dx \\ &= \int_\Omega \frac{\partial F}{\partial u}(x,u) v\, dx = \int_\Omega f(x,u) v\, dx. \end{aligned} \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we reach the desired conclusion. □

Now, let us consider the following auxiliary functional:

(4.5)

$$\tilde{I}(u) := \frac{c_{n,s,p}}{2} \iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy + \frac{1}{p}\int_\Omega |u|^p\, dx + \frac{1}{p} \int_{\mathbb{R}^n\setminus\Omega} \beta |u|^p\, dx - \int_\Omega K(x,u)\, dx,$$

where

$$K(x,u) := \int_0^u k(x,t)\, dt, \tag{4.6}$$

and

$$k(x,t) := \begin{cases} f(x,0) & t \le 0, \\ f(x,t) + t^{p-1} & t > 0. \end{cases} \tag{4.7}$$

As for I, $\tilde{I}$ is well-defined on $X^{s,p}_\beta$ as well. Moreover, if $u \geq 0$ (a.e. in Ω), then

$$\tilde{I}(u) = I(u). \tag{4.8}$$

Notice that $\tilde{I}$ has the advantage of being (essentially) the sum of the $p-$power of the norm in $X^{s,p}_\beta$ plus a nonlinear term with $p-$growth. In addition I and $\tilde{I}$ coincide on positive functions.

Lemma 4.3. *Let us assume (f1), (f2), (f3), (β_1) and*

$$\lambda_1(-a_\infty, \beta, s, p) > 0. \tag{4.9}$$

Then the functional $\tilde{I}$ defined in (4.5) is coercive.

Proof. We argue by contradiction assuming that the functional $\tilde{I}$ is not coercive. Then, there exist a sequence of functions $\{u_n\}_{n\in\mathbb{N}} \subset X^{s,p}_\beta$ and a positive constant $M > 0$ such that

$$\|u_n\|_{X^{s,p}_\beta} \to +\infty \quad \text{as } n \to +\infty, \tag{4.10}$$

and

$$\tilde{I}(u_n) < M \quad \text{for every } n \in \mathbb{N}. \tag{4.11}$$

Now, from (4.6) and (4.7) there exists $c > 0$ such that

$$\int_\Omega K(x,u)\,dx \le c\int_\Omega \left(|u| + \frac{|u|^p}{p}\right)dx \le c(\|u\|^p_{L^p(\Omega)} + 1),$$

where in the last step we used Hölder inequality and the boundedness of the set Ω. Therefore, from (4.5) (applied to u_n), we get that

$$\frac{c_{n,s,p}}{2}[u_n]^p + \frac{1}{p}\|u_n\|^p_{L^p(\Omega)} + \frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega}\beta|u_n|^p\,dx \le c(\|u_n\|^p_{L^p(\Omega)} + 1), \tag{4.12}$$

where $c > 0$ now depends on M as well. In particular, by (4.10), this yields that $\|u_n\|_{L^p(\Omega)} \to +\infty$ as $n \to +\infty$.
We now set

$$y_n := \frac{u_n}{\|u_n\|_{L^p(\Omega)}} \in X^{s,p}_\beta, \quad n \in \mathbb{N}.$$

From (4.12), and by the very definition of y_n, we have that

$$\frac{c_{n,s,p}}{2}\|u_n\|^p_{L^p(\Omega)}[y_n]^p + \frac{1}{p}\|u_n\|^p_{L^p(\Omega)} + \frac{\|u_n\|^p_{L^p(\Omega)}}{p}\int_{\mathbb{R}^n\setminus\Omega}\beta|y_n|^p\,dx \le c(\|u_n\|^p_{L^p(\Omega)} + 1),$$

for every $n \in \mathbb{N}$. Therefore, dividing by $\|u_n\|_{L^p(\Omega)}$ and reassembling the constants, we get

$$C(1 + [y_n]^p) + \frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega}\beta|y_n|^p\,dx \le c\left(1 + \frac{1}{\|u_n\|^p_{L^p(\Omega)}}\right),$$

for every $n \in \mathbb{N}$ and some $C > 0$. Since $\|u_n\|_{L^p(\Omega)} \to +\infty$, this implies that $\|y_n\|_{X^{s,p}_\beta}$ is a bounded sequence. By Lemma 2.1 and Lemma 2.2, (possibly passing to a subsequence) there exists $y \in X^{s,p}_\beta$ such that

$$y_n \rightharpoonup y \quad \text{in } X^{s,p}_\beta,\ y_n \to y \quad \text{in } L^p(\Omega) \quad \text{and} \quad \int_{\mathbb{R}^n\setminus\Omega}\beta|y|^p\,dx \le \liminf_{n\to\infty}\int_{\mathbb{R}^n\setminus\Omega}\beta|y_n|^p\,dx \tag{4.13}$$

Moreover, $\|y\|_{L^p(\Omega)} = 1$.
By (4.11) and the definition of $\tilde{I}$ we get that

$$\begin{aligned}\frac{1}{p}\left(\|y_n\|^p_{L^p(\Omega)} + [y_n]^p\right) + \frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega}\beta|y_n|^p\,dx &\le \frac{M}{\|u_n\|^p_{L^p(\Omega)}} + \int_\Omega \frac{K(x,u_n)}{\|u_n\|^p_{L^p(\Omega)}}\,dx\\ &\le \frac{M}{\|u_n\|^p_{L^p(\Omega)}} + \int_{\{u_n>0\}}\left(\frac{F(x,u_n^+)}{\|u_n\|^p_{L^p(\Omega)}} + \frac{y_n}{p}\right)dx + \int_{\{u_n<0\}}\frac{f(x,0)u_n}{\|u_n\|^p_{L^p(\Omega)}}\,dx.\end{aligned}$$

Using (1.10) we obtain

$$\frac{1}{p}\left(\|y_n\|^p_{L^p(\Omega)}+[y_n]^p\right)\le\frac{M}{\|u_n\|^p_{L^p(\Omega)}}+\frac{1}{p}\|y_n^+\|^p_{L^p(\Omega)}+\int_\Omega\frac{F(x,u_n^+)}{\|u_n\|^p_{L^p(\Omega)}}\,dx, \tag{4.14}$$

where $v^+:=\max\{v,0\}$. If $\{u_n^+\}\subset L^p(\Omega)$ is bounded, then since $y_n^+=\frac{u_n^+}{\|u_n\|_p}$ for all $n\in\mathbb{N}$ and using the fact that $\|u_n\|_{L^p(\Omega)}\to\infty$ we get $y_n^+\to 0$ in $L^p(\Omega)$, hence $y\le 0$ and the contradiction follows exaclt as in [18, Proof of Proposition 4]. If $\{u_n^+\}\subset L^p(\Omega)$ is unbounded we may assume that $\|u_n^+\|_{L^p(\Omega)}\to\infty$ and proceeding again as in [18, Proof of Proposition 4] we get

$$\frac{1}{p}\left(\|y_n^+\|^p_{L^p(\Omega)}+[y_n^+]^p\right)+\frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega}\beta|y_n^+|^p\,dx\le\frac{M}{\|u_n^+\|^p_{L^p(\Omega)}}+\int_\Omega\frac{F(x,u_n^+)}{\|u_n^+\|^p_{L^p(\Omega)}}\,dx, \tag{4.15}$$

and

$$\limsup_{n\to\infty}\int_\Omega\frac{F(x,u_n^+)}{\|u_n^+\|^p_{L^p(\Omega)}}\,dx\le\frac{1}{p}\int_{\{y^+\neq 0\}}a_\infty(x)(y^+)^p\,dx \tag{4.16}$$

where a_∞ is as in (1.7). Letting $n\to\infty$ in (4.15) and using (4.13) and (4.16) we get

$$[y^+]^p+\int_{\mathbb{R}^n\setminus\Omega}\beta|y^+|^p\,dx\le\int_\Omega a_\infty(x)(y^+)^p\,dx. \tag{4.17}$$

If $y^+=0$, then from (4.13),(4.14) and the fact that

$$\lim_{n\to\infty}\int_{\{y^+=0\}}\frac{F(x,u_n^+)}{\|u_n^+\|^p_{L^p(\Omega)}}\,dx=0$$

we conclude $y=0$ which contradicts $\|y\|_{L^p(\Omega)}=1$. Therefore, $y^+\neq 0$ and from (4.17) we have $\lambda_1(-a_\infty,\beta,p)\le 0$ which is in contradiction with (4.9). $\square$

Lemma 4.4. *If (f1), (f2), (f3) and (β_1) hold, then $\tilde{I}$ is sequentially weakly lower semicontinuous (in short s.w.l.s.c.).*

Proof. Let $I_1,I_2:X^{s,p}_\beta\to\mathbb{R}$ be the C^1-functionals defined by

$$I_1(u):=\frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy+\frac{1}{p}\int_\Omega|u|^p\,dx+\frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega}\beta|u|^p\,dx$$

and

$$I_2(u):=-\int_\Omega K(x,u)\,dx.$$

It is easy to see that there exist two positive constants $\tilde{c}_1,\tilde{c}_2>0$ such that for every $u\in X^{s,p}_\beta$, $\tilde{c}_1\|u\|_{X^{s,p}_\beta}\le I_1(u)^{1/p}\le\tilde{c}_2\|u\|_{X^{s,p}_\beta}$. Moreover, proceeding exactly as in Lemma 2.1 we get that $I_1^{1/p}$ is a norm, which implies that $I_1^{1/p}$ is s.w.l.s.c. along with I_1. Since $\tilde{I}=I_1+I_2$, to prove the sequential weak lower semicontinuity of $\tilde{I}$, we need to show that I_2 is sequentially weakly lower semicontinuous. To this end, let $S\in\mathbb{R}$ and consider the set

$$L_S:=\{u\in X^{s,p}_\beta\mid I_2(u)\le S\}$$

We need to show that L_S is sequentially weakly closed. So, let $\{u_n\}_n\subset L_S$ and assume that $u_n\rightharpoonup u$ hence by Lemma 2.2, $u_n\to u$ in $L^p(\Omega)$. This implies

$$u_n^+\to u^+\text{ and }u^-\to u^-\text{ in }L^p(\Omega)$$

and, possibly extracting a subsequence,

$$u_n^+(x) \to u^+(x) \text{ a.e. } x \in \Omega. \tag{4.18}$$

Proceeding exactly as in [18, Proposition 5] we get

$$S \geq -\int_\Omega K(x,u_n)\,dx = -\int_\Omega F(x,u_n^+)\,dx - \frac{1}{p}\|u_n^+\|^p_{L^p(\Omega)} - \int_\Omega f(x,0)(-u_n^-)\,dx; \tag{4.19}$$

$$\frac{1}{p}\|u_n^+\|^p_{L^p(\Omega)} \to \frac{1}{p}\|u^+\|^p_{L^p(\Omega)}; \tag{4.20}$$

$$\int_\Omega f(x,0)(-u_n^-)\,dx \to \int_\Omega f(x,0)(-u^-)\,dx. \tag{4.21}$$

Also, from (4.18) and Fatou's Lemma,

$$\liminf_{n\to\infty}\left(-\int_\Omega F(x,u_n^+)\,dx\right) = -\limsup_{n\to\infty}\int_\Omega F(x,u_n^+)\,dx \geq -\int_\Omega F(x,u^+)\,dx. \tag{4.22}$$

Letting $n \to \infty$ in (4.19) and using (4.20),(4.21) and (4.22), we get

$$S \geq -\int_\Omega F(x,u^+)\,dx - \frac{1}{p}\|u^+\|^p_{L^p(\Omega)} - \int_\Omega f(x,0)(-u^-)\,dx;$$

□

We are now ready to prove a sufficient condition for the existence of a positive solution to problem (1.5).

Proposition 4.5. *If (f1), (f2), (f3), (β_1) and*

$$\lambda_1(-a_0,\beta,s,p) < 0 < \lambda_1(-a_\infty,\beta,s,p)$$

hold, then problem (1.5) *admits a positive solution.*

Proof. By Lemma 4.3 and Lemma 4.4 we know that there exists $u_0 \in X^{s,p}_\beta$ such that

$$\tilde{I}(u_0) = \inf\left\{\tilde{I}(u) : u \in X^{s,p}_\beta\right\}.$$

Our first task is to show that it is possible to assume $u_0 \geq 0$. To this aim, let us assume that u_0 is sign-changing. By Lemma 2.3, $u_0^+ \in X^{s,p}_\beta$. Recalling (4.8),

$$\begin{aligned}
\tilde{I}(u_0^+) &= I(u_0^+) \\
&= \frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{|u_0^+(x)-u_0^+(y)|^p}{|x-y|^{n+sp}}dxdy - \int_\Omega F(x,u_0^+)\,dx + \frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega}\beta|u_0^+|^p\,dx \\
&\leq \frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{|u_0(x)-u_0(y)|^p}{|x-y|^{n+sp}}dxdy - \int_\Omega F(x,u_0^+)\,dx + \frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega}\beta|u_0^+|^p\,dx \\
&\quad + \frac{1}{p}\int_\Omega |u_0|^p\,dx - \int_\Omega f(x,0)(-u_0^-)\,dx \\
&= \tilde{I}(u_0),
\end{aligned}$$

where

$$u_0^- := \max\{-u_0, 0\} \geq 0.$$

Therefore u_0^+ is a non-negative solution of (1.5). Now, to simplify the notation, let us write directly u_0 in place of u_0^+. We want to show that actually $u_0 > 0$. By Theorem 2.6, we know that

$$\text{either } u_0 > 0 \text{ or } u_0 \equiv 0 \text{ in } \mathbb{R}^n$$

therefore to conclude it suffices to prove that $\tilde{I}(u_0) < 0$. By (1.13) there exists $\phi \in X_\beta^{s,p}$ such that $\int_\Omega |\phi|^p \, dx = 1$ and

$$[\phi]^p + \int_{\mathbb{R}^n \setminus \Omega} \beta |\phi|^p < \int_{\{\phi \neq 0\}} a_0 \, \phi^p \tag{4.23}$$

We can assume w.l.o.g. that $\phi > 0$. We claim that we can also assume $\phi \in L^\infty(\mathbb{R}^n)$. Let $k \in \mathbb{N}$ and consider $\phi_k = \min\{\phi, k\} \in L^\infty(\mathbb{R}^n)$, by Lemma 2.3 $\phi_k \in X_\beta^{s,p}$ and

$$\begin{aligned}
\tilde{I}(\phi_k) &= I(\phi_k) \\
&= \frac{c_{n,s,p}}{2} \iint_{\mathbb{R}^{2n} \setminus (\mathbf{C}\Omega)^2} \frac{|\phi_k(x) - \phi_k(y)|^p}{|x-y|^{n+sp}} dxdy - \int_\Omega F(x, \phi_k) \, dx + \frac{1}{p} \int_{\mathbb{R}^n \setminus \Omega} \beta |\phi_k|^p \, dx \\
&\leq \frac{c_{n,s,p}}{2} \iint_{\mathbb{R}^{2n} \setminus (\mathbf{C}\Omega)^2} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{n+sp}} dxdy - \int_\Omega F(x, \phi) \, dx + \frac{1}{p} \int_{\mathbb{R}^n \setminus \Omega} \beta |\phi|^p \, dx \\
&= I(\phi) = \tilde{I}(\phi)
\end{aligned}$$

and the claim follows. We note that

$$\liminf_{u \to 0} \frac{F(x,u)}{u^p} \geq \frac{a_0(x)}{p}$$

and proceeding as in [6, Proof of (15)] we get

$$\liminf_{\varepsilon \to 0} \int_{\{\phi \neq 0\}} \frac{F(x, \varepsilon\phi)}{\varepsilon^p} \geq \frac{1}{p} \int_{\{\phi \neq 0\}} a_0 \phi^p.$$

Thus

$$\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^n} \frac{F(x, \varepsilon\phi)}{\varepsilon^p} \geq \frac{1}{p} \int_{\{\phi \neq 0\}} a_0 \phi^p.$$

Therefore using (4.23) we conclude that

$$[\phi]^p + \int_{\mathbb{R}^n \setminus \Omega} \beta |\phi|^p - p \int_{\mathbb{R}^n} \frac{F(x, \varepsilon\phi)}{\varepsilon^p} < 0$$

for any $\varepsilon > 0$ small enough, which is $I(\varepsilon\phi) < 0$ and the thesis follows.

□

Theorem 4.6. *If (f1), (f2), (f3) and (β_1) hold, then Problem (1.5) admits at most one positive solution.*

Proof. Let u_1, u_2 be two weak positive solutions of (1.5). For $\varepsilon > 0$, we define the truncations $u_{i,\varepsilon}$ as in (2.1).

For $\varepsilon > 0$, we define the functions

$$\varphi_{1,\varepsilon} := \frac{u_{2,\varepsilon}^p}{(u_1 + \varepsilon)^{p-1}} - u_{1,\varepsilon},$$

and

$$\varphi_{2,\varepsilon} := \frac{u_{1,\varepsilon}^p}{(u_2 + \varepsilon)^{p-1}} - u_{2,\varepsilon},$$

By Lemma 2.3, we know that $\varphi_{i,\varepsilon} \in X_\beta^{s,p}$ for $i = 1, 2$.
Now, set

$$J_p(t) := |t|^{p-2} t.$$

We test (1.11) with $\varphi_{1,\varepsilon}$ and $\varphi_{2,\varepsilon}$ and we add the resulting identities, getting

$$
\begin{aligned}
(4.24)\quad & \frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{J_p(u_1(x)-u_1(y))\left(\frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}}(x)-\frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}}(y)\right)}{|x-y|^{n+sp}}dxdy\\
&-\frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{J_p(u_1(x)-u_1(y))\,(u_{1,\varepsilon}(x)-u_{1,\varepsilon}(y))}{|x-y|^{n+sp}}dxdy\\
&+\frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{J_p(u_2(x)-u_2(y))\left(\frac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}}(x)-\frac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}}(y)\right)}{|x-y|^{n+sp}}dxdy\\
&-\frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{J_p(u_1(x)-u_1(y))\,(u_{2,\varepsilon}(x)-u_{2,\varepsilon}(y))}{|x-y|^{n+sp}}dxdy\\
&=\int_\Omega f(x,u_1)\left(\frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}}-u_{1,\varepsilon}\right)+f(x,u_2)\left(\frac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}}-u_{2,\varepsilon}\right)dx\\
&-\int_{\mathbb{R}^n\setminus\Omega}\beta\left(|u_1|^{p-2}u_1\left(\frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}}-u_{1,\varepsilon}\right)+|u_2|^{p-2}u_2\left(\frac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}}-u_{2,\varepsilon}\right)\right)dx.
\end{aligned}
$$

We stress that

$$J_p(u_i(x)-u_i(y))=J_p((u_i+\varepsilon)(x)-(u_i+\varepsilon)(y)),\quad \text{for } i=1,2.$$

Now, applying the discrete Picone's inequality (see e.g. [5, Proposition 2.2]) and the fact that $t\to\min\{|t|,1/\varepsilon\}$ is 1−Lipschitz, we obtain

$$J_p((u_1+\varepsilon)(x)-(u_1+\varepsilon)(y))\left(\frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}}(x)-\frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}}(y)\right)\le|u_2(x)-u_2(y)|^p$$

and

$$J_p((u_2+\varepsilon)(x)-(u_2+\varepsilon)(y))\left(\frac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}}(x)-\frac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}}(y)\right)\le|u_1(x)-u_1(y)|^p.$$

Proceeding as in [5] and recalling Remark 1.1, we pass to the limit in (4.24). This yields

$$
\begin{aligned}
(4.25)\qquad & \frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{J_p(u_1(x)-u_1(y))\left(\frac{u_2^p}{u_1^{p-1}}(x)-\frac{u_2^p}{u_1^{p-1}}(y)\right)}{|x-y|^{n+sp}}dxdy\\
&-\frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{(u_1(x)-u_1(y))^p}{|x-y|^{n+sp}}dxdy\\
&+\frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{J_p(u_2(x)-u_2(y))\left(\frac{u_1^p}{u_2^{p-1}}(x)-\frac{u_1^p}{u_2^{p-1}}(y)\right)}{|x-y|^{n+sp}}dxdy\\
&-\frac{c_{n,s,p}}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathbf{C}\Omega)^2}\frac{(u_2(x)-u_2(y))^p}{|x-y|^{n+sp}}dxdy\\
&\geq\int_\Omega f(x,u_1)\left(\frac{u_2^p}{u_1^{p-1}}-u_1\right)+f(x,u_2)\left(\frac{u_1^p}{u_2^{p-1}}-u_2\right)dx\\
&=-\int_\Omega\left(\frac{f(x,u_1)}{u_1^{p-1}}-\frac{f(x,u_2)}{u_2^{p-1}}\right)(u_1^p-u_2^p)\,dx.
\end{aligned}
$$

Using the Picone's inequality in the left-hand side of (4.25) we obtain

$$
0\geq-\int_\Omega\left(\frac{f(x,u_1)}{u_1^{p-1}}-\frac{f(x,u_2)}{u_2^{p-1}}\right)(u_1^p-u_2^p)\,dx
$$

and the conclusion follows using the fact that $t\to\frac{f(x,t)}{t^{p-1}}$ is decreasing on $(0,\infty)$. □

Let us now prove the necessity conditions, adapting the analogous proofs in [6].

Lemma 4.7. *Let $u\in X_\beta^{s,p}$ be a solution of problem (1.5). Then*

$$
\lambda_1(-a_0,\beta,s,p)<0.
$$

Proof. By definition

$$
\lambda_1(-a_0,\beta,s,p)=\inf_{v\in X_\beta^{s,p},v\neq0}\frac{\frac{c_{n,s,p}}{2}[v]_{s,p}^p-\int_\Omega a_0|v|^pdx+\int_{\mathbb{R}^n\setminus\Omega}\beta|v|^pdx}{\int_\Omega|v|^pdx},
$$

then taking $v=u$ we get

$$
\lambda_1(-a_0,\beta,s,p)\leq\frac{\frac{c_{n,s,p}}{2}[u]_{s,p}^p-\int_\Omega a_0|u|^pdx+\int_{\mathbb{R}^n\setminus\Omega}\beta|u|^pdx}{\int_\Omega|u|^pdx}.
$$

On the other hand, since u solves (1.5), we have that

$$
\frac{c_{n,s,p}}{2}[u]_{s,p}^p+\int_{\mathbb{R}^n\setminus\Omega}\beta|u|^pdx=\int_\Omega f(x,u)u\,dx.
$$

By Theorem 2.6, $u > 0$ in Ω. Therefore,

$$\int_\Omega f(x,u)u\,dx < \int_\Omega a_0 u^p dx$$

and the conclusion follows. $\square$

In the following Lemma we show the necessity of the analog of (1.3) in our case. We stress that in this case both the boundedness of u and the linearity of the operator are crucial assumptions.

Lemma 4.8. *Let $u \in X^{s,2}_\beta$ be a bounded solution of problem (1.5) with $p = 2$. Then*

$$\lambda_1(-a_\infty, \beta, s, 2) > 0.$$

Proof. Set

$$\tilde{a}(x) := \frac{f(x, \|u\|_{L^\infty(\Omega)} + 1)}{\|u\|_{L^\infty(\Omega)} + 1}.$$

Notice that $\tilde{a} \in L^\infty(\Omega)$ by (f3).

Now, set $\Lambda = \lambda_1(-\tilde{a}, \beta, s, 2)$ and let be a solution of

$$\begin{cases} (-\Delta)^s_2 u - \tilde{a}(x)\psi = \Lambda\psi & \text{in } \Omega,\\ \quad > 0 & \text{in } \Omega,\\ \mathcal{N}_{s,2}\psi + \beta(x)\psi = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

Notice that by Proposition 3.1, such a ψ is uniquely determined up to a multiplicative constant. Now, use ψ as test function in (1.5) and u as test function in the equation solved by ψ and subtract obtaining

$$\int_\Omega \tilde{a}u\psi\,dx = \int_\Omega f(x,u)\psi\,dx - \Lambda\int_\Omega \psi u\,dx.$$

By the monotonicity property of f, we get

$$\int_\Omega f(x,u)\psi\,dx > \int_\Omega a_\infty(x)u\psi\,dx,$$

so that $\Lambda > 0$. Moreover, since $\tilde{a}_\infty(x) \geq a(x)$ for a.e. $x \in \Omega$), we get that

$$0 < \Lambda = \lambda_1(-\tilde{a}, \beta, s, 2) \leq \lambda_1(-a_\infty, \beta, s, 2),$$

and the conclusion follows. $\square$

The proof of Theorem 1.2 now follows collecting all the previous results.

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