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Abstract

We provide an efficiently computable characterization of two important properties describing stable and unstable complex behaviours as equicontinuity and sensitivity to the initial conditions for one-dimensional linear cellular automata (LCA) over $(\mathbb{Z}/m\mathbb{Z})^n$ (Theorem 9), a large and important class of cellular automata (CA) which are able to exhibit the complex behaviours of general CA and are used in applications. We stress that the setting of LCA over $(\mathbb{Z}/m\mathbb{Z})^n$ with $n > 1$ is more expressive, it gives rise to much more complex dynamics, and it is more difficult to deal with than the already investigated case $n = 1$. The proof techniques from [23, 28] used when $n = 1$ for obtaining an easy to check characterization of dynamical properties can no longer be exploited when $n > 1$ for achieving the same goal. Indeed, in order to get the efficiently computable characterization provided by Theorem 9 we need to prove a nontrivial result of abstract algebra about the finiteness of matrix semigroups (Theorem 8) which is also of interest in its own: if $\mathbb{K}$ is any finite commutative ring and $\mathbb{L}$ is any $\mathbb{K}$-algebra, then for every pair A, B of $n \times n$ matrices over $\mathbb{L}$ having the same characteristic polynomial, it holds that the set $\{A^0, A^1, A^2, \ldots\}$ is finite if and only if the set $\{B^0, B^1, B^2, \ldots\}$ is finite too. A further ingredient we provide to reach our goal is the result stated in Theorem 6, i.e., the generalization to $(\mathbb{Z}/m\mathbb{Z})^n$ of the efficiently computable criterion that allows deciding sensitivity and equicontinuity for the subclass of LCA over $(\mathbb{Z}/p^k\mathbb{Z})^n$ (with p prime) defined by a matrix in Frobenius normal form.

Keywords: Cellular Automata, Linear Cellular Automata, Decidability, Complex Systems

1. Introduction

Cellular automata (CA) are widely known formal models for studying and simulating complex systems. They are used in many disciplines ranging from physics to biology, stepping through sociology, ecology and many others. In computer science they are used for designing security schemes, random number generation, image processing, etc. This extensive use is essentially due to three main ingredients: the huge variety of distinct dynamical behaviours; the emergence of complex behaviours from simple local interactions (defined by a local rule); the ease of implementation (even at a hardware level). In practical applications one needs to know if the CA used for modelling a system has or not some specific property and, in particular, the properties describing stable and unstable behaviours are often required.

Unfortunately this need can be an issue. Indeed, a strong result states (roughly speaking) that all non-trivial dynamical behaviours are undecidable [24]. From this seminal result, a long sequence followed (see [4, 18, 21],

☆A preliminary version of the results of this paper have been presented at the international conference ICALP 2020. The ICALP paper [13] does not contain the proof of the important result stated in Theorem 8 (Section 4 is entirely devoted to that proof) . Moreover, it does not deal with the "efficient computability" aspects that are the focus of the present paper.

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just to cite some of them). The situation changes if the CA alphabet has the algebraic structure of n-th power $\mathbb{K}^n$ of the finite ring $\mathbb{K} = \mathbb{Z}/m\mathbb{Z}$, viewed then as a module over $\mathbb{K}$, and the superposition principle holds, i.e., the CA global rule is a homomorphism (see for instance [23, 28, 27, 25, 8, 7] for the case $n = 1$), giving rise to *linear cellular automata (LCA)* over $(\mathbb{Z}/m\mathbb{Z})^n$. In other terms, LCA are CA having $(\mathbb{Z}/m\mathbb{Z})^n$ as alphabet and local rule defined by $n \times n$ matrices with elements in $\mathbb{Z}/m\mathbb{Z}$. We want to stress that they form a large and important class of CA which are able to exhibit the complex behaviours of general CA and are used in applications (the latter especially with $n > 1$, see for instance [29, 2, 25][1]).

However, there are few results regarding efficiently computable characterizations, i.e., checkable in polynomial time, of the dynamical properties for LCA over $(\mathbb{Z}/m\mathbb{Z})^n$ with $n > 1$. Actually, the setting $n > 1$, which is more expressive and gives rise to much more complex dynamics than $n = 1$ (see, for instance [12]), is more difficult to deal with. The proof techniques from [23, 28] used when $n = 1$ for obtaining efficiently computable characterizations of dynamical and ergodic properties can no longer be exploited when $n > 1$ for achieving the same goal. For a generic n, only injectivity and surjectivity had been characterized (in terms of easy to check conditions on the matrix associated with the LCA [6, 25]). Just very recently, we have provided a decidable characterization of ergodicity [11] (and all properties, as for instance topological transitivity and mixing, that turned out to be equivalent to it for LCA). Furthermore, in [3] authors have proved that, among other properties, sensitivity and equicontinuity are decidable for the wider class of group CA, but, as noted by themselves, "the existing characterizations in the literature typically provide easy to check conditions on the local rule of the cellular automaton for the considered properties, while algorithms extracted from our proofs are impractical and only serve the purpose of proving decidability".

In this paper we study sensitivity to the initial conditions and equicontinuity, where the former is the well-known basic component and essence of the chaotic behaviour of a discrete time dynamical system, while the latter represents a strong form of stability. We show an efficiently computable characterization of these properties for LCA over $(\mathbb{Z}/m\mathbb{Z})^n$(Theorem 9). In order to get such a characterization we need to prove a nontrivial result of abstract algebra about the finiteness of matrix semigroups (Theorem 8) which is also of interest in its own: if $\mathbb{K}$ is any finite commutative ring and $\mathbb{L}$ is any $\mathbb{K}$-algebra, then for every pair A, B of $n \times n$ matrices over $\mathbb{L}$ having the same characteristic polynomial, it holds that the set $\{A^0, A^1, A^2, \ldots\}$ is finite if and only if the set $\{B^0, B^1, B^2, \ldots\}$ is finite too. Let us point out that proving a new result in algebra in order to show that another new one holds in theoretical computer science is rather unusual and, we believe, interesting. As a matter of fact, in order to provide an efficiently computable characterization of sensitivity and equicontinuity for LCA over $(\mathbb{Z}/m\mathbb{Z})^n$, such an algebra result allows us to exploit, and this is the last ingredient we provide in Theorem 6, the extension to $(\mathbb{Z}/m\mathbb{Z})^n$ (for any integer $m > 1$) of our earlier result stating that these properties are decidable by means of an efficiently computable criterion for LCA over $(\mathbb{Z}/p^k\mathbb{Z})^n$ (for any prime p) defined by matrices in Frobenius normal form [14]. Finally, we want to stress that, beside an obvious theoretical value, Theorem 9 also has a practical one since LCA are often required to exhibit a strongly stable or strongly unstable behaviour (depending on the real-world situation they are modelling) in order they can be successfully used in applications.

The paper is structured as follows. Next section introduces all the necessary background and formal definitions. Section 3 recalls the known results about LCA over $(\mathbb{Z}/m\mathbb{Z})^n$ and presents the new ones, including the nontrivial algebra result about matrices with finitely many distinct powers (Theorem 8). Section 4 is entirely devoted to the proof of such a result. In the last section we draw our conclusion and provide some perspectives.

2. Background on DTDS and Cellular Automata

We begin by reviewing some general notions about discrete time dynamical systems and cellular automata.

A *discrete time dynamical system* (DTDS) is a pair $(\mathcal{X}, \mathcal{F})$, where $\mathcal{X}$ is any set equipped with a distance function d (i.e., $(\mathcal{X}, d)$ is a metric space) and $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a map that is continuous on $\mathcal{X}$ according to the topology induced by d.

[1] Such applications are based on non-uniform (or hybrid) CA, i.e., variants of CA where cells use different local rules (see [15, 17, 16]), in particular on the periodic linear ones over $\mathbb{Z}/m\mathbb{Z}$. We stress that linear periodic non-uniform CA of period n are homeomorphic to LCA over $(\mathbb{Z}/m\mathbb{Z})^n$.

Let $(\mathcal{X},\mathcal{F})$ be a DTDS. We say that it is *sensitive to the initial conditions* (or simply *sensitive*) if there exists $\varepsilon > 0$ such that for any $x \in \mathcal{X}$ and any $\delta > 0$ there is an element $y \in \mathcal{X}$ such that $0 < d(y,x) < \delta$ and $d(\mathcal{F}^k(y),\mathcal{F}^k(x)) > \varepsilon$ for some $k \in \mathbb{N}$. The system $(\mathcal{X},\mathcal{F})$ is said to be *equicontinuous* if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \mathcal{X}$, $d(x,y) < \delta$ implies that $\forall k \in \mathbb{N}$, $d(\mathcal{F}^k(x),\mathcal{F}^k(y)) < \varepsilon$. As dynamical properties, sensitivity and equicontinuity represent the main features of unstable and stable dynamical systems, respectively. The former is the well-known basic component and essence of the chaotic behavior of discrete time dynamical systems, while the latter is a strong form of stability.

We now recall some general notions about cellular automata.

Let S be a finite set. A configuration over S is a map from $\mathbb{Z}$ to S. We consider the following *space of configurations* $S^{\mathbb{Z}} = \{\boldsymbol{c} |\ \boldsymbol{c} : \mathbb{Z} \to S\}$. Each element $\boldsymbol{c} \in S^{\mathbb{Z}}$ can be visualized as an infinite one-dimensional cell lattice in which each cell $i \in \mathbb{Z}$ contains the element $\boldsymbol{c}_i \in S$.

Let $r \in \mathbb{N}$ and $\delta : S^{2r+1} \to S$ be any map. We say that r is the radius of δ.

Definition 1 (Cellular Automaton). A *one-dimensional CA based on a radius r local rule δ* is a pair $(S^{\mathbb{Z}}, F)$, where $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is the *global transition map* defined as follows:

$$\forall \boldsymbol{c} \in S^{\mathbb{Z}},\ \forall i \in \mathbb{Z}, \quad F(\boldsymbol{c})_i = \delta\left(\boldsymbol{c}_{i-r}, \ldots, \boldsymbol{c}_{i+r}\right). \tag{1}$$

We stress that the local rule δ completely determines the global rule F of a CA.

In order to study the dynamical properties of one-dimensional CA, we introduce a distance over the space of the configurations. Namely, $S^{\mathbb{Z}}$ is equipped with the Tychonoff distance d defined as follows

$$\forall \boldsymbol{c}, \boldsymbol{c}' \in S^{\mathbb{Z}}, \quad d(\boldsymbol{c}, \boldsymbol{c}') = \begin{cases} 0, & \text{if} \quad \boldsymbol{c} = \boldsymbol{c}', \\ 2^{-\min\{i \in \mathbb{N} : \boldsymbol{c}_i \neq \boldsymbol{c}'_i \text{ or } \boldsymbol{c}_{-i} \neq \boldsymbol{c}'_{-i}\}} & \text{otherwise} \ . \end{cases}$$

It is easy to verify that metric topology induced by d coincides with the product topology induced by the discrete topology on $S^{\mathbb{Z}}$. With this topology, $S^{\mathbb{Z}}$ is a compact and totally disconnected space and the global transition map F of any CA $(S^{\mathbb{Z}}, F)$ turns out to be (uniformly) continuous. Therefore, any CA itself is also a discrete time dynamical system. Moreover, any map $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is the global transition rule of a CA if and only if F is (uniformly) continuous and $F \circ \sigma = \sigma \circ F$, where $\sigma : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is the *shift map* defined as $\forall \boldsymbol{c} \in S^{\mathbb{Z}}$, $\forall i \in \mathbb{Z}$, $\sigma(\boldsymbol{c})_i = \boldsymbol{c}_{i+1}$. From now, when no misunderstanding is possible, we identify a CA with its global rule.

2.1. Linear Cellular Automata

We now recall the notion of linear CA. We stress that, whenever the term *linear* is involved, the alphabet S is $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{Z}/m\mathbb{Z}$ for some positive integer m. Both $\mathbb{K}^n$ and $(\mathbb{K}^n)^{\mathbb{Z}}$ become $\mathbb{K}$-modules in the obvious (i.e., entrywise) way.

A local rule $\delta : (\mathbb{K}^n)^{2r+1} \to \mathbb{K}^n$ of radius r is said to be *linear* if it is defined by $2r+1$ matrices $A_{-r}, \ldots, A_0, \ldots, A_r \in \mathbb{K}^{n \times n}$ as follows:

$$\forall (x_{-r}, \ldots, x_0, \ldots, x_r) \in (\mathbb{K}^n)^{2r+1}, \quad \delta(x_{-r}, \ldots, x_0, \ldots, x_r) = \sum_{i=-r}^{r} A_i \cdot x_i \ .$$

Definition 2 (Linear Cellular Automata (LCA)). A *linear CA (LCA)* over $\mathbb{K}^n$ is a CA based on a linear local rule.

Let $\mathbb{K}^n[X, X^{-1}]$ and $\mathbb{K}^n[[X, X^{-1}]]$ denote the set of *Laurent polynomials* and the set of *Laurent series*, respectively, with coefficients in $\mathbb{K}^n$. Before proceeding, let us recall that such formalisms have been successfully used to study the dynamical behaviour of LCA in the case $n = 1$ [23, 28]. Indeed, global rules and configurations are represented by Laurent polynomials and Laurent series, respectively, and the application of a global rule turns into a polynomial-series multiplication. In the more general case of LCA over $\mathbb{K}^n$, a configuration $\boldsymbol{c} \in (\mathbb{K}^n)^{\mathbb{Z}}$ can be associated with the Laurent series

$$\boldsymbol{P}_{\boldsymbol{c}}(X) = \sum_{i \in \mathbb{Z}} \boldsymbol{c}_i X^i = \begin{bmatrix} c^1(X) \\ \vdots \\ c^n(X) \end{bmatrix} = \begin{bmatrix} \sum_{i \in \mathbb{Z}} c_i^1 X^i \\ \vdots \\ \sum_{i \in \mathbb{Z}} c_i^n X^i \end{bmatrix} \in \left(\mathbb{K}[[X, X^{-1}]]\right)^n \cong \mathbb{K}^n[[X, X^{-1}]] \ .$$

Then, if F is the global rule of a LCA defined by $A_{-r},\dots,A_0,\dots,A_r$, one finds

$$\boldsymbol{P}_{F(\boldsymbol{c})}(X) = A \cdot \boldsymbol{P}_{\boldsymbol{c}}(X)$$

where

$$A = \sum_{i=-r}^{r} A_i X^{-i} \in \mathbb{K}[X,X^{-1}]^{n\times n}$$

is the *the matrix associated with the LCA* F. In this way, for any integer $k>0$ the matrix associated with F^k is A^k, and then $\boldsymbol{P}_{F^k(\boldsymbol{c})}(X) = A^k \cdot \boldsymbol{P}_{\boldsymbol{c}}(X)$. Clearly, the matrix A is nothing but a Laurent polynomial when $n=1$.

A matrix $A \in \mathbb{K}[X,X^{-1}]^{n\times n}$ is in *Frobenius normal form* if

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \mathfrak{a}_0 & \mathfrak{a}_1 & \mathfrak{a}_2 & \dots & \mathfrak{a}_{n-2} & \mathfrak{a}_{n-1} \end{bmatrix} \tag{2}$$

where each $\mathfrak{a}_i \in \mathbb{K}[X,X^{-1}]$. Recall that the coefficients of the characteristic polynomial of A are just the elements $\mathfrak{a}_i$ of the n-th row of A (up to sign).

Definition 3 (Frobenius LCA). A LCA $((\mathbb{K}^n)^{\mathbb{Z}}, F)$ is said to be a *Frobenius LCA* if the matrix $A \in \mathbb{K}[X,X^{-1}]^{n\times n}$ associated with F is in Frobenius normal form.

3. Results

We now deal with sensitivity and equicontinuity for LCA over $\mathbb{K}^n$. First of all, we remind that a dichotomy between sensitivity and equicontinuity holds for LCA. Moreover, these properties are characterized by the behaviour of the powers of the matrix associated with a LCA.

Proposition 4 ([14]). *Let $\left((\mathbb{K}^n)^{\mathbb{Z}}, F\right)$ be a LCA over $\mathbb{K}^n$ and let A be the matrix associated with F. The following statements are equivalent:*

1. *F is sensitive to the initial conditions;*
2. *F is not equicontinuous;*
3. $\left|\{A^1, A^2, A^3, \dots\}\right| = \infty$.

An immediate consequence of Proposition 4 is that any decidable characterization of sensitivity to the initial conditions in terms of the matrix associated with a LCA over $\mathbb{K}^n$ would also provide a decidable characterization of equicontinuity always in terms of that matrix. In the sequel, we are going to show that such a characterization actually exists and it is efficiently computable.

First of all, we remind that an easy to check characterization of sensitivity and equicontinuity was provided for the class of Frobenius LCA over the alphabet $(\mathbb{Z}/p^k\mathbb{Z})^n$, where p^k is a power of a prime number p.

Theorem 5 (Theorem 31 in [14]). *Consider any Frobenius LCA F over $(\mathbb{Z}/p^k\mathbb{Z})^n$ where p is a prime number and k is a positive integer. Let A be the matrix associated with F. Let $\mathfrak{a}_i \in (\mathbb{Z}/p^k\mathbb{Z})\left[X,X^{-1}\right]$ be the coefficients of the characteristic polynomial of A. The following equivalence holds: F is sensitive to the initial conditions if and only if at least one $\mathfrak{a}_i$ is the Laurent polynomial associated with a sensitive LCA over $\mathbb{Z}/p^k\mathbb{Z}$.*

We now derive an efficiently computable characterization of sensitivity and equicontinuity for the class of Frobenius LCA over the whole $\mathbb{K}^n$.

Theorem 6. *Consider any Frobenius LCA F over $\mathbb{K}^n$ and let A be the matrix associated with F. Let $\mathfrak{a}_i \in \mathbb{K}[X, X^{-1}]$ be the coefficients of the characteristic polynomial of A. The following equivalence holds: F is sensitive to the initial conditions (resp., equicontinuous) if and only if at least one (resp., each) $\mathfrak{a}_i$ is the Laurent polynomial associated with a sensitive (resp., equicontinuous) LCA over $\mathbb{K}$.*

Proof. We are going to prove the equivalence regarding sensitivity. Let $m = p_1^{k_1} \cdots p_l^{k_l}$ be the prime factor decomposition of m and, for any $s \in \{1, \ldots, l\}$, let $(A \mod {p_s}^{k_s})$ denote the matrix obtained by A taking all its entries modulo ${p_s}^{k_s}$. By an immediate generalization of [7, Lemma 3.2] to LCA over $(\mathbb{Z}/m\mathbb{Z})^n$ and by Theorem 5, it follows that F is sensitive to the initial conditions if and only if for some j the Frobenius LCA over $(\mathbb{Z}/{p_j}^{k_j}\mathbb{Z})^n$ with associated matrix $(A \mod {p_j}^{k_j})$ is sensitive to the initial conditions if and only if for some j at least one coefficient of the characteristic polynomial of $(A \mod {p_j}^{k_j})$ is the Laurent polynomial associated with a sensitive LCA over $\mathbb{Z}/{p_j}^{k_j}\mathbb{Z}$. Since the modulo operation well behaves with respect to the computation of the characteristic polynomial, i.e., for each $s \in \{1, \ldots, l\}$, $(\mathfrak{a}_s \mod {p_j}^{k_j})$ is equal to the s-th coefficient of the characteristic polynomial of $(A \mod {p_j}^{k_j})$, the latter proposition holds if and only if for some j at least one $(\mathfrak{a}_i \mod {p_j}^{k_j})$ is the Laurent polynomial associated with a sensitive LCA over $\mathbb{Z}/{p_j}^{k_j}\mathbb{Z}$ if and only if at least one $\mathfrak{a}_i$ is the Laurent polynomial associated with a sensitive LCA over $\mathbb{K}$. Therefore, the equivalence regarding sensitivity has been proved. The one about equicontinuity trivially follows from Proposition 4. □

Remark 7. Theorem 6 also provides an efficiently computable characterization of sensitivity and equicontinuity for the class of Frobenius LCA over $\mathbb{K}^n$. Indeed, the main dynamical properties for LCA over $\mathbb{K}$, including sensitivity and equicontinuity, are decidable by means of efficiently computable criteria that can be checked in polynomial time in $\log m$ and in the radius of the local rule [28].

In order to provide an efficiently computable characterization of equicontinuity and sensitivity for the whole class of LCA over $\mathbb{K}^n$, we need to prove the following algebra result whose proof is reported in Section 4.

Notation. Let $\mathbb{K}$ be any commutative ring. Let $n \in \mathbb{N}$. Let A be an $n \times n$-matrix over $\mathbb{K}$. We denote by χ_A the *characteristic polynomial* of A which is as usual defined as the polynomial $\det(tI_n - A) \in \mathbb{K}[t]$, where I_n stands for the $n \times n$ identity matrix and $tI_n - A$ is considered as an $n \times n$-matrix over the polynomial ring $\mathbb{K}[t]$.

Theorem 8. *Let $\mathbb{K}$ be any finite commutative ring. Let $\mathbb{L}$ be a commutative $\mathbb{K}$-algebra. Let $n \in \mathbb{N}$. Let A and B be two $n \times n$-matrices over $\mathbb{L}$ such that $\chi_A = \chi_B$. Then, the set $\{A^0, A^1, A^2, \ldots\}$ is finite if and only if the set $\{B^0, B^1, B^2, \ldots\}$ is finite.*

We are now able to prove the main result of the paper.

Theorem 9. *The following efficiently computable characterization of sensitivity and equicontinuity holds for LCA over $\mathbb{K}^n$. Let G be any LCA over $\mathbb{K}^n$ and let A be the matrix associated with G. The LCA G is sensitive to the initial conditions (resp., equicontinuous) if and only if the Frobenius LCA F over $\mathbb{K}^n$ such that $\chi_A = \chi_B$ is sensitive (resp., equicontinuous) too, where B is the matrix (in Frobenius normal form) associated with F.*

Proof. The thesis follows from Theorem 8, Proposition 4, Theorem 6, Remark 7, and from the fact that χ_A can be efficiently computed. □

For a sake of completeness, we recall that injectivity and surjectivity are decidable for LCA over $\mathbb{K}^n$ and, in particular, with an easy to check characterization. This result follows from a characterization of these properties in terms of the determinant of the matrix associated with a LCA and from efficiently computable criteria that allow deciding injectivity and surjectivity for LCA over $\mathbb{K}$ (for the latter, see [23]).

Theorem 10 ([6, 25])**.** *Injectivity and surjectivity are decidable for LCA over $\mathbb{K}^n$. In particular, a LCA F over $\mathbb{K}^n$ is injective (resp., surjective) if and only if the determinant of the matrix associated with F is the Laurent polynomial associated with an injective (resp., surjective) LCA over $\mathbb{K}$.*

The decidability of topological transitivity, ergodicity, and other mixing and ergodic properties for LCA over $\mathbb{K}^n$ has been recently proved in [11], where authors essentially showed the decidability of ergodicity for LCA over $\mathbb{K}^n$ and exploited the equivalence among all the mixing and ergodic properties for the wider class of additive CA over a finite abelian group [12].

Theorem 11 ([11])**.** *Let F be an LCA over $\mathbb{K}^n$. The following properties are both equivalent and decidable: (1) F is topologically transitive; (2) F is ergodic; (3) F is surjective and for every integer $k > 0$ it holds that $F^k - I$ is surjective; (4) F is topologically mixing; (5) F is weak topologically transitive; (6) F is totally transitive; (7) F is weakly ergodic mixing; (8) F is ergodic mixing.*

Finally, we stress that by virtue of Theorem 9 (besides, resp., Theorem 10 and 11), we have been able to lift the decidability of sensitivity and equicontinuity (besides, resp., injectivity, surjectivity, and all the above mentioned mixing and ergodic properties) from LCA over $\mathbb{K}^n$ to the wider class of additive CA over a finite abelian group [13, 10].

4. On matrices with finitely many distinct powers: proof of Theorem 8

The goal of this section is to prove Theorem 8. Let us start by illustrate it by means of the following example.

Example 12.

(a) Let $\mathbb{K} = \mathbb{Z}/4\mathbb{Z}$, $\mathbb{L} = (\mathbb{Z}/4\mathbb{Z})[x]$ (a polynomial ring), $n = 2$, $A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, $\chi_A = (t-1)^2 = \chi_B$. Hence, Theorem 8 yields that the set $\{A^0, A^1, A^2, \ldots\}$ is finite if and only if the set $\{B^0, B^1, B^2, \ldots\}$ is finite. Indeed, both of these sets are finite: The former has 4 elements, while the latter has 1.

(b) Now, let $\mathbb{K} = \mathbb{Q}$, $\mathbb{L} = \mathbb{Q}$, $n = 2$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, $\chi_A = (t-1)^2 = \chi_B$. The ring $\mathbb{K}$ is not finite, so Theorem 8 does not apply here. And we see why: The set $\{B^0, B^1, B^2, \ldots\}$ is finite, but the set $\{A^0, A^1, A^2, \ldots\}$ is not.

In order to prove Theorem 8, we proceed as follows. We first review in Section 4.1 the needed concepts and properties about *integrality over a commutative ring*. Then, in Section 4.2 we deal with the *characterization of integral matrices*. Finally, Section 4.3 contains the proof of Theorem 8 just after Proposition 24 which is essential for that proof since it shows the equivalence between the finiteness of the set $\{A^0, A^1, A^2, \ldots\}$ from the statement of Theorem 8 and the integrality of the matrix A itself.

Before proceeding, let us briefly discuss what rings $\mathbb{L}$ Theorem 8 applies to. Denote the unity of any ring $\mathbb{A}$ by $1_{\mathbb{A}}$. Consider any commutative ring $\mathbb{L}$. It is not difficult to show that there exist a finite commutative ring $\mathbb{K}$ and a $\mathbb{K}$-algebra structure on $\mathbb{L}$ if and only if there exists a positive integer m such that $m \cdot 1_{\mathbb{L}} = 0$ (the proof relies on the application of Lagrange's Theorem to the finite group $(\mathbb{K}, +)$ and the fact that the canonical ring homomorphism $\mathbb{Z} \to \mathbb{L}$, $a \mapsto a \cdot 1_{\mathbb{L}}$ factors through the quotient ring $\mathbb{Z}/m\mathbb{Z}$). As a consequence, we can restate Theorem 8 as follows:

Corollary 13. *Let $\mathbb{L}$ be a commutative ring. Assume that there exists a positive integer m such that $m \cdot 1_{\mathbb{L}} = 0$. Let $n \in \mathbb{N}$. Let A and B be two $n \times n$-matrices over $\mathbb{L}$ such that $\chi_A = \chi_B$. Then, the set $\{A^0, A^1, A^2, \ldots\}$ is finite if and only if the set $\{B^0, B^1, B^2, \ldots\}$ is finite.*

Remark 14. A converse of this corollary holds as well: Let $\mathbb{L}$ be a commutative ring for which there is **no** positive integer m such that $m \cdot 1_{\mathbb{L}} = 0$. Let $n \geq 2$ be an integer. Then, there exist two $n \times n$-matrices A and B over $\mathbb{L}$ such that $\chi_A = \chi_B$ and the set $\{A^0, A^1, A^2, \ldots\}$ is infinite but the set $\{B^0, B^1, B^2, \ldots\}$ is finite. Such matrices can easily be constructed by imitation of Example 12 **(b)**.

In the following, semigroups will always be written multiplicatively: That is, if M is a semigroup, then the operation of M will be written as multiplication (i.e., we will write ab for the image of $(a, b) \in M \times M$ under this operation).

4.1. Integrality basics

Our proof will rely on some basic properties of integrality over a commutative ring. This concept is defined as follows:

Definition 15. Let $\mathbb{K}$ be a commutative ring. Let $\mathbb{L}$ be a $\mathbb{K}$-algebra (not necessarily commutative). An element $u \in \mathbb{L}$ is said to be *integral over* $\mathbb{K}$ if and only if there exists a monic polynomial $f \in \mathbb{K}[t]$ such that $f(u) = 0$.

Recall that a polynomial is said to be *monic* if its leading coefficient is 1. Definition 15 generalizes [22, Definition 2.1.1] from commutative ring extensions to arbitrary algebras, and generalizes [26, Definition (10.21)] from commutative $\mathbb{K}$-algebras $\mathbb{L}$ to arbitrary $\mathbb{K}$-algebras $\mathbb{L}$.

Philosophically, there is a similarity between integral elements of a $\mathbb{K}$-algebra, and "finite-order" elements of a semigroup (i.e., elements a such that the set $\left\{a^1, a^2, a^3, \ldots\right\}$ is finite). In Proposition 24, we shall see a direct connection between these two concepts, but even before that, the similarity is helpful as a guide.

Definition 16. Let $\mathbb{K}$ be a commutative ring. Let M be a $\mathbb{K}$-module, and let $n \in \mathbb{N}$.

(a) If $m_1, m_2, \ldots, m_n$ are n elements of M, then we let $\langle m_1, m_2, \ldots, m_n \rangle_{\mathbb{K}}$ denote the $\mathbb{K}$-submodule of M spanned by $m_1, m_2, \ldots, m_n$. This $\mathbb{K}$-submodule is called the $\mathbb{K}$-*linear span* of $m_1, m_2, \ldots, m_n$. A similar notation will be used for spans of infinitely many elements.

(b) We say that the $\mathbb{K}$-module M is *n-generated* if and only if there exist n elements $m_1, m_2, \ldots, m_n \in M$ such that $M = \langle m_1, m_2, \ldots, m_n \rangle_{\mathbb{K}}$.

We notice that a $\mathbb{K}$-module M is finitely generated if and only if there is some $n \in \mathbb{N}$ such that M is n-generated.

The following fact provides several criteria for when an element of a commutative $\mathbb{K}$-algebra is integral over $\mathbb{K}$:

Theorem 17 (Theorem 1.1 in [20])**.** *Let $\mathbb{K}$ be a commutative ring. Let $\mathbb{L}$ be a commutative $\mathbb{K}$-algebra. Let $n \in \mathbb{N}$. Let $u \in \mathbb{L}$. Then, the following assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are equivalent:*

- *Assertion $\mathcal{A}$: There exists a monic polynomial $f \in \mathbb{K}[t]$ of degree n such that $f(u) = 0$.*
- *Assertion $\mathcal{B}$: There exist an $\mathbb{L}$-module C and an n-generated $\mathbb{K}$-submodule U of C such that $uU \subseteq U$ and such that every $v \in \mathbb{L}$ satisfying $vU = 0$ satisfies $v = 0$. (Here, we are making use of the fact that each $\mathbb{L}$-module canonically becomes a $\mathbb{K}$-module, since $\mathbb{L}$ is a $\mathbb{K}$-algebra.)*
- *Assertion $\mathcal{C}$: There exists an n-generated $\mathbb{K}$-submodule U of $\mathbb{L}$ such that $1 \in U$ and $uU \subseteq U$.*
- *Assertion $\mathcal{D}$: We have $\mathbb{K}[u] = \left\langle u^0, u^1, \ldots, u^{n-1} \right\rangle_{\mathbb{K}}$.*

Note that Theorem 17 is just one of several "determinantal tricks" used in studying integrality over rings. See [5, Chapter V, Section 1.1, Theorem 1] or [9, Theorem 8.1.6] for another. We shall only use the implications $\mathcal{B} \Longrightarrow \mathcal{A}$ and $\mathcal{A} \Longrightarrow \mathcal{D}$ of Theorem 17.

We now draw the following conclusion from Theorem 17:

Corollary 18. *Let $\mathbb{K}$ be a commutative ring. Let $\mathbb{L}$ be a commutative $\mathbb{K}$-algebra. Let $u \in \mathbb{L}$. Let C be an $\mathbb{L}$-module. Let U be a finitely generated $\mathbb{K}$-submodule of C such that $uU \subseteq U$. Assume that every $v \in \mathbb{L}$ satisfying $vU = 0$ satisfies $v = 0$. (Again, we are making use of the fact that each $\mathbb{L}$-module canonically becomes a $\mathbb{K}$-module.)*

Then, $u \in \mathbb{L}$ is integral over $\mathbb{K}$.

Proof. The $\mathbb{K}$-module U is finitely generated. In other words, it is n-generated for some $n \in \mathbb{N}$. Consider this n. Thus, Assertion $\mathcal{B}$ of Theorem 17 is satisfied. Hence, Assertion $\mathcal{A}$ of Theorem 17 is satisfied as well, i.e., there exists a monic polynomial $f \in \mathbb{K}[t]$ of degree n such that $f(u) = 0$. Therefore, u is integral over $\mathbb{K}$. □

We conclude this part by recalling the following result which will be useful in the sequel:

Theorem 19 (implication (2) $\Longrightarrow$ (3) of Theorem (10.28) in [26])**.** *Let $\mathbb{K}$ be a commutative ring. Let $\mathbb{L}$ be a commutative $\mathbb{K}$-algebra. Let $u_1, u_2, \ldots, u_m$ be a finite list of elements of $\mathbb{L}$. Assume that these m elements $u_1, u_2, \ldots, u_m$ are all integral over $\mathbb{K}$, and generate $\mathbb{L}$ as a $\mathbb{K}$-algebra. Then, the $\mathbb{K}$-module $\mathbb{L}$ is finitely generated.*

4.2. Characterizing integral matrices

The following theorem characterizes integral matrices.

Theorem 20 ((a) $\Longleftrightarrow$ (c) of Proposition 17, Section 1.6, Chapter V in [5]). *Let $\mathbb{K}$ be a commutative ring. Let $n \in \mathbb{N}$. Let $\mathbb{L}$ be a commutative $\mathbb{K}$-algebra. Let A be an $n \times n$-matrix over $\mathbb{L}$. The matrix A is integral over $\mathbb{K}$ (as an element of the $\mathbb{K}$-algebra $\mathbb{L}^{n\times n}$) if and only if each coefficient of the characteristic polynomial $\chi_A \in \mathbb{L}[t]$ is integral over $\mathbb{K}$.*

As a consequence of Theorem 20, we get the following result.

Corollary 21. *Let $\mathbb{K}$ be a commutative ring. Let $n \in \mathbb{N}$. Let $\mathbb{L}$ be a commutative $\mathbb{K}$-algebra. Let A be an $n \times n$-matrix over $\mathbb{L}$. Assume that A is integral over $\mathbb{K}$ (as an element of the $\mathbb{K}$-algebra $\mathbb{L}^{n\times n}$). Let $\mathbb{M}$ be the $\mathbb{K}$-subalgebra of $\mathbb{L}$ generated by the coefficients of the characteristic polynomial $\chi_A \in \mathbb{L}[t]$. Then, $\mathbb{M}$ is a finitely generated $\mathbb{K}$-module.*

Proof. Let $u_1, u_2, \ldots, u_m$ be the coefficients of the polynomial χ_A. These coefficients are integral over $\mathbb{K}$ (by Theorem 20), and generate $\mathbb{M}$ as a $\mathbb{K}$-algebra (by the definition of $\mathbb{M}$); thus, in particular, they are elements of $\mathbb{M}$. Hence, Theorem 19 (applied to $\mathbb{M}$ instead of $\mathbb{L}$) yields that the $\mathbb{K}$-module $\mathbb{M}$ is finitely generated. □

4.3. Proof of Theorem 8

We need two more lemmata about finite generation of certain modules:

Lemma 22. *Let $\mathbb{K}$ be a finite commutative ring. Let M be a finitely generated $\mathbb{K}$-module. Then, M is finite (as a set).*

Proof. The $\mathbb{K}$-module M is finitely generated. In other words, there exist finitely many vectors $a_1, a_2, \ldots, a_m \in M$ that generate M as a $\mathbb{K}$-module. Consider these $a_1, a_2, \ldots, a_m$. Thus, each element of M is a $\mathbb{K}$-linear combination of $a_1, a_2, \ldots, a_m$. Since $\mathbb{K}$ is finite, there exist only finitely many $\mathbb{K}$-linear combinations of $a_1, a_2, \ldots, a_m$. Hence, there are only finitely many elements of M. In other words, M is finite. □

Lemma 23. *Let $\mathbb{K}$ be a commutative ring. Let $f \in \mathbb{K}[t]$ be a monic polynomial. Then, the $\mathbb{K}$-module $\mathbb{K}[t]/(f)$ is finitely generated.*

Proof. Much more can be said: For each $u \in \mathbb{K}[t]$, we let $\overline{u}$ denote the projection of u onto $\mathbb{K}[t]/(f)$. Then, the $\mathbb{K}$-module $\mathbb{K}[t]/(f)$ is free with basis $\left(\overline{t^0}, \overline{t^1}, \ldots, \overline{t^{n-1}}\right)$, where $n = \deg f$. This is a well-known fact[2] and follows easily from "Euclidean division of polynomials". Of course, this entails that the $\mathbb{K}$-module $\mathbb{K}[t]/(f)$ is finitely generated. □

The following fact will bring us very close to Theorem 8.

Proposition 24. *Let $\mathbb{K}$ be a finite commutative ring. Let $n \in \mathbb{N}$. Let $\mathbb{L}$ be a commutative $\mathbb{K}$-algebra. Let A be an $n \times n$-matrix over $\mathbb{L}$. Then, the following three assertions are equivalent:*

- *Assertion $\mathcal{U}$: The set $\left\{A^0, A^1, A^2, \ldots\right\}$ is finite.*
- *Assertion $\mathcal{V}$: The matrix A is integral over $\mathbb{K}$ (as an element of the $\mathbb{K}$-algebra $\mathbb{L}^{n\times n}$).*
- *Assertion $\mathcal{W}$: There exists a positive integer m such that the polynomial $t^{2m} - t^m$ is a multiple of χ_A in $\mathbb{L}[t]$.*

Proof. We shall prove the implications $\mathcal{U} \Longrightarrow \mathcal{V}$ and $\mathcal{V} \Longrightarrow \mathcal{W}$ and $\mathcal{W} \Longrightarrow \mathcal{U}$:

$\mathcal{U} \Longrightarrow \mathcal{V}$: Assume that Assertion $\mathcal{U}$ holds. The set $\left\{A^0, A^1, A^2, \ldots\right\}$ is closed under multiplication. Thus, this set (equipped with multiplication) is a semigroup. Furthermore, this set is finite (since Assertion $\mathcal{U}$ holds), and thus is a finite semigroup. By [30, Corollary 1.2] or [19, Proposition 6.31], there exists a positive integer m such that $A^m = A^{2m}$. Consider this m. Let $g \in \mathbb{K}[t]$ be the polynomial $t^{2m} - t^m$. Then, g is monic (since $m > 0$) and satisfies

[2] See, e.g., [1, Chapter III, Proposition 4.6] for an equivalent version of this fact (restated in terms of an isomorphism $\mathbb{K}[t]/(f) \to \mathbb{K}^{\oplus n}$).

$g(A) = A^{2m} - A^m = 0$ (since $A^m = A^{2m}$). Hence, there exists a monic polynomial $f \in \mathbb{K}[t]$ such that $f(A) = 0$ (namely, $f = g$). In other words, A is integral over $\mathbb{K}$, i.e., Assertion $\mathcal{V}$ holds.

$\mathcal{V} \Longrightarrow \mathcal{W}$: Assume that Assertion $\mathcal{V}$ holds. Let $\mathbb{M}$ be the $\mathbb{K}$-subalgebra of $\mathbb{L}$ generated by the coefficients of the characteristic polynomial $\chi_A \in \mathbb{L}[t]$. Then, the coefficients of χ_A belong to this $\mathbb{K}$-subalgebra $\mathbb{M}$; thus, $\chi_A \in \mathbb{M}[t]$. Furthermore, Corollary 21 shows that $\mathbb{M}$ is a finitely generated $\mathbb{K}$-module. Thus, Lemma 22 (applied to $M = \mathbb{M}$) shows that $\mathbb{M}$ is finite (as a set).

The polynomial $\chi_A \in \mathbb{M}[t]$ is monic. Thus, the $\mathbb{M}$-module $\mathbb{M}[t]/(\chi_A)$ is finitely generated (by Lemma 23, applied to $\mathbb{M}$ and χ_A instead of $\mathbb{K}$ and f). Thus, Lemma 22 (applied to $\mathbb{M}$ and $\mathbb{M}[t]/(\chi_A)$ instead of $\mathbb{K}$ and M) shows that $\mathbb{M}[t]/(\chi_A)$ is finite (as a set). This ring $\mathbb{M}[t]/(\chi_A)$ becomes a semigroup when equipped with its multiplication. This semigroup $\mathbb{M}[t]/(\chi_A)$ is finite (since we have just shown that $\mathbb{M}[t]/(\chi_A)$ is finite).

For each $u \in \mathbb{M}[t]$, we let $\overline{u}$ denote the projection of u onto $\mathbb{M}[t]/(\chi_A)$. By [30, Corollary 1.2] or [19, Proposition 6.31], there exists a positive integer m such that $\overline{t}^m = \overline{t}^{2m}$. Consider this m. Then, $\overline{t^m} = \overline{t}^m = \overline{t}^{2m} = \overline{t^{2m}}$; in other words, we have the congruence $t^m \equiv t^{2m} \bmod \chi_A$ in the ring $\mathbb{M}[t]$. In other words, the polynomial $t^{2m} - t^m$ is a multiple of χ_A in $\mathbb{M}[t]$. Hence, the polynomial $t^{2m} - t^m$ is a multiple of χ_A in $\mathbb{L}[t]$ (since $\mathbb{M}[t]$ is a subring of $\mathbb{L}[t]$). Thus, Assertion $\mathcal{W}$ holds.

$\mathcal{W} \Longrightarrow \mathcal{U}$: Assume that Assertion $\mathcal{W}$ holds. Let m be a positive integer such that the polynomial $t^{2m} - t^m$ is a multiple of χ_A in $\mathbb{L}[t]$. Note that $2m$ and m are positive integers satisfying $2m > m$. Consider the ring $\mathbb{L}^{n\times n}$ as a semigroup (equipped with its multiplication).

Now, there exists a polynomial $g \in \mathbb{L}[t]$ such that $t^{2m} - t^m = \chi_A \cdot g$ (since the polynomial $t^{2m} - t^m$ is a multiple of χ_A in $\mathbb{L}[t]$). Consider this g. Evaluating both sides of the polynomial identity $t^{2m} - t^m = \chi_A \cdot g$ at A, by the Cayley–Hamilton theorem, we obtain

$$A^{2m} - A^m = \chi_A(A) \cdot g(A) = 0 \cdot g(A) = 0$$

In other words, $A^{2m} = A^m$. Since $\mathbb{L}^{n\times n}$ is a semigroup, we get $\left\{A^1, A^2, A^3, \ldots\right\} = \left\{A^1, A^2, \ldots, A^{2m-1}\right\}$ Thus, the set $\left\{A^1, A^2, A^3, \ldots\right\}$ is finite. Hence, the set $\left\{A^0, A^1, A^2, \ldots\right\}$ is also finite. In other words, Assertion $\mathcal{U}$ holds. □

We can now prove Theorem 8:

Proof of Theorem 8. Proposition 24 (or, more precisely, the equivalence of the Assertions $\mathcal{U}$ and $\mathcal{W}$ in this proposition) shows that the set $\left\{A^0, A^1, A^2, \ldots\right\}$ is finite if and only if there exists a positive integer m such that the polynomial $t^{2m} - t^m$ is a multiple of χ_A in $\mathbb{L}[t]$. Since $\chi_A = \chi_B$, the set $\left\{A^0, A^1, A^2, \ldots\right\}$ is finite if and only if there exists a positive integer m such that the polynomial $t^{2m} - t^m$ is a multiple of χ_B in $\mathbb{L}[t]$. Therefore, the following logical equivalence holds: the set $\left\{A^0, A^1, A^2, \ldots\right\}$ is finite if and only if the set $\left\{B^0, B^1, B^2, \ldots\right\}$ is finite. □

5. Conclusions

We have provided an efficiently computable characterization of sensitivity to the initial conditions and equicontinuity for one-dimensional LCA over $\mathbb{K}^n$. To get such a characterization we have proved an algebra result about the finiteness of matrix semigroups which is also of interest in its own.

Providing an efficiently computable characterization for other interesting dynamical properties such as expansivity and strong transitivity for LCA over $\mathbb{K}^n$ is the first step for further researches in this domain.

Furthermore, an important research direction consists in generalizing our results (in terms of efficiently computable characterizations), on one hand, to D-dimensional LCA over $\mathbb{K}^n$, and, on the other hand, to additive CA over a possibly non abelian group. This would also allow to build more robust methods based on such CA in several applications.

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