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STABILITY OF THE MEAN VALUE FORMULA FOR HARMONIC FUNCTIONS IN LEBESGUE SPACES

GIOVANNI CUPINI, ERMANNO LANCONELLI

ABSTRACT. Let D be an open subset of $\mathbb{R}^n$ with finite measure, and let $x_0 \in D$. We introduce the *p-Gauss gap* of D w.r.t. x_0 to measure how far are the averages over D of the harmonic functions $u \in L^p(D)$ from $u(x_0)$. We estimate from below this gap in terms of the *ball gap* of D w.r.t. x_0, i.e., the normalized Lebesgue measure of $D \backslash B$, being B the biggest ball centered at x_0 contained in D. From these *stability* estimates of the mean value formula for harmonic functions in L^p-spaces, we straightforwardly obtain *rigidity* properties of the Euclidean balls. We also prove a continuity result of the p-Gauss gap in the Sobolev space $W^{1,p'}$, where p' is the conjugate exponent of p.

1. INTRODUCTION

Let D be an open subset of $\mathbb{R}^n$, $n \geq 2$, and let $B(x_0, r)$ be the open Euclidean ball with center x_0 and radius $r > 0$. If $\overline{B(x_0, r)} \subseteq D$, by the Gauss mean value Theorem,

$$u(x_0) = \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} u(y)\, dy, \qquad \forall u \in \mathcal{H}(D),$$

where $\mathcal{H}(D)$ denotes the linear space of the harmonic functions in D and $|B(x_0, r)|$ stands for the Lebesgue measure of $B(x_0, r)$. By the dominated convergence theorem, it follows that

$$u(x_0) = \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} u(y)\, dy, \qquad \forall u \in \mathcal{H}(B(x_0, r)) \cap L^1(B(x_0, r)). \tag{1.1}$$

In literature the *stability* and the *rigidity* properties of (1.1) have been studied. The question of *rigidity*, a sort of an inverse problem for (1.1), is the oldest one treated in literature and is related to the following question:

if D is an open set with finite Lebesgue measure containing x_0, such that the mean integral of harmonic functions in L^1 on D equals the value of these functions at x_0, then is D a ball centered at x_0?

The historical development of this problem, together with a comprehensive collection of results, is contained in the excellent survey [21] by Netuka and Veselý. Here we only quote a theorem by Kuran, who definitely gave a positive answer to the previous question, providing a short and elegant proof, see [18]. In his paper Kuran introduced a harmonic test function (see (1.5)) which will play a crucial role in the present paper. We also remark that a domain satisfying the mean value property for any harmonic functions is the simplest instance of a so-called quadrature domain. Quadrature domains have been extensively studied ever since the 1960's. A good source of reference is the survey article [17].

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The other question, very recently introduced in literature, is that of the *stability* of (1.1):

if the mean integral of every harmonic functions in L^1 on an open set D is "almost" equal to the value of these functions at x_0 in D, then is D "almost" a ball with center x_0?

The answer to this question is affirmative and can be found in the recent paper [11] in a joint collaboration with N. Fusco and X. Zhong.

In the present paper we face the problem of stability for harmonic functions in L^p with $p \in]1, \infty]$.

Our *L^p-stability theorems* straightforwardly imply rigidity results which, in the case $1 < p < \frac{n}{n-1}$, were proved by Goldstein, Haussmann and Rogge in [16, Theorem 3 (B)]. With a direct proof, modelled on the one of our stability results, we also obtain a rigidity result for test functions in $\mathcal{H}(\overline{D})$, in the same spirit of [16, Theorem 1].

We remark that our technique to prove the stability result Theorem 4.1 does not seem suitable to obtain a similar result for the Gauss gap related to the surface average. An interesting stability result in this direction has been obtained in dimension $n = 2$ by Agostiniani and Magnanini in [1]. We point out that many stability problems in various settings have been investigated. We limit ourselves to mention the papers [3], [4], [5], [7], [8], [9], [10], [13], [14], [15], [19], [20], [22].

1.1. **Stability results.** Our stability results are proved in Section 4. To describe them we need to mathematically formalize the two "almost" appearing in the naive formulation of the problem.

Given an open set $D \subseteq \mathbb{R}^n$ of finite Lebesgue measure ($|D| < \infty$), we denote $\mathcal{H}^p(D)$ the space of the harmonic functions in D that are p-summable in D with respect the Lebesgue measure, i.e.,

$$\mathcal{H}^p(D) := \mathcal{H}(D) \cap L^p(D).$$

In $\mathcal{H}^p(D)$ we introduce the following norm:

$$\|u\|_{\widetilde{L}^p(D)} := \left(⨍_D |u(x)|^p \, dx \right)^{\frac{1}{p}} \qquad \text{if } p < \infty$$

and

$$\|u\|_{\widetilde{L}^\infty(D)} := \|u\|_{L^\infty(D)},$$

where, as usual,

$$⨍_D \cdot \, dx := \frac{1}{|D|} \int_D \cdot \, dx.$$

We now introduce the *p-Gauss mean value gap*, that we use to measure how close is the mean integral of the harmonic functions in L^p on the set D to the value of these functions at x_0 in D.

Given an open set D of finite measure, $x_0 \in D$ and $p \in [1, \infty]$, we define the *p-Gauss mean value gap of D w.r.t. x_0* as

$$\mathcal{G}_p(D, x_0) := \sup_{u \in \mathcal{H}^p(D) \setminus \{0\}} \frac{\left| u(x_0) - ⨍_D u(x) \, dx \right|}{\|u\|_{\widetilde{L}^p(D)}}. \tag{1.2}$$

We devote Section 2 to discuss the main properties of this function, in particular we will prove that the p-Gauss gap is always finite and that the supremum is attained if $1 < p \leq \infty$, see Proposition 2.2.

Taking into account that in the stability problem for the mean value formula the point $x_0 \in D$ plays a privileged role, to measure how far is D from being a ball centered at x_0 we will use the following coefficient:

$$\mathcal{B}(D, x_0) := \frac{|D \setminus B(x_0, r_{x_0})|}{|D|}, \qquad \text{where } r_{x_0} := \operatorname{dist}(x_0, \partial D), \tag{1.3}$$

that we will call the *ball gap* of D with respect to x_0.

The stability question is the following:

$$\textit{fixed } p \in [1, \infty] \textit{ is it true that } \lim_{\mathcal{G}_p(D,x_0)\to 0} \mathcal{B}(D, x_0) = 0?$$

If $p = 1$ the answer to this question is affirmative, since as it is proved in [11],

$$c(n)\mathcal{B}(D, x_0) \leq \mathcal{G}_1(D, x_0). \tag{1.4}$$

Now, in Section 4 we deal with $1 < p \leq \infty$.

A dichotomy (both regarding the exponent and the regularity of the domain) depending on the exponent p appears and the threshold exponent is given by $\frac{n}{n-1}$.

In Theorem 4.1 we consider the case $1 < p < \frac{n}{n-1}$ and we prove that

$$c(n, p)\mathcal{B}(D, x_0) \leq \mathcal{G}_p(D, x_0) \quad \forall D \subset \mathbb{R}^n, D \text{ open set}, \ |D| < \infty.$$

This estimate, similar to the one obtained for $p = 1$, is sharp since for ellipsoids centered at x_0 the p-Gauss gap can be bounded also from above by the ball gap (see Proposition 4.2). The strategy of the proof of Theorem 4.1, similar to that successfully used for $p = 1$, is based on the use of the test function used by Kuran to prove his rigidity result: given the open D, $x_0 \in D$ and $\bar{x} \in \partial D \cap \partial B(x_0, r_{x_0})$, where, as above, r_{x_0} is the radius of the greatest ball centered at x_0 contained in D, the function considered by Kuran is

$$h_{\bar{x}}(x) = 1 + |x_0 - \bar{x}|^{n-2} \frac{|x - x_0|^2 - |x_0 - \bar{x}|^2}{|x - \bar{x}|^n} \qquad x \in \mathbb{R}^n \setminus \{\bar{x}\}. \tag{1.5}$$

We will refer to this function as the *Kuran's function* and in Section 3 we list and prove its main properties, the main one being that $h_{\bar{x}}$ is in $\mathcal{H}^p(D)$ for any $1 \leq p < \frac{n}{n-1}$, enabling us to use $h_{\bar{x}}$ to estimate from below the p-Gauss gap.

If $\frac{n}{n-1} \leq p \leq \infty$ a stability inequality is given in Theorem 4.3. This result is much more delicate and requires a deeper analysis to overcome the lack of the right summability property of $h_{\bar{x}}$. Under a suitable exterior cone condition on D, a bound for $\mathcal{G}_p(D, x_0)$ of (substantially) Hölder type is proved if $p > \frac{n}{n-1}$, and one of log-Lipschitz type for $p = \frac{n}{n-1}$. While our stability result for $p < \frac{n}{n-1}$ is sharp, the sharpness of our estimate for $p \geq \frac{n}{n-1}$ is still an open problem.

1.2. **$W^{1,p'}$-continuity of the p-Gauss gap.** For the 1-Gauss gap in [11] it is proved that $C^{1,\alpha}$-convergence of open sets to a Euclidean ball, for any $\alpha \in]0, 1[$, forces the 1-Gauss gap to go to zero and an example shows that this result is no more true if the $C^{1,\alpha}$-convergence of open sets is replaced by the weaker $W^{1,q}$-convergence for every $q > 1$.

In Section 5, we prove that $W^{1,p'}$-convergence of domains to a Euclidean ball forces the p-Gauss gap to go to zero if $1 < p < \frac{n}{n-1}$. Here, as usual, p' denotes the the conjugate exponent of p; i.e. p' is the real number such that $\frac{1}{p} + \frac{1}{p'} = 1$. We refer to Theorem 5.1 for the precise statement. We point out that the proof of this result relies on a continuity result for the p-Gauss gap for every real $p > 1$, see Proposition 5.2. The main tools to prove this proposition are the *densities with the mean value property* defined in [2], see also [12], and the deep L^p-estimate for the gradient of the solutions to the Dirichlet problem on Reifenberg-flat domains, see [6].

1.3. **Rigidity results.** Section 6 is devoted to the rigidity question. As remarked in [11], the estimate from below of the 1-*Gauss gap* with the *ball gap*, see (1.4), immediately implies Kuran's result, i.e.,

if D is an open set in $\mathbb{R}^n$, $|D| < \infty$, $x_0 \in D$, and the mean value formula holds true

$$u(x_0) = \fint_D u(x)\,dx \tag{1.6}$$

for every $u \in \mathcal{H}(D) \cap L^1(D)$, then D is a ball centered at x_0.

This implication (stability $\Rightarrow$ rigidity) turns out to be true also for harmonic functions in L^p. Indeed, the stability result Theorem 4.1 implies that if (1.6) holds for every $u \in \mathcal{H}^p(D)$, $1 < p < \frac{n}{n-1}$, then D is a ball centered at x_0. This rigidity result, Corollary 6.1, has been proved in a direct way, in [16].

For $p \geq \frac{n}{n-1}$ we looked for a direct proof of a rigidity result, so to avoid the exterior cone condition assumed in the corresponding stability result, Theorem 4.3. This independent proof allows to prove the following result:

[Theorem 6.2] *Let D be an open set in $\mathbb{R}^n$, $|D| < \infty$, $x_0 \in D$, and assume that* (1.6) *holds for every $u \in \mathcal{H}(\overline{D})$. If $D = \operatorname{int} \overline{D}$ then D is a ball centered at x_0.*

This result is in the spirit of [16, Theorem 1], where a similar rigidity result is proved for bounded open sets and test functions in $\mathcal{H}(\mathbb{R}^n)$, see Section 6 below for details.

We remark that, since

$$\mathcal{H}(\overline{D}) \subset \bigcap_{1 \leq p \leq \infty} \mathcal{H}^p(D),$$

then Theorem 6.2 implies a rigidity result where the class of test functions is $\mathcal{H}^p(D)$, for $\frac{n}{n-1} \leq p \leq \infty$.

The plan of the paper is the following: in Sections 2 and 3 we discuss some properties of the p-Gauss gap and of Kuran's function, respectively. In Section 4 we state and prove our stability results, in Section 5 we present the $W^{1,p'}$-continuity result for the p-Gauss gap. The last section is devoted to the rigidity issue.

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2. THE p-GAUSS GAP

In this section we discuss some properties of the p-Gauss gap defined in (1.2), $p \in [1, \infty]$.

It is easy to verify that $\mathcal{G}_p(D, x_0)$ is translation and scale invariant; i.e., for every $y \in \mathbb{R}^n$ and for every $\lambda > 0$,

$$\mathcal{G}_p(D, x_0) = \mathcal{G}_p(y + D, y + x_0) \qquad \text{and} \qquad \mathcal{G}_p(\lambda D, \lambda x_0) = \mathcal{G}_p(D, x_0). \tag{2.1}$$

This immediately follows by the translations and dilations invariance of harmonicity and Lebesgue measure.

The following lemma is a straightforward consequence of Hölder's inequality.

Lemma 2.1. *Given an open set $D \subseteq \mathbb{R}^n$ of finite measure and $x_0 \in D$ then for every $1 \le p < q < \infty$*

$$G_\infty(D, x_0) \le G_q(D, x_0) \le \mathcal{G}_p(D, x_0) \le \mathcal{G}_1(D, x_0) \le 1 + \frac{|D|}{|B(x_0, r_{x_0})|},$$

where $r_{x_0} := \operatorname{dist}(x_0, \partial D)$.

Proof. The last inequality is trivial and it also appears in [11]. Let us discuss the other inequalities.

By Holder's inequality, for every $1 \le p < q < \infty$ and every $u \in L^q(D)$

$$\left(\int_D |u(x)|^p \, dx\right)^{\frac{1}{p}} \le \left(\int_D |u(x)|^q \, dx\right)^{\frac{1}{q}} |D|^{\frac{1}{p} - \frac{1}{q}},$$

therefore

$$\|u\|_{\widetilde{L}^p(D)} \le \|u\|_{\widetilde{L}^q(D)}.$$

Since $L^q(D) \subseteq L^p(D)$ we easily get

$$G_q(D, x_0) \le \mathcal{G}_p(D, x_0).$$

Let us now prove that $G_\infty(D, x_0) \le G_q(D, x_0)$ if $1 \le q < \infty$. For every $u \in L^\infty(D)$

$$\int_D |u|^q \, dx \le \|u\|^q_{L^\infty(D)} |D| = \|u\|^q_{\widetilde{L}^\infty(D)} |D|,$$

that implies

$$\|u\|_{\widetilde{L}^q(D)} \le \|u\|_{\widetilde{L}^\infty(D)}.$$

Since $L^\infty(D) \subseteq L^q(D)$ we get $G_\infty(D, x_0) \le G_q(D, x_0)$. □

In the following proposition, we prove that if $1 < p \le \infty$ then the p-Gauss gap, that is defined as a supremum, is attained.

Proposition 2.2. *Given an open set $D \subseteq \mathbb{R}^n$ of finite measure and $x_0 \in D$, then for every $p \in]1, \infty]$*

$$\mathcal{G}_p(D, x_0) = \max_{u \in \mathcal{H}^p(D), \|u\|_{\widetilde{L}^p(D)} = 1} \left| u(x_0) - \fint_D u(x) \, dx \right|.$$

Proof. If $\mathcal{G}_p(D, x_0) = 0$ we have nothing to prove.

Assume $\mathcal{G}_p(D, x_0) > 0$.

Let (u_j) be a sequence in $\mathcal{H}(D)$, such that

$$\|u_j\|_{\widetilde{L}^p(D)} = 1, \tag{2.2}$$

and

$$\mathcal{G}_p(D, x_0) = \lim_{j \to \infty} \left| u_j(x_0) - \fint_D u_j(x) \, dx \right|.$$

Since $1 < p \le \infty$, by (2.2) there exists $\hat{u} \in L^p(D)$, such that, up to subsequences:

$$(u_j) \text{ weakly converges to } \hat{u} \text{ in } L^p(D) \quad \text{if } 1 < p < \infty,$$
$$(u_j) \text{ $*$-weakly converges to } \hat{u} \text{ in } L^\infty(D) \quad \text{if } p = \infty.$$

Moreover, by (2.2) and by the weak lower semicontinuity of the Lebesgue norms,

$$\|\hat{u}\|_{\widetilde{L}^p(D)} \le 1 \quad \text{for every } 1 < p \le \infty. \tag{2.3}$$

On the other hand, since the Laplacian is a hypoelliptic partial differential operator, we may assume $\hat{u}$ harmonic in D. Hence $\hat{u} \in \mathcal{H}^p(D)$.

Let us consider the linear functional $F : \mathcal{H}^p(D) \to \mathbb{R}$,

$$F(u) := u(x_0) - \fint_D u(x)\, dx.$$

Since for any open Euclidean ball B centered at x_0 and contained in D

$$F(u) = \fint_B u(x)\, dx - \fint_D u(x)\, dx \qquad \forall u \in \mathcal{H}^p(D),$$

then F is continuous with respect to the weak convergence, if $1 < p < \infty$, and with respect to the $*$-weak convergence, if $p = \infty$. Therefore, for every $p \in]1, \infty]$

$$\left| \hat{u}(x_0) - \fint_D \hat{u}(x)\, dx \right| = |F(\hat{u})| = \lim_{j\to\infty} |F(u_j)| = \mathcal{G}_p(D, x_0) > 0.$$

This implies $\hat{u} \not\equiv 0$ and, by (2.3),

$$\mathcal{G}_p(D, x_0) \le \frac{\left| \hat{u}(x_0) - \fint_D \hat{u}\, dx \right|}{\|\hat{u}\|_{\widetilde{L}^p}}.$$

Hence, by the very definition of $\mathcal{G}_p(D, x_0)$, see (1.2), we conclude. □

3. The Kuran's function

Fix $x_0, \alpha \in \mathbb{R}^n$, $x_0 \neq \alpha$, $n \ge 2$.

As in Kuran [18], we define $h_\alpha : \mathbb{R}^n \setminus \{\alpha\} \to \mathbb{R}$,

$$h_\alpha(x) := 1 + |x_0 - \alpha|^{n-2} \frac{|x - x_0|^2 - |x_0 - \alpha|^2}{|x - \alpha|^n}. \tag{3.1}$$

This function has the following properties:

Lemma 3.1. *The following properties of the Kuran's function* (3.1) *hold true:*

(i) *for every* $x \in \mathbb{R}^n \setminus \{\alpha\}$

$$h_\alpha(x) = 1 + |x_0 - \alpha|^{n-2} \left(\frac{1}{|x - \alpha|^{n-2}} + 2 \sum_{j=1}^n (\alpha - x_0)_j \frac{(x - \alpha)_j}{|x - \alpha|^n} \right),$$

(ii) $h_\alpha \in \mathcal{H}(\mathbb{R}^n \setminus \{\alpha\})$,

(iii) $h_\alpha(x_0) = 0$,

(iv) $h_\alpha > 1$ *in* $\mathbb{R}^n \setminus \overline{B(x_0, |\alpha - x_0|)}$,

(v) $h_\alpha \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ *for every* $p \in [1, \frac{n}{n-1}[$.

Proof. The proof of items (iii) and (iv) is trivial. As far as (i), (ii) and (v) are concerned, it is enough to notice that

$$\begin{aligned}\frac{|x-x_0|^2-|x_0-\alpha|^2}{|x-\alpha|^n} &= \frac{|x-\alpha|^2+2\langle x-\alpha,\alpha-x_0\rangle}{|x-\alpha|^n}\\ &= \frac{1}{|x-\alpha|^{n-2}}+2\sum_{j=1}^{n}(\alpha-x_0)_j\frac{(x-\alpha)_j}{|x-\alpha|^n} \qquad (3.2)\end{aligned}$$

and to observe that, up to constants, the last right hand side is the sum of the fundamental solution with pole at α of the Laplace operator (if $n \geq 3$, otherwise, if $n = 2$ of a constant function) and a linear combination of its first order derivatives. Moreover, by this formulation we immediately conclude that $h_\alpha \in L^p_{\text{loc}}(\mathbb{R}^n)$ for every $p \in [1, \frac{n}{n-1}[$.

□

Given an open set D of finite measure, $x_0 \in D$, we denote r_{x_0} the radius of the greatest ball centered at x_0 contained in D, i.e.

$$r_{x_0} := \operatorname{dist}(x_0, \partial D).$$

If $\alpha \in \partial D \cap \partial B(x_0, r_{x_0})$ then the norm in $L^p(D)$, $1 \leq p < \frac{n}{n-1}$, of Kuran's function h_α can be estimated by a constant only depending on n, p and $|D|$.

Proposition 3.2. *Consider an open set $D \subseteq \mathbb{R}^n$ of finite measure and $x_0 \in D$.*

For every $1 \leq p < \frac{n}{n-1}$, there exists a constant $c(n, p, |D|)$ only depending on n, p and $|D|$, such that

$$\|h_\alpha\|_{L^p(D)} \leq c(n, p, |D|) \qquad \forall \alpha \in \partial D \cap \partial B(x_0, r_{x_0}).$$

To prove this result we will use some integral estimates of the function $x \mapsto \frac{1}{|x-\alpha|}$.

Lemma 3.3. *Consider an open set $D \subseteq \mathbb{R}^n$ of finite measure and $x_0 \in D$. For every $\alpha \in \mathbb{R}^n$ and for every $q \in [0, n[$ there exists a positive constant c, depending only on n, q and $|D|$, such that*

$$\int_D \frac{1}{|x-\alpha|^q}\,dx \leq c(n, q, |D|).$$

Proof. It is easy to show that there exists a positive constant c, depending only on n and q, such that

$$\int_{D\cap B(\alpha,1)} \frac{1}{|x-\alpha|^q}\,dx \leq c(n, q).$$

Indeed,

$$\int_{D\cap B(\alpha,1)} \frac{1}{|x-\alpha|^q}\,dx \leq \int_{B(\alpha,1)} \frac{1}{|x-\alpha|^q}\,dx = \int_{B(0,1)} \frac{1}{|x|^q}\,dx$$

and the last integral is finite, because $q < n$, and it is bounded by a positive constant depending only on n and q.

On the other hand, for every $x \in D \setminus B(\alpha, 1)$, we have that $|x - \alpha| \geq 1$, therefore

$$\int_{D\setminus B(\alpha,1)} \frac{1}{|x-\alpha|^q}\,dx \leq |D \setminus B(\alpha, 1)|\,dx \leq |D|.$$

This concludes the proof. □

Thanks to the estimate in Lemma 3.3, we can prove Proposition 3.2.

Proof of Proposition 3.2. Let α be in $\partial D \cap \partial B(x_0, r_{x_0})$. By (3.2)

$$|h_\alpha(x)| \le 1 + |x_0 - \alpha|^{n-2} \left(\frac{1}{|x-\alpha|^{n-2}} + 2\frac{|\alpha - x_0|}{|x-\alpha|^{n-1}} \right).$$

Taking into account that $|\alpha - x_0| = r_{x_0}$, the radius of the greatest inner ball in D centered at x_0, then

$$|\alpha - x_0| = r_{x_0} \le \left(\frac{|D|}{\omega_n} \right)^{\frac{1}{n}}, \tag{3.3}$$

where ω_n is the Lebesgue measure of the unit ball of $\mathbb{R}^n$. Therefore

$$|h_\alpha(x)| \le 1 + \left(\frac{|D|}{\omega_n} \right)^{\frac{n-2}{n}} \left[\frac{1}{|x-\alpha|^{n-2}} + \left(\frac{|D|}{\omega_n} \right)^{\frac{1}{n}} \frac{2}{|x-\alpha|^{n-1}} \right].$$

By Lemma 3.3 the conclusion easily follows. □

4. Stability results

In this section we establish stability results for the Gauss mean value formula of harmonic functions in L^p. Precisely, we will estimate the *p-Gauss gap* with a function depending on the *ball gap*, see (1.2) and (1.3) for their definitions.

The first result deals with the case $1 \le p < \frac{n}{n-1}$. The case $p = 1$ has been yet considered by the authors in a joint paper with Fusco and Zhong, see [11].

Theorem 4.1 ($1 \le p < \frac{n}{n-1}$). *Let $D \subset \mathbb{R}^n$ be an open set of finite Lebesgue measure, $x_0 \in D$.*

If $1 \le p < \frac{n}{n-1}$ then there exists a positive constant c, only depending on n and p, such that

$$c\,\mathcal{B}(D, x_0) \le \mathcal{G}_p(D, x_0). \tag{4.1}$$

Proof. If $p = 1$ this result has been proved in [11]. Here we are left to consider the case $1 < p < \frac{n}{n-1}$.

Since the right and left hand sides of (4.1) are translations and dilations invariant, see (2.1), we may assume $x_0 = 0$ and $|D| = 1$.

Let $r_0 = \operatorname{dist}(0, \partial D)$ and let α be a point in $\partial D \cap \partial B(0, r_0)$.

Consider the Kuran's function $h_\alpha : \mathbb{R}^n \setminus \{\alpha\} \to \mathbb{R}$,

$$h_\alpha(x) := 1 + |\alpha|^{n-2} \frac{|x|^2 - |\alpha|^2}{|x-\alpha|^n}, \qquad x \in \mathbb{R}^n \setminus \{\alpha\}.$$

By Lemma 3.1, Proposition 3.2 and taking into account that $|D| = 1$, we have

$$\|h_\alpha\|_{L^p(D)} = \|h_\alpha\|_{\widetilde{L}^p(D)} \le c(n,p),$$

where $c(n,p)$ is a constant only depending on n and p.

Therefore, by Lemma 3.1 (iii),

$$\mathcal{G}_p(D, 0) \ge \frac{\left| h(0) - \int_D h_\alpha(x)\,dx \right|}{\|h_\alpha\|_{L^p(D)}} \ge \frac{1}{c(n,p)} \left| \int_D h_\alpha(x)\,dx \right|.$$

Since $B(0,r_0)\subseteq D$, by the Gauss mean value Theorem and properties (iii) and (iv) in Lemma 3.1, we have

$$\left|\int_D h_\alpha(x)\,dx\right| = \left|\int_{D\setminus B(0,r_0)} h_\alpha(x)\,dx\right| = \int_{D\setminus B(0,r_0)} h_\alpha(x)\,dx \geq |D\setminus B(0,r_0)|. \tag{4.2}$$

So we have proved that

$$\mathcal{G}_p(D,0)\geq \frac{1}{c(n,p)}|D\setminus B(0,r_0)| = \frac{1}{c(n,p)}\frac{|D\setminus B(0,r_0)|}{|D|}.$$

Hence, (4.1) follows. □

The estimate from below of the Gauss mean value gap in Theorem 4.1 is sharp in the following sense.

Proposition 4.2. *Consider the family of ellipsoids*

$$D_\varepsilon := \left\{x\in\mathbb{R}^n\ :\ (\varepsilon x_1)^2 + x_2^2+\cdots+x_n^2<1\right\},\qquad \varepsilon\in]\frac{1}{2},1[.$$

For every $p\in[1,\frac{n}{n-1}[$ there exist two constants $c_1(n,p)$, and $c_2(n)$, both independent of ε, such that

$$c_1(n,p)\frac{|D_\varepsilon\setminus B(0,r_0)|}{|D_\varepsilon|}\leq \mathcal{G}_p(D_\varepsilon,x_0)\leq c_2(n)\frac{|D_\varepsilon\setminus B(0,r_0)|}{|D_\varepsilon|}.$$

Proof. By [11], there exists a constant $c>0$, independent of ε, depending only on n, such that

$$\mathcal{G}_1(D_\varepsilon,0)\leq c(n)\frac{|D_\varepsilon\setminus B(0,1)|}{|D_\varepsilon|}. \tag{4.3}$$

Collecting Lemma 2.1, Theorem 4.1, and (4.3), we get the thesis. □

We remark that another common way to measure the distance of a measurable set $D\subset\mathbb{R}^n$, $|D|<\infty$, from a ball is provided by the so called *Fraenkel asymmetry*, defined as follows:

$$\alpha(D) := \inf_{x\in\mathbb{R}^n}\frac{|D\triangle B(x,r_D)|}{|D|},$$

where r_D is the radius of a ball with the same measure of D and

$$|D\triangle B(x,r_D)| := |D\setminus B(x,r_D)| + |B(x,r_D)\setminus D|.$$

Since $|D\triangle B(x_0,r_D)| = 2|D\setminus B(x_0,r_D)|\leq 2|D\setminus B(x_0,r_{x_0})|$, the stability estimate (4.1) implies that

if D is an open set of finite measure and $x_0\in D$, then for every $1\leq p<\frac{n}{n-1}$

$$\frac{c}{2}\alpha(D)\leq\ \mathcal{G}_p(D,x_0),$$

where c is the constant in (4.1).

If we assume the extra condition that the open set D has a suitable exterior cone property, we can obtain a stability estimate of $\mathcal{G}_p$ for $p\geq\frac{n}{n-1}$.

Precisely, we say that

$K(\bar{x},\theta,R)$ is a *cone exterior to D with vertex at $\bar{x}\in\partial D$*

if there exist $\theta \in \left[0, \frac{\pi}{2}\right[$ and $R > 0$ such that

$$K(\bar{x}, \theta, R) := \bar{x} + T(\Sigma(\theta, R)), \qquad K(\bar{x}, \theta, R) \cap \overline{D} = \{\bar{x}\},$$

where

$$\Sigma(\theta, R) := \{(y_1, \dots, y_{n-1}, y_n) \in \overline{B(0,R)} \,:\, \tan(\theta)\sqrt{y_1^2 + \cdots + y_{n-1}^2} \leq y_n\}$$

and T is a rotation in $\mathbb{R}^n$. We will call *axis* of the cone K the set

$$\{x \in K(\bar{x}, \theta, R) \,:\, x = \bar{x} + T(\underbrace{0, \cdots, 0}_{n-1}, t) \text{ with } t > 0\}.$$

Theorem 4.3 ($\frac{n}{n-1} \leq p \leq \infty$)**.** *Let $D \subset \mathbb{R}^n$ be an open set of finite measure and $x_0 \in D$.*

Denoted $r_{x_0} := \operatorname{dist}(x_0, \partial D)$ assume that there exists $\bar{x}$ in $\partial D \cap \partial B(x_0, r_{x_0})$ vertex of a cone $K(\bar{x}, \theta, R)$ exterior to D.

Then, for every $\gamma \in]0,1[$, the following inequalities hold:

$$\text{If } \frac{n}{n-1} < p \leq \infty: \quad \mathcal{G}_p(D, x_0) \geq c\,\mathcal{B}(D, x_0) \min\{(|D|^{-\frac{1}{n}} R)^\gamma, \mathcal{B}(D, x_0)\}^{\frac{1}{\gamma}(n-1-\frac{n}{p})}, \tag{4.4}$$

$$\text{if } p = \frac{n}{n-1}: \quad \mathcal{G}_p(D, x_0) \geq c\,\mathcal{B}(D, x_0) \left(\log \frac{\kappa}{\min\{(|D|^{-\frac{1}{n}} R)^\gamma, \mathcal{B}(D, x_0)\}} \right)^{-\frac{n-1}{n}}, \tag{4.5}$$

with constants $c, \kappa > 0$, c depending only on n, p, θ, γ, and κ depending only on n and γ.

The right hand side in (4.5) *is to be interpreted as* 0 *if* $\mathcal{B}(D, x_0) = 0$.

We remark that, while our stability result for $p < \frac{n}{n-1}$ is sharp, the sharpness of our estimates (4.4) and (4.5) is still an open problem.

To prove the result above we need some integral estimates for the function $x \mapsto \frac{1}{|x-\alpha|}$.

Lemma 4.4. *Let $D \subset \mathbb{R}^n$ be an open set, $|D| = 1$, and $0 \in D$.*

Denoted

$$r_0 := \operatorname{dist}(0, \partial D),$$

assume that there exists

$$\bar{x} \in \partial D \cap \partial B(0, r_0) \quad \textit{and} \quad K(\bar{x}, \theta, R) \ \textit{cone exterior to } D. \tag{4.6}$$

For any α on the axis of $K(\bar{x}, \theta, R)$, such that

$$|\alpha - \bar{x}| \leq \frac{1}{2}, \tag{4.7}$$

the following estimates hold:

(a) *there exists a positive constant c only depending on n, θ such that*

$$\int_D \frac{1}{|x-\alpha|^n}\, dx \leq c(n, \theta) \log \frac{1}{|\bar{x} - \alpha|}, \tag{4.8}$$

(b) *if $q > n$ then there exists a positive constant c only depending on n, q, θ such that*

$$\int_D \frac{1}{|x-\alpha|^q}\, dx \leq c(n, q, \theta) |\bar{x} - \alpha|^{n-q}, \tag{4.9}$$

(c) *if $0 \leq q < n$ then there exists a positive constant c only depending on n, q such that*

$$\int_D \frac{1}{|x-\alpha|^q}\, dx \leq c(n, q). \tag{4.10}$$

Proof. Let us define

$$\rho_\alpha := \sup\{r > 0 \, : \, B(\alpha, r) \subseteq K(\bar{x}, \theta, R)\}. \tag{4.11}$$

The definition of α and ρ_α, together with (4.6), (4.7), (4.11), imply

$$\rho_\alpha = |\alpha - \bar{x}| \cos\theta \le \mathrm{dist}(\alpha, \partial D) \le |\alpha - \bar{x}| \le \frac{1}{2}. \tag{4.12}$$

These inequalities, together with (4.6) and (4.11), imply

$$\begin{aligned}\int_D \frac{1}{|x-\alpha|^n}\,dx &\le \int_{D\setminus B(\alpha,1)} \frac{1}{|x-\alpha|^n}\,dx + \int_{B(\alpha,1)\setminus B(\alpha,\rho_\alpha)} \frac{1}{|x-\alpha|^n}\,dx \\ &\le |D \setminus B(\alpha,1)| + n\omega_n \log\frac{1}{\rho_\alpha} \le 1 + n\omega_n\left(\log\frac{1}{|\alpha-\bar{x}|} + \log\frac{1}{\cos\theta}\right).\end{aligned}$$

By (4.7)

$$1 + n\omega_n \log\frac{1}{\cos\theta} \le \frac{1 + n\omega_n \log\frac{1}{\cos\theta}}{\log 2} \log\frac{1}{|\alpha - \bar{x}|},$$

therefore (4.8) follows.

If $q > n$, by (4.11) and the first equality in (4.12) we get

$$\int_D \frac{1}{|x-\alpha|^q}\,dx \le \int_{\mathbb{R}^n\setminus B(\alpha,\rho_\alpha)} \frac{1}{|x-\alpha|^q}\,dx = c(n,q,\theta)|\bar{x}-\alpha|^{n-q},$$

that is (4.9) holds.

If $0 \le q < n$ we remark that

$$\int_D \frac{1}{|x-\alpha|^q}\,dx \le \int_{D\setminus B(\alpha,1)} \frac{1}{|x-\alpha|^q}\,dx + \int_{B(\alpha,1)} \frac{1}{|x-\alpha|^q}\,dx \le 1 + c(n,q)$$

and also (4.10) is proved. □

In the proof of Theorem 4.3 we will also use the following estimate.

Lemma 4.5. *For every* $\gamma, \kappa \in]0,1]$ *and* $c, R > 0$

$$\left(\frac{\min\{2^{1-\gamma} c\, R^\gamma, \kappa\}}{2\max\{2^\gamma, c\}}\right)^{\frac{1}{\gamma}} \le \min\{\frac{1}{2}, \frac{R}{2}\}.$$

Proof. Let us denote

$$F(\gamma, R, c, \kappa) := \left(\frac{\min\{2^{1-\gamma} c\, R^\gamma, \kappa\}}{2\max\{2^\gamma, c\}}\right)^{\frac{1}{\gamma}}.$$

We first consider the case $2^{1-\gamma} c\, R^\gamma \le \kappa$.

We get

$$F(\gamma, R, c, \kappa) \le \left(\frac{2^{1-\gamma}c}{2^{1+\gamma}}\right)^{\frac{1}{\gamma}} R \le \left(\frac{2^{1-\gamma}c}{2^{1+\gamma}}\right)^{\frac{1}{\gamma}} \left(\frac{\kappa}{2^{1-\gamma}c}\right)^{\frac{1}{\gamma}} \le \frac{1}{2}\kappa^{\frac{1}{\gamma}} \le \frac{1}{2}. \tag{4.13}$$

Let us now prove that $F(\gamma, R, c, \kappa) \le \frac{R}{2}$. We have

$$F(\gamma, R, c, \kappa) \le \left(\frac{2^{1-\gamma}c}{2c}\right)^{\frac{1}{\gamma}} R = \frac{R}{2}.$$

As far as the first case is concerned, the proof of the claim is concluded.

Let us consider the remaining case, i.e. $\kappa < 2^{1-\gamma} c\, R^{\gamma}$.

We have

$$F(\gamma, R, c, \kappa) \le \left(\frac{\kappa}{2^{1+\gamma}}\right)^{\frac{1}{\gamma}} \le \left(\frac{1}{2^{1+\gamma}}\right)^{\frac{1}{\gamma}} \le \frac{1}{2}.$$

Let us now prove that $F(\gamma, R, c, \kappa) \le \frac{R}{2}$. By using the assumption,

$$F(\gamma, R, c, \kappa) \le \left(\frac{\kappa}{2c}\right)^{\frac{1}{\gamma}} \le \left(\frac{2^{-\gamma} c}{c}\right)^{\frac{1}{\gamma}} R = \frac{R}{2}.$$

The proof is concluded. □

We are ready to prove Theorem 4.3.

Proof of Theorem 4.3. If $|D \setminus B(x_0, r_{x_0})| = 0$ there is nothing to prove.

Let us assume $|D \setminus B(x_0, r_{x_0})| > 0$.

By the translation invariance of the left and right hand sides of (4.4) and (4.5), without loss of generality we can assume $x_0 = 0$. Denoted

$$r_0 := \operatorname{dist}(0, \partial D),$$

by assumption there exist

$$\bar{x} \in \partial D \cap \partial B(0, r_0) \quad \text{and} \quad K(\bar{x}, \theta, R) \ \text{cone exterior to } D. \tag{4.14}$$

For any $\alpha \in K(\bar{x}, \theta, R)$,

$$\alpha = \bar{x} + T(\underbrace{0, \cdots, 0}_{n-1}, t) \quad \text{with } t \in]0, \frac{R}{2}], \tag{4.15}$$

we define

$$\rho_\alpha := \sup\{r > 0 \,:\, B(\alpha, r) \subseteq K(\bar{x}, \theta, R)\}. \tag{4.16}$$

A trivial computation shows that

$$\rho_\alpha = |\alpha - \bar{x}| \cos\theta. \tag{4.17}$$

We split the proof into steps.

Step I.

Let us consider the Kuran's function $h_\alpha : \mathbb{R}^n \setminus \{\alpha\} \to \mathbb{R}$,

$$h_\alpha(x) := 1 + |\alpha|^{n-2} \frac{|x|^2 - |\alpha|^2}{|x - \alpha|^n}, \quad x \in \mathbb{R}^n \setminus \{\alpha\}.$$

Since $\alpha \notin \overline{D}$, then $h_\alpha \in \mathcal{H}(D) \cap C(\overline{D})$ (see Lemma 3.1) and $h \in L^p(D)$, for every $p \in [1, \infty]$. Let us prove an estimate of the $\widetilde{L}^p$-norm of h_α in D.

We claim that:

if $|D| = 1$ and $|\alpha - \bar{x}| \le \frac{1}{2}$ there exists a positive constant c depending only on n, p, θ, such that

$$\|h_\alpha\|_{\widetilde{L}^p(D)} \le \frac{c(n, p, \theta)}{\varphi(|\bar{x} - \alpha|)}, \tag{4.18}$$

where

$$\varphi(|\bar{x}-\alpha|) := \left[\begin{array}{ll} |\bar{x}-\alpha|^{n-1-\frac{n}{p}} & \text{if } \frac{n}{n-1} < p \le \infty \\ \left(\log \dfrac{1}{|\bar{x}-\alpha|}\right)^{-\frac{n-1}{n}} & \text{if } p = \frac{n}{n-1} \end{array}\right. \tag{4.19}$$

with the position $\frac{1}{\infty} = 0$.

Let us prove the claim, starting from the case $\frac{n}{n-1} \le p < \infty$.

By Lemma 3.1 (i), (3.3) and taking into account that

$$|\alpha| \le |\alpha - \bar{x}| + |\bar{x}| \le 1 + r_0 \le 1 + \left(\frac{|D|}{\omega_n}\right)^{\frac{1}{n}} = 1 + \left(\frac{1}{\omega_n}\right)^{\frac{1}{n}}, \tag{4.20}$$

we have, if $n \ge 3$,

$$\begin{aligned} \fint_D |h_\alpha(x)|^p\,dx &= \int_D |h_\alpha(x)|^p\,dx \\ &\le c(n,p)\left[1 + \int_D \left(\frac{1}{|x-\alpha|^{p(n-2)}} + \frac{1}{|x-\alpha|^{p(n-1)}}\right)dx\right] \end{aligned} \tag{4.21}$$

and, analogously, if $n = 2$

$$\fint_D |h_\alpha(x)|^p\,dx \le c(p)\left(1 + \int_D \frac{1}{|x-\alpha|^p}\,dx\right). \tag{4.22}$$

To estimate the right hand sides of (4.21) and (4.22) we use Lemma 4.4.

Collecting (4.21), (4.22), (4.8), (4.9) and (4.10), we easily conclude.

Let us now consider the case $p = \infty$.

We claim that:

if $|D| = 1$ and $|\alpha - \bar{x}| \le \frac{1}{2}$ there exists a positive constant c depending only on n and θ, such that

$$\sup_D |h_\alpha| \le \frac{c(n,\theta)}{|\bar{x}-\alpha|^{n-1}}. \tag{4.23}$$

Let us prove the claim.

If $n \ge 3$, using (4.20) we get

$$\begin{aligned} \sup_D |h_\alpha| &\le 1 + c(n)\sup_D \left[\left(\frac{1}{|x-\alpha|}\right)^{n-2} + \left(\frac{1}{|x-\alpha|}\right)^{n-1}\right] \\ &\le c(n)\left[1 + \left(\frac{1}{\operatorname{dist}(\alpha,\partial D)}\right)^{n-2} + \left(\frac{1}{\operatorname{dist}(\alpha,\partial D)}\right)^{n-1}\right]. \end{aligned}$$

Taking into account (4.12) we get

$$\sup_D |h_\alpha| \le \frac{c(n)}{\operatorname{dist}(\alpha,\partial D)^{n-1}} \le \frac{c(n,\theta)}{|\bar{x}-\alpha|^{n-1}}$$

and (4.23) is proved.

In analogous way we can prove (4.23) for $n = 2$.

Step II.

Let us assume $|D| = 1$.

We claim that

for every $\gamma \in]0,1[$ *there exists* $\bar{c}(n,\gamma) > 0$ *such that*

$$\left| \fint_D h_\alpha(x)\, dx \right| \geq \mathcal{B}(D,0) - \bar{c}(n,\gamma)|\bar{x} - \alpha|^\gamma \qquad \forall \alpha \in B(\bar{x},1). \tag{4.24}$$

Define

$$H(\alpha) := \fint_D \Gamma(x-\alpha)\, dx = \int_D \Gamma(x-\alpha)\, dx = (\Gamma * \chi_D)(\alpha),$$

where Γ is the fundamental solution of the Laplace operator with pole at 0.

By Lemma 3.1 (i), there exist a_n, b_n dimensional constants, such that

$$\fint_D h_\alpha\, dx = \int_D h_\alpha\, dx = 1 + |\alpha|^{n-2} \left(a_n H(\alpha) - b_n \langle \nabla H(\alpha), \alpha \rangle \right) \quad \text{if } n \geq 3 \tag{4.25}$$

and

$$\fint_D h_\alpha\, dx = 2 - b_n \langle \nabla H(\alpha), \alpha \rangle \quad \text{if } n = 2. \tag{4.26}$$

Let us now suppose $n \geq 3$ and split $H(\alpha)$ as follows

$$H(\alpha) = \int_{D \cap B(\bar{x},2)} \Gamma(x-\alpha)\, dx + \int_{D \setminus B(\bar{x},2)} \Gamma(x-\alpha)\, dx =: H_1(\alpha) + H_2(\alpha).$$

The function H_2 is in $C^\infty(B(\bar{x},1))$ and

$$\|H_2\|_{C^1(B(\bar{x},1))} \leq c(n),$$

where $c(n)$ is a positive constant only depending on the dimension n.

On the other hand, by $|D| = 1$,

$$\|\chi_{D \cap B(\bar{x},2)}\|_{L^q(\mathbb{R}^n)} \leq 1 \qquad \forall q \in [1,\infty].$$

Therefore by Calderon-Zygmund's Theorem the function

$$\alpha \mapsto H_1(\alpha) = (\Gamma * \chi_{D \cap B(\bar{x},2)})(\alpha)$$

is in $W^{2,p}(B(\bar{x},2))$ for every $p \in [1,\infty[$ with $W^{2,p}$-norm only depending on n and p.

This information, together with the Sobolev-Morrey's embedding Theorem and (4.25), implies that

for every $\gamma \in]0,1[$ *there exists* $c(n,\gamma) > 0$ *such that*

$$\left| \fint_D h_\alpha(x)\, dx - \fint_D h_{\bar{x}}(x)\, dx \right| \leq c(n,\gamma)|\bar{x} - \alpha|^\gamma \qquad \forall \alpha \in B(\bar{x},1). \tag{4.27}$$

The same conclusion holds true if $n = 2$. Indeed,

$$\begin{aligned}
\langle \nabla H(\alpha), \alpha \rangle &= \sum_{i=1}^n \alpha_i \frac{\partial H}{\partial \alpha_i}(\alpha) \\
&= -\sum_{i=1}^n \alpha_i \left(\int_{D \cap B(\bar{x},2)} \frac{\partial \Gamma}{\partial x_i}(x-\alpha)\, dx + \int_{D \setminus B(\bar{x},2)} \frac{\partial \Gamma}{\partial x_i}(x-\alpha)\, dx \right) \\
&= \langle \alpha, \nabla(\Gamma * \chi_{D \cap B(\bar{x},2)})(\alpha) \rangle + \langle \alpha, \nabla(\Gamma * \chi_{D \setminus B(\bar{x},2)})(\alpha) \rangle,
\end{aligned}$$

and then we argue as in the previous case.

By (4.27) and

$$\left|\fint_D h_{\bar{x}}(x)\,dx\right| \geq \frac{|D \setminus B(0,r_0)|}{|D|} = |D \setminus B(0,r_0)|,$$

see (4.2), we conclude.

Step III: The p-Gauss gap estimate for $|D| = 1$.

Let us assume $|D| = 1$.

For every $\gamma \in]0,1[$ we now choose $\alpha = \alpha(\gamma, r_0) \in K(\bar{x}, \theta, R)$. We assume that α is on the axis of the cone, i.e.,

$$\alpha := \bar{x} + T(0, \cdots, 0, \tilde{t}(\gamma, r_0)), \tag{4.28}$$

where $\tilde{t}(\gamma, r_0)$ is defined as follows:

$$\tilde{t}(\gamma, r_0) := \left(\frac{\min\{2a(n,\gamma)R^\gamma, |D \setminus B(0,r_0)|\}}{2\max\{2^\gamma, \bar{c}(n,\gamma)\}}\right)^{\frac{1}{\gamma}}, \tag{4.29}$$

where $\bar{c}(n,\gamma)$ is the constant in (4.24) and

$$a(n,\gamma) := \frac{\bar{c}(n,\gamma)}{2^\gamma}. \tag{4.30}$$

By Lemma 4.5, used with $\kappa = |D \setminus B(0,r_0)|$ and $c = \bar{c}(n,\gamma)$,

$$|\alpha - \bar{x}| \leq \min\{\frac{1}{2}, \frac{R}{2}\}.$$

We use this to prove that there exists a constant c depending only on n, p, θ, γ, and a constant κ only depending on γ such that:

$$\text{if } \frac{n}{n-1} < p \leq \infty: \qquad \mathcal{G}_p(D,0) \geq c\,\mathcal{B}(D,0)\min\{R^\gamma, \mathcal{B}(D,0)\}^{\frac{n-1-\frac{n}{p}}{\gamma}}, \tag{4.31}$$

$$\text{if } p = \frac{n}{n-1}: \qquad \mathcal{G}_p(D,0) \geq c\,\mathcal{B}(D,0)\left(\log\frac{\kappa}{\min\{R^\gamma, \mathcal{B}(D,0)\}}\right)^{-\frac{n-1}{n}}. \tag{4.32}$$

We remark that the ball gap of D with respect to 0 is $|D \setminus B(0,r_0)|$, because we are assuming $|D| = 1$.

Since $h_\alpha \in \mathcal{H}(D) \cap L^p(D)$ and $h_\alpha(0) = 0$, then by definition of p-Gauss gap,

$$\mathcal{G}_p(D,0) \geq \frac{\left|\fint_D h_\alpha(x)\,dx\right|}{\|h_\alpha\|_{\widetilde{L}^p(D)}}.$$

Using (4.18) and (4.24) we get

$$\mathcal{G}_p(D,0) \geq c(n,p,\theta)\left(|D \setminus B(0,r_0)| - \bar{c}(n,\gamma)|\bar{x}-\alpha|^\gamma\right)\varphi(|\bar{x}-\alpha|), \tag{4.33}$$

where (see (4.19))

$$\varphi(|\bar{x}-\alpha|) := \left[\begin{array}{ll} |\bar{x}-\alpha|^{n-1-\frac{n}{p}} & \text{if } \frac{n}{n-1} < p \leq \infty \\ \left(\log\frac{1}{|\bar{x}-\alpha|}\right)^{-\frac{n-1}{n}} & \text{if } p = \frac{n}{n-1}. \end{array}\right.$$

Let us prove an estimate from below of the right hand side of (4.33).

By (4.29) and recalling the notation (4.30) we have

$$\bar{c}(n,\gamma)|\alpha-\bar{x}|^{\gamma}=\frac{2\bar{c}(n,\gamma)}{2\max\{2^{\gamma},\bar{c}(n,\gamma)\}}\min\left\{a(n,\gamma)R^{\gamma},\frac{|D\setminus B(0,r_0)|}{2}\right\}\leq\frac{|D\setminus B(0,r_0)|}{2}.$$

Therefore,

$$|D\setminus B(0,r_0)|-\bar{c}(n,\gamma)|\bar{x}-\alpha|^{\gamma}\geq\frac{|D\setminus B(0,r_0)|}{2}. \tag{4.34}$$

If $\frac{n}{n-1}<p\leq\infty$ then

$$\begin{aligned}\varphi(|\bar{x}-\alpha|)&=\min\{2^{1-\gamma}\bar{c}(n,\gamma)R^{\gamma},|D\setminus B(0,r_0)|\}^{\frac{n-1-\frac{n}{p}}{\gamma}}\\&\geq c(n,p,\gamma)\min\{R^{\gamma},|D\setminus B(0,r_0)|\}^{\frac{n-1-\frac{n}{p}}{\gamma}}.\end{aligned}$$

This inequality, together with (4.34), implies (4.31).

If $p=\frac{n}{n-1}$ then

$$\log\frac{1}{|\bar{x}-\alpha|}=\frac{1}{\gamma}\log\frac{2\max\{2^{\gamma},\bar{c}(n,\gamma)\}}{\min\{2^{1-\gamma}\bar{c}(n,\gamma)R^{\gamma},|D\setminus B(0,r_0)|\}}\leq\frac{1}{\gamma}\log\frac{\kappa(n,\gamma)}{\min\{R^{\gamma},|D\setminus B(0,r_0)|\}},$$

where $\kappa(n,\gamma)$ is a constant only depending on n and γ. By this inequality, together with (4.34), we obtain that (4.32) holds.

Step IV: Conclusion.

The inequalities (4.4) and (4.5) have been proved in the previous step under the assumption $|D|=1$, see (4.31) and (4.32). Let us now remove this assumption by using dilations.

Let D be an open set with finite Lebesgue measure and let $0\in D$. Define

$$D_\lambda:=\{\lambda x\,:\,x\in D\},\qquad \lambda:=\left(\frac{1}{|D|}\right)^{\frac{1}{n}}.$$

Then $|D_\lambda|=1$ and $0\in D_\lambda$. Moreover, if $K(\bar{x},\theta,R)$ is a cone exterior to D with vertex at $\bar{x}\in\partial D$, then

$$K_\lambda:=\{\lambda x\,:\,x\in K(\bar{x},\theta,R)\}=K(\lambda\bar{x},\theta,\lambda R)$$

is a cone exterior to D_λ with vertex at $\lambda\bar{x}\in\partial D_\lambda$. Notice that the *opening* of the cone K_λ is independent of λ. Then inequalities (4.31) and (4.32) hold true, with the same constants c and κ, by replacing D with D_λ and R with λR. On the other hand, the p-Gauss gap and the ball gap are scale invariant, i.e.,

$$\mathcal{G}_p(\lambda D,0)=\mathcal{G}_p(D,0)\quad\text{and}\quad\mathcal{B}(D_\lambda,0)=\mathcal{B}(D,0).$$

Therefore inequalities (4.4) and (4.5) follow. □

Since a convex open set D in $\mathbb{R}^n$ of finite Lebesgue measure has a cone exterior to D with vertex at any point of the boundary, with any height R, then a straightforward corollary of Theorem 4.3 is the following.

Corollary 4.6. *Let $D\subset\mathbb{R}^n$ be an open convex set of finite measure and $x_0\in D$.*

Then, for every $\gamma\in]0,1[$, the following inequalities hold:

$$\textit{If}\ \frac{n}{n-1}<p\leq\infty:\quad\mathcal{G}_p(D,x_0)\geq c\,\mathcal{B}(D,x_0)^{1+\frac{1}{\gamma}(n-1-\frac{n}{p})},$$

$$\textit{if } p = \frac{n}{n-1}\text{:} \qquad \mathcal{G}_p(D, x_0) \geq c\,\mathcal{B}(D, x_0)\left(\log \frac{\kappa}{\mathcal{B}(D, x_0)}\right)^{-\frac{n-1}{n}}, \tag{4.35}$$

with constants $c, \kappa > 0$, c depending only on n, p, θ, γ, and κ depending only on n and γ.

The right hand side in (4.35) *is to be interpreted as* 0 *if* $\mathcal{B}(D, x_0) = 0$.

5. $W^{1,p'}$-CONTINUITY OF THE p-GAUSS GAP

In this section we consider $1 < p < \infty$, and, as usual, p' denotes the conjugate exponent of p, i.e. p' is the real number such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Our main result is that if $p \in]1, \frac{n}{n-1}[$ then the $W^{1,p'}$-convergence of domains, in a sense specified by Theorem 5.1, to an Euclidean ball forces the p-Gauss gap to go to zero. We refer to [11] for a related result about the $C^{1,\alpha}$-convergence of domains.

Theorem 5.1. *Consider the ball $B(0, 2)$ in $\mathbb{R}^n$, $n \geq 2$, and a function $d \in C^{1,\alpha}(B(0, 2))$, $\alpha \in]0, 1[$. Let*

$$D := \{x \in B(0, 2) \, : \, d(x) < 1\}$$

be such that

$$\partial D = \{x \in B(0, 2) \, : \, d(x) = 1\}$$

and

$$B(0, 1/2) \subseteq D \subseteq B(0, 3/2). \tag{5.1}$$

Let

$$d_e : \mathbb{R}^n \to \mathbb{R}, \quad d_e(x) := |x|^2.$$

Then for every $p \in]1, \frac{n}{n-1}[$ there exists a positive constant c, only depending on n, p and the $C^{1,\alpha}$-norm of d in $B(0, 2)$, such that

$$\mathcal{G}_p(D, 0) \leq c \, \|d - d_e\|_{W^{1,p'}(B(0,2))} \, .$$

This result is a straightforward consequence of the Sobolev-Morrey embedding Theorem and of the following proposition, that holds true for any $p \in]1, \infty[$.

Proposition 5.2. *Under the same notation and assumptions on D of Theorem 5.1, let p be a real number, $p \in]1, \infty[$.*

Then there exists a positive constant c, only depending on n, p and the $C^{1,\alpha}$-norm of d in $B(0, 2)$, such that

$$\mathcal{G}_p(D, 0) \leq c \left(\|d - d_e\|_{W^{1,p'}(B(0,2))} + \|d - d_e\|_{C(B(0,2))} \right). \tag{5.2}$$

Proof. We give a proof in $\mathbb{R}^n$, $n \geq 3$. The case $n = 2$ can be handled exactly in the same way.

For the sake of simplicity, we will denote $B_{\frac{1}{2}}$, B_1 and B_2 the balls in $\mathbb{R}^n$ centered at 0 with radius $\frac{1}{2}$, 1 and 2, respectively.

Hereafter Γ denotes the fundamental solution of the classical Laplace operator with pole at 0, G_D stands for $G_D(\cdot, 0)$, the Green function of D with pole at 0 and $G_{B_1} : \mathbb{R}^n \to \mathbb{R}$, is an extension of the Green function of B_1 with pole at 0, precisely

$$G_{B_1}(x) = \Gamma(x) - \Gamma(1) \qquad x \in \mathbb{R}^n. \tag{5.3}$$

Let us consider $\varphi : [0, \infty[\to \mathbb{R}$,

$$\varphi(t) := \frac{n}{n-2} \frac{(\Gamma(1))^{\frac{n}{n-2}}}{(\Gamma(1)+t)^{1+\frac{n}{n-2}}}.$$

A trivial computation shows that

$$\int_0^\infty \varphi(t)\, dt = 1.$$

For any open set D, let us define the function $w_D : D \to \mathbb{R}$ by

$$w_D := \varphi(G_D)|\nabla G_D|^2. \tag{5.4}$$

It is easy to check that

$$w_{B_1} = \frac{1}{\omega_n} = \frac{1}{|B_1|}. \tag{5.5}$$

As proved in [2], see also [12], w_D is a density with the mean value property for D at 0; i.e.,

$$u(0) = \int_D u(x) w_D(x)\, dx \qquad \forall u \in \mathcal{H}(D) \cap L^1(D). \tag{5.6}$$

We now turn to the proof, that we split into steps.

Step I.

Let $\mathcal{U}_p := \{u \in \mathcal{H}(D) \cap L^p(D) \ : \ ⨍_D |u(x)|^p\, dx = 1\}$. Then, by (5.6),

$$\mathcal{G}_p(D, 0) = \sup_{u \in \mathcal{U}_p} \left| u(0) - ⨍_D u(x)\, dx \right| = \sup_{u \in \mathcal{U}_p} \left| \int_D u(x) \left(w_D - \frac{1}{|D|} \right) dx \right|.$$

Hence

$$\begin{aligned}
\mathcal{G}_p(D, 0) &\le \sup_{u \in \mathcal{U}_p} \|u\|_{L^p(D)} \big\| w_D - \frac{1}{|D|} \big\|_{L^{p'}(D)} \\
&= |D|^{\frac{1}{p}} \big\| w_D - \frac{1}{|D|} \big\|_{L^{p'}(D)} \le |B_2|^{\frac{1}{p}} \big\| w_D - \frac{1}{|D|} \big\|_{L^{p'}(D)} \\
&\le c(n,p) \left(\big\| w_D - \frac{1}{|B_1|} \big\|_{L^{p'}(D)} + \frac{||D| - |B_1||}{|B_1||D|^{\frac{1}{p}}} \right),
\end{aligned} \tag{5.7}$$

where in the last inequality we used the triangle inequality.

Step II.

In this step we provide an estimate of the last term at the right hand side of (5.7).

We claim that

$$\frac{||D| - |B_1||}{|B_1||D|^{\frac{1}{p}}} \le c(n,p) \|d - d_e\|_{C(B_2)}, \tag{5.8}$$

where $c(n,p) > 0$.

To prove this claim we notice that

$$\frac{1}{|B_1||D|^{\frac{1}{p}}} \le \frac{1}{|B_1||B_{\frac{1}{2}}|^{\frac{1}{p}}}.$$

To conclude, we use estimates proved in [11].

If $\|d-d_e\|_{C(B_2)} \geq \frac{1}{2}$,

$$||D|-|B_1|| \leq |B_2| \leq c(n)\|d-d_e\|_{C(B_2)}. \tag{5.9}$$

As far as the case $\|d-d_e\|_{C(B_2)} < \frac{1}{2}$ is concerned, it is proved in [11] that

$$\big||D|-|B_1|\big| \leq c(n)\|d-d_e\|_{C(B_2)} \tag{5.10}$$

for some $c(n)>0$ depending only on the dimension n, precisely

$$c(n) = \frac{n\omega_n}{2}\left[1+\big(\frac{n}{2}-1\big)\sup_{\eta\in]0,1[}(1+\eta)^{\frac{n}{2}-2}\right].$$

By (5.9) and (5.10) we get (5.8).

Step III.

In this step we estimate the first term at the right hand side of (5.7).

We claim that

$$\left\|w_D-\frac{1}{|B_1|}\right\|_{L^{p'}(D)} \leq c(n,p,\|d\|_{C^{1,\alpha}(B_2)})\|\Gamma(1)-h\|_{W^{1,p'}(D)}, \tag{5.11}$$

where h solves the Dirichlet problem

$$\begin{cases} \Delta h = 0 & \text{in } D \\ h=\Gamma & \text{on } \partial D \end{cases}$$

and $c(n,p,\|d\|_{C^{1,\alpha}(B_2)})$ is a positive constant that only depends on the dimension n, p and on the $C^{1,\alpha}$-norm of d in B_2.

To prove this, we refer to [11]. Indeed, by using (5.4) and (5.5), it is proved in [11] that the following inequality holds in D:

$$\big|w_D-\frac{1}{|B_1|}\big| \leq c(n,\|d\|_{C^{1,\alpha}(B_2)})\left(\big|G_D-G_{B_1}\big|+\big|\nabla G_D|-|\nabla G_{B_1}|\big|\right),$$

where c only depends on the dimension n and on the $C^{1,\alpha}$-norm of d in B_2.

Taking into account that (5.3) implies $G_D-G_{B_1}=\Gamma(1)-h$ we get

$$\big|w_D-\frac{1}{|B_1|}\big| \leq c(n,\|d\|_{C^{1,\alpha}(B_2)})\left(\big|\Gamma(1)-h\big|+\big|\nabla(h-\Gamma(1))\big|\right).$$

and the claim trivially follows.

Step IV.

Collecting (5.7), (5.8) and (5.11) we get

$$\mathcal{G}_p(D,0) \leq c(n,p,\|d\|_{C^{1,\alpha}(B_2)})\Big(\|\Gamma(1)-h\|_{W^{1,p'}(D)}+C(n)\|d-d_e\|_{C(B_2)}\Big). \tag{5.12}$$

We now prove that

$$\|\Gamma(1)-h\|_{W^{1,p'}(D)} \leq c(n,p,\|d\|_{C^{1,\alpha}(B_2)})\|d-d_e\|_{W^{1,p'}(B_2)}. \tag{5.13}$$

This inequality, together with (5.12), will prove (5.2).

To prove (5.13) we first observe that $h-\Gamma(1)$ solves

$$\begin{cases} \Delta(h-\Gamma(1)) = 0 & \text{in } D \\ h-\Gamma(1)=\Phi(\Gamma-\Gamma(1)) & \text{on } \partial D, \end{cases}$$

where $\Phi \in C_0^\infty(B_2)$ is such that

$$\Phi = 1 \text{ on } \partial D \quad \text{and} \quad \Phi = 0 \text{ in } B_{\frac{1}{2}}.$$

Now, if $x \in \partial D$ (i.e. $d(x) = 1$) we have

$$\begin{aligned}\Phi(x)(\Gamma(x) - \Gamma(1)) &= \Phi(x)\Gamma(x)(1 - |x|^{n-2}) = \frac{\Phi(x)\Gamma(x)}{1 + |x|^{n-2}}(1 - |x|^{2(n-2)}) \\ &= \frac{\Phi(x)\Gamma(x)}{1 + |x|^{n-2}}(d^{n-2}(x) - d_e^{n-2}(x)) = \Psi(x)(d(x) - d_e(x)),\end{aligned}$$

where, for every $x \in B_2$,

$$\Psi(x) = \frac{\Phi(x)\Gamma(x)}{1 + |x|^{n-2}}(d^{n-3}(x) + \cdots + d_e^{n-3}(x)).$$

Obviously $\Psi \in C_0^{1,\alpha}(B_2)$ and the function $v := h - \Gamma(1)$ satisfies

$$\begin{cases} \Delta v = 0 & \text{in } D \\ u = \psi & \text{on } \partial D, \end{cases}$$

with $\psi = (d - d_e)\Psi$. Notice that $\psi \in C^{1,\alpha}(B_2)$.

Let us consider $v := \psi - u$. This function is a weak solution in $W_0^{1,2}(D)$ to

$$\begin{cases} \Delta v = \operatorname{div}(\nabla \psi) & \text{in } D \\ v = 0 & \text{on } \partial D. \end{cases}$$

Since ∂D is $C^{1,\alpha}$, then D is a (δ, R)-Reifenberg flat domain, therefore by [6, Theorem 1.5] v actually belongs to $W_0^{1,q}(D)$ for any $q \in]1, \infty[$ and

$$\|\nabla v\|_{L^q(D)} \leq c\|\nabla \psi\|_{L^q(D)}$$

with c independent of v and ψ. Therefore, since

$$\|\psi\|_{W^{1,q}(D)} \leq \sup_{B_2}(|\Psi| + |\nabla \Psi|)\|d - d_e\|_{W^{1,q}(D)}, \tag{5.14}$$

we obtain

$$\|v\|_{W^{1,q}(D)} \leq c(q)\|\nabla \psi\|_{L^q(D)} \leq c(q)\|\Psi\|_{C^1(B_2)}\|d - d_e\|_{W^{1,q}(D)}.$$

By definition of v and using (5.14) once again, we conclude that for any $q > 1$

$$\|u\|_{W^{1,q}(D)} \leq \|v\|_{W^{1,q}(D)} + \|\psi\|_{W^{1,q}(D)} \leq c(q)\|\Psi\|_{C^1(B_2)}\|d - d_e\|_{W^{1,q}(D)}.$$

By using these inequalities with $q = p'$ we get (5.13). This concludes the proof. □

6. Rigidity results

As a corollary of Theorem 4.1 we get a rigidity result proved in [16].

Corollary 6.1 (Theorem 3 (B), [16])**.** *Let $D \subseteq \mathbb{R}^n$ be an open set with finite Lebesgue measure. Let $1 \leq p < \frac{n}{n-1}$. Suppose that there exists $x_0 \in D$ such that*

$$u(x_0) = \fint_D u(x)\,dx \qquad \forall u \in \mathcal{H}(D) \cap L^p(D).$$

Then D is a Euclidean ball centered at x_0.

Proof. By assumption, for every $u \in \mathcal{H}(D) \cap L^p(D)$, $u \neq 0$,

$$\frac{\left| u(x_0) - \fint_D u(x)\, dx \right|}{\|u\|_{\widetilde{L}^p(D)}} = 0.$$

Therefore $\mathcal{G}_p(D, x_0) = 0$, that implies, by (4.1) in Theorem 4.1, $|D \setminus B(x_0, r_{x_0})| = 0$, with $r_{x_0} := \operatorname{dist}(x_0, \partial D)$. Since D is an open set, we conclude that $D = B(x_0, r_{x_0})$. □

In [16] it is also proved the following result (see [16, Theorem 1]):

Let $D \subseteq \mathbb{R}^n$ be an open bounded set such that $\mathbb{R}^n \setminus \overline{D}$ is connected and $D = \operatorname{int} \overline{D}$. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that

$$u(x_0) = \fint_D u(x)\, dx \qquad \forall u \in \mathcal{H}(\mathbb{R}^n).$$

Then D is a ball centered at x_0.

Now we prove a related result for open sets $D \subseteq \mathbb{R}^n$ with finite Lebesgue measure and test functions in $\mathcal{H}(\overline{D})$, where

$$\mathcal{H}(\overline{D}) := \{u \in \mathcal{H}(D_0) \ : \ D_0 \subseteq \mathbb{R}^n \text{ is an open set, } \overline{D} \subset D_0\}.$$

Theorem 6.2. *Let $D \subseteq \mathbb{R}^n$ be an open set such that $|D| < \infty$ and $D = \operatorname{int} \overline{D}$. Suppose that there exists $x_0 \in D$ such that*

$$u(x_0) = \fint_D u(x)\, dx \qquad \forall u \in \mathcal{H}(\overline{D}). \tag{6.1}$$

Then D is a ball centered at x_0.

Before proving this result we prove a characterization of the assumption $D = \operatorname{int} \overline{D}$.

Lemma 6.3. *Let $D \subseteq \mathbb{R}^n$ be an open set. Then the following are equivalent.*

(i) $D = \operatorname{int} \overline{D}$

(ii) $\partial D = \partial \overline{D}$.

Proof. That (i) implies (ii) comes from the following chain of equalities:

$$\partial \overline{D} = \overline{D} \setminus \operatorname{int} \overline{D} = \overline{D} \setminus D = \partial D.$$

Let us now prove that (ii) implies (i). Since trivially $D \subseteq \operatorname{int} \overline{D}$, we only need to prove $\operatorname{int} \overline{D} \subseteq D$. By contradiction, assume $x \in \operatorname{int} \overline{D}$, but $x \notin D$. Then

$$x \in \overline{D} \setminus D = \partial D = \partial \overline{D} = \overline{D} \setminus \operatorname{int} \overline{D},$$

a contradiction. □

Proof of Theorem 6.2. By the translation and dilation invariance of harmonicity, we can assume without loss of generality that $0 \in D$ and that (6.1) holds for $x_0 = 0$ and $|D| = 1$.

For every $\alpha \in \mathbb{R}^n \setminus \overline{D}$ consider the Kuran's function $h_\alpha : \mathbb{R}^n \setminus \{\alpha\} \to \mathbb{R}$,

$$h_\alpha(x) := 1 + |\alpha|^{n-2} \frac{|x|^2 - |\alpha|^2}{|x - \alpha|^n}, \quad x \in \mathbb{R}^n \setminus \{\alpha\}.$$

Since $\alpha \notin \overline{D}$, then $h_\alpha \in \mathcal{H}(\overline{D})$ (see Lemma 3.1). Therefore, by (6.1) and by taking into account that $h_\alpha(0) = 0$, we get

$$0 = \left| \fint_D h_\alpha(x)\, dx \right| \qquad \forall \alpha \in \mathbb{R}^n \setminus \overline{D}. \tag{6.2}$$

By Step II in the proof of Theorem 4.3, see (4.24), if $r_0 = \operatorname{dist}(0, \partial D)$ and $\bar{x}$ is a point in $\partial B(0, r_0) \cap \partial D$, then

there exists $\bar{c}(n) > 0$ *such that*

$$\left| \fint_D h_\alpha(x)\, dx \right| \geq \mathcal{B}(D, 0) - \bar{c}(n)|\bar{x} - \alpha|^{\frac{1}{2}} \qquad \forall \alpha \in B(\bar{x}, 1). \tag{6.3}$$

By (6.2) and (6.3) we have

$$|D \setminus B(0, r_0)| \leq \bar{c}(n)|\bar{x} - \alpha|^{\frac{1}{2}} \qquad \forall \alpha \in B(\bar{x}, 1) \setminus \overline{D}. \tag{6.4}$$

Notice that, by Lemma 6.3, $B(\bar{x}, 1)\backslash\overline{D}$ is not empty and there exists a sequence (α_j) in $B(\bar{x}, 1)\backslash\overline{D}$ that is convergent to $\bar{x}$. By applying estimate (6.4) to each α_j and letting j go to ∞, we conclude that

$$|D \setminus B(0, r_0)| = 0.$$

Since D is an open set, we conclude that $D = B(0, r_0)$. □

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GIOVANNI CUPINI, ERMANNO LANCONELLI: DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI BOLOGNA, PIAZZA DI PORTA SAN DONATO 5, 40126 - BOLOGNA, ITALY

Email address: giovanni.cupini@unibo.it

Email address: ermanno.lanconelli@unibo.it