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GLOBAL ESTIMATES IN SOBOLEV SPACES FOR HOMOGENEOUS HÖRMANDER SUMS OF SQUARES

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ABSTRACT. Let $\mathcal{L} = \sum_{j=1}^{m} X_j^2$ be a Hörmander sum of squares of vector fields in space $\mathbb{R}^n$, where any X_j is homogeneous of degree 1 with respect to a family of non-isotropic dilations in space. In this paper we prove global estimates and regularity properties for $\mathcal{L}$ in the X-Sobolev spaces $W_X^{k,p}(\mathbb{R}^n)$, where $X = \{X_1, \dots, X_m\}$. In our approach, we combine local results for general Hörmander sums of squares, the homogeneity property of the X_j's, plus a global lifting technique for homogeneous vector fields.

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1. Introduction and statement of the result

Let $X_1, \dots, X_m$ be a set of smooth and linearly independent[1] vector fields on $\mathbb{R}^n$, satisfying the following assumptions:

(H.1) there exists a family of (non-isotropic) dilations $\{\delta_\lambda\}_{\lambda>0}$ of the form

$$\delta_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad \delta_\lambda(x) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_n} x_n),$$

where $1 = \sigma_1 \le \dots \le \sigma_n$ are integers such that the X_i's are δ_λ-homogeneous of degree 1:

$$X_j(f \circ \delta_\lambda) = \lambda\,(X_j f) \circ \delta_\lambda, \quad \forall\, \lambda > 0,\ f \in C^\infty(\mathbb{R}^n), \quad j = 1, \dots, m;$$

In what follows, we denote by $q := \sum_{j=1}^{m} \sigma_j$ the so-called homogeneous dimension of $(\mathbb{R}^n, \delta_\lambda)$.

(H.2) $X_1, \dots, X_m$ satisfy Hörmander's rank condition at 0, i.e.,

$$\dim\{Y(0) : Y \in \mathrm{Lie}(X)\} = n,$$

where $\mathrm{Lie}(X)$ is the smallest Lie sub-algebra of the Lie algebra of the smooth vector fields on $\mathbb{R}^n$ which contains $X := \{X_1, \dots, X_m\}$.

Some remarks on our assumptions are in order. Assumption (H.1) implies that, if

$$X_j = \sum_{k=1}^{n} b_{j,k}(x)\,\partial_{x_k},$$

then $b_{j,k}(x)$ must be a polynomial function, δ_λ-homogeneous of degree $\sigma_k - 1$. Incidentally, this straightforwardly implies that

$$b_{j,k}(x) = b_{j,k}(x_1, \dots, x_{k-1}) \quad \text{for any } j \le m \text{ and } k \le n, \tag{1.1}$$

or, more precisely, $b_{j,k}(x)$ depends on those x_i's such that $\sigma_i \le \sigma_k - 1$. From (1.1) we infer that the formal adjoint of X_j is $-X_j$. Let us fix some notation. For any multi-index $I = (i_1, \dots, i_k)$ with $i_1, \dots, i_k \in \{1, 2, \dots, m\}$, we let

$$X_I = X_{i_1} X_{i_2} \cdots X_{i_k}, \quad X_{[I]} = [[X_{i_1}, X_{i_2}], \dots, X_{i_k}], \qquad |I| = k. \tag{1.2}$$

[1] The linear independence of the X_i's is meant with respect to the vector space of the smooth vector fields on $\mathbb{R}^n$; this must not be confused with the linear independence of the vectors $X_1(x), \dots, X_m(x)$ in $\mathbb{R}^n$ (when $x \in \mathbb{R}^n$): the latter is sufficient but not necessary to the former linear independence. Thus, $X_1 = \partial_{x_1}$ and $X_2 = x_1\,\partial_{x_2}$ are linearly independent vector fields, even if $X_1(0, x_2) \equiv (1, 0)$ and $X_2(0, x_2) \equiv (0, 0)$ are dependent vectors of $\mathbb{R}^2$.

When $k = 1$ and $I = (i_1)$, we agree to let $X_I = X_{i_1}$. It is easy to check that, by (H.1), the operators X_I and $X_{[I]}$ are δ_λ-homogeneous of degree $|I|$. The δ_λ-homogeneity of the vector field $X_{[I]}$ is equivalent to the identity

$$X_{[I]}(\delta_\lambda(x)) = \lambda^{-|I|}\delta_\lambda(X_{[I]}(x)), \quad \forall\, \lambda > 0,\ x \in \mathbb{R}^n. \tag{1.3}$$

Remark 1.1 (Global Hörmander condition)**.** We observe that, by (H.1) and (H.2), the validity of Hörmander's rank condition at 0 implies its validity at any other point $x \in \mathbb{R}^n$. Indeed, the iterated (left nested) brackets $X_{[I]}$ span $\mathrm{Lie}(X)$. Hence, by (H.2), we can find a family $X_{[I_1]}, \dots, X_{[I_n]}$ such that $X_{[I_1]}(0), \dots, X_{[I_n]}(0)$ is a basis of $\mathbb{R}^n$. Thus, the matrix-valued function

$$z \mapsto \mathbf{M}(z) := \big(X_{[I_1]}(z) \cdots X_{[I_n]}(z)\big)$$

is non-singular at $z = 0$; therefore, there exists a neighborhood Ω of 0 such that $\det(\mathbf{M}(z)) \neq 0$ for every $z \in \Omega$. Fixing $x \in \mathbb{R}^n$ and taking a small $0 < \lambda \ll 1$ such that $\delta_\lambda(x) \in \Omega$, we have

$$\begin{aligned} 0 \neq \det\big(\mathbf{M}\big(\delta_\lambda(x)\big)\big) &\overset{(1.3)}{=} \det\Big(\lambda^{-|I_1|}\,\delta_\lambda\big(X_{[I_1]}(x)\big) \cdots \lambda^{-|I_n|}\,\delta_\lambda\big(X_{[I_n]}(x)\big)\Big) \\ &= \lambda^{-|I_1|-\cdots-|I_n|}\,\det\Big(\delta_\lambda\big(X_{[I_1]}(x)\big) \cdots \delta_\lambda\big(X_{[I_n]}(x)\big)\Big). \end{aligned}$$

This implies that the vectors $\delta_\lambda\big(X_{[I_1]}(x)\big), \dots, \delta_\lambda\big(X_{[I_n]}(x)\big)$ form a basis of $\mathbb{R}^n$, so that the same is true of $X_{[I_1]}(x), \dots, X_{[I_n]}(x)$, since the linear map δ_λ is an isomorphism of $\mathbb{R}^n$. This proves that $X_1, \dots, X_m$ satisfy Hörmander's rank condition at any $x \in \mathbb{R}^n$.

Thus, by Hörmander's Theorem [11], the *homogeneous sums of squares*

$$\mathcal{L} = \sum_{j=1}^m X_j^2$$

is C^∞-hypoelliptic on every open set $\Omega \subseteq \mathbb{R}^n$, which means that every distributional solution u of an equation $Lu = f$ in Ω is smooth on every sub-domain $\Omega' \subseteq \Omega$ where f is smooth. From (1.1) we also infer that L is formally self-adjoint. Note that the case $q = 2$ implies that $\mathcal{L}$ is a strictly elliptic constant-coefficient operator on $\mathbb{R}^2$, so that it is not restrictive to assume that $q > 2$.

Example 1.2. In $\mathbb{R}^2$, let us consider

$$X_1 = \partial_{x_1}, \quad X_2 = x_1\,\partial_{x_2}; \qquad \delta_\lambda(x_1, x_2) = (\lambda x_1, \lambda^2 x_2).$$

Condition (H.1) is easily checked. Here $n = 2$, $q = 3$ and

$$\mathcal{L} = X_1^2 + X_2^2 = \partial_{1,1} + x_1^2\,\partial_{2,2}.$$

Condition (H.2) holds because X_1 and $[X_1, X_2] = \partial_{x_2}$ give a basis of $\mathbb{R}^2$ at any point.

Example 1.3. More generally, in $\mathbb{R}^2$, let

$$X_1 = \partial_{x_1}, \quad X_2 = x_1^k\,\partial_{x_2}; \qquad \delta_\lambda(x_1, x_2) = (\lambda x_1, \lambda^{k+1} x_2).$$

Again, (H.1) is easy to check. Here $n = 2$, $q = k + 2$ and

$$\mathcal{L} = X_1^2 + X_2^2 = \partial_{1,1} + x_1^{2k}\,\partial_{2,2}.$$

Condition (H.2) holds true as well because X_1 and

$$\frac{1}{k!}[X_1, [X_1, \dots [X_1, X_2]]] = \partial_{x_2} \qquad \text{(bracket of length } k+1\text{)}$$

span $\mathbb{R}^2$ at any point.

Example 1.4. In $\mathbb{R}^n$, let us consider

$$X_1 = \partial_{x_1}, \quad X_2 = x_1\partial_{x_2} + x_2\partial_{x_3} + \cdots + x_{n-1}\partial_{x_n}; \qquad \delta_\lambda(x) = (\lambda x_1, \lambda^2 x_2, \cdots, \lambda^n x_n).$$

(H.1) is easily checked. Note that $q = n(n+1)/2 > n$ and

$$\mathcal{L} = X_1^2 + X_2^2 = \partial_{1,1} + (x_1\partial_{x_2} + x_2\partial_{x_3} + \ldots + x_{n-1}\partial_{x_n})^2.$$

Condition (H.2) holds because

$$\begin{aligned}\partial_{x_1} &= X_1\\ \partial_{x_2} &= [X_1, X_2]\\ &\vdots\\ \partial_{x_n} &= [[[X_1, X_2], X_2], \ldots, X_2] \quad \text{(bracket of length } n).\end{aligned}$$

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Following the notation in (1.2), the Sobolev spaces with respect to the system of vector fields X are defined, for $p \in (1, \infty)$ and $k \in \mathbb{N} \cup \{0\}$, by setting

$$W_X^{k,p}(\Omega) := \Big\{ u \in L^p(\Omega) : X_I u \in L^p(\Omega), \text{ for any } I \text{ with } |I| \le k \Big\},$$

endowed with the norm

$$\|u\|_{W_X^{k,p}(\Omega)} := \sum_{|I| \le k} \|X_I u\|_{L^p(\Omega)}.$$

Here the derivatives $X_I u$ exist, *a priori*, in the weak sense at least. It is understood that $X_I u = u$ only for $I = 0$; in particular, $(W_X^{k,p}(\Omega), \|\cdot\|_{W_X^{k,p}(\Omega)})$ reduces to the usual normed space $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ when $k = 0$.

We are interested in establishing global regularity results in the scale of these Sobolev spaces for homogeneous sums of squares $\mathcal{L}$. Namely, our main result is the following:

Theorem 1.5 (Global regularity for homogeneous sums of squares)**.** *Let $\mathcal{L}$ be as above, under assumptions* (H.1)-(H.2) *on the vector fields $X_1, \ldots, X_m$.*

Let also $p \in (1, \infty)$ and let k be a nonnegative integer. Then, there exists $\Lambda = \Lambda_{k,p} > 0$ such that, if $u \in L^p(\mathbb{R}^n)$ and $Lu \in W_X^{k,p}(\mathbb{R}^n)$ (which means that the distribution $\mathcal{L}u$ can be identified with a function in $W_X^{k,p}(\mathbb{R}^n)$), then $u \in W_X^{k+2,p}(\mathbb{R}^n)$ and

$$\|u\|_{W_X^{k+2,p}(\mathbb{R}^n)} \le \Lambda_{k,p} \Big(\|\mathcal{L}u\|_{W_X^{k,p}(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \Big). \tag{1.4}$$

This theorem will be proved in section 3, throughout Theorems 3.2 and 3.3.

Theorem 1.5 is well known if the sum of squares $\mathcal{L}$ is not just δ_λ-homogeneous of degree 2, but also left invariant with respect to a Lie group operation; more precisely, if $\mathcal{L}$ is a *sub-Laplacian on a Carnot group*: in this case the above result is due to Folland, see [9, Thm. 6.1]. Let us review the definition of this key concept, since it will play an important role in the following:

Definition 1.6. We say that $\mathbb{G} = (\mathbb{R}^N, *, D_\lambda)$ is a (homogeneous) Carnot group if:

(1) $*$ is a Lie group operation in $\mathbb{R}^N$ (that we qualify as "translations") and, for some fixed positive integer exponents $\alpha_1, \ldots, \alpha_N$, the maps

$$D_\lambda(x) = (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_N} x_N) \quad \text{for } \lambda > 0$$

form a family of group automorphisms (that we qualify as "dilations").

(2) Let X_i (for $i = 1, 2, \ldots, N$) be the only left invariant vector field which agrees with ∂_{x_i} at the origin; moreover, let H be the set of the vector fields among $X_1, \ldots, X_N$ which are D_λ-homogeneous of degree 1; then the set H satisfies Hörmander's condition at the origin (hence, by left-invariance, at every point of $\mathbb{R}^N$).

In this case, if $H = \{Z_1, \ldots, Z_m\}$, the sub-Laplacian operator on $\mathbb{G}$ defined by $\Delta_{\mathbb{G}} = \sum_{j=1}^m Z_j^2$ is D_λ-homogeneous of degree 2, left invariant, and C^∞-hypoelliptic.

For a technical reason that will become apparent in a moment (see (2.2)), we do not require that the exponents α_k's of the dilations D_λ be increasingly ordered (as is done e.g., in [4]).

In the more general case of the so-called "sums of squares of Hörmander's vector fields", defined on some domain $\Omega \subseteq \mathbb{R}^n$ but not necessarily homogeneous with respect to any family of dilations, nor necessarily left invariant with respect to any Lie-group translations, a regularity result such as Theorem 1.5 is known only in a local form. Namely, Rothschild-Stein proved the following:

Theorem A (Interior regularity for Hörmander sum of squares, [13, Thm. 16])**.** *Let $X_1, \dots, X_m$ be a system of smooth vector fields satisfying Hörmander's condition in some domain $\Omega \subseteq \mathbb{R}^n$, and let $L = \sum_{i=1}^m X_i^2$. Finally, let k be a nonnegative integer and $p \in (1, \infty)$.*

Then the following facts hold:

(i) *if u is any distribution in Ω with $Lu \in W_X^{k,p}(\Omega)$, then $u \in W_{X,\mathrm{loc}}^{k+2,p}(\Omega)$;*

(ii) *for any domains $\Omega' \Subset \Omega'' \Subset \Omega$, it is possible to find a constant $c_{k,p} > 0$ such that*

$$\|u\|_{W_X^{k+2,p}(\Omega')} \le c_{k,p} \Big\{ \|Lu\|_{W_X^{k,p}(\Omega'')} + \|u\|_{L^p(\Omega'')} \Big\}, \tag{1.5}$$

for every distribution u in Ω with $Lu \in W_X^{k,p}(\Omega)$.

Incidentally, we note that for general Hörmander operators $\sum_{i=1}^m X_i^2 + X_0$ with drift term X_0 (with $X_0, X_1, \dots, X_m$ satisfying Hörmander's condition in Ω), only the basic estimate (1.5) for $k = 0$ is known, while a complete regularity theory in the scale of Sobolev spaces $W_X^{k,p}$ is so far lacking.[2]

Coming back to the case of the sums of squares $L = \sum_{i=1}^m X_i^2$, if the vector fields $X_1, \dots, X_m$ satisfy Hörmander's condition in $\Omega = \mathbb{R}^n$, it is quite natural to ask whether the result of Theorem A can be improved to that of Theorem 1.5 without assuming the Carnot group structure. However, only a few results in this direction seem to be known, so far. Bramanti, Cupini, Lanconelli, Priola in [7] have studied a class of Ornstein-Uhlenbeck operators of the kind

$$Lu = \sum_{i,j=1}^m a_{i,j} u_{x_i, x_j} + \sum_{i,j=1}^N b_{i,j} x_i u_{x_j} \quad \text{in } \mathbb{R}^N,$$

with $m < N$, $(a_{i,j})_{i,j=1}^m$ a constant, symmetric, positive-definite matrix, and $(b_{i,j})_{i,j=1}^N$ a constant matrix satisfying a suitable structure assumption. This operator can be rewritten in the form of a Hörmander operator $Lu = \sum_{i=1}^m X_i^2 u + X_0$ on the whole of $\mathbb{R}^N$; however, this L is neither left invariant nor (in general) homogeneous with respect to any family of dilations. For these operators the following global estimates are proved (just in the basic case $k = 0$)

$$\sum_{i=1}^m \|u_{x_i, x_j}\|_{L^p(\mathbb{R}^N)} + \|X_0 u\|_{L^p(\mathbb{R}^N)} \le c \left\{ \|Lu\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)} \right\}, \quad \text{for } 1 < p < \infty.$$

Apart from this result, and its extension to continuous variable coefficients $a_{i,j}$ contained in [8], no global Sobolev estimates for classes of Hörmander operators which do not fulfill Folland's assumptions of both left-invariance and homogeneity seem to be known.

Therefore the present result Theorem 1.5 seems to be interesting in its own right, although its proof is not difficult. The simple idea is to apply Rothschild-Stein's local Sobolev estimates, and then to exploit the dilations to get global ones. In doing this, however, one also requires some global interpolation inequalities for Sobolev norms, which are so far available in the case of Carnot groups only. Establishing these inequalities in the present context is possible in view of some deep result dealing with a global lifting of homogeneous vector fields to a higher dimensional Carnot group. This lifting result is a powerful tool, first developed by Folland [10] and, in the form that we actually need, by two of us, [2]. We start (in Section 2) by reviewing this lifting procedure, then we establish suitable interpolation inequalities, and finally (in Section 3) we prove our main result.

2. Lifting and interpolation inequalities

The following result is proved in [2], by using Folland's lifting in [10] plus a convenient change of variable turning the lifting into an explicit projection.

Theorem 2.1 (Global Lifting)**.** *Assume that $X = \{X_1, \dots, X_m\}$ satisfy* (H.1) *and* (H.2)*. Let $N := \dim(\mathrm{Lie}\{X\})$. We denote the points of $\mathbb{R}^N \equiv \mathbb{R}^n \times \mathbb{R}^s$ by (x, ξ) (if $N = n$, we agree that the ξ variable does not appear). Then, the following facts hold:*

[2] Rothschild-Stein [13] state the result, but with no proof, and the methods in [13] do not seem to adapt easily to the drift case. We have not been able to locate any proof of Theorem A for $\sum_{i=1}^m X_i^2 + X_0$ in the existing literature.

(1) *There exist a Carnot group* $\mathbb{G} = (\mathbb{R}^N, *, D_\lambda)$ *and a system* $\{\widetilde{X}_1, \dots, \widetilde{X}_m\}$ *of Lie-generators of* $\mathrm{Lie}(\mathbb{G})$ *such that* $\widetilde{X}_i$ *is a lifting of* X_i *for every* $i = 1, \dots, m$*, that is:*

$$\widetilde{X}_i(x,\xi) = X_i(x) + R_i(x,\xi), \tag{2.1}$$

where $R_i(x,\xi)$ *is a smooth vector field operating only in the variable* $\xi \in \mathbb{R}^s$*, with coefficients possibly depending on* (x,ξ)*.*

(2) *The dilations* $\{D_\lambda\}_{\lambda>0}$ *(which make the* $\widetilde{X}_i$*'s homogeneous of degree* 1*) and the dilations* $\{\delta_\lambda\}_{\lambda>0}$ *(which make the* X_i*'s homogeneous of degree* 1*) are related as follows:*

$$D_\lambda(x,\xi) = (\delta_\lambda(x), \delta^*_\lambda(\xi)), \tag{2.2}$$

with $\delta^*_\lambda(\xi) = (\lambda^{\tau_1}\xi_1, \dots, \lambda^{\tau_s}\xi_s)$*, for suitable integers* $\tau_s \geq \dots \geq \tau_1 \geq 1$*.*

Remark 2.2 (The case $N = n$)**.** Since X is a Hörmander system in $\mathbb{R}^n$, one has $N \geq n$. As a matter of fact, Theorem 2.1 has been proved in [2] under the assumption $N > n$. By a recent result in [1], Theorem 2.1 also holds in the case $N = n$. Indeed, if the latter holds, we have that:

- $\mathrm{Lie}\{X\}$ is an n-dimensional Lie algebra of analytic vector fields in $\mathbb{R}^n$ (analyticity follows from the fact that the X_j's have polynomial component functions, due to (H.1));
- X is a Hörmander system, due to (H.2) (see also Remark 1.1);
- any vector field $Y \in \mathrm{Lie}\{X\}$ is complete, i.e., the integral curves of Y are defined on the whole of $\mathbb{R}$ (this can be easily proved as a consequence of (H.1) and (1.1)).

Under these three conditions, a result in [1] proves that $\mathrm{Lie}\{X\}$ coincides with the Lie algebra of a Lie group $\mathbb{G}$ on $\mathbb{R}^n$. As a matter of fact, under assumption (H.1), this Lie group $\mathbb{G}$ turns out to be a homogeneous Carnot group with dilations δ_λ (see e.g., [3, Chapter 16]), so that Theorem 2.1 holds without the need to perform any further lifting.

Remark 2.3 (Rothschild-Stein's lifting vs. Folland's lifting)**.** The first famous result about the lifting of vector fields was proved by Rothschild-Stein in [13]. They showed that every system of Hörmander's vector fields can be lifted, locally, to a higher dimensional system of free Hörmander's vector fields, which can be locally approximated, in a suitable sense, by the generators of a Carnot group. In the above Theorem 2.1, instead, the initial system is directly lifted to the generators of a Carnot group $\mathbb{G}$, the process being performed globally, while $\mathbb{G}$ needs not be a free group. These advantages are made possible by the homogeneity of the original vector fields.

Example 2.4. Let us consider the vector fields X_1, X_2 in Example 1.2. The associated Carnot group according to Theorem 2.1 is $\mathbb{G} = (\mathbb{R}^3, *, D_\lambda)$ with

$$D_\lambda(x_1, x_2, \xi_1) = (\lambda x_1, \lambda^2 x_2, \lambda\,\xi_1),$$

while the composition law is

$$(x_1, x_2, \xi_1) * (x_1', x_2', \xi_1') = (x_1 + x_1', x_2 + x_2' + x_1\xi_1', \xi_1 + \xi_1').$$

Furthermore, the vector fields $\widetilde{X}_1, \widetilde{X}_2$ lifting X_1 and X_2 are

$$\widetilde{X}_1 = \partial_{x_1}, \qquad \widetilde{X}_2 = x_1\,\partial_{x_2} + \partial_{\xi_1}. \tag{2.3}$$

The operator $\mathcal{L} = X_1^2 + X_2^2$ lifts to the sub-Laplacian $\Delta_{\mathbb{G}} = \widetilde{X}_1^2 + \widetilde{X}_2^2$. The latter is (modulo a change of variable) the Kohn-Laplacian on the first Heisenberg group.

Example 2.5. Let us consider the vector fields X_1, X_2 in Example 1.3, in the case when $k = 2$. The associated Carnot group according to Theorem 2.1 is $\mathbb{G} = (\mathbb{R}^4, *, D_\lambda)$ with

$$d_\lambda(x_1, x_2, \xi_1, \xi_2) = (\lambda x_1, \lambda^3 x_2, \lambda\,\xi_1, \lambda^2\,\xi_2),$$

and the composition law $(x_1, x_2, \xi_1, \xi_2) * (x_1', x_2', \xi_1', \xi_2')$ is

$$\Big(x_1 + x_1', x_2 + x_2' + x_1(x_1 + x_1')\xi_1' + 2x_1\xi_2', \xi_1 + \xi_1', \xi_2 + \xi_2' + \tfrac{1}{2}(x_1\xi_1' - x_1'\xi_1)\Big).$$

The vector fields $\widetilde{X}_1, \widetilde{X}_2$ lifting X_1 and X_2 are

$$\widetilde{X}_1 = \partial_{x_1} - \frac{\xi_1}{2}\,\partial_{\xi_2}, \qquad \widetilde{X}_2 = x_1^2\,\partial_{x_2} + \partial_{\xi_1} + \frac{x_1}{2}\,\partial_{\xi_2}.$$

Following the notation in Theorem 2.1, in the lifted space $\mathbb{R}^N$ we can consider the Sobolev spaces $W^{k,p}_{\widetilde{X}}$, where $\widetilde{X} = \{\widetilde{X}_1, \ldots, \widetilde{X}_m\}$. On the other hand, when $\widetilde{X}_i$ acts on a function f only depending on the variables x, one simply gets

$$\widetilde{X}_i f(x) = X_i f(x), \quad i = 1, \ldots, m.$$

This suggests that these Sobolev spaces simply project onto the spaces $W^{k,p}_X$. However, when computing L^p norms, some care must be taken about the domain of the functions involved. In Proposition 2.8 we shall compare L^p norms in suitable balls of the original space and in the lifted variables. Let us first fix some notation and basic facts.

The dilations δ_λ in $\mathbb{R}^n$ induce a *homogeneous norm* $\|\cdot\|$ in $\mathbb{R}^n$ as follows: by definition, we let $\|0\| = 0$, and, for every $x \in \mathbb{R}^n \setminus \{0\}$, we define $\|x\|$ as the unique positive number such as

$$\left|\delta_{1/\|x\|}(x)\right| = 1,$$

where $|\cdot|$ stands for the Euclidean norm. This definition makes sense since, for every $x \neq 0$, the function $(0, \infty) \ni \lambda \mapsto |\delta_\lambda(x)|$ is continuous, strictly increasing, and its image set is $(0, \infty)$.

Remark 2.6. Let $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ denote, as usual, the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$. Then $\|\cdot\|$ is characterized by any of the following equivalent conditions:

(1) for any $\lambda > 0$, the level set $\{x \in \mathbb{R}^n : \|x\| = \lambda\}$ coincides with $\delta_\lambda(\mathbb{S}^{n-1})$ (the latter being the ellipsoid with semi-axes $\lambda^{\sigma_1}, \ldots, \lambda^{\sigma_n}$) which is the set described by the equation

$$\frac{x_1^2}{\lambda^{2\sigma_1}} + \frac{x_2^2}{\lambda^{2\sigma_2}} + \cdots + \frac{x_n^2}{\lambda^{2\sigma_n}} = 1; \tag{2.4}$$

(2) $\|\cdot\|$ coincides with the unique map $u : \mathbb{R}^n \to [0, \infty)$ which is δ_λ-homogeneous of degree 1 and such that

$$u(x) = 1 \quad \text{if and only if} \quad |x| = 1;$$

(3) for any $x \neq 0$, $\|x\|$ is the reciprocal of the unique positive solution t to the algebraic equation

$$x_1^2\, t^{2\sigma_1} + \cdots + x_n^2\, t^{2\sigma_n} = 1;$$

(4) for any $x \neq 0$, $\|x\|$ is the reciprocal of the unique $\lambda > 0$ for which the δ_λ-line through x, that is the set $\{\delta_\lambda(x) : \lambda > 0\}$, intersects the sphere $\mathbb{S}^{n-1}$.

Thus $\|\cdot\|$ enjoys the following properties:

$$\begin{gathered}\|x\| \geq 0 \quad \text{and} \quad (\|x\| = 0 \Leftrightarrow x = 0), \\ \|\delta_\lambda(x)\| = \lambda\, \|x\| \quad \text{for every } \lambda > 0 \text{ and every } x \in \mathbb{R}^n.\end{gathered}$$

Also, since the exponents σ_i appearing in the dilations are positive integers, the function $x \mapsto \|x\|$ is smooth outside the origin. (This can be seen by applying the Implicit Function Theorem to the function $f(\lambda, x) = |\delta_\lambda(x)|^2 - 1$).

Analogously we can define in $\mathbb{R}^N$ and in $\mathbb{R}^s$ two homogeneous norms by means of the dilations $\{D_\lambda\}_{\lambda>0}$ and $\{\delta^*_\lambda\}_{\lambda>0}$ introduced in Theorem 2.1, and these homogeneous norms enjoy similar properties of the ones established for the pair $(\mathbb{R}^n, \{\delta_\lambda\}_{\lambda>0})$. By a small abuse of notation we shall denote with the same symbol $\|\cdot\|$ these three homogeneous norms defined in $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^s$. They are related by the following facts (which holds by point 2 in Theorem 2.1):

$$\|(x,\xi)\| \geq \|(x,0)\| = \|x\|; \qquad \|(x,\xi)\| \geq \|(0,\xi)\| = \|\xi\|. \tag{2.5}$$

We will define the following balls centered at the origins of $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^s$ respectively:

$$\begin{aligned} B_r(0) &= \{x \in \mathbb{R}^n : \|x\| < r\}, \\ \widetilde{B}_r(0) &= \left\{(x,\xi) \in \mathbb{R}^N : \|(x,\xi)\| < r\right\}, \\ B^*_r(0) &= \{\xi \in \mathbb{R}^s : \|\xi\| < r\}, \end{aligned}$$

and we note that, due to (2.5), $B_r(0)$ is the projection of $\widetilde{B}_r(0)$ via the canonical projection of $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^s$ onto $\mathbb{R}^n$. It is not difficult to prove that

$$B_r(0) = \left\{ x \in \mathbb{R}^n : \frac{x_1^2}{r^{2\sigma_1}} + \cdots + \frac{x_n^2}{r^{2\sigma_n}} < 1 \right\}, \tag{2.6}$$

$$\widetilde{B}_r(0) = \left\{ (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^s : \frac{x_1^2}{r^{2\sigma_1}} + \cdots + \frac{x_n^2}{r^{2\sigma_n}} + \frac{\xi_1^2}{r^{2\tau_1}} + \cdots + \frac{\xi_s^2}{r^{2\tau_s}} < 1 \right\}, \tag{2.7}$$

$$B_r^*(0) = \left\{ \xi \in \mathbb{R}^s : \frac{\xi_1^2}{r^{2\tau_1}} + \cdots + \frac{\xi_s^2}{r^{2\tau_s}} < 1 \right\}, \tag{2.8}$$

which means that $B_r(0)$, $\widetilde{B}_r(0)$ and $B_r^*(0)$ are the bounded open sets whose boundaries are the ellipsoids with equations analogous to (2.4) (relative to the dilations δ_r, D_λ and δ_r^* respectively). Equivalently, if $D, \widetilde{D}, D^*$ denote (respectively) the open Euclidean balls with center at the origin and radius 1 in $\mathbb{R}^n, \mathbb{R}^N, \mathbb{R}^s$ (respectively), then, for any $r > 0$ one has

$$B_r(0) = \delta_r(D), \qquad \widetilde{B}_r(0) = D_r(\widetilde{D}), \qquad B_r^*(0) = \delta_r^*(D^*).$$

Starting from (2.6)-to-(2.8) one can prove that (for any $r > 0$)

$$\widetilde{B}_r(0) \subseteq B_r(0) \times B_r^*(0), \tag{2.9}$$

$$\widetilde{B}_r(0) \supseteq B_{r/2}(0) \times B_{r/2}^*(0). \tag{2.10}$$

Indeed, (2.9) is a consequence of

$$\max\left\{ \frac{x_1^2}{r^{2\sigma_1}} + \cdots + \frac{x_n^2}{r^{2\sigma_n}}, \frac{\xi_1^2}{r^{2\tau_1}} + \cdots + \frac{\xi_s^2}{r^{2\tau_s}} \right\} \leq \frac{x_1^2}{r^{2\sigma_1}} + \cdots + \frac{x_n^2}{r^{2\sigma_n}} + \frac{\xi_1^2}{r^{2\tau_1}} + \cdots + \frac{\xi_s^2}{r^{2\tau_s}} < 1,$$

whereas (2.10) is a consequence of

$$1 \geq \sum_{j=1}^n \frac{x_j^2}{(r/2)^{2\sigma_j}} = \sum_{j=1}^n \frac{2^{2\sigma_j} x_j^2}{r^{2\sigma_j}} \geq 2^{2\sigma_1} \sum_{j=1}^n \frac{x_j^2}{r^{2\sigma_j}} > 2 \sum_{j=1}^n \frac{x_j^2}{r^{2\sigma_j}},$$

together with an analogous inequality involving ξ's and τ's; here we also used

$$1 \leq \sigma_1 \leq \cdots \leq \sigma_n, \quad 1 \leq \tau_1 \leq \cdots \leq \tau_s.$$

Throughout the paper, we shall occasionally use the simplified notation B_r for any set $B_r(0)$.

Example 2.7. Consider the vector fields X_1, X_2 in Example 2.4. The dilations in $\mathbb{R}^2$ and in the lifted space $\mathbb{R}^3$ are respectively

$$\delta_\lambda(x_1, x_2) = (\lambda x_1, \lambda^2 x_2), \qquad D_\lambda(x_1, x_2, \xi_1) = (\lambda x_1, \lambda^2 x_2, \lambda \xi_1).$$

Thus, by using for example the characterization (3) in Remark 2.6, one can obtain the explicit expressions for the homogeneous norms in the un-lifted and lifted spaces:

$$\|(x_1, x_2)\| = \frac{1}{\sqrt{2}} \sqrt{\sqrt{x_1^4 + 4\,x_2^2} + x_1^2}\,,$$

$$\|(x_1, x_2, \xi_1)\| = \frac{1}{\sqrt{2}} \sqrt{\sqrt{(x_1^2 + \xi_1^2)^2 + 4\,x_2^2} + x_1^2 + \xi_1^2}\,.$$

We have the following result, concerning L^p-norms in $B_r(0)$ and $\widetilde{B}_r(0)$:

Lemma 2.8. *With the above notation, for any function $u(x)$ of n variables defined in $B_r(0)$, let us define the corresponding function $\widetilde{u}$ of N variables by setting*

$$\widetilde{u}(x,\xi) = u(x), \quad (x,\xi) \in B_r(0) \times \mathbb{R}^s. \tag{2.11}$$

Then, for every $p \in [1, \infty)$ and $r > 0$, we have

$$c_1 \|u\|_{L^p(B_{r/2}(0))} \leq \|\widetilde{u}\|_{L^p(\widetilde{B}_r(0))} \leq c_2 \|u\|_{L^p(B_r(0))}, \tag{2.12}$$

where (denoting by meas *the Lebesgue measure in $\mathbb{R}^s$)*

$$c_1 = c_1(r,p) = \mathrm{meas}(B_{r/2}^*(0))^{1/p}, \qquad c_2 = c_2(r,p) = \mathrm{meas}(B_r^*(0))^{1/p}.$$

Note that (2.12) makes sense, since $\widetilde{B}_r(0) \subset B_r(0) \times \mathbb{R}^s$, due to (2.5). From Lemma 2.8 and (2.1), we immediately infer that (if $\widetilde{u}$ is as in (2.11))

$$u \in W^{k,p}_X(B_r(0)) \iff \widetilde{u} \in W^{k,p}_{\widetilde{X}}(\widetilde{B}_r(0)). \tag{2.13}$$

Indeed, from (2.11) we get that $\widetilde{X}_I \widetilde{u} = \widetilde{X_I u}$ on $\widetilde{B}_r(0)$, for any multi-index I.

Proof. We have the following computation, based on (2.9):

$$\|\widetilde{u}\|^p_{L^p(\widetilde{B}_r(0))} = \iint_{\widetilde{B}_r(0)} |u(x)|^p \,\mathrm{d}x\,\mathrm{d}\xi \le \iint_{B_r(0)\times B^*_r(0)} |u(x)|^p \,\mathrm{d}x\,\mathrm{d}\xi = c_2(r) \int_{B_r(0)} |u(x)|^p \,\mathrm{d}x,$$

where $c_2(r)$ is the Lebesgue measure in $\mathbb{R}^s$ of $B^*_r(0)$. On the other hand, by (2.10),

$$\|\widetilde{u}\|^p_{L^p(\widetilde{B}_r(0))} \ge \iint_{B_{r/2}(0)\times B^*_{r/2}(0)} |u(x)|^p \,\mathrm{d}x\,\mathrm{d}\xi = c_1(r) \int_{B_{r/2}(0)} |u(x)|^p \,\mathrm{d}x,$$

where $c_1(r)$ is the Lebesgue measure in $\mathbb{R}^s$ of $B^*_{r/2}(0)$. This completes the proof. $\square$

With the above result at hand, we can now prove the following useful:

Proposition 2.9 (Global interpolation inequality)**.** *For every $p \in (1,\infty)$ there exists $\mathbf{c}_p > 0$ such that, for every $u \in W^{2,p}_X(\mathbb{R}^n)$ and every $\varepsilon > 0$, one has*

$$\|X_i u\|_{L^p(\mathbb{R}^n)} \le \varepsilon \,\|X_i^2 u\|_{L^p(\mathbb{R}^n)} + \frac{\mathbf{c}_p}{\varepsilon}\,\|u\|_{L^p(\mathbb{R}^n)} \quad \textit{for } i = 1, 2, \ldots, m. \tag{2.14}$$

Proof. For simplicity, we write $B_r, \widetilde{B}_r$ instead of $B_r(0), \widetilde{B}_r(0)$.

If, as usual, $\widetilde{X}_i$ is the lifted vector field of X_i in the Carnot group $\mathbb{G}$, by known interpolation inequalities in Carnot groups (see [6, Thm. 21]), we know that (for some constant $\widetilde{c}_p > 0$)

$$\|\widetilde{X}_i v\|_{L^p(\widetilde{B}_{1/2})} \le \sigma\,\|\widetilde{X}_i^2 v\|_{L^p(\widetilde{B}_1)} + \frac{\widetilde{c}_p}{\sigma}\|v\|_{L^p(\widetilde{B}_1)} \quad \text{for every } v \in W^{2,p}_{\widetilde{X}}(\widetilde{B}_1) \text{ and every } \sigma \in (0,1).$$

Let us apply this inequality to a function $v = \widetilde{w}$, where w depends only on x: by Lemma 2.8 (see also (2.13)), for every $w \in W^{2,p}_X(B_1)$ and any $\sigma \in (0,1)$ we get

$$\|X_i w\|_{L^p(B_{1/4})} \le c'_p\Big(\sigma\,\|X_i^2 w\|_{L^p(B_1)} + \frac{\widetilde{c}_p}{\sigma}\,\|w\|_{L^p(B_1)}\Big), \quad \text{where } c'_p := \frac{\mathrm{meas}(B^*_{1/4}(0))^{1/p}}{\mathrm{meas}(B^*_1(0))^{1/p}}.$$

Next, let us apply the last inequality to $w(x) := u(\delta_R(x))$, where $u \in W^{2,p}_X(B_R)$. We find:

$$R^{1-q/p}\|X_i u\|_{L^p(B_{R/4})} \le c'_p\, R^{2-q/p}\,\sigma\|X_i^2 u\|_{L^p(B_R)} + \frac{c''_p\,R^{-q/p}}{\sigma}\|u\|_{L^p(B_R)},$$

for every $u \in W^{2,p}_X(B_R(0))$ (and $c''_p := c'_p\,\widetilde{c}_p$). After dividing by $R^{1-q/p}$, this gives

$$\|X_i u\|_{L^p(B_{R/4})} \le c'_p\,R\,\sigma\|X_i^2 u\|_{L^p(B_R)} + \frac{c''_p}{R\,\sigma}\|u\|_{L^p(B_R)}, \quad \text{for every } u \in W^{2,p}_X(B_R). \tag{2.15}$$

For every fixed $\varepsilon > 0$ and every $R > 2\,\varepsilon/c'_p$, let us take $\sigma = \varepsilon/(c'_p R) < 1/2$ in (2.15): we obtain

$$\|X_i u\|_{L^p(B_{R/4})} \le \varepsilon\|X_i^2 u\|_{L^p(B_R)} + \frac{\mathbf{c}_p}{\varepsilon}\|u\|_{L^p(B_R)} \qquad (\text{with } \mathbf{c}_p = c'_p c''_p), \tag{2.16}$$

for every $u \in W^{2,p}_X(B_R)$. Hence, given $u \in W^{2,p}_X(\mathbb{R}^n)$, letting $R \to \infty$ in (2.16) (and noticing that $\mathbf{c}_p$ is independent of u and R), we get at once (2.14). $\square$

Notation 2.10. Henceforth, we shall use the following compact notation (where $i \ge 1$ is integer):

$$\|Du\|_{L^p(\Omega)} = \sum_{j=1}^m \|X_j u\|_{L^p(\Omega)}, \qquad \|D^i u\|_{L^p(\Omega)} = \sum_{|I|=i} \|X_I u\|_{L^p(\Omega)}.$$

We also let $D^0 u = u$. Notice that $\|u\|_{W^{k,p}_X(\Omega)} = \sum_{i=0}^k \|D^i u\|_{L^p(\Omega)}$.

In the sequel, we shall also need the following local version of the interpolation inequality:

Proposition 2.11. *For fixed $p \in (1,\infty)$, $R>0$ and $u \in W^{2,p}_X(B_R(0))$, let*

$$\Phi_k(u) := \sup_{\sigma\in(0,1)} \Big\{ \big((1-\sigma)R\big)^k \, \|D^k u\|_{L^p(B_{\sigma R}(0))} \Big\}, \qquad \text{for } k=0,1,2. \tag{2.17}$$

There exists $\alpha_p>0$ independent of u and R such that, for every $\varepsilon\in(0,1]$, one has

$$\Phi_1(u) \le \varepsilon\, \Phi_2(u) + \frac{\alpha_p}{\varepsilon}\, \Phi_0(u). \tag{2.18}$$

In order to prove Proposition 2.11, we need the following:

Lemma 2.12 (Radial cutoff functions)**.** *For every $r_1, r_2 \in (0,\infty)$, with $r_1<r_2$, there exists a cut-off function $\phi\in C^\infty_0(\mathbb{R}^n)$, valued in $[0,1]$, with the following properties:*

(i) *$\phi\equiv 1$ on $B_{r_1}(0)$;*
(ii) *$\phi \equiv 0$ outside $B_{r_2}(0)$;*
(iii) *for any $j\in\mathbb{N}$ there exists a constant $\varrho_j>0$, independent of r_1 and r_2, such that*

$$\|D^j\phi\|_{L^\infty(\mathbb{R}^n)} \le \frac{\varrho_j}{(r_2-r_1)^j}. \tag{2.19}$$

Proof. We leave it to the reader to check that the following choice of ϕ does the job:

$$\phi(x) = \chi\bigg(\frac{\|x\|}{2\,(r_2-r_1)} - \frac{r_1+r_2}{4(r_2-r_1)}\bigg),$$

where $\chi:\mathbb{R}\to\mathbb{R}$ is a C^∞-function with the following properties: ϕ is decreasing, $\phi\equiv 1$ on $(-\infty,-1/4]$, $\phi\equiv 0$ on $[1/4,\infty)$. (The smoothness of ϕ is a consequence of the fact that $\|\cdot\|$ is smooth outside the origin.) □

It is worthwhile noting that, in the present context, we are able to build cut-off functions adapted to any ball *centered at the origin* (but not at any point).

Proof of Proposition 2.11. We arbitrarily take $\sigma\in(0,1)$ and we let $\phi\in C^\infty_0(\mathbb{R}^n)$ be a cut-off function as in Lemma 2.12, with $r_1:=\sigma R$ and $r_2:=\sigma' R$ (where $\sigma':=(1+\sigma)/2<1$).

Since, by assumption, $u\in W^{2,p}_X(B_R)$, it is straightforward to check that $v:=\phi\, u\in W^{2,p}_X(\mathbb{R}^n)$ (note that $v\equiv u$ on $B_{\sigma R}$). Thus, if δ is *any* positive real number, from Proposition 2.9 we obtain

$$\begin{aligned} \|Du\|_{L^p(B_{\sigma R})} &= \sum_{i=1}^m \|X_i u\|_{L^p(B_{\sigma R})} = \sum_{i=1}^m \|X_i v\|_{L^p(B_{\sigma R})} \le \sum_{i=1}^m \|X_i v\|_{L^p(\mathbb{R}^n)} \\ &\overset{(2.14)}{\le} \delta \sum_{i=i}^m \|X_i^2 v\|_{L^p(\mathbb{R}^n)} + \frac{m\,\mathbf{c}_p}{\delta}\, \|v\|_{L^p(\mathbb{R}^n)}, \end{aligned} \tag{2.20}$$

where $\mathbf{c}_p>0$ is a suitable constant independent of u,δ and σ. We then observe that, by taking into account the properties of ϕ in Lemma 2.12, one has

$$\|v\|_{L^p(\mathbb{R}^n)} = \|\phi\, u\|_{L^p(\mathbb{R}^n)} \le \|u\|_{L^p(B_{\sigma' R})}; \tag{2.21}$$

moreover, for every index $i\in\{1,\dots,m\}$, we also have

$$\begin{aligned} \|X_i^2 v\|_{L^p(\mathbb{R}^n)} &= \|X_i^2(\phi\, u)\|_{L^p(\mathbb{R}^n)} = \Big\| u\, X_i^2\phi + 2\,(X_i\phi)\,(X_i u) + \phi\, X_i^2 u \Big\|_{L^p(B_{\sigma' R})} \\ &\le \frac{4\varrho_2}{\big((1-\sigma)R\big)^2}\, \|u\|_{L^p(B_{\sigma' R})} + \frac{4\varrho_1}{(1-\sigma)R}\, \|X_i u\|_{L^p(B_{\sigma' R})} + \varrho_0 \|X_i^2 u\|_{L^p(B_{\sigma' R})}; \end{aligned} \tag{2.22}$$

here, $\varrho_0,\varrho_1,\varrho_2$ are the constants appearing in (2.19), which are independent of σ and R. Multiplying both sides of (2.20) by $(1-\sigma)R>0$, and using estimates (2.21)-(2.22), we get

$$\begin{aligned} (1-\sigma)R\, \|Du\|_{L^p(B_{\sigma R})} &\le \varrho_0\,\delta\,(1-\sigma)R\, \|D^2 u\|_{L^p(B_{\sigma' R})} + 4\varrho_1\,\delta\, \|Du\|_{L^p(B_{\sigma' R})} \\ &\quad + m\,(\mathbf{c}_p + 4\varrho_2) \left\{ \frac{\delta}{(1-\sigma)R} + \frac{(1-\sigma)R}{\delta} \right\} \|u\|_{L^p(B_{\sigma' R})}. \end{aligned}$$

Setting $\theta_p := \varrho_0 + 4\,\varrho_1 + m\,(\mathbf{c}_p + 4\varrho_2)$, this gives

$$\begin{aligned}(1-\sigma)R\,\|Du\|_{L^p(B_{\sigma R})} &\le \theta_p\,\delta\,(1-\sigma)R\,\|D^2u\|_{L^p(B_{\sigma' R})} + \theta_p\,\delta\,\|Du\|_{L^p(B_{\sigma' R})} \\ &\quad + \theta_p\left\{\frac{\delta}{(1-\sigma)R} + \frac{(1-\sigma)R}{\delta}\right\}\|u\|_{L^p(B_{\sigma' R})},\end{aligned} \tag{2.23}$$

Now, if $\varepsilon \in (0,1]$ is arbitrarily fixed, since (2.23) holds for every $\delta > 0$, we can choose in particular

$$\delta = \delta_\varepsilon := \frac{(1-\sigma)R\,\varepsilon}{8\,\theta_p} > 0.$$

Thanks to this choice of δ, (2.23) becomes

$$\begin{aligned}&(1-\sigma)R\,\|Du\|_{L^p(B_{\sigma R})} \le \\ &\frac{\varepsilon}{8}\big((1-\sigma)R\big)^2\,\|D^2u\|_{L^p(B_{\sigma' R})} + \frac{\varepsilon}{8}\,(1-\sigma)R\,\|Du\|_{L^p(B_{\sigma' R})} + \left(\frac{\varepsilon}{8} + \frac{8\,\theta_p^2}{\varepsilon}\right)\|u\|_{L^p(B_{\sigma' R})}.\end{aligned} \tag{2.24}$$

Bearing in mind that

$$\sigma' = \frac{1+\sigma}{2} \in (0,1), \qquad \text{so that} \qquad (1-\sigma)R = 2(1-\sigma')R, \tag{2.25}$$

the above (2.24) can be rewritten as

$$\begin{aligned}&(1-\sigma)R\,\|Du\|_{L^p(B_{\sigma R})} \le \\ &\frac{\varepsilon}{2}\big((1-\sigma')R\big)^2\,\|D^2u\|_{L^p(B_{\sigma' R})} + \frac{\varepsilon}{4}\,(1-\sigma')R\,\|Du\|_{L^p(B_{\sigma' R})} + \left(\frac{\varepsilon}{8} + \frac{8\,\theta_p^2}{\varepsilon}\right)\|u\|_{L^p(B_{\sigma' R})}.\end{aligned}$$

Taking the supremum over $\sigma \in (0,1)$ on both sides of the latter inequality, one gets

$$\Phi_1(u) \le \frac{\varepsilon}{2}\,\Phi_2(u) + \frac{\varepsilon}{4}\Phi_1(u) + \left(\frac{\varepsilon}{8} + \frac{8\,\theta_p^2}{\varepsilon}\right)\Phi_0(u).$$

As a consequence, since $\varepsilon \in (0,1]$, we obtain

$$\frac{3}{4}\,\Phi_1(u) \le \left(1 - \frac{\varepsilon}{4}\right)\Phi_1(u) \le \frac{\varepsilon}{2}\,\Phi_2(u) + \left(\frac{\varepsilon}{8} + \frac{8\,\theta_p^2}{\varepsilon}\right)\Phi_0(u),$$

from which we derive that

$$\Phi_1(u) \le \frac{2\,\varepsilon}{3}\,\Phi_2(u) + \frac{3}{4}\left(\frac{\varepsilon}{8} + \frac{8\,\theta_p^2}{\varepsilon}\right)\Phi_0(u) \le \varepsilon\,\Phi_2(u) + \frac{\alpha_p}{\varepsilon}\,\Phi_0(u),$$

where $\alpha_p := \dfrac{3(64\,\theta_p^2 + 1)}{32}$, which is the desired (2.18). □

3. Global estimates and regularity results

In this last section we provide the proof of our main result, Theorem 1.5. To begin with, we establish the following lemma, of independent interest.

Lemma 3.1. *Let $p \in (1,\infty)$ and let k be a nonnegative integer. There exists a positive constant $\Theta_{k,p} > 0$, only depending on k and p, such that*

$$\|D^{i+2}u\|_{L^p(\mathbb{R}^n)} \le \Theta_{k,p}\,\|D^i(\mathcal{L}u)\|_{L^p(\mathbb{R}^n)} \qquad \text{for } i \in \{0,\dots,k\}, \tag{3.1}$$

for every function $u \in W_X^{k+2,p}(\mathbb{R}^n)$. As usual, $\mathcal{L}u = \sum_{j=1}^m X_j^2 u$.

Proof. Let $i \in \{0,\dots,k\}$ be fixed. For every $R > 0$, we consider the function $v_R := u \circ \delta_R$. Since, by assumption, u belongs to $W_X^{k+2,p}(\mathbb{R}^n)$ (and δ_R is linear), it is easy to see that

$$v_R \in W_X^{k+2,p}(\mathbb{R}^n) \subseteq W_X^{i+2,p}(\mathbb{R}^n).$$

Thus, since $\mathcal{L} = \sum_{j=1}^m X_j^2$ is a Hörmander sum of squares in $\mathbb{R}^n$, we are entitled to apply Theorem A for $v_R \in W_X^{i+2,p}(\mathbb{R}^n)$, with $\Omega' := B_1(0)$ and $\Omega'' := B_2(0)$, obtaining (for some $c_{i,p} > 0$)

$$\|D^{i+2}v_R\|_{L^p(B_1)} \le \|v_R\|_{W_X^{i+2,p}(B_1)} \le c_{i,p}\left\{\|\mathcal{L}v_R\|_{W_X^{i,p}(B_2)} + \|v_R\|_{L^p(B_2)}\right\}. \tag{3.2}$$

We now observe that, since $X_1,\dots,X_m$ are δ_λ-homogeneous of degree 1, one has

$$\begin{aligned}
&\|\mathcal{L}v_R\|_{W^{i,p}_X(B_2)} = \sum_{|I|\leq i} R^{2+|I|}\,\Big\|\big(X_I(\mathcal{L}u)\big)\circ\delta_R\Big\|_{L^p(B_2)},\\
&\|D^{i+2}v_R\|_{L^p(B_1)} = R^{i+2}\sum_{|I|=i+2}\Big\|\big(X_I u\big)\circ\delta_R\Big\|_{L^p(B_1)};
\end{aligned}\tag{3.3}$$

thus, by inserting (3.3) in (3.2), we obtain

$$R^{i+2-q/p}\,\|D^{i+2}u\|_{L^p(B_R)} \leq c_{i,p}\left\{\sum_{j=0}^{i} R^{2+j-q/p}\,\|D^j(\mathcal{L}u)\|_{L^p(B_{2R})} + R^{-q/p}\,\|u\|_{L^p(B_{2R})}\right\}.$$

Finally, since this last inequality clearly implies that

$$\|D^{i+2}u\|_{L^p(B_R)} \leq c_{i,p}\left\{\sum_{j=0}^{i} R^{j-i}\,\|D^j(\mathcal{L}u)\|_{L^p(B_{2R})} + R^{-i-2}\,\|u\|_{L^p(B_{2R})}\right\},$$

upon letting $R\to\infty$, we derive (remind that $u\in W^{k+2,p}_X(\mathbb{R}^n)$ and that $c_{i,p}$ is independent of R)

$$\|D^{i+2}u\|_{L^p(\mathbb{R}^n)} \leq c_{i,p}\,\|D^i(\mathcal{L}u)\|_{L^p(\mathbb{R}^n)}.$$

This readily gives the desired (3.1) with $\Theta_{k,p} := \max_{i=0,\dots,k} c_{i,p}$. $\square$

With Lemma 3.1 at hand, we can prove the following global estimates for $\mathcal{L}$.

Theorem 3.2 (Global $W^{k+2,p}_X$-estimates for $\mathcal{L}$)**.** *Let $p\in(1,\infty)$ and let k be a nonnegative integer. There exists a constant $\Lambda_{k,p}>0$ such that, if $u\in W^{k+2,p}_X(\mathbb{R}^n)$, then*

$$\|u\|_{W^{k+2,p}_X(\mathbb{R}^n)} \leq \Lambda_{k,p}\left\{\|\mathcal{L}u\|_{W^{k,p}_X(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\right\}.\tag{3.4}$$

Proof. By crucially exploiting Lemma 3.1, we have the estimate

$$\begin{aligned}
\|u\|_{W^{k+2,p}(\mathbb{R}^n)} &= \|u\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \sum_{i=0}^{k}\|D^{i+2}u\|_{L^p(\mathbb{R}^n)}\\
&\overset{(3.1)}{\leq} \|u\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \Theta_{k,p}\sum_{i=0}^{k}\|D^i(\mathcal{L}u)\|_{L^p(\mathbb{R}^n)}\\
&= \|u\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \Theta_{k,p}\,\|\mathcal{L}u\|_{W^{k,p}_X(\mathbb{R}^n)}.
\end{aligned}\tag{3.5}$$

On the other hand, by using the global interpolation inequality (2.14) (with $\varepsilon=1$), we have

$$\begin{aligned}
\|Du\|_{L^p(\mathbb{R}^n)} &= \sum_{j=1}^{m}\|X_j u\|_{L^p(\mathbb{R}^n)} \leq \sum_{j=1}^{m}\|X_j^2 u\|_{L^p(\mathbb{R}^n)} + m\,\mathbf{c}_p\,\|u\|_{L^p(\mathbb{R}^n)}\\
&\leq \|D^2u\|_{L^p(\mathbb{R}^n)} + m\,\mathbf{c}_p\,\|u\|_{L^p(\mathbb{R}^n)} \leq \qquad \big(\text{by Lemma 3.1 with } i=0\big)\\
&\leq \Theta_{k,p}\,\|\mathcal{L}u\|_{L^p(\mathbb{R}^n)} + m\,\mathbf{c}_p\,\|u\|_{L^p(\mathbb{R}^n)}.
\end{aligned}\tag{3.6}$$

Gathering together (3.5) and (3.6), we obtain (3.4) (with $\Lambda_{k,p} = \max\{2\,\Theta_{k,p}, m\,\mathbf{c}_p+1\}$). $\square$

We now turn to demonstrate the last ingredient for the proof of Theorem 1.5:

Theorem 3.3 (Global Sobolev regularity theorem for $\mathcal{L}$)**.** *Let $p\in(1,\infty)$ and let k be a nonnegative integer. Suppose that $u\in L^p(\mathbb{R}^n)$ is such that $\mathcal{L}u\in W^{k,p}_X(\mathbb{R}^n)$ (meaning that the distribution $\mathcal{L}u$ can be identified with a function belonging to $W^{k,p}_X(\mathbb{R}^n)$).*

Then $u\in W^{k+2,p}_X(\mathbb{R}^n)$.

By combining Theorems 3.2 and 3.3, we can readily provide the

Proof of Theorem 1.5. Let $u \in L^p(\mathbb{R}^n)$ be such that $\mathcal{L}u \in W^{k,p}_X(\mathbb{R}^n)$ (for some $p \in (1,\infty)$ and some integer $k \geq 0$). On account of Theorem 3.3, we have that

$$u \in W^{k+2,p}_X(\mathbb{R}^n);$$

as a consequence, by Theorem 3.2 we have

$$\|u\|_{W^{k+2,p}_X(\mathbb{R}^n)} \leq \Lambda_{k,p}\Big\{\|\mathcal{L}u\|_{W^{k,p}_X(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\Big\},$$

for a suitable constant $\Lambda_{k,p} > 0$ independent on u. This ends the proof. $\square$

We are left with the

Proof of Theorem 3.3. Let u be as in the assertion of Theorem 3.3. By Theorem A, $u \in W^{k+2,p}_{X,\mathrm{loc}}(\mathbb{R}^n)$; thus, to prove the theorem it suffices to show that

$$\|D^i u\|_{L^p(\mathbb{R}^n)} < \infty \qquad \text{for every } i = 1,\dots,k+2. \tag{3.7}$$

To prove (3.7), we proceed by steps.

STEP I: We begin by proving that (3.7) holds for $i = 2$.

To this end, let $R > 0$ be arbitrarily fixed, let $\sigma \in (0,1)$ and let $\phi \in C^\infty_0(\mathbb{R}^n)$ be a cut-off function as in Lemma 2.12, with $r_1 := \sigma R$ and $r_2 := \sigma' R$ (where $\sigma' := (1+\sigma)/2 < 1$). Since $v := u\,\phi$ belongs to $W^{k+2,p}_X(\mathbb{R}^n) \subseteq W^{2,p}_X(\mathbb{R}^n)$, we can apply Theorem 3.2 (with $k = 0$) to v, obtaining

$$\|D^2(u\,\phi)\|_{L^p(\mathbb{R}^n)} \leq \|u\,\phi\|_{W^{2,p}_X(\mathbb{R}^n)} \leq \Lambda_{0,p}\Big\{\|\mathcal{L}(u\,\phi)\|_{L^p(\mathbb{R}^n)} + \|u\,\phi\|_{L^p(\mathbb{R}^n)}\Big\}.$$

From this, by taking into account properties (i)-to-(iii) of ϕ in Lemma 2.12, we get

$$\begin{aligned}
\|D^2u\|_{L^p(B_{\sigma R})} &= \|D^2(u\,\phi)\|_{L^p(B_{\sigma R})} \leq \|D^2(u\,\phi)\|_{L^p(\mathbb{R}^n)} \leq \Lambda_{0,p}\Big\{\|\mathcal{L}(u\,\phi)\|_{L^p(\mathbb{R}^n)} + \|u\,\phi\|_{L^p(\mathbb{R}^n)}\Big\}\\
&\leq \Lambda_{0,p}\left\{\left\|\phi\,\mathcal{L}u + 2\sum_{j=1}^m X_j u\, X_j\phi + u\,\mathcal{L}\phi\right\|_{L^p(B_{\sigma' R})} + \|u\phi\|_{L^p(B_{\sigma' R})}\right\}\\
&\leq \gamma_p\left\{\|\mathcal{L}u\|_{L^p(B_{\sigma' R})} + \frac{1}{(1-\sigma)R}\|Du\|_{L^p(B_{\sigma' R})} + \left(\frac{1}{(1-\sigma)^2R^2} + 1\right)\|u\|_{L^p(B_{\sigma' R})}\right\},
\end{aligned}$$

where $\gamma_p > 0$ is a constant only depending on p and on $\varrho_0, \varrho_1, \varrho_2$ in (3.1) (hence, γ_p is independent of R and σ). We multiply both far sides of the above inequality by $(1-\sigma)^2R^2 > 0$,

$$\begin{aligned}
&\big((1-\sigma)R\big)^2\,\|D^2u\|_{L^p(B_{\sigma R})} \leq\\
&\gamma_p\Big\{R^2\,\|\mathcal{L}u\|_{L^p(B_{\sigma' R})} + (1-\sigma)\,R\,\|Du\|_{L^p(B_{\sigma' R})} + (1+R^2)\,\|u\|_{L^p(B_{\sigma' R})}\Big\}.
\end{aligned}$$

Due to the arbitrariness of σ, remembering the definition of $\Phi_i(u)$ (with $i = 0,1,2$) in (2.17) and using the local interpolation inequality in Proposition 2.11, we get (see also (2.25))

$$\begin{aligned}
\Phi_2(u) &\leq \gamma_p\Big\{R^2\,\|\mathcal{L}u\|_{L^p(B_R)} + 2\,\Phi_1(u) + (1+R^2)\,\|u\|_{L^p(B_R)}\Big\}\\
&\quad\big(\text{by (2.14) with } 0 < \varepsilon < \min\{1,(2\gamma_p)^{-1}\}\big)\\
&\leq \gamma_p\left\{R^2\,\|\mathcal{L}u\|_{L^p(B_R)} + 2\,\varepsilon\,\Phi_2(u) + \left(1 + R^2 + \frac{2\,\alpha_p}{\varepsilon}\right)\|u\|_{L^p(B_R)}\right\}.
\end{aligned}$$

As a consequence (isolating $\sigma = 1/2$ in the definition of $\Phi_2(u)$), we obtain

$$\begin{aligned}
\|D^2u\|_{L^p(B_{R/2})} &= \frac{4}{R^2}\left(\frac{R^2}{4}\,\|D^2u\|_{L^p(B_{R/2})}\right) \leq \frac{4}{R^2}\,\Phi_2(u)\\
&\leq \frac{4\,\gamma_p}{1-2\,\varepsilon\,\gamma_p}\left\{\|\mathcal{L}u\|_{L^p(B_R)} + \left(\frac{1}{R^2} + 1 + \frac{2\,\alpha_p}{\varepsilon\,R^2}\right)\|u\|_{L^p(B_R)}\right\}.
\end{aligned}$$

Finally, letting $R \to \infty$ (and remembering that γ_p does not depend on R), one has

$$\|D^2u\|_{L^p(\mathbb{R}^n)} \leq \frac{4\,\gamma_p}{1-2\,\varepsilon\,\gamma_p}\Big\{\|\mathcal{L}u\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\Big\},$$

and this proves that $\|D^2u\|_{L^p(\mathbb{R}^n)} < \infty$ (since, by assumption, both u and $\mathcal{L}u$ belong to $L^p(\mathbb{R}^n)$).

STEP II: We now prove that (3.7) holds for $i = 1$. To this end, let $R > 0$ be arbitrarily fixed. Since $u \in W^{k+2,p}_{X,\mathrm{loc}}(\mathbb{R}^n)$, we know that $u \in W^{2,p}_{X,\mathrm{loc}}(B_R)$. In due course of the proof of Proposition 2.9, we have proved that, if R is sufficiently large, it holds that (see (2.16) with $\varepsilon = 1$)

$$\|X_i u\|_{L^p(B_{R/4})} \le \|X_i^2 u\|_{L^p(B_R)} + \mathbf{c}_p\, \|u\|_{L^p(B_R)};$$

as a consequence, we infer that

$$\begin{aligned}\|Du\|_{L^p(B_{R/4})} &= \sum_{j=1}^m \|X_i u\|_{L^p(B_{R/4})} \le \sum_{j=1}^m \|X_i^2 u\|_{L^p(B_R)} + m\,\mathbf{c}_p\, \|u\|_{L^p(B_R)} \\ &\le \|D^2u\|_{L^p(B_R)} + m\,\mathbf{c}_p\, \|u\|_{L^p(B_R)}.\end{aligned}$$

By letting $R \to \infty$ (and remembering that $\mathbf{c}_p$ does not depend on R), we get

$$\|Du\|_{L^p(\mathbb{R}^n)} \le \|D^2u\|_{L^p(\mathbb{R}^n)} + m\mathbf{c}_p\, \|u\|_{L^p(\mathbb{R}^n)},$$

and this proves that $\|Du\|_{L^p(\mathbb{R}^n)} < \infty$, as $u \in L^p(\mathbb{R}^n)$ and, by Step I, $\|D^2u\|_{L^p(\mathbb{R}^n)} < \infty$.

STEP III: In this last step we show that (3.7) holds for every $i = 3, \dots, k+2$.

To this end, we first perform a (finite) induction argument on $i \in \{0, \dots, k\}$ to prove the existence of a constant $\kappa_i > 0$, only depending on i (and on k and p), such that

$$\|D^{i+2}u\|_{L^p(B_h)} \le \kappa_i \Big\{\|\mathcal{L}u\|_{W^{i,p}_X(B_{h+1+i})} + \|Du\|_{L^p(B_{h+1+i})} + \|u\|_{L^p(B_{h+1+i})}\Big\}, \quad \forall\, h \in \mathbb{N}. \tag{3.8}$$

Let us start with the case $i = 0$. For any fixed $h \in \mathbb{N}$, we choose a cut-off function $\phi_h \in C_0^\infty(\mathbb{R}^n)$ as in Lemma 2.12, with $r_1 := h$ and $r_2 := h+1$, and we define $v_h := u\phi_h$. Since we already know that $u \in W^{k+2,p}_{X,\mathrm{loc}}(\mathbb{R}^n)$, we have $v_h \in W^{k+2,p}_X(\mathbb{R}^n)$; as a consequence, by Lemma 3.1 (with $i = 0$),

$$\|D^2 v_h\|_{L^p(\mathbb{R}^n)} \le \Theta_{k,p}\, \|\mathcal{L}v_h\|_{L^p(\mathbb{R}^n)}.$$

From this, taking into account the properties of ϕ_h in Lemma 2.12, we have (notice that $r_2 - r_1 = 1$)

$$\begin{aligned}\|D^2u\|_{L^p(B_h)} &= \|D^2v_h\|_{L^p(B_h)} \le \|D^2 v_h\|_{L^p(\mathbb{R}^n)} \le \Theta_{k,p}\, \|\mathcal{L}v_h\|_{L^p(\mathbb{R}^n)} \\ &\le \Theta_{k,p} \Big\| u\,\mathcal{L}\phi_h + 2\sum_{j=1}^m X_j u\, X_j\phi_h + \phi_h\, \mathcal{L}u\Big\|_{L^p(B_{h+1})} \\ &\le \kappa_1 \Big\{\|\mathcal{L}u\|_{L^p(B_{h+1})} + \|u\|_{L^p(B_{h+1})} + \|Du\|_{L^p(B_{h+1})}\Big\},\end{aligned}$$

where $\kappa_1 > 0$ is a constant only depending on the bounds $\varrho_0, \varrho_1, \varrho_2$ in (2.19) (hence, κ_1 does not depend on h). This is precisely the desired (3.8) with $i = 0$.

Let us now take $j \in \{0, \dots, k-1\}$ and, assuming that (3.8) holds for $i = 0, \dots, j$, let us prove that (3.8) is fulfilled for i replaced by $j+1$. Arguing as above, with the very same ϕ_h, by applying Lemma 3.1 to the function $v_h = u\phi_h$ (and with $i = j+1 \le k$), we obtain

$$\begin{aligned}&\|D^{j+3}u\|_{L^p(B_h)} \le \|D^{j+3}v_h\|_{L^p(\mathbb{R}^n)} \overset{(3.1)}{\le} \Theta_{k,p}\, \|D^{j+1}(\mathcal{L}v_h)\|_{L^p(\mathbb{R}^n)} \\ &\quad \le \Theta_{k,p} \Big\| D^{j+1}\Big(u\,\mathcal{L}\phi_h + 2\sum_{l=1}^m X_l u\, X_l\phi_h + \phi_h\,\mathcal{L}u\Big)\Big\|_{L^p(B_{h+1})} \\ &\quad \le \Theta'_{k,p} \Big\{\|D^{j+1}(\mathcal{L}u)\|_{L^p(B_{h+1})} + \sum_{l=0}^{j+2} \|D^l u\|_{L^p(B_{h+1})}\Big\} \\ &\quad = \Theta'_{k,p} \Big\{\|D^{j+1}(\mathcal{L}u)\|_{L^p(B_{h+1})} + \|u\|_{L^p(B_{h+1})} + \|Du\|_{L^p(B_{h+1})} + \sum_{i=0}^{j} \|D^{i+2}u\|_{L^p(B_{h+1})}\Big\} = (\star),\end{aligned}$$

where $\Theta'_{k,p} > 0$ is a suitable constant independent of h. On the other hand, since we are assuming that (3.8) holds for any $0 \leq i \leq j$ (and for every $h \geq 1$), we have

$$\begin{aligned}\|D^{i+2}u\|_{L^p(B_{h+1})} &\leq \kappa_i \Big\{ \|\mathcal{L}u\|_{W_X^{i,p}(B_{h+2+i})} + \|Du\|_{L^p(B_{h+2+i})} + \|u\|_{L^p(B_{h+2+i})} \Big\} \\ &\leq \kappa_i \Big\{ \|\mathcal{L}u\|_{W_X^{j,p}(B_{h+2+j})} + \|Du\|_{L^p(B_{h+2+j})} + \|u\|_{L^p(B_{h+2+j})} \Big\}.\end{aligned}$$

By using this last estimate, we obtain

$$\begin{aligned}(\star) &\leq \Theta'_{k,p} \left(1 + \sum_{i=0}^{j} \kappa_i\right) \cdot \Big\{ \|D^{j+1}(\mathcal{L}u)\|_{L^p(B_{h+2+j})} + \|u\|_{L^p(B_{h+2+j})} + \|Du\|_{L^p(B_{h+2+j})} \\ &\qquad\qquad + \|\mathcal{L}u\|_{W_X^{j,p}(B_{h+2+j})} + \|Du\|_{L^p(B_{h+2+j})} + \|u\|_{L^p(B_{h+2+j})} \Big\} \\ &\leq \kappa_{j+1} \Big\{ \|\mathcal{L}u\|_{W_X^{j+1,p}(B_{h+2+j})} + \|Du\|_{L^p(B_{h+2+j})} + \|u\|_{L^p(B_{h+2+j})} \Big\},\end{aligned}$$

where we have introduced the constant (independent of h) $\kappa_{j+1} := 2\,\Theta'_{k,p}\big(1 + \sum_{i=0}^{j} \kappa_i\big)$. This is precisely the desired (3.8) with i replaced by $j+1$ and we are done.

Letting $h \to \infty$ in (3.8), one gets

$$\|D^{i+2}u\|_{L^p(\mathbb{R}^n)} \leq \kappa_i \Big\{ \|\mathcal{L}u\|_{W_X^{i,p}(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \Big\}, \quad \text{for } i = 0, \ldots, k.$$

Since the right-hand side is finite due to Step II (and the assumption), we infer $\|D^{i+2}u\|_{L^p(\mathbb{R}^n)} < \infty$ for $i = 0, \ldots, k$, and the proof is complete. □

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