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MULTIPLE BOUNDED VARIATION SOLUTIONS FOR A PRESCRIBED MEAN CURVATURE EQUATION WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. We prove the existence of multiple positive BV-solutions of the Neumann problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = a(x)f(u) & \text{in } (0,1), \\ u'(0) = u'(1) = 0, \end{cases}$$

where $a(x) > 0$ and f belongs to a class of nonlinear functions whose prototype example is given by $f(u) = -\lambda u + u^p$, for $\lambda > 0$ and $p > 1$. In particular, $f(0) = 0$ and f has a unique positive zero, denoted by u_0. Solutions are distinguished by the number of intersections (in a generalized sense) with the constant solution $u = u_0$. We further prove that the solutions found have continuous energy and we also give sufficient conditions on the nonlinearity to get classical solutions. The analysis is performed using an approximation of the mean curvature operator and the shooting method.

1. Introduction

In the last decades, a great deal of research has been devoted to the study of nonlinear boundary value problems associated with the mean curvature equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + g(x,u) = 0, \qquad x \in \Omega \subset \mathbb{R}^N, \tag{1.1}$$

both in the ODE case ($N = 1$) and in the PDE one ($N \geq 2$); see, among many others, [5, 14, 21, 22, 30, 41, 47, 51, 54] and the references therein. Besides this well known interpretation from Differential Geometry, this equation also appears in several contexts from Mathematical Physics, such as reaction-diffusion processes with saturation at high regimes [12, 29], capillarity phenomena for incompressible fluids [20, 28], modeling of the human cornea [15, 16, 17, 50]. From the genuinely mathematical point of view, the investigation of equation (1.1) leads to a variety of challenging technical issues, since, due to the strongly nonlinear character of the differential operator, it becomes necessary to take into account weaker notions of solutions, possibly exhibiting jump discontinuities.

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Along this line of research, in this paper we look for positive solutions of the following one-dimensional Neumann problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = a(x)f(u) & \text{in } (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.2}$$

where $a \in C^1([0,1])$, $a > 0$ in $[0,1]$, and $f \in C^1([0,\infty))$ is a nonlinear term whose prototype example is given by

$$f(s) = -\lambda s + s^p, \qquad \lambda > 0,\ p > 1. \tag{1.3}$$

In particular, problem (1.2) has a unique positive constant solution $u \equiv u_0$ and we are interested in studying existence, multiplicity and some qualitative properties of non-constant positive solutions of (1.2) that oscillate around u_0.

The choice for this nonlinear term is partially inspired by some recent results, dealing with the radial Neumann problem (in an annulus or in a ball) associated with the semilinear equation

$$-\Delta u = f(u),$$

see [3, 4, 6, 7, 10, 40], and with the Minkowski-curvature equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + f(u) = 0,$$

see [8, 9]. In the above papers, it is shown that, for a large class of nonlinear terms f including (1.3), non-constant positive radial solutions oscillating around u_0 can be provided: more precisely, radial solutions u having exactly k intersections with u_0 exist if $f'(u_0)$ is greater than the k-th non-zero eigenvalue of the radial Neumann problem for $-\Delta u = \lambda u$. On growing of the value $f'(u_0)$, a high multiplicity of solutions thus appears, confirming a conjecture first given in [6]; in all these papers, solutions are meant in the classical sense. Notice also that the Minkowski-curvature operator behaves as Δu when ∇u is small, this being the reason why in both cases the condition required on $f'(u_0)$ to guarantee the existence of non-constant solutions is related to the eigenvalues of $-\Delta u = \lambda u$, cf. [9, Theorem 1.1].

The aim of this paper is to provide a similar solvability pattern for the boundary value problem (1.2). As in the recent papers [35, 36, 37, 38, 39], and since we will take advantage of some regularity results proved therein, we choose here to work in a purely one-dimensional setting; however, to avoid trivialities, we assume that a non-constant weight $a(x)$ can appear in front of the nonlinear term f. Notice that, since the mean curvature operator linearizes as u'' for u' small, here the eigenvalues of the weighted problem $-u'' = \lambda a(x)u$ are expected to play a role.

As already anticipated, the main difficulties in considering problem (1.2) are due to the possible lack of regularity of the solutions. We work indeed with Bounded Variation (BV, for short) solutions to (1.2), a variational notion of solutions basically going back to the works of A. Lichnewsky and R. Temam [19, 31, 32, 33, 34, 55] and E. Giusti and M. Miranda in [23, 24, 42, 43] and now commonly used in this context; we recall the precise definition at the beginning of Section 2 for the reader's convenience. The crucial point, here, is to define, for a (possibly discontinuous) BV-solution, a suitable notion of intersection with the constant u_0, so as to provide multiplicity. To the best of our knowledge, a similar issue has never been faced in the context of mean curvature equations, for which only few high multiplicity results are known [18, 26, 44, 45, 46, 48].

We now state our result precisely. First, we introduce the following structural assumptions on the nonlinear term: for some positive constant $u_0 > 0$, it holds

(f_{eq}) $f(0) = f(u_0) = 0$;

(f_{sgn}) $f(s) < 0$ if $s \in (0, u_0)$, $f(s) > 0$ if $s \in (u_0, \infty)$.

Moreover, we will also suppose that f satisfies one of the following two conditions:

(f_{ap}) there exists $\bar{u} > u_0$ such that

$$\int_{u_0}^{\bar{u}} f(s)ds = C_a \int_{u_0}^{0} f(s)ds,$$

where $C_a := \max\left\{ \frac{a(1)}{a(0)} \exp\left(\int_0^1 \frac{a'^-(x)}{a(x)}dx \right), \frac{a(0)}{a(1)} \exp\left(\int_0^1 \frac{a'^+(x)}{a(x)}dx \right) \right\}$;

($f_{\mathrm{ap}})'$ $\|a\|_{L^1(0,1)} \max f^- < 1$.

Clearly enough, the above conditions are completely unrelated. In particular, (f_{ap}) is a condition on the behavior of f at infinity; it is surely satisfied when $\int_{u_0}^{+\infty} f(s)ds = +\infty$ so that, in particular, the model nonlinearity (1.3) fulfills it for every $\lambda > 0$ and $p > 1$. Incidentally, notice also that, in case a is monotone, the constant C_a reduces to $\max_{[0,1]} a / \min_{[0,1]} a$. On the other hand, $(f_{\mathrm{ap}})'$ concerns only the behavior of f in $[0, u_0]$, cf. (f_{sgn}). Assumption $(f_{\mathrm{ap}})'$ is inspired from some arguments in [49], see also [53]. It is satisfied by every function f satisfying (f_{sgn}), at the cost of asking that $\|a\|_{L^1(0,1)}$ is small enough. Moreover, in view of [35, Lemma 3.1], $(f_{\mathrm{ap}})'$ seems to be quite natural when looking for solutions with $u(0) < u_0$.

Second, we define the energy $\mathcal{E}$ for a non-negative solution of (1.2) by formally letting

$$\mathcal{E}(x) := 1 - \frac{1}{\sqrt{1 + (u'(x))^2}} + a(x)F(u(x)), \tag{1.4}$$

where $F(u) := \int_{u_0}^{u} f(s)ds$. Of course, in principle this definition is meaningless for a BV-function; however, it will be clear from the statement of the result that $\mathcal{E}$ is well-defined for every value of x in the interval $[0, 1]$ up to a finite number of points where u is discontinuous (if u is continuous but not differentiable at some point $\bar{x}$, it must be $|u'(\bar{x})| = +\infty$ and the definition of $\mathcal{E}$ has to be intended in the limit sense, i.e., $\mathcal{E}(\bar{x}) = 1 + a(\bar{x})F(u(\bar{x}))$.

Finally, we introduce, for every $k \in \mathbb{N}$, λ_k as the k-th eigenvalue of $-u'' = \lambda a(x)u$ in $(0, 1)$ with Neumann boundary conditions; namely, $0 = \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ and $\lambda_k \to \infty$ as $k \to \infty$. Moreover, we denote by $u'(x^-)$ and $u'(x^+)$ the left and the right Dini derivatives of u at $x \in (0, 1)$, respectively, and we mean that a BV-function u is positive in $(0, 1)$ if $\operatorname{ess\,inf}_{(0,1)} u > 0$.

We can then state our result as follows.

Theorem 1.1. *Let $a \in C^1([0, 1])$ be such that $a(x) > 0$ for every $x \in [0, 1]$. Let $f \in C^1([0, \infty))$ satisfy (f_{eq}), (f_{sgn}) and either (f_{ap}) or $(f_{\mathrm{ap}})'$. Moreover, let us suppose that, for some $k \in \mathbb{N}$,*

$$f'(u_0) > \lambda_{k+1}.$$

Then, there exist at least $2k$ distinct non-constant positive BV-solutions $u_1, \ldots, u_{2k}$ of (1.2). Furthermore, *(I) for every $j = 1, \dots, k$, there exist exactly $j + 2$ points $0 = x_{j,0} < x_{j,1} < \cdots < x_{j,j} < x_{j,j+1} = 1$, such that*

(a) $u_j \in C^2([0, x_{j,1})) \cap C^2((x_{j,j}, 1])$, $u_j \in C^2((x_{j,i}, x_{j,i+1}))$ for $i = 1, \dots, j-1$ and $u_j'(0) = 0 = u_j'(1)$;

(b) $(-1)^{i+1}(u_j(x) - u_0) > 0$ for every $x \in (x_{j,i}, x_{j,i+1})$ for $i = 0, \dots, j$;

(c) for every $i = 1, \dots, j$ one of the following two statements holds true:

(i) $u_j(x_{j,i}) = u_0$ and $u_j \in C^2((x_{j,i-1}, x_{j,i+1}))$;

(ii) $u_j(x_{j,i}^-) \le u_0 \le u_j(x_{j,i}^+)$ and $u_j'(x_{j,i}^-) = +\infty = u_j'(x_{j,i}^+)$ if i is odd, $u_j(x_{j,i}^+) \le u_0 \le u_j(x_{j,i}^-)$ and $u_j'(x_{j,i}^-) = -\infty = u_j'(x_{j,i}^+)$ if i is even,

(II) for every $\ell = 1, \dots, k$, there exist exactly $\ell + 2$ points $0 = x_{\ell,0} < x_{\ell,1} < \cdots < x_{\ell,\ell} < x_{\ell,\ell+1} = 1$, such that

(a) $u_{2k+1-\ell} \in C^2([0, x_{\ell,1})) \cap C^2((x_{\ell,\ell}, 1])$, $u_{2k+1-\ell} \in C^2((x_{\ell,i}, x_{\ell,i+1}))$ for $i = 1, \dots, \ell - 1$ and $u_\ell'(0) = 0 = u_\ell'(1)$;

(b) $(-1)^{i+1}(u_{2k+1-\ell}(x) - u_0) < 0$ for every $x \in (x_{\ell,i}, x_{\ell,i+1})$ for $i = 0, \dots, \ell$;

(c) for every $i = 1, \dots, \ell$ one of the following two statements holds true:

(i) $u_{2k+1-\ell}(x_{\ell,i}) = u_0$ and $u_{2k+1-\ell} \in C^2((x_{\ell,i-1}, x_{\ell,i+1}))$;

(ii) $u_{2k+1-\ell}(x_{\ell,i}^-) \le u_0 \le u_{2k+1-\ell}(x_{\ell,i}^+)$ and $u_{2k+1-\ell}'(x_{\ell,i}^-) = +\infty = u_{2k+1-\ell}'(x_{\ell,i}^+)$ if i is even,
$u_{2k+1-\ell}(x_{\ell,i}^+) \le u_0 \le u_{2k+1-\ell}(x_{\ell,i}^-)$ and $u_{2k+1-\ell}'(x_{\ell,i}^-) = -\infty = u_{2k+1-\ell}'(x_{\ell,i}^+)$ if i is odd,

(III) for every $j = 1, \dots, 2k$ the energy $\mathcal{E}_j$ corresponding to the solution u_j can be extended by continuity to $[0, 1]$.

We observe that in part (I) of this theorem, we describe the oscillating solutions having $u(0) < u_0$, while part (II) deals with solutions having $u(0) > u_0$. Moreover, in both cases (I)-(c)-(i) and (I)-(c)-(ii), for every $i = 1, \dots, j$ the function $u_j(\cdot) - u_0$ changes sign exactly once in the interval $(x_{j,i-1}, x_{j,i+1})$, at point $x_{j,i}$. When the case (c)-(ii) occurs, we will informally refer to $x_{j,i}$ as a *generalized intersection point* of u with u_0. In this way, we can summarize parts (I)-(c)-(i) and (I)-(c)-(ii) of Theorem 1.1 stating that u_j has exactly j generalized intersections with u_0, that occur at points $x_{j,1}, \dots, x_{j,j}$. A similar remark can be done also for parts (II)-(c)-(i) and (II)-(c)-(ii) of the statement. Notice also that, by regularity, every solution u satisfies $u(x) \ge \operatorname{ess\,inf}_{(0,1)} u > 0$ for every continuity point $x \in [0, 1]$. We also emphasize Part (III) of Theorem 1.1, ensuring that the solutions found have continuous energy: this is not always the case for a general BV-solution and it will be obtained as a consequence of our method of proof.

The necessity of taking into account possibly discontinuous solutions with generalized intersections with u_0 is well-recognized even in the autonomous case, $a(x) \equiv a$. In such a case the equation in (1.2) can be equivalently written as the planar Hamiltonian system

$$u' = \frac{v}{\sqrt{1 - v^2}}, \qquad v' = -af(u), \tag{1.5}$$

and solutions lie on level sets of the energy

$$H(u, v) = 1 - \sqrt{1 - v^2} + aF(u),$$

where, again, $F(u) = \int_{u_0}^u f(s)ds$; incidentally, notice that this agrees with our previous definition of the energy $\mathcal{E}$, that is, $H(u(x), v(x)) = \mathcal{E}(x)$ if (u, v) is a solution of (1.5). An elementary phase-plane analysis shows that the point $(u_0, 0)$ is a global minimum of H and, hence, a local center for the system, being surrounded by classical closed orbits with periods tending to $2\pi/\sqrt{f'(u_0)}$, as the solution shrinks to u_0. These orbits give rise to natural candidate solutions to (1.2): precisely, if (u, v) is a closed orbit around $(u_0, 0)$ satisfying $(u(0), v(0)) \in \mathbb{R}^+ \times \{0\}$, then u satisfies the Neumann boundary conditions if and only if the half-period is of the type $1/\ell$ for some $\ell \in \mathbb{N}$; in such a case, the solution u is classical and the number of intersections with u_0 is exactly ℓ. Discontinuous BV-solutions, on the other hand, have to be found at energy levels giving rise to disconnected level sets: due to the specific form of H, it is easy to check that, for an orbit (u, v) intersecting the positive u-semiaxis at a point $(\tilde{u}, 0)$, this happens if and only if $aF(\tilde{u}) \geq 1$. In this regard, it is worth recalling that the BV-solutions provided by Theorem 1.1, even when discontinuous, still have a continuous energy. Therefore, in the autonomous case BV-solutions are obtained by switching to different connected components of the same energy level set. See also Figure 1.

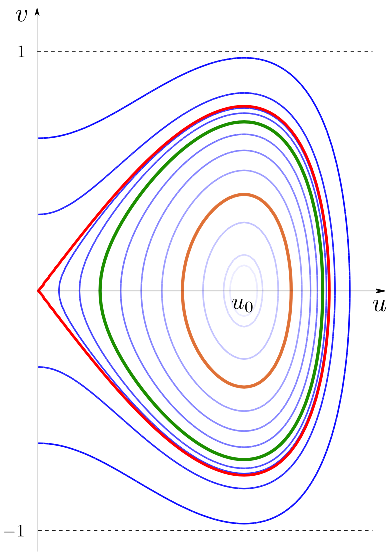
The proof of this theorem is quite long and we prefer to describe here the strategy and the main ideas behind it. In order to do this, it is useful to write the differential operator driving the equation in (1.2) as $(\varphi(u'))'$ with $\varphi(s) := s/\sqrt{1+s^2}$: it thus becomes apparent that the possible lack of regularity of the solutions is a consequence of the boundedness of $\varphi(\mathbb{R})$.

To overcome this difficulty, we approximate φ with the sequence $(\varphi_n)_n$ of C^1-functions that coincide with φ in $[-n, n]$ and are affine in $\mathbb{R} \setminus [-n, n]$. We first study, for every $n \in \mathbb{N}$, the approximated problem (3.1) governed by the operator $-(\varphi_n(u'))'$ (cf. [5, 38] for a similar strategy) and we prove that, if $f'(u_0) > \lambda_{k+1}$, each approximated problem (3.1) has $2k$ classical positive solutions. This multiplicity result is obtained via shooting method. In particular, we consider the Cauchy problem $u(0) = d$, $u'(0) = 0$ associated to the equation $-(\varphi_n(u'))' = a(x)f(u)$ of the approximated problem (3.1) and look for values of $d \in \mathbb{R}^+$ for which the solution u_d of the Cauchy problem satisfies $u_d'(1) = 0$, thus solving also (3.1). To this aim, we count the number of half-turns performed by the solution in the phase plane around the equilibrium $(u_0, 0)$, as done in [7, 10] for a p-Laplacian problem. We observe that neither (f_{ap}) nor $(f_{\mathrm{ap}})'$ is required for the multiplicity result of the approximated problem.

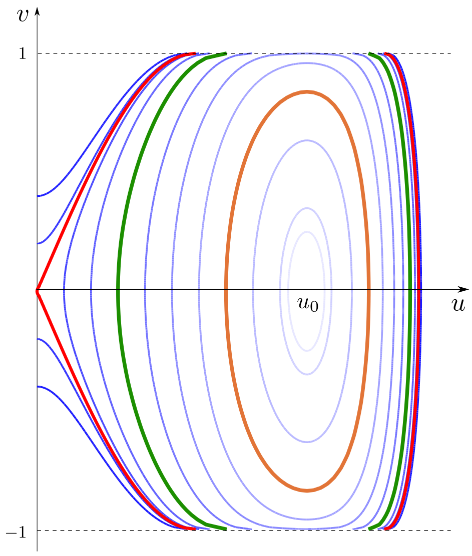
We then prove that each sequence of solutions of the approximated problems (with a fixed number of intersections with u_0) converges in some sense to a BV-solution of the original problem (1.2). To this aim, under either of the assumptions (f_{ap}) and $(f_{\mathrm{ap}})'$, we prove that these sequences are bounded in $W^{1,1}(0, 1)$, cf. Lemmas 4.1 and 4.2, and so they weakly-* converge to BV-functions, up to subsequences. Once we have a limit function for every sequence, we prove in Proposition 4.3 that those functions actually are BV-solutions of (1.2).

The most delicate point is now to distinguish the $2k$ BV-solutions that we obtained with this approximation procedure. We manage to do that, by proving that each limit function inherits the oscillatory behavior of the approximating solutions. This is based on the Propositions 4.5 and 4.6, that ensure that the convergence is actually much stronger (viz. uniform) away from the intersection points. As a consequence, we get that the BV-solutions of (1.2) are allowed to jump only at generalized intersection points.

We finally prove that the BV-solutions inherit also another important property from the approximating solutions, that is the continuity of the energy. This is



(a)



(b)

Figure 1. Phase-portraits of the planar Hamiltonian system (1.5), for different suitable choices of a and f (incidentally, let us recall that $|v| = 1$ if and only if $|u'| = +\infty$). On the left, all Neumann solutions to the equation are classical ones, corresponding to closed orbits around the equilibrium point $(u_0, 0)$ (two of them are painted in green and orange color). On the right, on the contrary, due to the breakdown of some energy levels as soon as $|v|$ reaches the value 1, both classical (the orange one, corresponding to a closed orbit) and discontinuous BV-solutions (the green one, corresponding to different connected components of the same energy level) appear. It is worth noticing that the existence of a classical homoclinic orbit to the saddle equilibrium point $(0, 0)$ (painted in red in the figure on the left) is a sufficient condition for all the Neumann solutions to be classical: based on this observation, we will establish a similar criterion for the non-autonomous case in Section 6.

mainly based on the preliminary Lemma 5.3 which ensures that, away from the generalized intersection points, the convergence is even C^1.

The paper is organized as follows. In Section 2, we give some preliminary known results and useful consequences about BV-solutions and the associated linear eigenvalue problem. In Section 3, we study the approximated problems via shooting method and prove some properties of the approximating solutions. Section 4 is devoted to the proof of Parts (I) and (II) of the main result of the paper, Theorem 1.1, while in Section 5 we establish Part (III) of the same result. Finally, in Section 6, we give some sufficient conditions on the nonlinearity f to get classical solutions of (1.2).

2. Preliminaries

In this section, we state some known results that will be useful in the subsequent sections.

2.1. **The notion of BV-solution.** Here we clarify the notion of solution used throughout the paper. Since we are interested in positive solutions, we introduce the notion of solution only for non-negative BV-functions u, so as to guarantee the well-posedeness of $f(u)$.

Definition 2.1. We say that a non-negative function $u \in BV(0,1)$ is a *bounded variation solution* (BV-solution) of problem (1.2) if for every $v \in BV(0,1)$

$$\int_0^1 \sqrt{1+|Dv|^2} \geq \int_0^1 \sqrt{1+|Du|^2} + \int_0^1 a(x)f(u)(v-u)dx, \tag{2.1}$$

where

$$\int_0^1 \sqrt{1+|Dv|^2} := \sup\left\{ \int_0^1 (vw_1' + w_2)dx \,:\, w_1,\, w_2 \in C_c^1(0,1), \|w_1^2 + w_2^2\|_\infty \leq 1 \right\}.$$

Equivalently, a non-negative function $u \in BV(0,1)$ is a BV-solution of (1.2) if it is a global minimizer of the functional $I_u : BV(0,1) \to \mathbb{R}$ defined as

$$I_u(v) := \int_0^1 \sqrt{1+|Dv|^2} - \int_0^1 a(x)f(u)v dx \quad \text{for every } v \in BV(0,1).$$

Incidentally, let us recall that, from [13, Proposition 2.36], $BV(0,1)$ embeds into $L^\infty(0,1)$ so that the integrals are well-defined.

We also notice that, by the results in [2], see also [35, Definition 2.3], the variational inequality (2.1) is equivalent to the Euler equation

$$\int_0^1 \frac{Du^a Dv^a}{\sqrt{1+(Du^a)^2}}dx + \int_0^1 \frac{Du^s}{|Du^s|}Dv^s = \int_0^1 a(x)f(u)v dx$$

for every $v \in BV(0,1)$ such that $|Dv^s|$ is absolutely continuous with respect to $|Du^s|$. In the previous equation $Dw = Dw^a dx + Dw^s$ is the Lebesgue-Nikodym decomposition of the Radon measure Dw in its absolutely continuous part $Dw^a dx$, with density function Dw^a, and its singular part Dw^s. Moreover, the symbol $|\mu|$ stands for the absolute variation of a Radon measure μ and finally $\frac{\mu}{|\mu|}$ is the density function of μ with respect to its absolute variation $|\mu|$.

Remark 2.2. Observe that by [35, Lemma 2.1], a non-negative $u \in W^{1,1}(0,1)$ is a weak solution of (1.2) (in the usual sense), if u satisfies (2.1) for all $v \in W^{1,1}(0,1)$. We recall that if $u \in W^{1,1}(0,1)$, $\int_0^1 \sqrt{1+|Du|^2} = \int_0^1 \sqrt{1+u'^2}dx$, cf. [35, (2.9)]. Moreover by [35, Lemma 2.3], a weak solution $u \in W^{1,1}(0,1)$ of (1.2) is also a BV-solution.

The following approximation result will be useful along the paper.

Lemma 2.3. *For every $u \in BV(0,1)$ there exists a sequence $(u_n) \subset W^{1,1}(0,1)$ such that*

(i) *$u_n \to u$ in $L^1(0,1)$;*
(ii) $\int_0^1 \sqrt{1+u_n'^2}dx \to \int_0^1 \sqrt{1+|Du|^2}$.

Proof. See [2, Facts 3.1 and 3.3]. □

Using the previous lemma, we can state the following proposition.

Proposition 2.4. *If a non-negative function $u \in BV(0,1)$ satisfies the inequality in (2.1) for every $v \in C^\infty([0,1])$, then it is a BV-solution of (1.2).*

Proof. We first prove that if $0 \leq u \in BV(0,1)$ satisfies the inequality in (2.1) for every $v \in W^{1,1}(0,1)$, then it is a BV-solution of (1.2). Let $v \in BV(0,1)$ and consider a sequence $(v_n) \subset W^{1,1}(0,1)$ that converges to v in the sense of (i)-(ii) of Lemma 2.3. We observe that, since $v_n \in W^{1,1}(0,1)$, $\int_0^1 \sqrt{1+|Dv_n|^2} = \int_0^1 \sqrt{1+v_n'^2}dx$, cf. [35, (2.9)]. Thus, the following inequality holds for every $n \in \mathbb{N}$

$$\int_0^1 \sqrt{1+v_n'^2}dx \geq \int_0^1 \sqrt{1+|Du|^2} + \int_0^1 a(x)f(u)(v_n - u)dx, \tag{2.2}$$

and by Lemma 2.3-(ii), $\int_0^1 \sqrt{1+v_n'^2}dx \to \int_0^1 \sqrt{1+|Dv|^2}$ as $n \to \infty$. Furthermore, in view of the L^1-convergence (Lemma 2.3-(i)) and of the embedding of $BV(0,1)$ in $L^\infty(0,1)$, we can pass to the limit as $n \to \infty$ in (2.2) and infer that (2.1) holds for v, concluding this first step of the proof.

Now, by [11, Theorem 8.7], for every $v \in W^{1,1}(0,1)$ there exists a sequence $(v_n) \subset C^\infty([0,1])$ such that $v_n \to v$ in $W^{1,1}(0,1)$. Therefore, since by assumption (2.1) holds for every v_n, passing to the limit as $n \to \infty$, we get that (2.1) holds for v. In view of the first part, this concludes the proof of the proposition. □

2.2. **Regularity and qualitative properties of BV-solutions.** The following result follows in a straightforward way from [35, Proposition 3.6].

Proposition 2.5. *Let u be a BV-solution of (1.2), then the following statements hold true.*

(i) $\displaystyle\int_0^1 a(x)f(u(x))dx = 0.$

(ii) *Let $(\alpha,\beta) \subset (0,1)$ be an interval such that $u(x) \in [u_0,\infty)$ for a.e. $x \in (\alpha,\beta)$ (resp., $u(x) \in [0,u_0]$ for a.e. $x \in (\alpha,\beta)$). Then, u is concave (resp., convex) in (α,β), and its restriction to (α,β) is of class $C^2((\alpha,\beta)) \cap W^{1,1}(\alpha,\beta)$ and satisfies $-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = a(x)f(u)$ for every $x \in (\alpha,\beta)$.*
Furthermore, if $\alpha = 0$, $u \in C^2([0,\beta))$ and $u'(0) = 0$, and similarly, if $\beta = 0$, $u \in C^2((\alpha,1])$ and $u'(1) = 0$.

(iii) *Let (α,β) and (β,γ) be any pair of adjacent subintervals of $(0,1)$ such that $u(x) \in [u_0,\infty)$ for a.e. $x \in (\alpha,\beta)$ and $u(x) \in [0,u_0]$ for a.e. $x \in (\beta,\gamma)$ (resp., $u(x) \in [0,u_0]$ for a.e. $x \in (\alpha,\beta)$ and $u(x) \in [u_0,\infty)$ for a.e. $x \in (\beta,\gamma)$). Then, either $u \in C^2((\alpha,\gamma))$, or $u(\beta^-) \geq u(\beta^+)$ and $u'(\beta^-) = -\infty = u'(\beta^+)$ (resp., $u(\beta^-) \leq u(\beta^+)$ and $u'(\beta^-) = +\infty = u'(\beta^+)$).*

Proof. Testing (2.1) with $v = u \pm 1$, it becomes clear that the only possibility is that (i) holds. In view of assumption (f_{sgn}), and taking into account the fact that a is continuous and positive, parts (ii) and (iii) are immediate consequences of [35, Proposition 3.6]. □

Remark 2.6. By Proposition 2.5-(i), in view of assumption (f_{sgn}), we get that if u is a BV-solution of (1.2), then neither of the following two conditions can be verified

- $u(x) \in (0,u_0)$ for a.e. $x \in (0,1)$;
- $u(x) \in (u_0,\infty)$ for a.e. $x \in (0,1)$.

As a consequence of Proposition 2.5-(ii), we can obtain the following result, which basically ensures that a BV-solution u of (1.2), if not identically equal to u_0, can assume the value u_0 only at points where $u(\cdot) - u_0$ changes sign.

Corollary 2.7. *Let u be a BV-solution of (1.2). If there exist $j+2$ points $0 = x_0 < x_1 < \cdots < x_j < x_{j+1} = 1$ such that $(-1)^{i+1}(u(x) - u_0) \geq 0$ for every $x \in (x_i, x_{i+1})$ for $i = 0, \ldots, j$ (resp., $(-1)^i(u(x)-u_0) \geq 0$ for every $x \in (x_i, x_{i+1})$ for $i = 0, \ldots, j$), then either $u \equiv u_0$ on $(0, 1)$ or $u(x) \neq u_0$ for every $x \in (0, 1) \setminus \{x_1, \ldots, x_j\}$.*

For the proof, we adopt the following elementary version of the strong maximum principle: if $u : (a, b) \to \mathbb{R}$ is a convex (resp., concave) function of class C^1 and if $x_0 \in (a, b)$ is a maximum (resp., minimum) point of u, then u is constant on (a, b).

Proof. Assume that there exists $x^* \in (x_i, x_{i+1})$ for some $i \in \{0, \ldots, j\}$ such that $u(x^*) = u_0$; moreover, to fix the ideas suppose that $u(x) \leq u_0$ for every $x \in (x_i, x_{i+1})$ so that, by part (ii) of Proposition 2.5, u is convex on such an interval. Therefore, the strong maximum principle yields $u(x) = u_0$ for every $x \in (x_i, x_{i+1})$. Hence, $u(x) \geq u_0$ for every $x \in (x_{i-1}, x_{i+2})$ (if $i = 0$ or $i = j$, we work on the intervals (x_i, x_{i+2}) or (x_{i-1}, x_{i+1})). By part (ii) of Proposition 2.5, u is concave on (x_{i-1}, x_{i+2}) and the strong maximum principle can be applied again to obtain $u \equiv u_0$ on (x_{i-1}, x_{i+2}). By repeating the argument, $u \equiv u_0$ on $(0, 1)$. □

2.3. **The associated eigenvalue problem.** We consider the eigenvalue problem associated to (1.2), namely

$$\begin{cases} -u'' = \lambda a(x) u & \text{in } (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{2.3}$$

where $a = a(x)$ is the same positive weight appearing in (1.2). For (2.3), the following classical result holds, cf. for instance [27].

Theorem 2.8. *The eigenvalues of (2.3) form a divergent, increasing sequence $0 = \lambda_1 < \lambda_2 < \ldots \lambda_k < \ldots$, $\lim_{k\to\infty} \lambda_k = \infty$. Moreover, every eigenvalue is simple. The eigenfunction that corresponds to the k-th eigenvalue λ_k has exactly $k - 1$ simple zeros in $(0, 1)$.*

For future use, we introduce the clockwise polar coordinates

$$\begin{cases} u(x) = \varrho(x) \cos \vartheta(x) \\ u'(x) = -\varrho(x) \sin \vartheta(x) \end{cases}$$

and we write the equation satisfied by the angular variable ϑ of a solution u of (2.3)

$$\vartheta'_\lambda(x) = \sin^2 \vartheta(x) + \lambda a(x) \cos^2 \vartheta(x) \quad \text{for } x \in (0, 1). \tag{2.4}$$

Notice that $\vartheta'_\lambda(x) > 0$ for every $x \in (0, 1)$, and so $\vartheta_\lambda = \vartheta_\lambda(x)$ is strictly increasing. Hence, the oscillatory behavior of the eigenfunctions of (2.3) described in Theorem 2.8 can be expressed in the following way: if $\lambda = \lambda_{k+1}$, the angular variable $\vartheta_{\lambda_{k+1}}$ that corresponds to the $(k + 1)$-th eigenfunction satisfies the identity

$$\vartheta_{\lambda_{k+1}}(1) - \vartheta_{\lambda_{k+1}}(0) = k\pi. \tag{2.5}$$

We further observe that also the map $\lambda \mapsto \vartheta_\lambda$ is strictly increasing, in the sense that if $\lambda < \mu$, then the following implication holds

$$\vartheta_\lambda(0) \leq \vartheta_\mu(0) \quad \Rightarrow \quad \vartheta_\lambda(x) < \vartheta_\mu(x) \text{ for every } x \in (0, 1), \tag{2.6}$$

cf. for instance [52, Theorem 4]. By convention, we choose an eigenfunction u satisfying $u(0) > 0$, and so we couple (2.4) with the initial condition $\vartheta_\lambda(0) = 0$ for every λ.

3. The approximated problem

Let $\varphi(s) := \frac{s}{\sqrt{1+s^2}}$ for every $s \in \mathbb{R}$. For every $n \in \mathbb{N}$, we introduce $\varphi_n : \mathbb{R} \to \mathbb{R}$ as the C^1-function such that

$$\varphi_n(s) = \begin{cases} \varphi(s) & \text{if } |s| \leq n, \\ \text{affine} & \text{if } |s| > n. \end{cases}$$

Proposition 3.1. *Let Φ and Φ_n be the primitives of φ and φ_n that vanish in zero, namely*

$$\Phi(s) := \int_0^s \varphi(\xi) d\xi = \sqrt{1+s^2} - 1 \quad \text{and} \quad \Phi_n(s) := \int_0^s \varphi_n(\xi) d\xi.$$

For every $n \in \mathbb{N}$, φ, φ_n, Φ, and Φ_n enjoy the following properties:

(a) $\varphi_n(s)s \geq \varphi_{n+1}(s)s \geq \varphi(s)s \geq 0$ *for every* $s \in \mathbb{R}$;
(b) φ_n *is increasing and* Φ_n *is convex;*
(c) $\Phi_n(s) \geq \Phi(s)$ *for every* $s \in \mathbb{R}$;
(d) $\Phi_n(s) \leq \varphi_n(s)s$ *for every* $s \in \mathbb{R}$ *and* $\lim_{s\to\pm\infty}(\varphi_n(s)s - \Phi_n(s)) = +\infty$;
(e) $s\varphi_n^{-1}(s) \geq s^2$ *for every* $s \in \mathbb{R}$.

Proof. By definition, φ and φ_n are odd functions, and so we can restrict the proof of the properties to $s \geq 0$. To prove (a), we observe that being φ concave, $\varphi_n \geq \varphi$ in $[0, \infty)$ for every n. Moreover since $\varphi'(s) = \frac{1}{(1+s^2)^{3/2}}$ is decreasing, by the definition $\varphi_n(s) \geq \varphi_{n+1}(s)$ for every n and for every $s \in [0, \infty)$. Property (b) is obvious by the definition of φ_n and by the fact that Φ_n is a primitive of φ_n. Property (c) follows immediately by (a), indeed for every n and s

$$\Phi_n(s) = \int_0^s \varphi_n(\xi) d\xi \geq \int_0^s \varphi(\xi) d\xi = \Phi(s).$$

To prove (d) we use that φ_n is monotone increasing, hence for every $s \geq 0$

$$\Phi_n(s) \leq \varphi_n(s) \int_0^s d\xi = \varphi_n(s)s.$$

Now, by the definitions of φ_n and Φ_n, there exist three suitable constants a_n, b_n, c_n with $a_n > 0$ such that $\varphi_n(s)s = (a_n s + b_n)s$ and $\Phi_n(s) = a_n \frac{s^2}{2} + b_n s + c_n$ for every $|s| \geq n$. Thus for every $s \geq n$

$$0 \leq \varphi_n(s)s - \Phi_n(s) = \frac{a_n}{2} s^2 - c_n \to +\infty \qquad \text{as } s \to \infty.$$

Finally, for property (e) we observe that since φ' is decreasing and $\varphi'(0) = \varphi_n'(0) = 1$, $\varphi_n(s) \leq s$ for every $s \geq 0$. Now, φ_n is strictly increasing, and so also its inverse φ_n^{-1} has the same monotonicity. Thus, for every $s \geq 0$

$$s = \varphi_n^{-1}(\varphi_n(s)) \leq \varphi_n^{-1}(s).$$

This concludes the proof. $\square$

For every $n \in \mathbb{N}$, we consider the approximated problem

$$\begin{cases} -(\varphi_n(u'))' = a(x)f(u) & \text{in } (0,1), \\ u > 0 & \text{in } (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{3.1}$$

where we recall that $a \in C^1([0,1])$, $a > 0$ in $[0,1]$, and on $f \in C^1([0,\infty))$ we require only the two assumptions (f_{eq}), (f_{sgn}) given in the Introduction. We now prove the following multiplicity result for (3.1).

Theorem 3.2. *Let n, $k \in \mathbb{N}$, and let $a \in C^1([0,1])$, $a > 0$ in $[0,1]$, and assume that $f \in C^1([0,\infty))$ satisfies (f_{eq}) and (f_{sgn}). If $f'(u_0) > \lambda_{k+1}$, then problem (3.1) admits at least $2k$ non-constant classical solutions $u_{n,1}, \ldots, u_{n,2k}$. Furthermore,*

(a) for every $j = 1, \ldots, k$, $u_{n,j}(0) < u_0$ and $u_{n,j} - u_0$ has exactly j zeros.

(b) for every $\ell = 1, \ldots, k$, $u_{n,2k+1-\ell}(0) > u_0$ and $u_{n,2k+1-\ell} - u_0$ has exactly ℓ zeros.

The proof of the above theorem relies on a shooting technique. As a first step, we introduce $\hat{f}$ as the continuous extension to zero of f on $(-\infty, 0)$ and we consider the problem

$$\begin{cases} -(\varphi_n(u'))' = a(x)\hat{f}(u) & \text{in } (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{3.2}$$

We also define the primitive of $\hat{f}$ vanishing at $s = u_0$, i.e. $\hat{F}(s) := \int_{u_0}^{s} \hat{f}(\xi)d\xi$. Notice that, by (f_{sgn}),

$$\hat{F}(s) \geq 0 \text{ for every } s \in \mathbb{R} \qquad \text{and} \qquad \hat{F}(s) = 0 \;\Leftrightarrow s = u_0, \tag{3.3}$$

$$\hat{F} \text{ is strictly decreasing on } [0, u_0] \quad \text{and} \quad \hat{F} \text{ is strictly increasing on } [u_0, +\infty). \tag{3.4}$$

As a useful tool to investigate the qualitative properties of the solutions, we introduce, for every $n \in \mathbb{N}$, the *energy* of a function u satisfying the equation in (3.2) as

$$E_n(x) := u'(x)\varphi_n(u'(x)) - \Phi_n(u'(x)) + a(x)\hat{F}(u(x)) \quad \text{for every } x \in [0,1]. \tag{3.5}$$

Notice that, from (3.3) together with Proposition 3.1-(d), it holds that $E_n(x) \geq 0$ for every $x \in [0,1]$.

We are now in a position to make our shooting procedure effective. For every $d \geq 0$, we consider the Cauchy problem

$$\begin{cases} -(\varphi_n(u'))' = a(x)\hat{f}(u) & \text{in } (0,1), \\ u(0) = d, \quad u'(0) = 0. \end{cases} \tag{3.6}$$

For (3.6), the following global existence, uniqueness, and continuous dependence result holds.

Lemma 3.3. *For every $d \in [0,+\infty)$, there exists a unique global solution u_d of (3.6) in $[0,1]$; moreover, u_d is of class $C^2([0,1])$. In addition, if $(d_j) \subset [0,\infty)$ is such that $d_j \to d \in [0,\infty)$ as $j \to \infty$, then*

$$u_{d_j} \to u_d \quad \textit{and} \quad \varphi_n(u'_{d_j}) \to \varphi_n(u'_d) \quad \textit{as } j \to \infty \tag{3.7}$$

uniformly in $[0,1]$.

Proof. We can rewrite the equation in (3.6) as the following equivalent first-order planar system

$$\begin{cases} u'(x) = \varphi_n^{-1}(v(x)) \\ v'(x) = -a(x)\hat{f}(u(x)) \end{cases} \tag{3.8}$$

and we consider the Cauchy problem with initial conditions

$$u(x^0) = u^0, \quad v(x^0) = v^0. \tag{3.9}$$

For any $(x^0, (u^0, v^0)) \in [0, 1] \times \mathbb{R}^2$, the existence and uniqueness of a local solution (u, v) of (3.8)-(3.9) is guaranteed by the Cauchy-Lipschitz Theorem.

Concerning the global existence for (3.6), suppose by contradiction that there exists $x^* \in (x^0, 1]$ such that (u, v) is not defined for $x \geq x^*$, then

$$\lim_{x \to (x^*)^-} (|u(x)| + |v(x)|) = \infty. \tag{3.10}$$

Now, if we consider the energy along the solution as defined in (3.5), we get, by Proposition 3.1-(d),

$$|E_n'(x)| = |a'(x)|\hat{F}(u(x)) \leq Ca(x)\hat{F}(u(x)) \leq CE_n(x),$$

where $C := \max_{[0,1]} |a'(x)|/ \min_{[0,1]} a(x)$. Therefore, by Gronwall's Lemma,

$$E_n(x) \leq e^C E_n(x^0) \quad \text{for every } x \in [x^0, x^*). \tag{3.11}$$

As $a(x)\hat{F}(u_d(x)) \geq 0$ by (3.3), from (3.11) we infer that

$$u'(x)\varphi_n(u'(x)) - \Phi_n(u'(x)) \leq e^C E_n(x_0) \quad \text{for every } x \in [x^0, x^*). \tag{3.12}$$

By Proposition 3.1-(d), this implies that $|u'|$ is bounded in $[x^0, x^*)$ and hence, also u and v. This contradicts (3.10).

In the same way, we exclude the case where there exists $x^* \in [0, x^0)$ such that (u, v) is not defined for $x \leq x^*$. This implies that the solution must be defined on [0,1].

Finally, the regularity of u is a consequence of the continuity of a and $\hat{f}$, while the proof of the continuous dependence is standard, once we have uniqueness and global existence. □

Furthermore, for (3.2) we prove the following maximum principle-type result, which guarantees that all non-constant solutions of (3.2) are positive.

Lemma 3.4. *If u is a classical solution of (3.2), then either $u \equiv -C$ for some $C \geq 0$, or $u > 0$ in $[0, 1]$.*

Proof. This can be easily deduced from the uniqueness of the Cauchy problem as if $\min u = u(x_0) = -C$ with $C \geq 0$, then $u'(x_0) = 0$. This implies that u and $-C$ are solutions of the Cauchy problem

$$\begin{cases} -(\varphi_n(w'))' = a(x)\hat{f}(w) & \text{in } (0, 1), \\ w(x_0) = -C, \quad w'(x_0) = 0. \end{cases}$$

By uniqueness, we conclude that $u \equiv -C$. □

By Lemmas 3.3 and 3.4, if for some $d \in [0, \infty)$ the solution u_d of (3.6) is non-constant and satisfies $u_d'(1) = 0$, then u_d solves (3.1). Therefore, our goal is to look for such initial data d. To this aim, set $v(x) := \varphi_n(u'(x))$, it is convenient to introduce the following system of clockwise polar coordinates around the point $(u, v) = (u_0, 0)$

$$\begin{cases} u(x) - u_0 = \rho(x)\cos(\theta(x)) \\ v(x) = -\rho(x)\sin(\theta(x)). \end{cases} \tag{3.13}$$

We remark that, in view of the uniqueness proved in Lemma 3.3, either $\rho \equiv 0$ or $\rho(x) \neq 0$ for all $x \in [0, 1]$. Moreover, for $d \in [0, \infty)$, if u_d solves (3.6), the corresponding angular variable θ_d satisfies the following differential equation in $(0, 1)$,

$$\begin{aligned} \theta_d'(x) &= \frac{1}{\rho_d^2(x)}\left[\varphi_n(u_d'(x))u_d'(x) + a(x)\hat{f}(u_d(x))(u_d(x) - u_0)\right] \\ &= \frac{1}{\rho_d^2(x)}\left[v_d(x)\varphi_n^{-1}(v_d(x)) + a(x)\hat{f}(u_d(x))(u_d(x) - u_0)\right] \end{aligned} \tag{3.14}$$

with initial conditions

$$\begin{cases} \theta_d(0) = \pi \quad \text{and} \quad \rho_d(0) = |d - u_0|, \quad \text{if } d \in [0, u_0), \\ \theta_d(0) = 0 \quad \text{and} \quad \rho_d(0) = |d - u_0|, \quad \text{if } d \in (u_0, +\infty). \end{cases} \tag{3.15}$$

We further observe that by (3.14) and (f_{sgn}), $\theta_d'(x) > 0$ for every $x \in [0, 1]$.

In the following lemma we prove that the hypothesis $f'(u_0) > \lambda_{k+1}$ allows us to estimate from below the number of half turns that a solution (u_d, v_d) of

$$\begin{cases} u'(x) = \varphi_n^{-1}(v(x)) \\ v'(x) = -a(x)\hat{f}(u(x)) \\ u(0) = d, \quad v(0) = 0 \end{cases} \tag{3.16}$$

performs around $(u_0, 0)$ if it is shot from a point $(d, 0)$ sufficiently close to $(u_0, 0)$.

Lemma 3.5. *If $f'(u_0) > \lambda_{k+1}$ for some $k \in \mathbb{N}$, then there exists $\bar{\delta} > 0$ such that*

$$\theta_d(1) - \theta_d(0) > k\pi \quad \textit{for every } d \in [u_0 - \bar{\delta}, u_0 + \bar{\delta}] \setminus \{u_0\}.$$

Proof. By the assumption on $f'(u_0)$, for every $\bar{\lambda} \in (\lambda_{k+1}, f'(u_0))$, there exists $\delta > 0$ such that

$$\hat{f}(s)(s - u_0) \geq \bar{\lambda}(s - u_0)^2 \quad \text{for every } s \text{ such that } |s - u_0| < \delta. \tag{3.17}$$

On the other hand, by the continuous dependence with respect to d of (3.6), in correspondence of δ there exists $\bar{\delta} > 0$ such that if $d \in [u_0 - \bar{\delta}, u_0 + \bar{\delta}] \setminus \{u_0\}$, $|u_d(x) - u_0| < \delta$ for every $x \in [0, 1]$. Therefore, by (3.14), (3.17), and Proposition 3.1-(e), we get for every $d \in [u_0 - \bar{\delta}, u_0 + \bar{\delta}] \setminus \{u_0\}$ and for every $x \in [0, 1]$

$$\theta_d'(x) \geq \frac{1}{\rho_d^2(x)}\left[v_d^2(x) + \bar{\lambda}a(x)(u_d(x) - u_0)^2\right] = \sin^2(\theta_d(x)) + \bar{\lambda}a(x)\cos^2(\theta_d(x)).$$

Hence, by (2.4) with $\lambda = \bar{\lambda}$ and using the Comparison Theorem for ODEs, we get

$$\theta_d(1) - \theta_d(0) \geq \vartheta_{\bar{\lambda}}(1) - \vartheta_{\bar{\lambda}}(0).$$

Being $\lambda_{k+1} < \bar{\lambda}$, by (2.6) and (2.5) we deduce

$$\theta_d(1) - \theta_d(0) > \vartheta_{\lambda_{k+1}}(1) - \vartheta_{\lambda_{k+1}}(0) = k\pi,$$

that concludes the proof. □

The next lemma ensures that solutions (u_d, v_d) with d large enough are too "slow" to make a half turn.

Lemma 3.6. *There exists $d^* > u_0$ such that the solution of (3.16) with $d = d^*$ satisfies $\theta_d(1) - \theta_d(0) < \pi$.*

Proof. Otherwise, let u be a solution of (3.16) with $d = d^* > u_0$ and $\theta_d(1) - \theta_d(0) \geq \pi$. Then, there exists $x_0 \in (0, 1]$ with $u'(x_0) = 0$ and, for all $x \in (0, x_0)$, $u'(x) < 0$. We observe that $u(x_0) < u_0$ as otherwise, for every $x \in (0, x_0)$, $u(x) > u_0$ and hence $u''(x) < 0$, contradicting $u'(0) = u'(x_0) = 0$. Moreover, we can prove that $u(x_0) > 0$ in the same way as in Lemma 3.4.

We now repeat the argument of the proof of Lemma 3.3 so as to obtain

$$E_n(x) \leq e^C E_n(x_0) \quad \text{for every } x \in [0, x_0],$$

where $C = \max_{[0,1]} |a'(x)| / \min_{[0,1]} a(x)$. From this, recalling (3.3)-(3.4) we get

$$u'(x)\varphi_n(u'(x)) - \Phi_n(u'(x)) < e^C a(x_0) F(u(x_0)) < e^C a(x_0) F(0) \quad \text{for all } x \in [0, x_0],$$

and hence, due to Proposition 3.1-(d),

$$|u'(x)| \leq K^* \quad \text{for all } x \in [0, x_0],$$

for a suitable $K^* > 0$ which does not depend on u. Then $u(x_0) \geq u(0) - K^* = d^* - K^*$, which is a contradiction if $d^* > u_0 + K^*$ (since $u(x_0) < u_0$). ☐

We are now ready to prove the main result of this section.

Proof of Theorem 3.2. Fix $n \in \mathbb{N}$. We first observe that the continuous dependence for (3.16) yields the continuity of the map $d \mapsto \theta_d(1)$, where θ_d is the angular variable of the solution (u_d, v_d).

First part: Existence of $u_{n,1}, \ldots, u_{n,k}$.

If $d = 0$, the solution (u_0, v_0) is identically equal to $(0, 0)$, hence $\theta_0(1) - \theta_0(0) = 0$. If $d \in [u_0 - \bar{\delta}, u_0)$, by Lemma 3.5, $\theta_d(1) - \theta_0(0) > k\pi$. Therefore, by continuity, there exist k values of d, denoted by $d_{n,1}, \ldots, d_{n,k}$ such that

$$0 < d_{n,1} < \cdots < d_{n,k} < u_0 \quad \text{and} \quad \theta_{d_{n,j}}(1) - \theta_{d_{n,j}}(0) = j\pi \ \text{ for every } j = 1, \ldots, k.$$

As a result, to such $d_{n,j}$'s correspond k solutions $u_{n,1}, \ldots, u_{n,k}$ of the problem (3.2). Moreover, since $\theta_{d_{n,j}}$ is monotone increasing in $(0, 1)$, cf. (3.14), for every $j = 1, \ldots, k$, there exist exactly j points $0 < x_{n,1} < \cdots < x_{n,j} < 1$ such that $\theta_{d_{n,j}}(x_{n,i}) = \left(i + \frac{1}{2}\right)\pi$ for $i = 1, \ldots, j$. This means that $u_{n,j}(x_{n,i}) = u_0$ for every $i = 1, \ldots, j$ and proves in particular that $u_{n,1}, \ldots, u_{n,k}$ are k distinct non-constant solutions of (3.2). Thus, by Lemma 3.4, they solve (3.1) and have the desired oscillatory behavior.

Second part: Existence of $u_{n,k+1}, \ldots, u_{n,2k}$.

The argument is exactly the same with d between u_0 and d^* using Lemmas 3.5 and 3.6. ☐

For further convenience, we prove here below an improved version of Lemma 3.6: precisely, we prove that we can take $d^* = \bar{u}$, when f satisfies the additional assumption (f_{ap}).

Lemma 3.7. *Let f satisfy also (f_{ap}), then the solution of (3.16) with $d = \bar{u}$ satisfies $\theta_d(1) - \theta_d(0) < \pi$.*

Proof. Otherwise, there exists $x_0 \in (0,1]$ with $u'(x_0) = 0$ and, for all $x \in (0, x_0)$, $u'(x) < 0$. As already observed in the proof of Lemma 3.6, it must be $0 < u(x_0) < u_0$.

Now, for every $x \in [0, x_0]$, we get

$$\begin{aligned} E_n(x) - E_n(x_0) &= \int_{x_0}^{x} a'(s)F(u(s))ds = \int_{x}^{x_0} \frac{a'^-(s) - a'^+(s)}{a(s)} a(s)F(u(s))ds \\ &\le \int_{x}^{x_0} \frac{a'^-(s)}{a(s)} E_n(s)ds. \end{aligned}$$

Therefore, by backward Gronwall's inequality, we obtain for all $x \in [0, x_0]$

$$E_n(x) \le E_n(x_0) \exp\Big(\int_{x}^{x_0} \frac{a'^-(s)}{a(s)} ds\Big).$$

In particular, recalling (3.3)-(3.4) we obtain

$$\begin{aligned} a(0)F(\bar u) = E_n(0) &\le \exp\Big(\int_0^{x_0} \frac{a'^-(x)}{a(x)} dx\Big) a(x_0)F(u(x_0)) \\ &< \exp\Big(\int_0^{x_0} \frac{a'^-(x)}{a(x)} dx\Big) a(x_0)F(0). \end{aligned}$$

As

$$\max_{x_0 \in [0,1]} \frac{a(x_0)}{a(0)} \exp\Big(\int_0^{x_0} \frac{a'^-(x)}{a(x)} dx\Big) = \frac{a(1)}{a(0)} \exp\Big(\int_0^{1} \frac{a'^-(x)}{a(x)} dx\Big),$$

this contradicts the choice of $\bar u$. □

Remark 3.8. In view of Lemma 3.7, we can ensure that, when f satisfies (f_{ap}), all the solutions given by Theorem 3.2 satisfy $u(0) < \bar u$. This is easily understood by checking the final part of the proof of Theorem 3.2.

In the rest of the section, we fix $j \in \{1, \dots, 2k\}$ and we consider a sequence of solutions $(u_{n,j})_n$ given by Theorem 3.2. We are going to establish, for such a sequence, an auxiliary property, which will play an important role in the next section. In what follows, since j is fixed, to simplify the notation we simply write u_n instead of $u_{n,j}$.

Proposition 3.9. *The sequence (u_n) does not have subsequences which converge uniformly to the constant function $u \equiv u_0$.*

Proof. As in the proof of Lemma 3.5, there exists $\delta > 0$ such that, if u is a solution of (3.1) with $|u(x) - u_0| < \delta$ for every $x \in [0,1]$, then the corresponding angular variable satisfies $\theta(1) - \theta(0) > k\pi$.

As $\theta_n(1) - \theta_n(0) = j\pi \le k\pi$ this implies that, for all $n \in \mathbb{N}$, $\max_{[0,1]} |u_n(x) - u_0| \ge \delta$. This proves the result. □

As a corollary of the above result, we can further obtain the following bound for the minimum and maximum of the solutions u_n.

Corollary 3.10. *There exists $\bar\varepsilon > 0$ such that for every $n \in \mathbb{N}$, and for every extremum point $\bar x \in [0,1]$ of u_n,*

$$|u_n(\bar x) - u_0| > \bar\varepsilon.$$

In particular, $u_n(\bar{x}) > u_0 + \bar{\varepsilon}$ if $\bar{x}$ is a relative maximizer, and $u_n(\bar{x}) < u_0 - \bar{\varepsilon}$ if $\bar{x}$ is a relative minimizer for u_n.

Proof. Suppose by contradiction that in correspondence of $\varepsilon = \frac{1}{m}$ there exists $n(m) \in \mathbb{N}$ such that $u_{n(m)}$ has an extremum point, denoted by x_m, such that

$$|u_{n(m)}(x_m) - u_0| \le \frac{1}{m}.$$

By uniqueness of the solution of the Cauchy problem $u_{n(m)}(x_m) \neq u_0$ as otherwise, x_m being an extremum, $u'_{n(m)}(x_m) = 0$ and we would have two distinct solutions of the Cauchy problem $u(x_m) = u_0$ and $u'(x_m) = 0$. As every u_n has exactly $j+1$ extremum points (counting also $x = 0$ and $x = 1$), the set $\{n(m) : m \in \mathbb{N}\}$ is unbounded. Thus, passing if necessary to a subsequence, $\lim_{m\to\infty} n(m) = \infty$. Moreover, since $(x_m) \subset [0,1]$, up to a subsequence, (x_m) converges to some point $\bar{x} \in [0,1]$.

We claim that the corresponding subsequence of $(u_{n(m)})_m$ converges uniformly to u_0: in view of Proposition 3.9, this will conclude the proof. To prove the claim, we consider the energy along the solution $u_{n(m)}$ defined in (3.5) and we argue as in the proof of Lemma 3.3 to get by Gronwall's Lemma

$$E_{n(m)}(x) \le e^C E_{n(m)}(x_m) \quad \text{for every } x \in [0,1], \tag{3.18}$$

with $C = \max_{[0,1]} |a'(x)| / \min_{[0,1]} a(x)$. On the other hand, letting $m \to \infty$,

$$|u_{n(m)}(x_m) - u_0| \to 0 \quad \text{and consequently} \quad E_{n(m)}(x_m) \to 0,$$

where we have used again that $u'_{n(m)}(x_m) = 0$. This, together with (3.18), gives

$$E_{n(m)} \to 0 \quad \text{uniformly in } [0,1] \quad \text{as } m \to \infty.$$

Using Proposition 3.1-(d) and the fact that a is positive and F is non-negative, the last convergence yields

$$\lim_{m\to\infty} \|F(u_{n(m)})\|_{L^\infty(0,1)} = 0.$$

By (3.3), we get that $u_{n(m)} \to u_0$ uniformly in $[0,1]$, that proves the claim.

The last part of the statement follows by the concavity/convexity of u_n. Indeed, we know that u_n solves $-\varphi'_n(u')u'' = a(x)f(u)$. Hence, being φ_n increasing, a positive and using (f_{sgn}), the solution u_n is concave (resp., convex) in intervals in which $u_n > u_0$ (resp., $u_n < u_0$). Therefore, since a concave (resp., convex) function cannot have a minimum (rep. maximum) unless it is constant, $u_n(\bar{x}) > u_0$ (resp., $u_n(\bar{x}) < u_0$) if $\bar{x}$ is a relative maximizer (resp., minimizer) and the proof is concluded. □

4. Proof of Part (I) - (II) of the main theorem

Let $f'(u_0) > \lambda_{k+1}$ for some $k \in \mathbb{N}$ and fix any $j \in \{1, \dots, 2k\}$. Consider the sequence $(u_{n,j})_n$ of solutions of (3.1) having ℓ intersections with u_0 (where $\ell = j$ or $2k+1-j$ according to $j \le k$ or $j > k$) whose existence has been proved in Theorem 3.2. As in the final part of the previous section, we will denote this sequence simply by (u_n).

We are going to show that the sequence (u_n) converges, in a suitable sense, to a BV-solution of (1.2), having exactly ℓ intersections (possibly in the generalized sense explained after the statement of Theorem 1.1) with the constant u_0. This will be the core of the proof of Part (I) - (II) of Theorem 1.1 (the other statements in

Part (I) - (II) follow as direct consequences of Proposition 2.5). We split the next arguments into some steps.

4.1. **A priori estimates.** In the next two lemmas, we prove that the sequence (u_n) is bounded in $W^{1,1}(0,1)$ when either $(f_{\rm ap})$ or $(f_{\rm ap})'$ are satisfied.

Lemma 4.1. *Under condition $(f_{\rm ap})$, the sequence (u_n) is bounded in $L^\infty(0,1)$: more precisely, for all $x \in [0,1]$, $0 < u_n(x) < \bar{u}$. Moreover, (u_n') is bounded in $L^1(0,1)$ and, consequently, (u_n) is bounded in $W^{1,1}(0,1)$.*

Proof. For the first part, observe that, for every n, there exists $\overline{x}_n \in [0,1]$ such that $u_n(\overline{x}_n) \in (0,u_0)$, $u_n'(\overline{x}_n) = 0$ and $u_n(\overline{x}_n) < u_n(x) < u_n(0)$ for all $x \in (0,\overline{x}_n)$. Moreover, by Remark 3.8 we also have $u_n(0) < \bar{u}$ for every n. We now denote by E_n the energy of u_n, i.e. (3.5) with $u = u_n$. For every $x \in [0,1]$ we obtain

$$E_n'(x) \leq \frac{a'^{+}(x) - a'^{-}(x)}{a(x)} E_n(x) \leq \frac{a'^{+}(x)}{a(x)} E_n(x).$$

Thus, by Gronwall's Lemma and using (3.3)-(3.4), we get for every $x \in [\overline{x}_n, 1]$

$$\begin{aligned} a(x)F(u_n(x)) \leq E_n(x) &\leq E_n(\overline{x}_n) \exp\Big(\int_{\overline{x}_n}^{x} \frac{a'^{+}(s)}{a(s)} ds \Big) \\ &\leq a(\overline{x}_n) \exp\Big(\int_{\overline{x}_n}^{x} \frac{a'^{+}(s)}{a(s)} ds \Big) F(0). \end{aligned}$$

As

$$\max_{\overline{x}_n, x \in [0,1]} \frac{a(\overline{x}_n)}{a(x)} \exp\Big(\int_{\overline{x}_n}^{x} \frac{a'^{+}(s)}{a(s)} ds \Big) = \frac{a(0)}{a(1)} \exp\Big(\int_0^1 \frac{a'^{+}(x)}{a(x)} dx \Big),$$

the first part of the thesis follows by the definition of $\bar{u}$ in $(f_{\rm ap})$.

As for the boundedness of (u_n') in $L^1(0,1)$, we proceed as follows. By Proposition 3.1-(c), we get

$$\int_0^1 |u_n'| dx \leq \int_0^1 \sqrt{1 + u_n'^2} dx \leq \int_0^1 (\Phi_n(u_n') + 1) dx. \tag{4.1}$$

Since u_n solves (3.1), u_n is a global minimizer of the functional

$$I_n(v) := \int_0^1 \Phi_n(v') dx - \int_0^1 a(x) f(u_n) v dx \quad v \in H^1(0,1), \tag{4.2}$$

by convexity of Φ_n and consequently of I_n. Therefore, for every $v \in H^1(0,1)$, by (4.1) we obtain

$$\int_0^1 |u_n'| dx \leq \int_0^1 \Phi_n(v') dx - \int_0^1 a(x) f(u_n)(v - u_n) dx + 1.$$

In particular, for $v \equiv 0$,

$$\int_0^1 |u_n'| dx \leq \int_0^1 a(x) f(u_n) u_n dx + 1 \leq M,$$

for a suitable constant M, being (u_n) bounded in $L^\infty(0,1)$ and f continuous.

In conclusion, (u_n) is bounded in $W^{1,1}(0,1)$ and so also in $BV(0,1)$. □

Lemma 4.2. *Under condition $(f_{\rm ap})'$, the sequence (u_n) is bounded in $W^{1,1}(0,1)$.*

Proof. Consider as test function in (3.1), the function $v_n = u_n - \max u_n$, we obtain by Proposition 3.1-(a)

$$\begin{aligned}
\int_0^1 \varphi(u_n')u_n'\,dx &\le \int_0^1 \varphi_n(u_n')u_n'\,dx = \int_0^1 \varphi_n(u_n')v_n'\,dx \\
&= \int_0^1 a(x)f(u_n(x))(u_n - \max u_n)\,dx \\
&\le \int_0^1 a(x)(\min f)(u_n - \max u_n)\,dx \\
&\le |\min u_n - \max u_n|\|a\|_{L^1(0,1)} \max f^- \\
&\le \int_0^1 |u_n'|\,dx\,\|a\|_{L^1(0,1)} \max f^-.
\end{aligned}$$

Let c such that, for all $s \in \mathbb{R}$, $\varphi(s)s - |s| \ge c$. Hence we deduce that

$$\int_0^1 |u_n'|\,dx - c \le \|a\|_{L^1(0,1)} \max f^- \int_0^1 |u_n'|\,dx.$$

As $\|a\|_{L^1(0,1)} \max f^- < 1$, this proves that $\|u_n'\|_{L^1(0,1)}$ is bounded.

The result follows then from the fact that $0 < \min u_n < u_0$ for every n, since

$$\|u_n\|_{L^\infty(0,1)} \le \min u_n + \sup_n \|u_n\|_{L^1(0,1)} < u_0 + \sup_n \|u_n\|_{L^1(0,1)} < \infty.$$

□

4.2. **Passing to the limit.** By Lemmas 4.1 and 4.2, under either of the assumptions $(f_{\rm ap})$ and $(f_{\rm ap})'$, (u_n) is bounded in $W^{1,1}(0,1)$. It then follows (see, for instance, [1, Theorem 3.23]) that, up to subsequences, the sequence (u_n) weakly-$*$ converges to u in $BV(0,1)$, i.e.,

$$u_n \to u \text{ in } L^1(0,1) \quad \text{and} \quad Du_n \overset{*}{\rightharpoonup} Du, \tag{4.3}$$

where $Du_n = u_n' dx$ and the last convergence means that $\int_0^1 \phi u_n' dx \to \int_0^1 \phi Du$ for every $\phi \in C_0(0,1)$.

We now prove that the limit function u is actually a solution of (1.2) in the BV-sense and that a stronger convergence holds. Similar arguments have been used for instance in [38, pp. 2367-2370].

Proposition 4.3. *The limit function u is a BV-solution of (1.2); moreover*

$$\int_0^1 |u_n'|dx \to \int_0^1 |Du|, \tag{4.4}$$

where $\int_0^1 |Du|$ denotes the measure of the interval $(0,1)$ with respect to $|Du|$.

Proof. We first prove that u is a BV-solution of (1.2). Since each u_n is positive in $[0,1]$, by (4.3) $u \ge 0$ in $(0,1)$. In view of Proposition 2.4, it suffices to show that the inequality

$$\int_0^1 \sqrt{1+v'^2}dx \ge \int_0^1 \sqrt{1+|Du|^2} + \int_0^1 a(x)f(u)(v-u)dx \tag{4.5}$$

holds for every $v \in C^\infty([0,1])$. To this aim, since u_n is a global minimizer of I_n in $H^1(0,1) \supset C^\infty([0,1])$ (cf. (4.2)), using Proposition 3.1-(c), we get for every $v \in C^\infty([0,1])$

$$\int_0^1 \sqrt{1+u_n'^2}dx \le \int_0^1 (\Phi_n(u_n')+1)dx \le \int_0^1 (\Phi_n(v')+1)dx - \int_0^1 a(x)f(u_n)(v-u_n)dx.$$

Passing to the upper limit on both sides we get

$$\limsup_{n\to\infty} \int_0^1 \sqrt{1+u_n'^2}dx \le \int_0^1 \sqrt{1+v'^2}dx - \int_0^1 a(x)f(u)(v-u)dx, \tag{4.6}$$

where we applied the Dominated Convergence Theorem to a subsequence of the right-hand side, recalling that $u_n \to u$ in $L^1(0,1)$, and we used that $\Phi_n \to \Phi$. We know that the functional $J : BV(0,1) \to \mathbb{R} : v \mapsto \int_0^1 \sqrt{1+|Dv|^2}$ is lower semicontinuous with respect to the L^1-convergence, cf. [25, Thm 14.2], thus by (4.3),

$$\int_0^1 \sqrt{1+|Du|^2} \le \liminf_{n\to\infty} \int_0^1 \sqrt{1+u_n'^2}dx, \tag{4.7}$$

which, together with (4.6), proves the thesis of this first part.

As for (4.4), by [2, Fact 3.1], it is enough to prove

$$\int_0^1 \sqrt{1+u_n'^2}dx \to \int_0^1 \sqrt{1+|Du|^2}.$$

To this end, we first observe that, since $C^\infty([0,1])$ is dense in $W^{1,1}(0,1)$, inequality (4.6) actually holds for every $v \in W^{1,1}(0,1)$. Now, by Lemma 2.3, there exists a sequence $(v_k) \subset W^{1,1}(0,1)$ such that $v_k \to u$ in $L^1(0,1)$ and $\int_0^1 \sqrt{1+v_k'^2}dx \to \int_0^1 \sqrt{1+|Du|^2}$. Thus, applying (4.6) to v_k and taking the limit as $k \to \infty$ we have

$$\limsup_{n\to\infty} \int_0^1 \sqrt{1+u_n'^2}dx \le \int_0^1 \sqrt{1+|Du|^2},$$

which, together with (4.7), concludes the proof. □

As a consequence of (4.4), we can establish a first important fact about the solution u. Precisely:

Proposition 4.4. *The solution u is not identically equal to u_0.*

Proof. Assume by contradiction that $u \equiv u_0$. Then, by (4.4), it holds that

$$\lim_{n\to\infty} \int_0^1 |u_n'|dx = \int_0^1 |Du_0| = 0.$$

Now, with obvious meaning of x_M and x_m,

$$\max_{x\in[0,1]} u_n - \min_{x\in[0,1]} u_n = u_n(x_M) - u_n(x_m) = \int_{x_m}^{x_M} u_n'(x)dx \le \int_0^1 |u_n'|dx,$$

hence, we can infer that $u_n \to u_0$ uniformly on $[0,1]$. Since this is excluded by Proposition 3.9, the proof is concluded. □

The next step of the proof will consist of course in showing that the number of generalized intersections with u_0 is preserved when passing to the limit. This will be done in the next subsection, using in an essential way the following two results, which basically ensure that the convergence of u_n to u is stronger in the subintervals of $(0,1)$ where (u_n') is bounded.

Proposition 4.5. *Let $v_n = \varphi_n(u_n')$. Then, up to a subsequence, (v_n) converges uniformly in $[0,1]$. As a consequence, denoted by v the limit function, $v \in C([0,1])$.*

Proof. We first observe that, by the regularity of u_n and of φ_n, $v_n = \varphi_n(u_n') \in C^1([0,1])$ for every n. If we prove that (v_n') is bounded in $C^1([0,1])$, by the Arzelà-Ascoli Theorem, we get the thesis. By Lemmas 4.1 and 4.2, we know that (u_n) is bounded in $L^\infty(0,1)$. By the second equation in (3.8), we get for every $n \in \mathbb{N}$

$$|v_n'(x)| \leq \|a\|_{L^\infty(0,1)} \max_{s\in[0,\sup_n \|u_n\|_{L^\infty}]} f(s) =: C_2 \quad \text{for every } x \in [0,1].$$

Consequently, as $v_n(0) = 0$, this gives $\|v_n\|_{C^1([0,1])} \leq 2C_2$ for every $n \in \mathbb{N}$ and concludes the proof. □

Proposition 4.6. *Let v be the limit function of (v_n) introduced in Proposition 4.5 and let $[\alpha,\beta] \subseteq [0,1]$. If $|v| \leq 1-\varepsilon$ in $[\alpha,\beta]$ for some $\varepsilon > 0$, then up to a subsequence, (u_n) converges uniformly to u in $[\alpha,\beta]$ and consequently $u \in C([\alpha,\beta])$.*

Proof. Let $|v| \leq 1-\varepsilon$ in $[\alpha,\beta]$, by the uniform convergence proved in Proposition 4.5, for n sufficiently large,

$$|v_n(x)| \leq 1-\varepsilon' \quad \text{for every } x \in [\alpha,\beta],$$

for some $\varepsilon' > 0$. Consequently, for n large,

$$\varphi_n^{-1}(v_n(x)) = \varphi^{-1}(v_n(x)) \quad \text{for all } x \in [\alpha,\beta].$$

Hence, for n large enough,

$$|u_n'(x)| = |\varphi^{-1}(v_n(x))| \leq \varphi^{-1}(1-\varepsilon') \quad \text{for every } x \in [\alpha,\beta] \tag{4.8}$$

Since by Lemmas 4.1 and 4.2 we already know that (u_n) is bounded in $L^\infty(0,1)$, (4.8) ensures that (u_n) is bounded in $C^1([\alpha,\beta])$ and so, it admits a subsequence that converges uniformly to u in $[\alpha,\beta]$. □

4.3. **Proof of (I) and (II) of Theorem 1.1.** For every n, let $x_{n,1},\dots,x_{n,\ell}$ be the ℓ intersection points of u_n with u_0, so that $0 < x_{n,1} < \cdots < x_{n,\ell} < 1$.

Step 1: There exists $\delta > 0$ such that, for all $n \in \mathbb{N}$,

(1.a) $x_{n,1} > \delta$*;*
(1.b) $x_{n,i+1} - x_{n,i} > \delta$ *for every* $i = 1,\dots,\ell-1$*;*
(1.c) $x_{n,\ell} < 1-\delta$*.*

By the uniform convergence of (v_n) to v (Proposition 4.5), as $u_n'(0) = v_n(0) = 0$ for every n, we have $v(0) = 0$ and there exist ε, $\delta > 0$ such that $|v| \leq 1-\varepsilon$ in $[0,\delta]$. Thus, by Proposition 4.6, u_n converges uniformly to u in $[0,\delta]$. Now, by Corollary 3.10, $|u_n(0) - u_0| \geq \bar\varepsilon$ for every n. Then also $|u(0) - u_0| \geq \bar\varepsilon$. Since u is continuous in $[0,\delta]$, there exist ε', $\delta' > 0$ such that $|u - u_0| \geq \varepsilon'$ in $[0,\delta']$. This proves (1.a) by the uniform convergence of u_n to u in $[0,\delta]$.

One can argue similarly to prove (1.c).

It remains to prove (1.b). For every n, $u_n(x_{n,i}) = u_n(x_{n,i+1}) = u_0$, hence, there exists $\bar{x}_n \in (x_{n,i}, x_{n,i+1})$ such that $u_n'(\bar{x}_n) = v_n(\bar{x}_n) = 0$. Up to a subsequence, $\lim_{n\to\infty} \bar{x}_n =: \bar{x} \in (0,1)$ and by Proposition 4.5, we get $v(\bar{x}) = 0$. By continuity of v, there exist ε, $\bar{\delta} > 0$ such that $|v| \leq 1 - \varepsilon$ in $(\bar{x} - \bar{\delta}, \bar{x} + \bar{\delta})$. Therefore, the convergence $u_n \to u$ is uniform in $(\bar{x} - \bar{\delta}, \bar{x} + \bar{\delta})$, by Proposition 4.6, and so, in particular, $u_n(\bar{x}_n) \to u(\bar{x})$ as $n \to \infty$. Since $\bar{x}_n$ is an extremum point for u_n, by Corollary 3.10, $|u_n(\bar{x}_n) - u_0| > \bar{\varepsilon}$ for every n, hence

$$|u(\bar{x}) - u_0| \geq \bar{\varepsilon} > 0. \tag{4.9}$$

By the uniform convergence of (u_n) to u on $(\bar{x} - \bar{\delta}, \bar{x} + \bar{\delta})$, we deduce the existence of $\delta \leq \bar{\delta}$ such that, for all $x \in [\bar{x} - \delta, \bar{x} + \delta]$, $|u_n(x) - u_0| \geq \bar{\varepsilon}/2$ which proves (1.b).

This concludes the proof of this first step.

Step 2: For every $i = 1, \ldots, \ell$, the sequence $(x_{n,i})$ has a limit denoted x_i. Let us prove it by recurrence on $i \in \{0, 1, \ldots, \ell\}$ denoting $x_{n,0} = 0$ and $x_{n,\ell+1} = 1$.

Assume by contradiction the existence of $i \in \{1, \ldots, \ell\}$ such that

$$\lim_{n\to\infty} x_{n,i-1} = x_{i-1} \quad \text{and} \quad \liminf_{n\to\infty} x_{n,i} =: x_{i,1} < \limsup_{n\to\infty} x_{n,i} =: x_{i,2}.$$

Let $(x_{n_k,i})$ be a subsequence converging to $x_{i,1}$ and $(x_{n_p,i})$ be a subsequence converging to $x_{i,2}$ and assume without loss of generality that

$$\begin{cases} u_n(x) > u_0, & \text{if } x \in (x_{n,i-1}, x_{n,i}), \\ u_n(x) < u_0, & \text{if } x \in (x_{n,i}, x_{n,i+1}). \end{cases}$$

Observe that, by Step 1, $x_{i+1,1} = \liminf_{n\to\infty} x_{n,i+1} \geq x_{i,1} + \delta$ and $x_{i,1} \geq x_{i-1} + \delta$.

Let $[a,b] \subset (x_{i,1}, \min(x_{i,2}, x_{i+1,1}))$. For n large enough, we have also $[a,b] \subset (x_{n_k,i}, \min(x_{n_p,i}, x_{n_k,i+1})) \subset (x_{n_p,i-1}, x_{n_p,i})$ and hence, for all $x \in [a,b]$,

$$u_{n_p}(x) > u_0 > u_{n_k}(x).$$

By the L^1 convergence of (u_n) to u we deduce that $u(x) = u_0$ for a.e. $x \in [a,b]$.

Moreover, arguing in the same way, we easily prove that $u(x) \geq u_0$ a.e. on $(x_{i-1}, \min(x_{i+1,1}, x_{i,2}))$. By Proposition 2.5, we then have that u is concave on $(x_{i-1}, \min(x_{i+1,1}, x_{i,2}))$ and arguing as in Corollary 2.7, as $u(x) = u_0$ for a.e. $x \in [a,b] \subset (x_{i-1}, \min(x_{i+1,1}, x_{i,2}))$, we deduce that $u = u_0$ on $(x_{i-1}, \min(x_{i+1,1}, x_{i,2}))$ which contradicts the existence of $\bar{x} \in [x_{i-1}, x_{i,2})$ such that $|u(\bar{x}) - u_0| \geq \bar{\varepsilon} > 0$.

Step 3: We have, for $i = 0, \ldots, \ell$

$$\begin{cases} (-1)^{i+1}(u(x) - u_0) \geq 0 \textit{ for every } x \in (x_i, x_{i+1}) \textit{ if } j \in \{1, \ldots, k\}, \\ (-1)^{i+1}(u(x) - u_0) \leq 0 \textit{ for every } x \in (x_i, x_{i+1}) \textit{ if } j \in \{k+1, \ldots, 2k\}. \end{cases} \tag{4.10}$$

The argument is the same as in Step 2.

Step 4: Conclusion. We are now in the assumptions of Corollary 2.7. Recalling that, by Proposition 4.4, u is not identically equal to u_0, we further infer that we have strict inequalities in (4.10). We have thus proved part b. of (I) (resp., (II)) of Theorem 1.1. At this point, parts a. and c. directly follow from Proposition 2.5.

5. Continuity of the energy - End of the proof of Theorem 1.1

As in the previous section, let $f'(u_0) > \lambda_{k+1}$ for some $k \in \mathbb{N}$ and fix any $j \in \{1, \dots, 2k\}$. Consider the sequence $(u_{n,j})_n$ of solutions of (3.1) given by Theorem 3.2. For simplicity, we will denote this sequence by (u_n). Furthermore, we denote by u the limit function of (u_n) and by $x_1, \dots, x_\ell$ the (generalized or not) intersection points of u with u_0.

We recall that the energy of (3.1) along u_n is given by

$$E_n(x) = u_n'(x)\varphi_n(u_n'(x)) - \Phi_n(u_n'(x)) + a(x)F(u_n(x)) \quad \text{for every } x \in [0,1]$$

and that the energy of (1.2) along u is given by

$$\mathcal{E}(x) := 1 - \frac{1}{\sqrt{1+(u'(x))^2}} + a(x)F(u(x)) \quad \text{for every } x \in D, \tag{5.1}$$

where $D := \{x \in [0,1] \, : \, u \text{ is continuous in } x\}$ and $F(u) = \int_{u_0}^{u} f(s)ds$. At the points of D where u has a vertical tangent, this definition has to be intended in the limit sense, i.e., $\mathcal{E}(x) = 1 + a(x)F(u(x))$.

The aim of this section is to prove that $\mathcal{E}$ can be continuously extended to the whole interval $[0,1]$. This will conclude the proof of Theorem 1.1. We start with some preliminary results.

Lemma 5.1. *Up to a subsequence, (E_n) converges uniformly in $[0,1]$. Consequently, denoted by E the limit function, E is continuous in $[0,1]$.*

Proof. Arguing as in the proof of Lemma 3.3, we get for every $n \in \mathbb{N}$

$$|E_n'(x)| \le C E_n(x) \quad \text{for every } x \in [0,1],$$

with $C = \max_{[0,1]} |a'(x)| / \min_{[0,1]} a(x)$, and so by Gronwall's Lemma

$$E_n(x) \le e^C E_n(0) \le e^C a(0) \max_{[0, \sup_n \|u_n\|_{L^\infty}]} F \quad \text{for every } x \in [0,1].$$

Therefore, by the Arzelà-Ascoli Theorem, up to a subsequence (E_n) converges uniformly in $[0,1]$ to some function E. The continuity of E then follows by the continuity of E_n for every n. □

Lemma 5.2. *Let v be the function defined in Proposition 4.5. We have $|v(x)| \le 1$ for every $x \in [0,1]$ and, for $\bar{x} \in [0,1] \setminus \{x_1, \dots, x_\ell\}$, $|v(\bar{x})| < 1$.*

Proof. Step 1: $|v(x)| \le 1$ for every $x \in [0,1]$.

Suppose by contradiction that $v(\bar{x}) > 1$ for some $\bar{x} \in [0,1]$, then for n large, $v_n(x) > 1$ in $[\bar{x} - \delta, \bar{x} + \delta]$ for some $\delta > 0$. This gives the contradiction with Lemmas 4.1 and 4.2, being

$$\int_{\bar{x}-\delta}^{\bar{x}+\delta} u_n'(\xi)d\xi = \int_{\bar{x}-\delta}^{\bar{x}+\delta} \varphi_n^{-1}(v_n(\xi))d\xi \ge 2n\delta \to \infty \quad \text{as } n \to \infty.$$

Step 2: For $\bar{x} \in [0,1] \setminus \{x_1, \dots, x_\ell\}$, $|v(\bar{x})| < 1$.

If $\bar{x} \in \{0, 1\}$, $u_n'(\bar{x}) = 0$ for every n, and the thesis is clearly verified. Let us consider the case $\bar{x} \in (0,1) \setminus \{x_1, \dots, x_\ell\}$. Suppose by contradiction that $|v(\bar{x})| \ge 1$, then $(|u_n'(\bar{x})|)$ is unbounded and so, up to a subsequence, $|u_n'(\bar{x})| \to \infty$ as $n \to \infty$. Put $x_0 := 0$ and $x_{\ell+1} := 1$, and let $i \in \{0, \dots, \ell\}$ be the integer such that $\bar{x} \in (x_i, x_{i+1})$. Fix $\varepsilon > 0$ such that $\varepsilon < \min\{\bar{x} - x_i, x_{i+1} - \bar{x}\}$. Using the same notation as in the proof of Theorem 1.1, since $x_{n,i} \to x_i$ and $x_{n,i+1} \to x_{i+1}$ as $n \to \infty$, for n large,

$\max\{|x_{n,i} - x_i|, |x_{n,i+1} - x_{i+1}|\} \leq \varepsilon$. By the equation in (3.1), we know that for n large, the functions u_n are either all convex or all concave in the whole interval $[x_i+\varepsilon, x_{i+1}-\varepsilon]$. Suppose, to fix the ideas, that u_n are all convex on $[x_i+\varepsilon, x_{i+1}-\varepsilon]$. Then, for n large, the following inequalities hold

$$\begin{aligned} u_n'(\bar{x}) \leq u_n'(x) \leq u_n'(x_{i+1} - \varepsilon) \qquad & \text{if } x \in [\bar{x}, x_{i+1} - \varepsilon], \\ u_n'(\bar{x}) \geq u_n'(x) \geq u_n'(x_i + \varepsilon) \qquad & \text{if } x \in [x_i + \varepsilon, \bar{x}]. \end{aligned}$$

If $u_n'(\bar{x}) \to +\infty$, by Fatou's Lemma,

$$\begin{aligned} \liminf_{n\to\infty} \int_{\bar{x}}^{x_{i+1}-\varepsilon} u_n'(x)dx &\geq \int_{\bar{x}}^{x_{i+1}-\varepsilon} \liminf_{n\to\infty} u_n'(x)dx \\ &\geq \int_{\bar{x}}^{x_{i+1}-\varepsilon} \liminf_{n\to\infty} u_n'(\bar{x})dx = +\infty. \end{aligned}$$

This contradicts the fact that (u_n') is bounded in $L^1(0,1)$. If $u_n'(\bar{x}) \to -\infty$, applying again Fatou's Lemma, we have

$$\liminf_{n\to\infty} \int_{x_i+\varepsilon}^{\bar{x}} |u_n'(x)|dx \geq \int_{x_i+\varepsilon}^{\bar{x}} \liminf_{n\to\infty} |u_n'(x)|dx \geq \int_{x_i+\varepsilon}^{\bar{x}} \liminf_{n\to\infty} |u_n'(\bar{x})|dx = +\infty.$$

This yields again a contradiction and concludes the proof in this case. In case u_n are all concave, the proof is analogous and we omit it. □

Lemma 5.3. *Let $[\alpha, \beta] \subset [0,1] \setminus \{x_1, \ldots, x_\ell\}$, with $0 \leq \alpha < \beta \leq 1$. Then, up to a subsequence, (u_n) converges to u in $C^1([\alpha, \beta])$.*

Proof. By Lemma 5.2 and by the continuity of v (see Proposition 4.5), we know that there exists $\varepsilon > 0$ such that $|v(x)| \leq 1-\varepsilon$ for every $x \in [\alpha, \beta]$. By the equation in (3.1),

$$|u_n''(x)| \leq \frac{\|a\|_{L^\infty(0,1)}|f(u_n(x))|}{|\varphi_n'(u_n'(x))|} \quad \text{for every } x \in [0,1]. \tag{5.2}$$

Proceeding as in the proof of Proposition 4.6, we have for n large

$$|u_n'(x)| \leq \varphi^{-1}(1-\varepsilon') \quad \text{for every } x \in [\alpha, \beta] \text{ and for some } \varepsilon' > 0. \tag{5.3}$$

Therefore, for n large $\varphi_n'(u_n') = \varphi'(u_n')$ in $[\alpha, \beta]$, and we obtain

$$|\varphi'(u_n'(x))| \geq |\varphi'(\varphi^{-1}(1-\varepsilon')| > 0 \quad \text{for every } x \in [\alpha, \beta].$$

On the other hand, by Lemmas 4.1 and 4.2, we know that $\sup_n \|u_n\|_{L^\infty(0,1)} < \infty$. Hence, from (5.2) we get for n large and for every $x \in [\alpha, \beta]$,

$$|u_n''(x)| \leq \frac{\|a\|_{L^\infty(0,1)} \max_{s\in[0,\sup_n \|u_n\|_{L^\infty}]} |f(s)|}{|\varphi'(\varphi^{-1}(1-\varepsilon'))|}. \tag{5.4}$$

Combining together (5.3) and (5.4), we can apply the Arzelà-Ascoli Theorem and conclude that, up to a subsequence, (u_n') converges uniformly to some function w in $[\alpha, \beta]$. On the other hand, by Proposition 4.6, we know that (u_n) converges uniformly to u in $[\alpha, \beta]$. Thus $w = u'$ in $[\alpha, \beta]$. □

Theorem 5.4. *$E(x) = \mathcal{E}(x)$ for every $x \in [0,1] \setminus \{x_1, \ldots, x_j\}$. In particular, $\mathcal{E}$ can be extended by continuity as $E(x)$ at every point $x \in [0,1] \setminus D$.*

Proof. Let $\bar{x} \in [0,1] \setminus \{x_1, \dots, x_\ell\}$. If $\bar{x} \in \{0,1\}$ the thesis is verified, since $u_n(0) \to u(0)$ and $u_n(1) \to u(1)$, cf. the proof of Theorem 1.1- Step 1 (1.a) and (1.c). Otherwise, let $\delta > 0$ be such that $[\bar{x}-\delta, \bar{x}+\delta] \subset [0,1] \setminus \{x_1, \dots, x_j\}$. By Lemma 5.3,

$$a(\cdot)F(u_n) \to a(\cdot)F(u) \quad \text{uniformly in } [\bar{x}-\delta, \bar{x}+\delta]$$

and

$$u_n' \varphi_n(u_n') - \Phi_n(u_n') \to 1 - \frac{1}{\sqrt{1+(u')^2}} \quad \text{uniformly in } [\bar{x}-\delta, \bar{x}+\delta].$$

Therefore, by Lemma 5.1 and the uniqueness of the limit,

$$E(x) = 1 - \frac{1}{\sqrt{1+(u'(x))^2}} + a(x)F(u(x)) \quad \text{for every } x \in [\bar{x}-\delta, \bar{x}+\delta].$$

□

6. Existence of classical solutions

In this section, we give a result ensuring that the solutions of (1.2) found by the previous approximation procedure are actually classical solutions. The precise statement is the following.

Theorem 6.1. *Let $k \in \mathbb{N}$, $a \in C^1([0,1])$, $a > 0$ in $[0,1]$ and assume that $f \in C^1([0,\infty))$ satisfy (f_{eq}), (f_{sgn}) and $f'(u_0) > \lambda_{k+1}$.*

If moreover

$$a(0) \exp\Big(\int_0^1 \frac{a'^+(x)}{a(x)} dx\Big) \int_{u_0}^0 f(s) ds < 1 \tag{6.1}$$

then, there exist at least k non-constant positive C^2-solutions $u_1, \dots, u_k$ of (1.2), having the properties stated in Theorem 1.1-(I).

On the other hand, if

$$a(0) \exp\Big(\int_0^1 \frac{a'^+(x)}{a(x)} dx\Big) \int_{u_0}^{+\infty} f(s) ds < 1 \tag{6.2}$$

then, there exist at least k non-constant positive C^2-solutions $u_{k+1}, \dots, u_{2k}$ of (1.2), having the properties stated in Theorem 1.1-(II).

Proof. Let us consider the first case, the second one is similar.

We are going to show that, for n sufficiently large, it holds that

$$|u_{n,i}'(x)| \le n, \quad \text{for all } x \in [0,1]. \tag{6.3}$$

Since $\varphi_n(s) = \varphi(s)$ for $|s| \le n$, this implies that, for n large enough, the C^2-function $u_{n,i}$ is a solution of (1.2), thus concluding the proof.

In order to prove (6.3), we first argue similarly as in the proof of Lemma 3.3 to find the estimate

$$E_n'(x) \le \frac{a'^+(x)}{a(x)} E_n(x), \quad \text{for all } x \in (0,1),$$

which in turn implies that

$$E_n(x) \le E_n(0) \exp\Big(\int_0^x \frac{a'^+(s)}{a(s)} ds\Big), \quad \text{for all } x \in (0,1).$$

From this, setting $K_n(s) := s\varphi_n(s) - \Phi_n(s)$, we obtain

$$\begin{aligned} K_n(u'_{n,i}(x)) &\le E_n(0) \exp\Big(\int_0^x \frac{a'^+(s)}{a(s)} ds\Big) - a(x)F(u_{n,i}(x)) \\ &= a(0)F(d_{n,i}) \exp\Big(\int_0^x \frac{a'^+(s)}{a(s)} ds\Big) - a(x)F(u_{n,i}(x)), \end{aligned}$$

where we have denoted $d_{n,i} = u_{n,i}(0)$. Thus, recalling (3.3), we infer that

$$K_n(u'_{n,i}(x)) \le a(0) \exp\Big(\int_0^x \frac{a'^+(s)}{a(s)} ds\Big) F(d_{n,i}), \quad \text{for all } x \in (0,1),$$

and finally, by assumption (6.1) and by (3.4),

$$K_n(u'_{n,i}(x)) \le 1 - \eta, \quad \text{for all } x \in (0,1), \tag{6.4}$$

where $\eta = 1 - a(0) \exp\big(\int_0^1 \frac{a'^+(x)}{a(x)} dx\big) F(0) > 0$.

On the other hand, since the function K_n is increasing for $s \ge 0$ and decreasing for $s \le 0$, a simple computation yields

$$K_n(s) \ge K_n(n) = 1 - \frac{1}{\sqrt{1+n^2}}, \quad \text{for } |s| \ge n. \tag{6.5}$$

Combining (6.4) and (6.5), the estimate (6.3) easily follows for $n \ge \sqrt{1/\eta^2 - 1}$. □

Remark 6.2. Observe that, in case (f_{ap}) holds, as we know that for $i \in \{k+1, \dots, 2k\}$, $d_{n,i} \in [u_0, \bar{u}]$, the condition (6.2) can be replaced by

$$a(0) \exp\Big(\int_0^1 \frac{a'^+(x)}{a(x)} dx\Big) \int_{u_0}^{\bar{u}} f(s) ds < 1.$$

In the same way, in case $(f_{\mathrm{ap}})'$ holds, let R be given by Lemma 4.2 such that, for all $n \in \mathbb{N}$, $\|u_n\|_{L^\infty(0,1)} \le R$. Then the condition (6.2) can be replaced by

$$a(0) \exp\Big(\int_0^1 \frac{a'^+(x)}{a(x)} dx\Big) \int_{u_0}^{R} f(s) ds < 1.$$

Remark 6.3. Considering the proof of Theorem 6.1, it seems natural that the solutions of (1.2) having a large number of intersections with u_0 are classical while the solutions having a low number of intersections are only BV-solutions.

Remark 6.4. In the autonomous case, that is $a(x) \equiv a$, in case (f_{ap}) holds, one has $\int_{u_0}^{\bar{u}} f(s) ds = \int_{u_0}^{0} f(s) ds$ and so, also in view of Remark 6.2, conditions (6.1) and (6.2) reduce to

$$a \int_{u_0}^{0} f(s) ds < 1. \tag{6.6}$$

The above condition has a clear dynamical interpretation. Indeed, it means that the planar system (1.5) admits a classical homoclinic orbit to the equilibrium point $(0,0)$ (incidentally, notice that $(\bar{u}, 0)$ is nothing but the intersection point of the homoclinic with the positive u-semiaxis). Since, as already discussed in Figure 1, all the Neumann solutions must lie inside the region bounded by this homoclinic orbit, it is immediately understood that they have to be classical solutions.

We also notice that, for a fixed nonlinear term f, condition (6.6) is always satisfied for a sufficiently small and never satisfied when a is sufficiently large. More explicit conditions can be given for particular choices of the function f. For instance, in the model example

$$f(s) = -\lambda s + s^p, \quad \text{with } p > 1 \text{ and } \lambda > 0,$$

it turns out that $u_0 = \lambda^{\frac{1}{p-1}}$ and a simple computation shows that (6.6) is satisfied if and only if

$$a\lambda^{\frac{p+1}{p-1}} \frac{p-1}{2(p+1)} < 1. \tag{6.7}$$

Assuming for simplicity $a = 1$, we thus see that, if $\lambda \leq 1$, (6.7) is automatically verified and all the solutions found in Theorem 1.1 are classical. On the contrary, if $\lambda > 1$, (6.7) is not automatic and it is in competition with the assumption required in Theorem 1.1 for the existence of at least one non-constant possibly discontinuous BV-solution of (1.2), i.e., $f'(u_0) > \lambda_2$, or equivalently

$$(p-1)\lambda > \lambda_2. \tag{6.8}$$

So, in this case, the intersection of the two assumptions (6.7) and (6.8) is given by

$$\frac{\lambda_2}{p-1} < \lambda < \left(\frac{2(p+1)}{p-1}\right)^{\frac{p-1}{p+1}} \tag{6.9}$$

which is certainly not empty for p large.

We finally observe that, if we set the problem (1.2) in the interval $(0, L)$, instead of $(0, 1)$, and we let $L \to \infty$, the eigenvalues $\lambda_k \to 0$. Thus, condition (6.9) is not empty also when L is sufficiently large.

Remark 6.5. Recall also that, by [35, Corollary 3.5], if

$$\int_0^1 a(x)|f(u(x))|\, dx < 1,$$

then the solution u is classical.

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