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# ON GOOD APPROXIMATIONS AND BOWEN-SERIES EXPANSION

LUCA MARCHESE

**ABSTRACT.** We consider the continued fraction expansion of real numbers under the action of a non-uniform lattice in  $\mathrm{PSL}(2, \mathbb{R})$  and prove metric relations between the convergents and a natural geometric notion of good approximations.

## 1. INTRODUCTION

Let  $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$  be the *upper half plane* and for  $p/q \in \mathbb{Q}$  let  $H_{p/q} \subset \mathbb{H}$  be the circle of diameter  $1/q^2$  tangent at  $p/q$ . Set  $H_\infty = \{z \in \mathbb{H} : \mathrm{Im}(z) > 1\}$  and consider the family  $\{H_{p/q} : p/q \in \mathbb{Q} \cup \{\infty\}\}$  of *Ford circles*, which are the orbit of  $H_\infty$  under the projective action of the *modular group*  $\mathrm{SL}(2, \mathbb{Z})$ , that is the group of  $2 \times 2$  matrices with coefficients  $a, b, c, d$  in  $\mathbb{Z}$  (notation refers to Equation (1.3) below). Any two circles are either disjoint or tangent, and Figure 1 shows that for any irrational  $\alpha$  there exist infinitely many  $p/q \in \mathbb{Q}$  with  $\alpha \in \Pi(H_{p/q})$ , that is  $|\alpha - p/q| < (1/2)q^{-2}$ , where  $\Pi(x + iy) := x$ . This defines the sequence

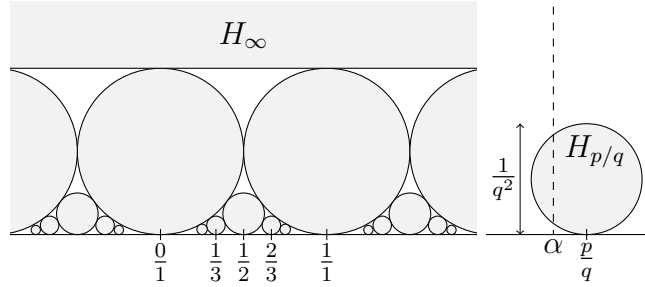


FIGURE 1. Balls  $G(H_k)$ ,  $k \in \mathbb{Z}$ , tangent to  $H_{p/q} = G(H_\infty)$ , where  $p/q = G \cdot \infty$ .

of *geometric good approximations* of  $\alpha$  as the sequence of  $p_n/q_n$  in  $\mathbb{Q}$  with  $\alpha \in \Pi(B_{p_n/q_n})$ . The same sequence arises from the continued fraction expansion  $\alpha = a_0 + [a_1, a_2, \dots]$  of  $\alpha$ , indeed the *convergents*  $p_n/q_n := a_0 + [a_1, \dots, a_n]$  satisfy:

$$(1.1) \quad |\alpha - p/q| < (1/2)q^{-2} \Rightarrow p/q = p_n/q_n \text{ for some } n \geq 1.$$

The first  $n + 1$  *partial quotients*  $a_1, \dots, a_{n+1}$  approximate  $\alpha$  with error given by

$$(1.2) \quad \frac{1}{2 + a_{n+1}} \leq q_n^2 \cdot |\alpha - p_n/q_n| \leq \frac{1}{a_{n+1}} \text{ for any } n \in \mathbb{N}.$$

*Rosen continued fractions* were introduced in [9], in relation to diophantine approximation for *Hecke groups*, proving in particular an extension of Equation (1.2), which was later

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improved by [7]. Equation (1.1) was extended to Rosen continued fraction in [5], where the sharp constant replacing  $1/2$  was obtained in [10]. In this note we consider diophantine approximation for a general non-uniform lattice Fuchsian group, in relation to the so-called *Bowen-Series expansion* of real numbers ([3]). Our Main Theorem 3.1 provides an extension of Equations (1.1) and (1.2) to this setting. This result is used in [6] to approximate the dimension of sets of *badly approximable points* by the dimension of dynamically defined regular Cantor sets. The study of the high part of *Markov and Lagrange spectra* is also a natural application, in the spirit of [11], [1] and [2]. In general, Theorem 3.1 applies to a large variety of problems in diophantine approximations, since it translates diophantine properties into ergodic properties of the Bowen-Series expansion.

Let  $\mathrm{SL}(2, \mathbb{C})$  be the group of matrices

$$(1.3) \quad G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with  $a, b, c, d \in \mathbb{C}$  and  $ad - bc = 1$ , where any such  $G$  acts on points  $z \in \mathbb{C} \cup \{\infty\}$  by

$$(1.4) \quad G \cdot z := \frac{az + b}{cz + d}.$$

Denote  $a = a(G)$ ,  $b = b(G)$ ,  $c = c(G)$  and  $d = d(G)$  the coefficients of  $G$  as in Equation (1.3). The group  $\mathrm{SL}(2, \mathbb{R})$  of  $G$  with coefficients  $a, b, c, d$  in  $\mathbb{R}$  acts by isometries on  $\mathbb{H}$  via Equation (1.4), and inherits a topology from the identification with the set of  $(a, b, c, d) \in \mathbb{R}^4$  with  $ad - bc = 1$ . A *Fuchsian group* is a discrete subgroup  $\Gamma < \mathrm{SL}(2, \mathbb{R})$ . Referring to [4], we say that  $\Gamma$  is a *lattice* if it has a *Dirichlet region*  $\Omega \subset \mathbb{H}$  with finite hyperbolic area. If  $\Omega$  is not compact, then the lattice  $\Gamma$  is said *non-uniform*. In this case the intersection  $\bar{\Omega} \cap \partial\mathbb{H}$  is a finite non-empty set, whose elements are called the vertices *at infinity* of  $\Omega$ . A point  $z \in \mathbb{R} \cup \{\infty\}$  is a parabolic fixed point for  $\Gamma$  if there exists  $P \in \Gamma$  parabolic with  $P(z) = z$ . Let  $\mathcal{P}_\Gamma$  be the set of parabolic fixed points of  $\Gamma$ , which is equal to the orbit under  $\Gamma$  of the vertices at infinity of  $\Omega$ . The set  $\mathcal{P}_\Gamma$  is dense in  $\mathbb{R}$ . Two points  $z_1$  and  $z_2$  in  $\mathcal{P}_\Gamma$  are *equivalent* if  $z_2 = G(z_1)$  for some  $G \in \Gamma$ . Any non-uniform lattice  $\Gamma$  has a finite number  $p \geq 1$  of equivalence classes  $[z_1], \dots, [z_p]$  of parabolic fixed points, called the *cusps* of  $\Gamma$ .

Let  $\Gamma$  be a non-uniform lattice with  $p \geq 1$  cusps. Fix a list  $\mathcal{S} = (A_1, \dots, A_p)$  of elements  $A_k \in \mathrm{SL}(2, \mathbb{R})$  such that the points

$$(1.5) \quad z_k = A_k \cdot \infty \quad \text{for} \quad k = 1, \dots, p$$

form a complete set  $\{z_1, \dots, z_p\} \subset \mathcal{P}_\Gamma$  of inequivalent parabolic fixed points. A natural choice for  $z_1, \dots, z_p$  is a maximal set of non-equivalent vertices at infinity of a fundamental domain. Any element of  $\mathcal{P}_\Gamma$  has the form  $G \cdot z_k$  for some  $G \in \Gamma$  and  $k = 1, \dots, p$ . We have horoballs

$$B_k := A_k(\{z \in \mathbb{H} : \mathrm{Im}(z) > 1\}) \quad \text{with} \quad k = 1, \dots, p,$$

each  $B_k$  being tangent to  $\mathbb{R} \cup \{\infty\}$  at  $z_k$ . We can have  $A_k = \mathrm{Id}$ , that is  $z_k = \infty$  and  $B_k = H_\infty$ . Thus  $G(B_k)$  is a ball tangent to the real line at  $G \cdot z_k$  for any  $G \in \Gamma$  with  $G \cdot z_k \neq \infty$ . These balls generalize Ford circles and we measure how their diameter shrinks to zero as  $G$  varies in  $\Gamma$  with the *denominator*

$$D(G \cdot z_k) := \begin{cases} 1/\sqrt{\mathrm{Diam}(G(B_k))} & \text{if } G \cdot z_k \neq \infty \\ 0 & \text{if } G \cdot z_k = \infty. \end{cases}$$

Recall that for any  $T > 0$  and any  $G \in \mathrm{SL}(2, \mathbb{R})$  with  $c(G) \neq 0$  we have

$$(1.6) \quad \mathrm{Diam}\left(G(\{z \in \mathbb{H} : \mathrm{Im}(z) > T\})\right) = \frac{1}{Tc^2(G)},$$

where we refer to the notation of Equation (1.3). Hence

$$(1.7) \quad D(G \cdot z_k) = |c(GA_k)| \quad \text{for any } G \cdot z_k \in \mathcal{P}_\Gamma.$$

In [8], Patterson proves that there exists a constant  $M = M(\Gamma, \mathcal{S}) > 0$  such that for any  $Q > 0$  big enough and any  $\alpha \in \mathbb{R}$  there exists  $G \in \Gamma$  and  $k \in \{1, \dots, p\}$  with

$$|\alpha - G \cdot z_k| \leq \frac{M}{D(G \cdot z_k)Q} \quad \text{and} \quad 0 < D(G \cdot z_k) \leq Q.$$

For  $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ ,  $\mathcal{S} = \{\mathrm{Id}\}$  and  $M = 1$  Patterson's Theorem gives the Classical Dirichlet Theorem. In general, for any  $\alpha \in \mathbb{R}$  we obtain infinitely many  $G \cdot z_k \in \mathcal{P}_\Gamma$  with

$$(1.8) \quad |\alpha - G \cdot z_k| \leq \frac{M}{D^2(G \cdot z_k)}.$$

The *Bowen-Series expansion* ([3]) provides a coding  $\alpha = [W_1, W_2, \dots]$  of a real number  $\alpha$ , where for  $r \geq 1$  we call *cuspidal words* the symbols  $W_r$ , which belong to a countable alphabet  $\mathcal{W}$  (definitions are in § 2 and § 3). Cuspidal words  $W \in \mathcal{W}$ , that were introduced in [1] and [2], label a subset of elements  $\{G_W : W \in \mathcal{W}\}$  of  $\Gamma$ , which generalize the role played in the theory of classical continued fractions by the matrices

$$\begin{pmatrix} 1 & a_{2k+1} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ a_{2k} & 1 \end{pmatrix} \quad \text{with} \quad a_{2k}, a_{2k+1} \in \mathbb{N}^* \text{ for any } k \in \mathbb{N}.$$

The coding is a continuous bijection  $\Sigma \rightarrow \mathbb{R}$ , where  $\Sigma \subset \mathcal{W}^\mathbb{N}$  is a subshift with *aperiodic transition matrix* (see [6]). For  $r \geq 1$  the first  $r$  symbols in the expansion of  $\alpha = [W_1, W_2, \dots]$  define  $\zeta_r = \zeta_r(W_1, \dots, W_r) \in \mathcal{P}_\Gamma$ , see Equation (3.5). This extends the classical notion of convergents  $p_n/q_n$  of  $\alpha$ . The main result of this note is Theorem 3.1 in § 3. We give the following preliminary statement (see also Remark 3.2).

**Main Theorem** (Theorem 3.1). *Fix  $\alpha = [W_1, W_2, \dots]$  which is not an element of  $\mathcal{P}_\Gamma$ . The convergents  $\zeta_r = \zeta_r(W_1, \dots, W_r)$  approximate  $\alpha$  with error given by an analogue of Equation (1.2). Moreover there exists a constant  $\epsilon_0 > 0$  such that any  $G \cdot z_k \in \mathcal{P}_\Gamma$  satisfying Equation (1.8) with  $M = \epsilon_0$  belongs to the sequence  $(\zeta_r)_{r \geq 1}$ .*

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## 2. THE BOWEN-SERIES EXPANSION

We follow § 3 in [6], which is itself based on § 2.4 in [1] and § 2 in [2]. The original construction is in [3], where it is defined a *Markov map*, which is *orbit equivalent* to the action of a given finitely generated Fuchsian group of the first kind. In our setting such Markov map corresponds to an *acceleration* of the map in Equation (2.7) below. This § 2 describes the coding by *cuspidal words*. The same description appears in [6], where it is

followed by the study of the combinatorial and metric properties of the subshift related to the coding. Consider the unit disc  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$  and the map

$$(2.1) \quad \varphi : \mathbb{H} \rightarrow \mathbb{D} \quad ; \quad \varphi(z) := \frac{z - i}{z + i}.$$

The conjugate of  $\mathrm{SL}(2, \mathbb{R})$  under  $\varphi$  is the group  $\mathrm{SU}(1, 1)$  of  $F \in \mathrm{GL}(2, \mathbb{C})$  with

$$(2.2) \quad F = \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \quad \text{with} \quad |\alpha|^2 - |\beta|^2 = 1.$$

Denote  $\alpha = \alpha(F)$  and  $\beta = \beta(F)$  the coefficients of  $F$  as in Equation (2.2).

**2.1. Isometric circles.** Consider  $F \in \mathrm{SU}(1, 1)$  and  $\alpha = \alpha(F)$ ,  $\beta = \beta(F)$  as in Equation (2.2). Assume  $\beta \neq 0$  and let  $\omega_F := -\bar{\alpha}/\beta$  be the pole of  $F$ . The *isometric circle*  $I_F$  of  $F$  is the euclidean circle centered at  $\omega_F$  with radius  $\rho(F) := |\beta|^{-1}$ , that is

$$I_F := \{\xi \in \mathbb{C} : |\xi - \omega_F| = \rho(F)\}.$$

We have  $F(I_F) = I_{F^{-1}}$ , where  $\rho(F) = \rho(F^{-1})$  and  $|\omega_{F^{-1}}| = |\omega_F|$ . See Theorem 3.3.2 in [4]. Moreover  $I_F \cap \mathbb{D}$  is a geodesic of  $\mathbb{D}$  for any  $F \in \mathrm{SU}(1, 1)$ , by Theorem 3.3.3 in [4]. Denote  $U_F$  the disc in  $\mathbb{C}$  with  $\partial U_F = I_F$ , that is the interior of  $I_F$ .

**2.2. Labelled ideal polygon.** Let  $\Gamma \subset \mathrm{SU}(1, 1)$  be a non-uniform lattice. According to [12], there exist a free subgroup  $\Gamma_0 < \Gamma$  with finite index  $[\Gamma_0 : \Gamma] < +\infty$ . See also § 2.2 of [6]. In particular  $\beta(F) \neq 0$  for any  $F \in \Gamma_0$ , referring to Equation (2.2), so that the isometric circle  $I_F$  and the disc  $U_F$  introduced in § 2.1 are defined. The origin  $0 \in \mathbb{D}$  is not a fixed point of any  $F \in \Gamma_0$  and Theorem 3.3.5 in [4] implies that the set

$$(2.3) \quad \Omega_0 := \mathbb{D} \setminus \overline{\bigcup_{F \in \Gamma_0} U_F}$$

is a Dirichlet region for  $\Gamma_0$ . Recall from [4] that  $\Omega_0$  is an hyperbolic polygon with an even number  $2d$  of sides, denoted by the letter  $s$ , and with  $2d$  vertices, denoted by the letter  $\xi$  (see also § 2.4 of [6]). All vertices of  $\Omega_0$  belong to  $\partial\mathbb{D}$ , because  $\Gamma_0$  is free. Any side  $s$  is a complete geodesic in  $\mathbb{D}$  and for any such  $s$  there exists a unique  $F \in \Gamma$  such that  $F(s)$  is another side of  $\Omega_0$  with  $F(s) \neq s$ . The sides  $s$  and  $F(s)$  are thus *paired*. See Figure 2. The set of pairings generates  $\Gamma_0$ , according to Theorem 3.5.4 in [4]. For a convenient labelling, consider two finite alphabets  $\mathcal{A}_0$  and  $\widehat{\mathcal{A}}_0$ , both with  $d$  elements, and a map

$$\iota : \mathcal{A}_0 \cup \widehat{\mathcal{A}}_0 \rightarrow \mathcal{A}_0 \cup \widehat{\mathcal{A}}_0 \quad \text{with} \quad \iota^2 = \mathrm{Id} \quad \text{and} \quad \iota(\mathcal{A}_0) = \widehat{\mathcal{A}}_0,$$

that is an involution of  $\mathcal{A}_0 \cup \widehat{\mathcal{A}}_0$  which exchanges  $\mathcal{A}_0$  with  $\widehat{\mathcal{A}}_0$ . Set  $\mathcal{A} := \mathcal{A}_0 \cup \widehat{\mathcal{A}}_0$  and for any  $a \in \mathcal{A}$ , denote  $\widehat{a} := \iota(a)$ .

- Label the sides of  $\Omega_0$  by the letters in  $\mathcal{A}$ , so that for any  $a \in \mathcal{A}$  the sides  $s_a$  and  $s_{\widehat{a}}$  are those which are paired by the action of  $\Gamma_0$ .
- For any pair of sides  $s_a$  and  $s_{\widehat{a}}$  as above, let  $F_a$  be the unique element of  $\Gamma_0$  such that

$$(2.4) \quad F_a(s_{\widehat{a}}) = s_a.$$

- For any  $a \in \mathcal{A}$  we have  $F_{\widehat{a}} = F_a^{-1}$ , and the latter form a set of generators for  $\Gamma_0$ .

In the following we denote  $\Omega_{\mathbb{D}} := \Omega_0 \subset \mathbb{D}$  the labelled ideal polygon defined above and  $\Omega_{\mathbb{H}} := \varphi^{-1}(\Omega_{\mathbb{D}}) \subset \mathbb{H}$  its preimage under the map in Equation (2.1).

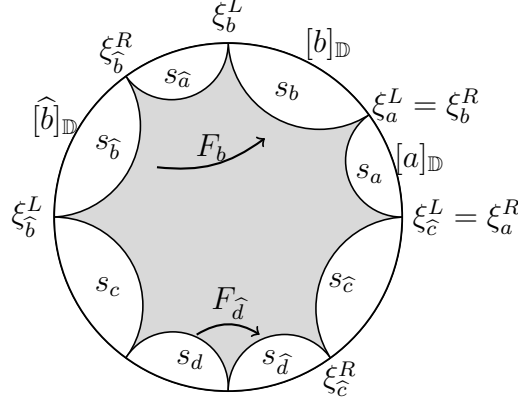


FIGURE 2. Ideal polygon labelled by  $\mathcal{A} = \{a, b, c, d, \hat{a}, \hat{b}, \hat{c}, \hat{d}\}$ .

**2.3. The boundary map.** Parametrize arcs  $J \subset \partial\mathbb{D}$  by  $t \mapsto e^{-it}$  with  $t \in (x, y)$ . Set  $\inf J := e^{-ix}$  and  $\sup J := e^{-iy}$ . We say that  $J$  is *right open* if  $\inf J \in J$  and  $\sup J \notin J$ . Let  $\Gamma_0 < \Gamma$  be a finite index free subgroup and  $\Omega_{\mathbb{D}}$  be an ideal polygon for  $\Gamma_0$  labelled by  $\mathcal{A}$ , as in § 2.2.

For  $a \in \mathcal{A}$  let  $F_a$  be the map in Equation (2.4). Let  $I_{F_a}$  be the isometric circle of  $F_a$  and  $U_{F_a}$  be its interior, as in § 2.1. Recall that  $s_{\hat{a}} = I_{F_a} \cap \mathbb{D}$  and  $s_a = I_{F_{\hat{a}}} \cap \mathbb{D}$ . Let  $[a]_{\mathbb{D}}$  be the right open arc of  $\partial\mathbb{D}$  cut by the side  $s_a$ , that is

$$[a]_{\mathbb{D}} := U_{F_{\hat{a}}} \cap \partial\mathbb{D}.$$

Set  $\xi_a^L := \inf[a]_{\mathbb{D}}$  and  $\xi_a^R := \sup[a]_{\mathbb{D}}$ . Figure 2 shows examples of such notation. In order to take account of the cyclic order in  $\partial\mathbb{D}$  of the arcs  $[a]_{\mathbb{D}}$ , fix  $a_0 \in \mathcal{A}$  and define a map  $o : \mathcal{A} \rightarrow \mathbb{Z}/2d\mathbb{Z}$  setting  $o(a_0) := 0$  and

$$(2.5) \quad o(b) = o(a) + 1 \pmod{2d} \quad \text{for } a, b \in \mathcal{A} \quad \text{with } \xi_a^R = \xi_b^L.$$

We have  $F_a(I_{F_a}) = I_{F_{\hat{a}}}$  for any  $a \in \mathcal{A}$ , thus  $F_a$  sends the complement of  $[\hat{a}]_{\mathbb{D}}$  to  $[a]_{\mathbb{D}}$ , that is

$$(2.6) \quad F_a(\partial\mathbb{D} \setminus [\hat{a}]_{\mathbb{D}}) = [a]_{\mathbb{D}}.$$

The *Bowen-Series map* is the map  $\mathcal{BS} : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$  defined by

$$(2.7) \quad \mathcal{BS}(\xi) := F_a^{-1}(\xi) \quad \text{iff } \xi \in [a]_{\mathbb{D}}.$$

The *boundary expansion* of a point  $\xi \in \partial\mathbb{D}$  is the sequence  $(a_k)_{k \in \mathbb{N}}$  of letters  $a_k \in \mathcal{A}$  with

$$(2.8) \quad \mathcal{BS}^k(\xi) \in [a_k]_{\mathbb{D}} \quad \text{for any } k \in \mathbb{N}.$$

By Equation (2.6), any such sequence satisfies the so-called *no backtracking Condition*:

$$(2.9) \quad a_{k+1} \neq \hat{a}_k \quad \text{for any } k \in \mathbb{N}.$$

A finite word  $(a_0, \dots, a_n)$  satisfying Condition (2.9) corresponds to a *factor* of the map  $\mathcal{BS} : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$ , that is a finite concatenation  $F_{a_n}^{-1} \circ \dots \circ F_{a_0}^{-1}$  arising from iterations of  $\mathcal{BS}$ . We call *admissible word*, or simply *word*, any finite or infinite word in the letters of  $\mathcal{A}$  satisfying Condition (2.9). We use the notation

$$F_{a_0, \dots, a_n} := F_{a_0} \circ \dots \circ F_{a_n} \in \Gamma_0.$$

Define the right open arc  $[a_0, \dots, a_n]_{\mathbb{D}}$  as the set of  $\xi \in \partial\mathbb{D}$  such that  $\mathcal{BS}^k(\xi) \in [a_k]_{\mathbb{D}}$  for any  $k = 0, \dots, n$ , that is

$$(2.10) \quad [a_0, \dots, a_n]_{\mathbb{D}} := F_{a_0, \dots, a_{n-1}}[a_n]_{\mathbb{D}} = F_{a_0, \dots, a_n}(\partial\mathbb{D} \setminus [\widehat{a_n}]_{\mathbb{D}}).$$

Two such arcs satisfy  $[a_0, \dots, a_n]_{\mathbb{D}} \subset [b_0, \dots, b_m]_{\mathbb{D}}$  if and only if  $m \geq n$  and  $a_k = b_k$  for any  $k = 0, \dots, n$ . It is easy to see that  $[a_0, \dots, a_n]_{\mathbb{D}}$  shrinks to a point as  $n \rightarrow \infty$ . See Lemma 3.1 in [6] for a proof. A sequence  $(a_k)_{k \in \mathbb{N}}$  satisfying Condition (2.9) corresponds to a point  $\xi = [a_0, a_1, \dots]_{\mathbb{D}}$  in  $\partial\mathbb{D}$ , where we use the notation

$$[a_0, a_1, \dots]_{\mathbb{D}} := \bigcap_{n \in \mathbb{N}} [a_0, \dots, a_n]_{\mathbb{D}}.$$

Conversely, if  $(a_k)_{k \in \mathbb{N}}$  is the boundary expansion of  $\xi \in \partial\mathbb{D}$ , then  $\xi = [a_0, a_1, \dots]_{\mathbb{D}}$ . The Bowen-Series map  $\mathcal{BS}$  is the shift on the space of admissible infinite words.

**2.4. Cuspidal words.** Consider the map  $o : \mathcal{A} \rightarrow \mathbb{Z}/2d\mathbb{Z}$  in Equation (2.5). The definitions in § 2.3 easily imply Lemma 2.1 below. See Lemma 3.2 in [6] for a proof.

**Lemma 2.1.** *Let  $(a_0, \dots, a_n)$  be a word satisfying Condition (2.9) with  $n \geq 1$  and  $a_0 = a_n$ . The map  $F_{a_0, \dots, a_{n-1}}$  is a parabolic element of  $\Gamma_0$  fixing  $\xi_{a_0}^R$  if and only if*

$$(2.11) \quad o(a_{k+1}) = o(\widehat{a_k}) - 1 \quad \text{for any } k = 0, \dots, n-1.$$

*The map  $F_{a_0, \dots, a_{n-1}}$  is a parabolic element of  $\Gamma_0$  fixing  $\xi_{a_0}^L$  if and only if*

$$(2.12) \quad o(a_{k+1}) = o(\widehat{a_k}) + 1 \quad \text{for any } k = 0, \dots, n-1.$$

Let  $W = (a_0, \dots, a_n)$  be an admissible word. We say that  $W$  is a *cuspidal word* if it is the initial factor of an admissible word  $(a_0, \dots, a_m)$  with  $m \geq n$  such that  $F_{a_0, \dots, a_m}$  is a parabolic element of  $\Gamma_0$  fixing a vertex of  $\Omega_{\mathbb{D}}$ .

- If  $n \geq 1$  and Equation (2.11) is satisfied, we say that  $W$  is a *right cuspidal word*. In this case we define its type by  $\varepsilon(W) := R$  and we set  $\xi_W := \xi_{a_0}^R$ .
- If  $n \geq 1$  and Equation (2.12) is satisfied, we say that  $W$  is a *left cuspidal word*. In this case we define its type by  $\varepsilon(W) := L$  and we set  $\xi_W := \xi_{a_0}^L$ .
- If  $n = 0$ , that is  $W = (a_0)$  has just one letter, the type  $\varepsilon(W)$  is not defined. We set by convention  $\xi_W := \xi_{a_0}^R$ .

If  $W = (a_0, \dots, a_n)$  is cuspidal with  $n \geq 1$ , Lemma 2.1 implies  $\xi_{a_k}^{\varepsilon(W)} = F_{a_k} \cdot \xi_{a_{k+1}}^{\varepsilon(W)}$  for any  $k = 0, \dots, n-1$  and it follows

$$(2.13) \quad \xi_W = \partial[a_0]_{\mathbb{D}} \cap \partial[a_0, a_1]_{\mathbb{D}} \cap \dots \cap \partial[a_0, \dots, a_n]_{\mathbb{D}},$$

that is the  $n+1$  arcs above share  $\xi_W$  as common endpoint (see also § 2.4 in [2] and § 4.3 in [1]). A sequence  $(a_n)_{n \in \mathbb{N}}$  is said *cuspidal* if any initial factor  $(a_0, \dots, a_n)$  with  $n \in \mathbb{N}$  is a cuspidal word, and *eventually cuspidal* if there exists  $k \in \mathbb{N}$  such that  $(a_{n+k})_{n \in \mathbb{N}}$  is a cuspidal sequence.

**2.5. The cuspidal acceleration.** If  $W = (b_0, \dots, b_m)$  and  $W' = (a_0, \dots, a_n)$  are words with  $a_0 \neq \widehat{b_m}$ , define the word  $W * W' := (b_0, \dots, b_m, a_0, \dots, a_n)$ . Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence satisfying Condition (2.9) and not eventually cuspidal.

**Initial step:** Set  $n(0) := 0$ . Let  $n(1) \in \mathbb{N}$  be the maximal integer  $n(1) \geq 1$  such that  $(a_0, \dots, a_{n(1)-1})$  is cuspidal, then set  $W_0 := (a_0, \dots, a_{n(1)-1})$ .



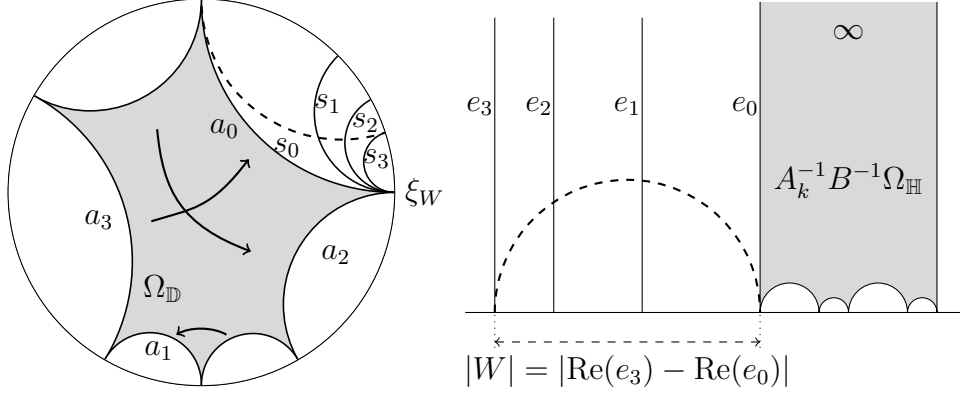


FIGURE 3. Geometric length  $|W|$  of a right cuspidal word  $W = (a_0, a_1, a_2, a_3)$ . The arrows inside  $\Omega_{\mathbb{D}}$  represent the action of  $F_{a_0}, F_{a_1}, F_{a_2}$ . The arcs  $s_0 := s_{a_0}$ ,  $s_1 := F_{a_0}(s_{a_1})$ ,  $s_2 := F_{a_0, a_1}(s_{a_2})$  and  $s_3 := F_{a_0, a_1, a_2}(s_{a_3})$  share the common vertex  $\xi_W$ , which is sent to  $\infty$  under the map  $A_k^{-1} B^{-1} \varphi^{-1}$ . Thus the arcs  $s_0, s_1, s_2, s_3$  in  $\mathbb{D}$  are sent to parallel vertical arcs  $e_i := \varphi^{-1}(s_i)$  in  $\mathbb{H}$ .

**Recursive step:** Fix  $r \geq 1$  and assume that the instants  $n(0) < \dots < n(r)$  and the cuspidal words  $W_0, \dots, W_{r-1}$  are defined. Define  $n(r+1) \geq n(r) + 1$  as the maximal integer such that  $[a_{n(r)}, \dots, a_{n(r+1)-1}]$  is cuspidal, then set

$$W_r := (a_{n(r)}, \dots, a_{n(r+1)-1}).$$

The sequence of words  $(W_r)_{r \in \mathbb{N}}$  is called the *cuspidal decomposition* of  $(a_n)_{n \in \mathbb{N}}$ . We have of course  $a_0, a_1, a_2 \dots = W_0 * W_1 * \dots$ . For any  $\xi = [a_0, a_1, \dots]_{\mathbb{D}}$ , if  $(W_r)_{r \in \mathbb{N}}$  is the cuspidal decomposition of  $(a_n)_{n \in \mathbb{N}}$ , we write

$$(2.14) \quad \xi = [a_0, a_1, \dots]_{\mathbb{D}} = [W_0, W_1, \dots]_{\mathbb{D}}.$$

**Remark 2.2.** If  $W_{r-1} := (a_{n(r-1)}, \dots, a_{n(r)-1})$  and  $W_r := (a_{n(r)}, \dots, a_{n(r+1)-1})$  are two consecutive cuspidal words in the cuspidal decomposition of a sequence  $(a_n)_{n \in \mathbb{N}}$  satisfying Condition (2.9), then the word  $(a_{n(r)-1}, a_{n(r)}, \dots, a_{n(r+1)-1})$  can be cuspidal.

### 3. THE MAIN THEOREM 3.1

The tools in § 2 induce a boundary expansion on  $\mathbb{R}$ . Let  $\Gamma_0 < \Gamma$  be the free subgroup and  $\Omega_{\mathbb{D}} \subset \mathbb{D}$  be the ideal polygon in § 2.2. Recall that  $\mathcal{P}_{\Gamma_0} = \Gamma_0(\Omega_{\mathbb{D}} \cap \partial\mathbb{D})$  by Theorem 4.2.5 in [4]. Since  $\Gamma_0$  has finite index in  $\Gamma$  then the two groups have the same set of parabolic fixed points, that is

$$(3.1) \quad \mathcal{P}_{\Gamma} = \Gamma_0(\Omega_{\mathbb{D}} \cap \partial\mathbb{D}).$$

**3.1. Geometric length of cuspidal words and main statement.** Fix  $\mathcal{S} = (A_1, \dots, A_p)$  as in Equation (1.5). Let  $\Omega_{\mathbb{H}} := \varphi^{-1}(\Omega_{\mathbb{D}}) \subset \mathbb{H}$  be the pre-image of  $\Omega_{\mathbb{D}}$  under the map in Equation (2.1). Any vertex  $\xi$  of  $\Omega_{\mathbb{D}}$  corresponds to an unique vertex  $\zeta = \varphi^{-1}(\xi)$  of  $\Omega_{\mathbb{H}}$ . For any such vertex  $\zeta$  consider  $B \in \Gamma$  and  $k \in \{1, \dots, p\}$  with

$$(3.2) \quad \zeta = BA_k \cdot \infty$$

Any side  $s_a$  of  $\Omega_{\mathbb{D}}$  corresponds to an unique side  $e_a := \varphi^{-1}(s_a)$  of  $\Omega_{\mathbb{H}}$ , where  $a \in \mathcal{A}$ . If  $BA_k \cdot \infty = B'A_j \cdot \infty$ , then  $j = k$ . Moreover  $B' = BP$ , where  $P \in \Gamma$  is parabolic fixing

$A_k \cdot \infty$ , where we recall that in any Fuchsian group  $\Gamma$  with cusps, if  $G \in \Gamma$  satisfies  $G \cdot \zeta = \zeta$  for some  $\zeta \in \mathcal{P}_\Gamma$ , then  $G$  is parabolic. Hence the map  $z \mapsto A_k^{-1} P A_k(z)$  is an horizontal translation in  $\mathbb{H}$ . If  $s$  and  $s'$  are geodesics in  $\mathbb{D}$  having  $\xi$  as common endpoint, then their pre-images in  $\mathbb{H}$  under  $\varphi \circ B \circ A_k$  are parallel vertical half lines whose distance does not depend on the choice of  $B$  in Equation (3.2). We have a well defined positive real number

$$\Delta(s, s', \xi) := |\operatorname{Re}(A_k^{-1} B^{-1} \varphi^{-1}(s)) - \operatorname{Re}(A_k^{-1} B^{-1} \varphi^{-1}(s'))|.$$

Fix a cuspidal word  $W = (a_0, \dots, a_n)$  and the vertex  $\xi_W$  of  $\Omega_{\mathbb{D}}$  associated to  $W$  in § 2.4. For  $n \geq 1$  Equation (2.13) implies that the geodesics  $s_{a_0}, F_{a_0}(s_{a_1}), \dots, F_{a_0, \dots, a_{n-1}}(s_{a_n})$  all have  $\xi_W$  as common endpoint. See Figure 3. Define the *geometric length*  $|W| \geq 0$  of  $W$  as

$$(3.3) \quad |W| := \begin{cases} \Delta(s_{a_0}, F_{a_0, \dots, a_{n-1}}(s_{a_n}), \xi_W) & \text{if } n \geq 1 \\ 0 & \text{if } n = 0. \end{cases}$$

For  $a \in \mathcal{A}$  set  $G_a = \varphi^{-1} \circ F_a \circ \varphi$ . Set  $G_{a_0, \dots, a_n} := G_{a_0} \circ \dots \circ G_{a_n}$  for any word  $(a_0, \dots, a_n)$  and  $G_{W_0, \dots, W_r} = G_{a_0, \dots, a_n}$  if  $(a_0, \dots, a_n) = W_0 * \dots * W_r$ . Define the interval

$$[a_0, \dots, a_n]_{\mathbb{H}} := \varphi^{-1}([a_0, \dots, a_n]_{\mathbb{D}}) = G_{a_0, \dots, a_n}(\partial\mathbb{H} \setminus [\hat{a}_n]_{\mathbb{H}}).$$

Set  $[a_0, a_1, \dots]_{\mathbb{H}} := \varphi^{-1}([a_0, a_1, \dots]_{\mathbb{D}})$ , that is encode  $\alpha \in \mathbb{R}$  by the same cutting sequence as  $\varphi(\alpha) \in \mathbb{D}$ . If  $(a_n)_{n \in \mathbb{N}}$  has cuspidal decomposition  $(W_r)_{r \in \mathbb{N}}$ , Equation (2.14) becomes

$$(3.4) \quad \alpha = [W_0, W_1, \dots]_{\mathbb{H}} := [a_0, a_1, \dots]_{\mathbb{H}}.$$

For  $r \in \mathbb{N}$  let  $W_r$  be the  $r$ -th cuspidal word. Set  $\zeta_{W_r} := \varphi^{-1}(\xi_{W_r})$ . The convergents of  $\alpha$  are

$$(3.5) \quad \zeta_r := G_{W_0, \dots, W_{r-1}} \cdot \zeta_{W_r} \quad ; \quad r \in \mathbb{N}.$$

For  $k = 1, \dots, p$  let  $\mu_k > 0$  be such that the primitive parabolic element  $P_k \in A_k \Gamma A_k^{-1}$  fixing  $\infty$  acts by  $P_k(z) = z + \mu_k$ . Set  $\mu := \max\{\mu_1, \dots, \mu_p\}$ .

**Theorem 3.1** (Main Theorem). *For any  $r \in \mathbb{N}$  with  $|W_r| > 0$  we have*

$$(3.6) \quad \frac{1}{|W_r| + 2\mu} \leq D(G_{W_0, \dots, W_{r-1}} \cdot \zeta_{W_r})^2 \cdot |\alpha - G_{W_0, \dots, W_{r-1}} \cdot \zeta_{W_r}| \leq \frac{1}{|W_r|}.$$

Moreover there exists  $\epsilon_0 > 0$  depending only on  $\Omega_{\mathbb{D}}$  and on  $\mathcal{S}$ , such that for any  $G \in \Gamma$  and  $k = 1, \dots, p$  with  $D(G \cdot z_k) \neq 0$  the condition

$$D(G \cdot z_k)^2 \cdot |\alpha - G \cdot z_k| < \epsilon_0$$

implies that there exists some  $r \in \mathbb{N}$  such that

$$(3.7) \quad G \cdot z_k = G_{W_0, \dots, W_{r-1}} \cdot \zeta_{W_r} \quad \text{where} \quad |W_r| > 0.$$

**Remark 3.2.** Equation (3.6) holds for any choice of  $\mathcal{S}$  as in Equation (1.5), and this follows because geometric length and denominators satisfy a form of equivariance under the choice of  $\mathcal{S}$ . Equation (3.7) shows that, for any choice of the subgroup  $\Gamma_0$ , all good enough approximations of a given  $\alpha$  belong to the sequence of its convergents.

**3.2. Reduced form of parabolic fixed points.** Fix  $G \cdot z_k \in \mathcal{P}_\Gamma$ . Recall Equation (3.1) and write elements of  $\Gamma_0$  in the generators  $\{G_a : a \in \mathcal{A}\}$ . There exists a unique admissible word  $b_0, \dots, b_m$  and a vertex  $\zeta$  of  $\Omega_{\mathbb{H}}$  which is not an endpoint of  $e_{\widehat{b_m}}$  such that

$$G \cdot z_k = G_{b_0, \dots, b_m} \cdot \zeta.$$

The representation above is called the *reduced form* of the parabolic fixed point  $G \cdot z_k$ . In the next Lemmas 3.3 and 3.4, let  $(b_0, \dots, b_m)$  be a non-trivial admissible word and let  $\zeta_0$  be a vertex of  $\Omega_{\mathbb{H}}$  which is not an endpoint of  $e_{\widehat{b_m}}$ , so that  $G_{b_0, \dots, b_m} \cdot \zeta_0$  is a parabolic fixed point written in its reduced form and different from  $\infty$ .

**Lemma 3.3.** *There exists a constant  $\kappa_1 > 0$ , depending only on  $\Omega_{\mathbb{H}}$ , such that*

$$|\zeta_0 - G_{b_0, \dots, b_m}^{-1} \cdot \infty| \geq \kappa_1,$$

*that is the vertex  $\zeta_0$  and the pole of  $G_{b_0, \dots, b_m}$  stay at distance uniformly bounded from below.*

*Proof.* We have  $G_{b_0, \dots, b_m}(\mathbb{R} \setminus [\widehat{b_m}]_{\mathbb{H}}) = [b_0, \dots, b_m]_{\mathbb{H}}$  By Equation (2.10). Since  $\infty$  does not belong to the interior of  $[b_0, \dots, b_m]_{\mathbb{H}}$  then the pole of  $G_{b_0, \dots, b_m}$  belongs to the closure of  $[\widehat{b_m}]_{\mathbb{H}}$ . The Lemma follows because  $\zeta_0$  is a vertex of  $\Omega_{\mathbb{H}}$  different from the endpoints of  $e_{\widehat{b_m}}$ .  $\square$

**Lemma 3.4.** *There exists a constant  $\kappa_2 > 0$ , depending only on  $\Omega_{\mathbb{H}}$  and on  $\mathcal{S}$ , such that the following holds.*

(1) *If  $\zeta_1$  is a vertex of  $\Omega_{\mathbb{H}}$  different from  $\zeta_0$ , then*

$$D(G_{b_0, \dots, b_m} \cdot \zeta_0) \geq \kappa_2 \cdot D(G_{b_0, \dots, b_m} \cdot \zeta_1).$$

(2) *If  $b_{m+1}$  satisfies  $b_{m+1} \neq \widehat{b_m}$  and  $\zeta_2$  is a vertex of  $\Omega_{\mathbb{H}}$  with  $G_{b_{m+1}} \cdot \zeta_2 \neq \zeta_0$ , then*

$$D(G_{b_0, \dots, b_m} \cdot \zeta_0) \geq \kappa_2 \cdot D(G_{b_0, \dots, b_m, b_{m+1}} \cdot \zeta_2).$$

*Proof.* We prove Part (1). Set  $G := G_{b_0, \dots, b_m}$ ,  $\zeta := G \cdot \zeta_0$ , and  $\zeta' := G \cdot \zeta_1$ . If  $\zeta' = \infty$  then the statement is trivially true. If  $D(G \cdot \zeta_1) \neq 0$ , let  $\zeta_0 = B_0 A_k \cdot \infty$  and  $\zeta_1 = B_1 A_j \cdot \infty$  as in Equation (3.2). Referring to Equation (1.3), let  $c, d$  be the entries of  $G$ . Let  $a_0, c_0$  and  $a_1, c_1$  be the entries of  $B_0 A_k$  and  $B_1 A_j$  respectively. We prove an upper bound for

$$\frac{D(G_{b_0, \dots, b_m} \cdot \zeta_1)}{D(G_{b_0, \dots, b_m} \cdot \zeta_0)} = \left| \frac{ca_1 + dc_1}{ca_0 + dc_0} \right|.$$

We cannot have  $c_0 = c_1 = 0$ , because  $\zeta_0 \neq \zeta_1$  and in particular  $\zeta_0, \zeta_1$  cannot be both equal to  $\infty$ . Moreover  $G \cdot \zeta_0, G \cdot \zeta_1$  are both different from  $\infty$ , thus condition  $c = 0$  implies  $c_0, c_1 \neq 0$ . Hence for  $c = 0$  Part (1) follows because the ratio above equals  $|c_1/c_0|$ , which varies in a finite set of values and is therefore bounded from above. If  $c, c_0, c_1 \neq 0$  then

$$\left| \frac{ca_1 + dc_1}{ca_0 + dc_0} \right| = \left| \frac{c_1}{c_0} \right| \cdot \left| \frac{(a_1/c_1) - (-d/c)}{(a_0/c_0) - (-d/c)} \right| = \left| \frac{c_1}{c_0} \right| \cdot \left| \frac{\zeta_1 - (G^{-1} \cdot \infty)}{\zeta_0 - (G^{-1} \cdot \infty)} \right|.$$

In this case Part (1) follows because  $|c_1/c_0|$  is bounded from above, and Lemma 3.3 gives a lower bound for the denominator of the second factor (the numerator is not bounded, but as it increases the ratio converges to 1). If  $c, c_0 \neq 0$  and  $c_1 = 0$  then Lemma 3.3 gives

$$\left| \frac{ca_1 + dc_1}{ca_0 + dc_0} \right| = \left| \frac{a_1}{c_0} \right| \cdot \left| \frac{1}{(a_0/c_0) - (-d/c)} \right| = \left| \frac{a_1}{c_0} \right| \cdot \left| \frac{1}{\zeta_0 - (G^{-1} \cdot \infty)} \right| \leq \left| \frac{a_1}{c_0 \cdot \kappa_1} \right|,$$

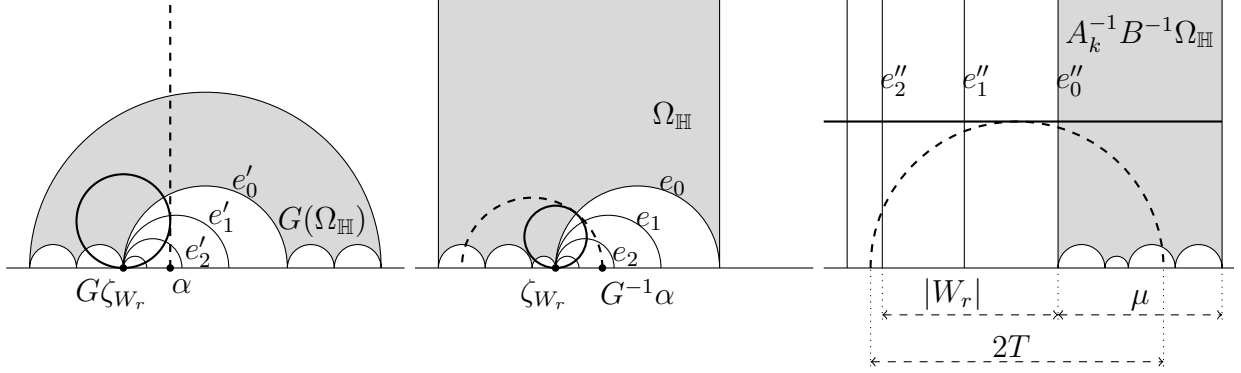


FIGURE 4. The  $r$ -th cuspidal word  $W_r = (a_0, a_1, a_2)$  of  $\alpha$  is the first cuspidal word of  $G^{-1} \cdot \alpha$ , where  $G = G_{W_0, \dots, W_{r-1}}$ . The vertex  $\zeta_{W_r}$  of  $\Omega_{\mathbb{H}}$  is common to the arcs  $e_0 = e_{a_0}$ ,  $e_1 := G_{a_0} e_{a_1}$  and  $e_2 := G_{a_0 a_1} e_{a_2}$ . The arcs  $e'_i = G e_i$  share the vertex  $G\zeta_{W_r}$ . The point  $\zeta_{W_r}$  is sent to  $\infty$ , and the arcs  $e_0, e_1, e_2$  are sent to the parallel vertical arcs  $e''_0, e''_1, e''_2$ . We have  $|W_r| = |\operatorname{Re}(e''_2) - \operatorname{Re}(e''_0)|$ .

and Part (1) follows observing that  $a_1/c_0$  varies in a finite set of values. Finally, if  $c, c_1 \neq 0$  and  $c_0 = 0$  then

$$\left| \frac{ca_1 + dc_1}{ca_0 + dc_0} \right| = \left| \frac{a_1}{a_0} - (-d/c) \frac{c_1}{a_0} \right| \leq \left| \frac{a_1}{a_0} \right| + |G^{-1} \cdot \infty| \left| \frac{c_1}{a_0} \right|.$$

In this case  $\zeta_0 = \infty$ , which is not an endpoint of  $[\widehat{b_m}]$ . Thus  $[\widehat{b_m}]$  is contained in the compact interval of  $\mathbb{R}$  delimited by the two parallel vertical segments of  $\Omega_{\mathbb{H}}$ . Hence  $|G^{-1} \cdot \infty|$  is uniformly bounded, because the pole  $G^{-1} \cdot \infty$  belongs to the closure of  $[\widehat{b_m}]$  (see proof of Lemma 3.3). Part (1) follows in this case too, and the proof is complete. Part (2) follows similarly, replacing  $\zeta_1$  by  $\zeta_* := G_{b_{m+1}} \cdot \zeta_2$  and observing that, since  $G_{b_{m+1}}$  varies in the finite set  $\{G_a : a \in \mathcal{A}\}$  then also the entries of  $X \in \operatorname{SL}(2, \mathbb{R})$  with  $G_{b_{m+1}} \cdot \zeta_2 = X \cdot \infty$  vary in a finite set. Moreover  $\zeta_0 \neq \zeta_*$ , and thus  $G \cdot \zeta_0 \neq G \cdot \zeta_*$ .  $\square$

**3.3. Proof of Theorem 3.1.** By a standard separation property of parabolic fixed points (a proof is in § A in [6]), there exists a constant  $S_0 > 0$ , depending only on  $\Gamma$  and on  $\mathcal{S}$ , such that for any  $G \cdot z_i$  and  $F \cdot z_j$  in  $\mathcal{P}_{\Gamma}$  with  $G \cdot z_i \neq F \cdot z_j$  we have

$$(3.8) \quad |G \cdot z_i - F \cdot z_j| \geq \frac{S_0}{D(G \cdot z_i)D(F \cdot z_j)}.$$

Let  $\alpha = [a_0, a_1, \dots]_{\mathbb{H}} = [W_0, W_1, \dots]_{\mathbb{H}}$  be the expansion of  $\alpha \in \mathbb{R}$  as in Equation (3.4).

**3.3.1. Proof of Equation (3.6).** Fix  $r \in \mathbb{N}$  with  $|W_r| > 0$ . Take  $k \in \{1, \dots, p\}$  and  $B \in \Gamma$  as in Equation (3.2), that is  $\zeta_{W_r} = BA_k \cdot \infty$ . As in Figure 4, let  $T > 0$  be such that the horoball

$$B_T := G_{W_0, \dots, W_{r-1}} BA_k (\{z \in \mathbb{H} : \operatorname{Im}(z) > T\})$$

is tangent at  $G_{W_0, \dots, W_{r-1}} \cdot \zeta_{W_r}$  with radius  $\rho(B_T) = |\alpha - G_{W_0, \dots, W_{r-1}} \cdot \zeta_{W_r}|$ . Equation (1.6) and Equation (1.7) give

$$D(G_{W_0, \dots, W_{r-1}} \cdot \zeta_{W_r})^2 \cdot |\alpha - G_{W_0, \dots, W_{r-1}} \cdot \zeta_{W_r}| = c^2(G_{W_0, \dots, W_{r-1}} BA_k) \cdot \frac{\operatorname{Diam}(B_T)}{2} = \frac{1}{2T}.$$

The geodesic in  $\mathbb{H}$  with endpoints  $(G_{W_0, \dots, W_{r-1}} BA_k)^{-1} \cdot \infty$  and  $(G_{W_0, \dots, W_{r-1}} BA_k)^{-1} \cdot \alpha$  is tangent to  $\{z \in \mathbb{H} : \text{Im}(z) > T\}$ . Equation (3.6) follows because Equation (3.3) gives

$$|W_r| \leq 2T \leq |W_r| + 2\mu.$$

**3.3.2. Proof of Equation (3.7).** Referring to § 3.2, let  $\zeta_0$  be the vertex of  $\Omega_{\mathbb{H}}$  and  $(b_0, \dots, b_m)$  be the admissible word such that the reduced form of the parabolic fixed point  $G \cdot z_k$  is

$$G \cdot z_k = G_{b_0, \dots, b_m} \cdot \zeta_0,$$

where  $\zeta_0$  is not an endpoint of  $e_{\widehat{b_m}}$  whenever  $(b_0, \dots, b_m)$  is not the empty word. Assume  $D(G \cdot z_k)^2 |\alpha - G \cdot z_k| < \epsilon_0$ , where the constant  $\epsilon_0 > 0$  will be determined later.

*Step (0).* Assume that  $(b_0, \dots, b_m)$  is the empty word, so that  $\zeta_0 = G \cdot z_k \neq \infty$ . Consider the extra assumption  $|W_0| > 0$  and  $\zeta_0 = \zeta_{W_0}$  on pairs  $(\alpha, \zeta_0)$ , where  $\zeta_{W_0} = \varphi^{-1}(\xi_{W_0})$  and  $\xi_{W_0}$  is the vertex of  $\Omega_{\mathbb{D}}$  associated to  $W_0$  as in § 2.4. Define  $\epsilon_0 > 0$  by

$$\epsilon_0 := \inf_{(\alpha, \zeta_0)} D(\zeta_0)^2 \cdot |\alpha - \zeta_0|,$$

where the infimum is taken over all pairs  $(\alpha, \zeta_0)$  not satisfying the extra assumption. With such  $\epsilon_0$ , the statement follows whenever  $(b_0, \dots, b_m)$  is the empty word.

*Step (1).* Now assume that  $(b_0, \dots, b_m)$  is not the empty word. Then  $G \cdot z_k$  is an interior point of  $[b_0, \dots, b_m]_{\mathbb{H}}$ . Let  $\zeta_1, \zeta_2$  be the endpoints of  $\widehat{[b_m]}$ , which are vertices of  $\Omega_{\mathbb{H}}$  different from  $\zeta_0$ . The endpoints of  $[b_0, \dots, b_m]_{\mathbb{H}}$  are  $\zeta'_i := G_{b_0, \dots, b_m} \cdot \zeta_i$  for  $i = 1, 2$ , according to Equation (2.10). Let  $N \geq -1$  be maximal with  $a_n = b_n$  for any  $n = 0, \dots, N$ , where the last condition is empty for  $N = -1$ , and where  $N \leq m$ . Observe that condition  $N \leq m - 1$  implies  $\alpha \notin [b_0, \dots, b_m]_{\mathbb{H}}$ , and therefore

$$\begin{aligned} |\alpha - G \cdot z_k| &\geq \min_{i=1,2} |\zeta'_i - G \cdot z_k| = \min_{i=1,2} |G_{b_0, \dots, b_m} \cdot \zeta_i - G_{b_0, \dots, b_m} \cdot \zeta_0| \\ &\geq \frac{S_0}{D(G_{b_0, \dots, b_m} \cdot \zeta_0)} \cdot \min_{i=1,2} \frac{1}{D(G_{b_0, \dots, b_m} \cdot \zeta_i)} \geq \frac{S_0 \kappa_2}{D(G_{b_0, \dots, b_m} \cdot \zeta_0)^2}, \end{aligned}$$

where the third inequality follows from Part (1) of Lemma 3.4 and the second from Equation (3.8). Therefore  $N = m$ , provided that  $\epsilon_0 < \kappa_2 S_0$ .

We proved  $[a_0, \dots, a_m]_{\mathbb{H}} = [b_0, \dots, b_m]_{\mathbb{H}}$ . Moreover  $G \cdot z_k$  does not belong to the interior of  $[a_0, \dots, a_m, a_{m+1}]_{\mathbb{H}}$ , since the latter is a subinterval of  $[b_0, \dots, b_m]_{\mathbb{H}}$  delimited by the image under  $G_{b_0, \dots, b_m}$  of two consecutive vertices of  $\Omega_{\mathbb{H}}$ . The same argument as in the first part of Step (1), which is left to the reader, shows that  $G \cdot z_k$  is an endpoint of  $[a_0, \dots, a_m, a_{m+1}]_{\mathbb{H}}$ .

*Step (2).* We show that  $G \cdot z_k = G_{b_0, \dots, b_m} \cdot \zeta_0$  is an endpoint of  $[a_0, \dots, a_{m+2}]_{\mathbb{H}}$ . Otherwise  $G \cdot z_k$  doesn't belong to the closure of  $[a_0, \dots, a_{m+2}]_{\mathbb{H}}$ . Since  $\alpha \in [a_0, \dots, a_{m+2}]_{\mathbb{H}}$  then

$$\begin{aligned} |\alpha - G \cdot z_k| &\geq |G_{b_0, \dots, b_m, a_{m+1}} \cdot \zeta_3 - G_{b_0, \dots, b_m} \cdot \zeta_0| \\ &\geq \frac{S_0}{D(G_{b_0, \dots, b_m} \cdot \zeta_0) D(G_{b_0, \dots, b_m, a_{m+1}} \cdot \zeta_3)} \geq \frac{S_0 \kappa_2}{D(G_{b_0, \dots, b_m} \cdot \zeta_0)^2}, \end{aligned}$$

where  $G_{b_0, \dots, b_m, a_{m+1}} \cdot \zeta_3$  is the endpoint of  $[a_0, \dots, a_{m+2}]_{\mathbb{H}}$  which is closest to  $G \cdot z_k$  and where  $\zeta_3$  is a vertex of  $\Omega_{\mathbb{H}}$  which is not an endpoint of  $e_{\widehat{a_{m+1}}}$ . We use Equation (3.8) and Part (2) of Lemma 3.4. The inequality is absurd by condition  $\epsilon_0 < \kappa_2 S_0$ .

*Step (3).* Let  $r$  be minimal such that  $(a_0, \dots, a_m)$  is an initial factor of  $W_0 * \dots * W_{r-1}$ . If  $(a_0, \dots, a_{m+2})$  is also an initial factor of  $W_0 * \dots * W_{r-1}$ , then  $G_{W_0, \dots, W_{r-1}} \cdot \xi_{W_{r-1}}$  is a common endpoint of the intervals  $[a_0, \dots, a_m]_{\mathbb{H}}$ ,  $[a_0, \dots, a_{m+1}]_{\mathbb{H}}$  and  $[a_0, \dots, a_{m+2}]_{\mathbb{H}}$ , according to Equation (2.13). Without loss of generality we have

$$G_{W_0, \dots, W_{r-1}} \cdot \xi_{W_{r-1}} = \inf[a_0, \dots, a_m]_{\mathbb{H}} = \inf[a_0, \dots, a_{m+1}]_{\mathbb{H}} = \inf[a_0, \dots, a_{m+2}]_{\mathbb{H}}.$$

The common endpoint is not  $G \cdot z_k$ , which belongs to the interior of  $[a_0, \dots, a_m]_{\mathbb{H}}$ . Thus Step (1) implies  $G \cdot z_k = \sup[a_0, \dots, a_{m+1}]_{\mathbb{H}}$ , which is absurd because  $G \cdot z_k$  is an endpoint of  $[a_0, \dots, a_{m+2}]_{\mathbb{H}}$  by Step (2). Hence  $W_0 * \dots * W_{r-1}$  is either equal to  $(a_0, \dots, a_m)$  or to  $(a_0, \dots, a_{m+1})$ . Moreover  $(a_{m+1}, a_{m+2})$  is a cuspidal word, because  $[a_0, \dots, a_{m+1}]_{\mathbb{H}}$  and  $[a_0, \dots, a_{m+2}]_{\mathbb{H}}$  share the endpoint  $G \cdot z_k$ .

- In case  $W_0 * \dots * W_{r-1} = (a_0, \dots, a_m)$  the word  $(a_{m+1}, a_{m+2})$  is an initial factor of  $W_r$ , that is  $|W_r| > 0$  and  $\zeta_0 = \zeta_{W_r}$ .

- In case  $W_0 * \dots * W_{r-1} = (a_0, \dots, a_{m+1})$  the word  $W' := (a_{m+1}) * W_r$  is also cuspidal (this is allowed by Remark 2.2). If  $|W_r| = 0$ , that is  $W_r = (a_{m+2})$ , then  $G \cdot z_k$  does not belong to the closure of  $[a_0, \dots, a_{m+3}]_{\mathbb{H}}$  and we get an absurd by

$$|\alpha - G \cdot z_k| \geq |G_{b_0, \dots, b_m} \cdot \zeta_0 - G_{b_0, \dots, b_m, a_{m+1}, a_{m+2}} \cdot \zeta_3| \geq \frac{S_0 \kappa_2}{D(G_{b_0, \dots, b_m} \cdot \zeta_0)^2},$$

where  $\zeta_3$  is a vertex of  $\Omega_{\mathbb{H}}$  and  $G_{b_0, \dots, b_m, a_{m+1}, a_{m+2}} \cdot \zeta_3$  is the endpoint of  $[a_0, \dots, a_{m+3}]_{\mathbb{H}}$  which is closest to  $G \cdot z_k$ . In the last inequality we reason as in Step (2), replacing  $\kappa_2$  by a smaller constant and extending Part (2) of Lemma 3.4 one more step, in order to compare  $D(G_{b_0, \dots, b_m} \cdot \zeta_0)$  and  $D(G_{b_0, \dots, b_m, a_{m+1}, a_{m+2}} \cdot \zeta_3)$ . Since  $W'$  is cuspidal with  $|W'| > 0$  we have  $\zeta_0 = \zeta_{W'}$ . But we have also  $\zeta_{W'} = G_{a_{m+1}} \cdot \zeta_{W_r}$ , which implies

$$G_{b_0, \dots, b_m} \cdot \zeta_0 = G_{a_0, \dots, a_m} \cdot G_{a_{m+1}} \cdot \zeta_{W_r} = G_{W_0, \dots, W_{r-1}} \cdot \zeta_{W_r}.$$

In both cases Equation (3.7) follows. The proof of Theorem 3.1 is complete.  $\square$

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