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ON GOOD APPROXIMATIONS AND BOWEN-SERIES EXPANSION

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Abstract. We consider the continued fraction expansion of real numbers under the action of a non-uniform lattice in $\mathrm{PSL}(2,\mathbb{R})$ and prove metric relations between the convergents and a natural geometric notion of good approximations.

1. Introduction

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the *upper half plane* and for $p/q \in \mathbb{Q}$ let $H_{p/q} \subset \mathbb{H}$ be the circle of diameter $1/q^2$ tangent at p/q. Set $H_\infty = \{z \in \mathbb{H} : \mathrm{Im}(z) > 1\}$ and consider the family $\{H_{p/q} : p/q \in \mathbb{Q}\cup\{\infty\}\}$ of *Ford circles*, which are the orbit of H_∞ under the projective action of the *modular group* $\mathrm{SL}(2,\mathbb{Z})$, that is the group of 2×2 matrices with coefficients a, b, c, d in $\mathbb{Z}$ (notation refers to Equation (1.3) below). Any two circles are either disjoint or tangent, and Figure 1 shows that for any irrational α there exist infinitely many $p/q \in \mathbb{Q}$ with $\alpha \in \Pi(H_{p/q})$, that is $|\alpha - p/q| < (1/2)q^{-2}$, where $\Pi(x + iy) := x$. This defines the sequence

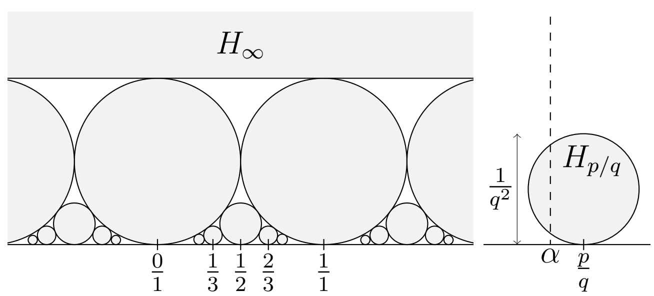

Figure 1. Balls $G(H_k)$, $k \in \mathbb{Z}$, tangent to $H_{p/q} = G(H_\infty)$, where $p/q = G \cdot \infty$.

of *geometric good approximations* of α as the sequence of p_n/q_n in $\mathbb{Q}$ with $\alpha \in \Pi(B_{p_n/q_n})$. The same sequence arises from the continued fraction expansion $\alpha = a_0 + [a_1, a_2, \dots]$ of α, indeed the *convergents* $p_n/q_n := a_0 + [a_1, \dots, a_n]$ satisfy:

$$|\alpha - p/q| < (1/2)q^{-2} \Rightarrow p/q = p_n/q_n \text{ for some } n \geq 1. \tag{1.1}$$

The first $n + 1$ *partial quotients* $a_1, \dots, a_{n+1}$ approximate α with error given by

$$\frac{1}{2 + a_{n+1}} \leq q_n^2 \cdot |\alpha - p_n/q_n| \leq \frac{1}{a_{n+1}} \text{ for any } n \in \mathbb{N}. \tag{1.2}$$

Rosen continued fractions where introduced in [9], in relation to diophantine approximation for *Hecke groups*, proving in particular an extension of Equation (1.2), which was later

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improved by [7]. Equation (1.1) was extended to Rosen continued fraction in [5], where the sharp constant replacing $1/2$ was obtained in [10]. In this note we consider diophantine approximation for a general non-uniform lattice Fuchsian group, in relation to the so-called *Bowen-Series expansion* of real numbers ([3]). Our Main Theorem 3.1 provides an extension of Equations (1.1) and (1.2) to this setting. This result is used in [6] to approximate the dimension of sets of *badly approximable points* by the dimension of dynamically defined regular Cantor sets. The study of the high part of *Markov and Lagrange spectra* is also a natural application, in the spirit of [11], [1] and [2]. In general, Theorem 3.1 applies to a large variety of problems in diophantine approximations, since it translates diophantine properties into ergodic properties of the Bowen-Series expansion.

Let $\mathrm{SL}(2,\mathbb{C})$ be the group of matrices

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.3}$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$, where any such G acts on points $z \in \mathbb{C} \cup \{\infty\}$ by

$$G \cdot z := \frac{az+b}{cz+d}. \tag{1.4}$$

Denote $a = a(G)$, $b = b(G)$, $c = c(G)$ and $d = d(G)$ the coefficients of G as in Equation (1.3). The group $\mathrm{SL}(2,\mathbb{R})$ of G with coefficients a, b, c, d in $\mathbb{R}$ acts by isometries on $\mathbb{H}$ via Equation (1.4), and inherits a topology from the identification with the set of $(a, b, c, d) \in \mathbb{R}^4$ with $ad - bc = 1$. A *Fuchsian group* is a discrete subgroup $\Gamma < \mathrm{SL}(2,\mathbb{R})$. Referring to [4], we say that Γ is a *lattice* if it has a *Dirichlet region* $\Omega \subset \mathbb{H}$ with finite hyperbolic area. If Ω is not compact, then the lattice Γ is said *non-uniform*. In this case the intersection $\overline{\Omega} \cap \partial\mathbb{H}$ is a finite non-empty set, whose elements are called the vertices *at infinity* of Ω. A point $z \in \mathbb{R} \cup \{\infty\}$ is a parabolic fixed point for Γ if there exists $P \in \Gamma$ parabolic with $P(z) = z$. Let $\mathcal{P}_\Gamma$ be the set of parabolic fixed points of Γ, which is equal to the orbit under Γ of the vertices at infinity of Ω. The set $\mathcal{P}_\Gamma$ is dense in $\mathbb{R}$. Two points z_1 and z_2 in $\mathcal{P}_\Gamma$ are *equivalent* if $z_2 = G(z_1)$ for some $G \in \Gamma$. Any non-uniform lattice Γ has a finite number $p \geq 1$ of equivalence classes $[z_1], \dots, [z_p]$ of parabolic fixed points, called the *cusps* of Γ.

Let Γ be a non-uniform lattice with $p \geq 1$ cusps. Fix a list $\mathcal{S} = (A_1, \dots, A_p)$ of elements $A_k \in \mathrm{SL}(2,\mathbb{R})$ such that the points

$$z_k = A_k \cdot \infty \quad \text{for} \quad k = 1, \dots, p \tag{1.5}$$

form a complete set $\{z_1, \dots, z_p\} \subset \mathcal{P}_\Gamma$ of inequivalent parabolic fixed points. A natural choice for $z_1, \dots, z_p$ is a maximal set of non-equivalent vertices at infinity of a fundamental domain. Any element of $\mathcal{P}_\Gamma$ has the form $G \cdot z_k$ for some $G \in \Gamma$ and $k = 1, \dots, p$. We have horoballs

$$B_k := A_k\big(\{z \in \mathbb{H} : \mathrm{Im}(z) > 1\}\big) \quad \text{with} \quad k = 1, \dots, p,$$

each B_k being tangent to $\mathbb{R} \cup \{\infty\}$ at z_k. We can have $A_k = \mathrm{Id}$, that is $z_k = \infty$ and $B_k = H_\infty$. Thus $G(B_k)$ is a ball tangent to the real line at $G \cdot z_k$ for any $G \in \Gamma$ with $G \cdot z_k \neq \infty$. These balls generalize Ford circles and we measure how their diameter shrinks to zero as G varies in Γ with the *denominator*

$$D(G \cdot z_k) := \begin{cases} 1/\sqrt{\mathrm{Diam}\big(G(B_k)\big)} & \text{if} \quad G \cdot z_k \neq \infty \\ 0 & \text{if} \quad G \cdot z_k = \infty. \end{cases}$$

Recall that for any $T > 0$ and any $G \in \mathrm{SL}(2,\mathbb{R})$ with $c(G) \neq 0$ we have

$$\mathrm{Diam}\Big(G\big(\{z \in \mathbb{H} : \mathrm{Im}(z) > T\}\big)\Big) = \frac{1}{Tc^2(G)}, \tag{1.6}$$

where we refer to the notation of Equation (1.3). Hence

$$D(G \cdot z_k) = |c(GA_k)| \quad \text{for any} \quad G \cdot z_k \in \mathcal{P}_\Gamma. \tag{1.7}$$

In [8], Patterson proves that there exists a constant $M = M(\Gamma, \mathcal{S}) > 0$ such that for any $Q > 0$ big enough and any $\alpha \in \mathbb{R}$ there exists $G \in \Gamma$ and $k \in \{1, \dots, p\}$ with

$$|\alpha - G \cdot z_k| \leq \frac{M}{D(G \cdot z_k)Q} \quad \text{and} \quad 0 < D(G \cdot z_k) \leq Q.$$

For $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, $\mathcal{S} = \{\mathrm{Id}\}$ and $M = 1$ Patterson's Theorem gives the Classical Dirichlet Theorem. In general, for any $\alpha \in \mathbb{R}$ we obtain infinitely many $G \cdot z_k \in \mathcal{P}_\Gamma$ with

$$|\alpha - G \cdot z_k| \leq \frac{M}{D^2(G \cdot z_k)}. \tag{1.8}$$

The *Bowen-Series expansion* ([3]) provides a coding $\alpha = [W_1, W_2, \dots]$ of a real number α, where for $r \geq 1$ we call *cuspidal words* the symbols W_r, which belong to a countable alphabet $\mathcal{W}$ (definitions are in § 2 and § 3). Cuspidal words $W \in \mathcal{W}$, that were introduced in [1] and [2], label a subset of elements $\{G_W : W \in \mathcal{W}\}$ of Γ, which generalize the role played in the theory of classical continued fractions by the matrices

$$\begin{pmatrix} 1 & a_{2k+1} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ a_{2k} & 1 \end{pmatrix} \quad \text{with} \quad a_{2k}, a_{2k+1} \in \mathbb{N}^* \text{ for any } k \in \mathbb{N}.$$

The coding is a continuous bijection $\Sigma \to \mathbb{R}$, where $\Sigma \subset \mathcal{W}^{\mathbb{N}}$ is a subshift with *aperiodic transition matrix* (see [6]). For $r \geq 1$ the first r symbols in the expansion of $\alpha = [W_1, W_2, \dots]$ define $\zeta_r = \zeta_r(W_1, \dots, W_r) \in \mathcal{P}_\Gamma$, see Equation (3.5). This extends the classical notion of convergents p_n/q_n of α. The main result of this note is Theorem 3.1 in § 3. We give the following preliminary statement (see also Remark 3.2).

Main Theorem (Theorem 3.1)**.** *Fix $\alpha = [W_1, W_2, \dots]$ which is not an element of $\mathcal{P}_\Gamma$. The convergents $\zeta_r = \zeta_r(W_1, \dots, W_r)$ approximate α with error given by an analogue of Equation* (1.2)*. Moreover there exists a constant $\epsilon_0 > 0$ such that any $G \cdot z_k \in \mathcal{P}_\Gamma$ satisfying Equation* (1.8) *with $M = \epsilon_0$ belongs to the sequence $(\zeta_r)_{r \geq 1}$.*

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2. The Bowen-Series expansion

We follow § 3 in [6], which is itself based on § 2.4 in [1] and § 2 in [2]. The original construction is in [3], where it is defined a *Markov map*, which is *orbit equivalent* to the action of a given finitely generated Fuchsian group of the first kind. In our setting such Markov map corresponds to an *acceleration* of the map in Equation (2.7) below. This § 2 describes the coding by *cusidal words*. The same description appears in [6], where it is followed by the study of the combinatorial and metric properties of the subshift related to the coding. Consider the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and the map

$$\varphi : \mathbb{H} \to \mathbb{D} \quad ; \quad \varphi(z) := \frac{z - i}{z + i}. \tag{2.1}$$

The conjugate of $\mathrm{SL}(2,\mathbb{R})$ under φ is the group $\mathrm{SU}(1,1)$ of $F \in \mathrm{GL}(2,\mathbb{C})$ with

$$F = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \quad \text{with} \quad |\alpha|^2 - |\beta|^2 = 1. \tag{2.2}$$

Denote $\alpha = \alpha(F)$ and $\beta = \beta(F)$ the coefficients of F as in Equation (2.2).

2.1. **Isometric circles.** Consider $F \in \mathrm{SU}(1,1)$ and $\alpha = \alpha(F)$, $\beta = \beta(F)$ as in Equation (2.2). Assume $\beta \neq 0$ and let $\omega_F := -\overline{\alpha}/\beta$ be the pole of F. The *isometric circle* I_F of F is the euclidean circle centered at ω_F with radius $\rho(F) := |\beta|^{-1}$, that is

$$I_F := \{\xi \in \mathbb{C} : |\xi - \omega_F|\}.$$

We have $F(I_F) = I_{F^{-1}}$, where $\rho(F) = \rho(F^{-1})$ and $|\omega_{F^{-1}}| = |\omega_F|$. See Theorem 3.3.2 in [4]. Moreover $I_F \cap \mathbb{D}$ is a geodesic of $\mathbb{D}$ for any $F \in \mathrm{SU}(1,1)$, by Theorem 3.3.3 in [4]. Denote U_F the disc in $\mathbb{C}$ with $\partial U_F = I_F$, that is the interior of I_F.

2.2. **Labelled ideal polygon.** Let $\Gamma \subset \mathrm{SU}(1,1)$ be a non-uniform lattice. According to [12], there exist a free subgroup $\Gamma_0 < \Gamma$ with finite index $[\Gamma_0 : \Gamma] < +\infty$. See also § 2.2 of [6]. In particular $\beta(F) \neq 0$ for any $F \in \Gamma_0$, referring to Equation (2.2), so that the isometric circle I_F and the disc U_F introduced in § 2.1 are defined. The origin $0 \in \mathbb{D}$ is not a fixed point of any $F \in \Gamma_0$ and Theorem 3.3.5 in [4] implies that the set

$$\Omega_0 := \overline{\mathbb{D} \setminus \bigcup_{F \in \Gamma_0} U_F} \tag{2.3}$$

is a Dirichlet region for Γ_0. Recall from [4] that Ω_0 is an hyperbolic polygon with an even number $2d$ of sides, denoted by the letter s, and with $2d$ vertices, denoted by the letter ξ (see also § 2.4 of [6]). All vertices of Ω_0 belong to $\partial\mathbb{D}$, because Γ_0 is free. Any side s is a complete geodesic in $\mathbb{D}$ and for any such s there exists an unique $F \in \Gamma$ such that $F(s)$ is another side of Ω_0 with $F(s) \neq s$. The sides s and $F(s)$ are thus *paired*. See Figure 2. The set of pairings generates Γ_0, according to Theorem 3.5.4 in [4]. For a convenient labelling, consider two finite alphabets $\mathcal{A}_0$ and $\widehat{\mathcal{A}_0}$, both with d elements, and a map

$$\iota : \mathcal{A}_0 \cup \widehat{\mathcal{A}_0} \to \mathcal{A}_0 \cup \widehat{\mathcal{A}_0} \quad \text{with} \quad \iota^2 = \mathrm{Id} \quad \text{and} \quad \iota(\mathcal{A}_0) = \widehat{\mathcal{A}_0},$$

that is an involution of $\mathcal{A}_0 \cup \widehat{\mathcal{A}_0}$ which exchanges $\mathcal{A}_0$ with $\widehat{\mathcal{A}_0}$. Set $\mathcal{A} := \mathcal{A}_0 \cup \widehat{\mathcal{A}_0}$ and for any $a \in \mathcal{A}$, denote $\widehat{a} := \iota(a)$.

- Label the sides of Ω_0 by the letters in $\mathcal{A}$, so that for any $a \in \mathcal{A}$ the sides s_a and $s_{\widehat{a}}$ are those which are paired by the action of Γ_0.
- For any pair of sides s_a and $s_{\widehat{a}}$ as above, let F_a be the unique element of Γ_0 such that

$$F_a(s_{\widehat{a}}) = s_a. \tag{2.4}$$

- For any $a \in \mathcal{A}$ we have $F_{\widehat{a}} = F_a^{-1}$, and the latter form a set of generators for Γ_0.

In the following we denote $\Omega_{\mathbb{D}} := \Omega_0 \subset \mathbb{D}$ the labelled ideal polygon defined above and $\Omega_{\mathbb{H}} := \varphi^{-1}(\Omega_{\mathbb{D}}) \subset \mathbb{H}$ its preimage under the map in Equation (2.1).

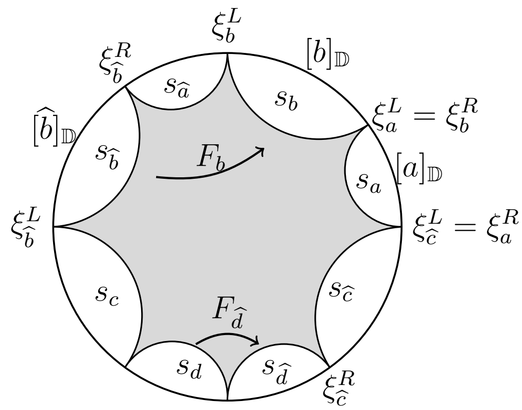

FIGURE 2. Ideal polygon labelled by $\mathcal{A} = \{a, b, c, d, \widehat{a}, \widehat{b}, \widehat{c}, \widehat{d}\}$.

2.3. **The boundary map.** Parametrize arcs $J \subset \partial\mathbb{D}$ by $t \mapsto e^{-it}$ with $t \in (x, y)$. Set $\inf J := e^{-ix}$ and $\sup J := e^{-iy}$. We say that J is *right open* if $\inf J \in J$ and $\sup J \notin J$. Let $\Gamma_0 < \Gamma$ be a finite index free subgroup and $\Omega_{\mathbb{D}}$ be an ideal polygon for Γ_0 labelled by $\mathcal{A}$, as in § 2.2.

For $a \in \mathcal{A}$ let F_a be the map in Equation (2.4). Let I_{F_a} be the isometric circle of F_a and U_{F_a} be its interior, as in § 2.1. Recall that $s_{\widehat{a}} = I_{F_a} \cap \mathbb{D}$ and $s_a = I_{F_{\widehat{a}}} \cap \mathbb{D}$. Let $[a]_{\mathbb{D}}$ be the right open arc of $\partial\mathbb{D}$ cut by the side s_a, that is

$$[a]_{\mathbb{D}} := U_{F_{\widehat{a}}} \cap \partial\mathbb{D}.$$

Set $\xi_a^L := \inf[a]_{\mathbb{D}}$ and $\xi_a^R := \sup[a]_{\mathbb{D}}$. Figure 2 shows examples of such notation. In order to take account of the cyclic order in $\partial\mathbb{D}$ of the arcs $[a]_{\mathbb{D}}$, fix $a_0 \in \mathcal{A}$ and define a map $o : \mathcal{A} \to \mathbb{Z}/2d\mathbb{Z}$ setting $o(a_0) := 0$ and

$$o(b) = o(a) + 1 \mod 2d \quad \text{for} \quad a, b \in \mathcal{A} \quad \text{with} \quad \xi_a^R = \xi_b^L. \tag{2.5}$$

We have $F_a(I_{F_a}) = I_{F_{\widehat{a}}}$ for any $a \in \mathcal{A}$, thus F_a sends the complement of $[\widehat{a}]_{\mathbb{D}}$ to $[a]_{\mathbb{D}}$, that is

$$F_a\big(\partial\mathbb{D} \setminus [\widehat{a}]_{\mathbb{D}}\big) = [a]_{\mathbb{D}}. \tag{2.6}$$

The *Bowen-Series map* is the map $\mathcal{BS} : \partial\mathbb{D} \to \partial\mathbb{D}$ defined by

$$\mathcal{BS}(\xi) := F_a^{-1}(\xi) \quad \text{iff} \quad \xi \in [a]_{\mathbb{D}}. \tag{2.7}$$

The *boundary expansion* of a point $\xi \in \partial\mathbb{D}$ is the sequence $(a_k)_{k\in\mathbb{N}}$ of letters $a_k \in \mathcal{A}$ with

$$\mathcal{BS}^k(\xi) \in [a_k]_{\mathbb{D}} \quad \text{for any } k \in \mathbb{N}. \tag{2.8}$$

By Equation (2.6), any such sequence satisfies the so-called *no backtracking Condition*:

$$a_{k+1} \neq \widehat{a_k} \text{ for any } k \in \mathbb{N}. \tag{2.9}$$

A finite word $(a_0, \ldots, a_n)$ satisfying Condition (2.9) corresponds to a *factor* of the map $\mathcal{BS} : \partial\mathbb{D} \to \partial\mathbb{D}$, that is a finite concatenation $F_{a_n}^{-1} \circ \cdots \circ F_{a_0}^{-1}$ arising from iterations of $\mathcal{BS}$. We call *admissible word*, or simply *word*, any finite or infinite word in the letters of $\mathcal{A}$ satisfying Condition (2.9). We use the notation

$$F_{a_0,\ldots,a_n} := F_{a_0} \circ \cdots \circ F_{a_n} \in \Gamma_0.$$

Define the right open arc $[a_0,\dots,a_n]_{\mathbb{D}}$ as the set of $\xi\in\partial\mathbb{D}$ such that $\mathcal{BS}^k(\xi)\in[a_k]_{\mathbb{D}}$ for any $k=0,\dots,n$, that is

$$[a_0,\dots,a_n]_{\mathbb{D}} := F_{a_0,\dots,a_{n-1}}[a_n]_{\mathbb{D}} = F_{a_0,\dots,a_n}\big(\partial\mathbb{D}\setminus[\widehat{a_n}]_{\mathbb{D}}\big). \tag{2.10}$$

Two such arcs satisfy $[a_0,\dots,a_n]_{\mathbb{D}}\subset[b_0,\dots,b_m]_{\mathbb{D}}$ if and only if $m\geq n$ and $a_k=b_k$ for any $k=0,\dots,n$. It is easy to see that $[a_0,\dots,a_n]_{\mathbb{D}}$ shrinks to a point as $n\to\infty$. See Lemma 3.1 in [6] for a proof. A sequence $(a_k)_{k\in\mathbb{N}}$ satisfying Condition (2.9) corresponds to a point $\xi=[a_0,a_1,\dots]_{\mathbb{D}}$ in $\partial\mathbb{D}$, where we use the notation

$$[a_0,a_1,\dots]_{\mathbb{D}} := \bigcap_{n\in\mathbb{N}}[a_0\dots,a_n]_{\mathbb{D}}.$$

Conversely, if $(a_k)_{k\in\mathbb{N}}$ is the boundary expansion of $\xi\in\partial\mathbb{D}$, then $\xi=[a_0,a_1,\dots]_{\mathbb{D}}$. The Bowen-Series map $\mathcal{BS}$ is the shift on the space of admissible infinite words.

2.4. **Cuspidal words.** Consider the map $o:\mathcal{A}\to\mathbb{Z}/2d\mathbb{Z}$ in Equation (2.5). The definitions in § 2.3 easily imply Lemma 2.1 below. See Lemma 3.2 in [6] for a proof.

Lemma 2.1. *Let $(a_0,\dots,a_n)$ be a word satisfying Condition (2.9) with $n\geq 1$ and $a_0=a_n$. The map $F_{a_0,\dots,a_{n-1}}$ is a parabolic element of Γ_0 fixing $\xi^R_{a_0}$ if and only if*

$$o(a_{k+1}) = o(\widehat{a_k})-1 \quad \textit{for any} \quad k=0,\dots,n-1. \tag{2.11}$$

The map $F_{a_0,\dots,a_{n-1}}$ is a parabolic element of Γ_0 fixing $\xi^L_{a_0}$ if and only if

$$o(a_{k+1}) = o(\widehat{a_k})+1 \quad \textit{for any} \quad k=0,\dots,n-1. \tag{2.12}$$

Let $W=(a_0,\dots,a_n)$ be an admissible word. We say that W is a *cuspidal word* if it is the initial factor of an admissible word $(a_0,\dots,a_m)$ with $m\geq n$ such that $F_{a_0,\dots,a_m}$ is a parabolic element of Γ_0 fixing a vertex of $\Omega_{\mathbb{D}}$.

- If $n\geq 1$ and Equation (2.11) is satisfied, we say that W is a *right cuspidal word*. In this case we define its type by $\varepsilon(W):=R$ and we set $\xi_W:=\xi^R_{a_0}$.
- If $n\geq 1$ and Equation (2.12) is satisfied, we say that W is a *left cuspidal word*. In this case we define its type by $\varepsilon(W):=L$ and we set $\xi_W:=\xi^L_{a_0}$.
- If $n=0$, that is $W=(a_0)$ has just one letter, the type $\varepsilon(W)$ is not defined. We set by convention $\xi_W:=\xi^R_{a_0}$.

If $W=(a_0,\dots,a_n)$ is cuspidal with $n\geq 1$, Lemma 2.1 implies $\xi^{\varepsilon(W)}_{a_k}=F_{a_k}\cdot\xi^{\varepsilon(W)}_{a_{k+1}}$ for any $k=0,\dots,n-1$ and it follows

$$\xi_W = \partial[a_0]_{\mathbb{D}}\cap\partial[a_0,a_1]_{\mathbb{D}}\cap\dots\cap\partial[a_0,\dots,a_n]_{\mathbb{D}}, \tag{2.13}$$

that is the $n+1$ arcs above share ξ_W as common endpoint (see also § 2.4 in [2] and § 4.3 in [1]). A sequence $(a_n)_{n\in\mathbb{N}}$ is said *cuspidal* if any initial factor $(a_0,\dots,a_n)$ with $n\in\mathbb{N}$ is a cuspidal word, and *eventually cuspidal* if there exists $k\in\mathbb{N}$ such that $(a_{n+k})_{n\in\mathbb{N}}$ is a cuspidal sequence.

2.5. **The cuspidal acceleration.** If $W=(b_0,\dots,b_m)$ and $W'=(a_0,\dots,a_n)$ are words with $a_0\neq\widehat{b_m}$, define the word $W*W':=(b_0,\dots,b_m,a_0,\dots,a_n)$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence satisfying Condition (2.9) and not eventually cuspidal.

Initial step: Set $n(0):=0$. Let $n(1)\in\mathbb{N}$ be the maximal integer $n(1)\geq 1$ such that $(a_0,\dots,a_{n(1)-1})$ is cuspidal, then set $W_0:=(a_0,\dots,a_{n(1)-1})$.

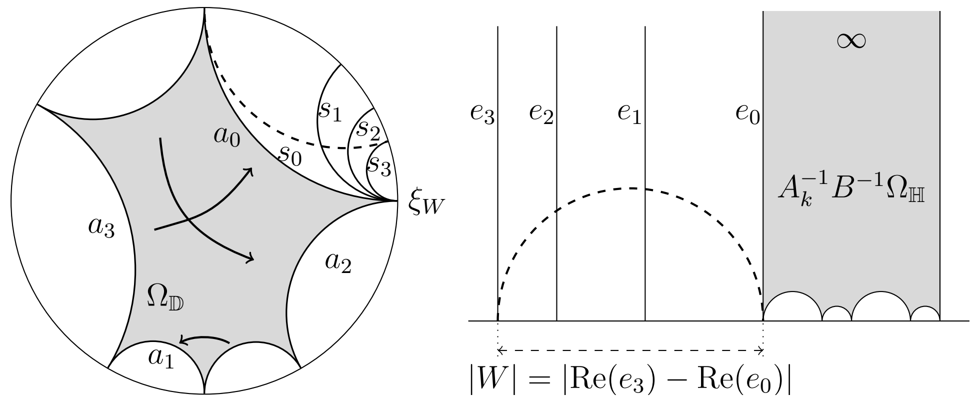

FIGURE 3. Geometric length $|W|$ of a right cuspidal word $W = (a_0, a_1, a_2, a_3)$. The arrows inside $\Omega_{\mathbb{D}}$ represent the action of $F_{a_0}, F_{a_1}, F_{a_2}$. The arcs $s_0 := s_{a_0}$, $s_1 := F_{a_0}(s_{a_1})$, $s_2 := F_{a_0,a_1}(s_{a_2})$ and $s_3 := F_{a_0,a_1,a_2}(s_{a_3})$ share the common vertex ξ_W, which is sent to ∞ under the map $A_k^{-1}B^{-1}\varphi^{-1}$. Thus the arcs s_0, s_1, s_2, s_3 in $\mathbb{D}$ are sent to parallel vertical arcs $e_i := \varphi^{-1}(s_i)$ in $\mathbb{H}$.

Recursive step: Fix $r \geq 1$ and assume that the instants $n(0) < \cdots < n(r)$ and the cuspidal words $W_0, \ldots, W_{r-1}$ are defined. Define $n(r+1) \geq n(r)+1$ as the maximal integer such that $[a_{n(r)}, \ldots, a_{n(r+1)-1}]$ is cuspidal, then set

$$W_r := (a_{n(r)}, \ldots, a_{n(r+1)-1}).$$

The sequence of words $(W_r)_{r\in\mathbb{N}}$ is called the *cuspidal decomposition* of $(a_n)_{n\in\mathbb{N}}$. We have of course $a_0, a_1, a_2 \cdots = W_0 * W_1 * \ldots$. For any $\xi = [a_0, a_1, \ldots]_{\mathbb{D}}$, if $(W_r)_{r\in\mathbb{N}}$ is the cuspidal decomposition of $(a_n)_{n\in\mathbb{N}}$, we write

$$\xi = [a_0, a_1, \ldots]_{\mathbb{D}} = [W_0, W_1, \ldots]_{\mathbb{D}}. \tag{2.14}$$

Remark 2.2. *If $W_{r-1} := (a_{n(r-1)}, \ldots, a_{n(r)-1})$ and $W_r := (a_{n(r)}, \ldots, a_{n(r+1)-1})$ are two consecutive cuspidal words in the cuspidal decomposition of a sequence $(a_n)_{n\in\mathbb{N}}$ satisfying Condition* (2.9)*, then the word $(a_{n(r)-1}, a_{n(r)}, \ldots, a_{n(r+1)-1})$ can be cuspidal.*

3. The main Theorem 3.1

The tools in § 2 induce a boundary expansion on $\mathbb{R}$. Let $\Gamma_0 < \Gamma$ be the free subgroup and $\Omega_{\mathbb{D}} \subset \mathbb{D}$ be the ideal polygon in § 2.2. Recall that $\mathcal{P}_{\Gamma_0} = \Gamma_0(\Omega_{\mathbb{D}} \cap \partial\mathbb{D})$ by Theorem 4.2.5 in [4]. Since Γ_0 has finite index in Γ then the two groups have the same set of parabolic fixed points, that is

$$\mathcal{P}_{\Gamma} = \Gamma_0(\Omega_{\mathbb{D}} \cap \partial\mathbb{D}). \tag{3.1}$$

3.1. **Geometric length of cuspidal words and main statement.** Fix $\mathcal{S} = (A_1, \ldots, A_p)$ as in Equation (1.5). Let $\Omega_{\mathbb{H}} := \varphi^{-1}(\Omega_{\mathbb{D}}) \subset \mathbb{H}$ be the pre-image of $\Omega_{\mathbb{D}}$ under the map in Equation (2.1). Any vertex ξ of $\Omega_{\mathbb{D}}$ corresponds to an unique vertex $\zeta = \varphi^{-1}(\xi)$ of $\Omega_{\mathbb{H}}$. For any such vertex ζ consider $B \in \Gamma$ and $k \in \{1, \ldots, p\}$ with

$$\zeta = BA_k \cdot \infty \tag{3.2}$$

Any side s_a of $\Omega_{\mathbb{D}}$ corresponds to an unique side $e_a := \varphi^{-1}(s_a)$ of $\Omega_{\mathbb{H}}$, where $a \in \mathcal{A}$. If $BA_k \cdot \infty = B'A_j \cdot \infty$, then $j = k$. Moreover $B' = BP$, where $P \in \Gamma$ is parabolic fixing

$A_k \cdot \infty$, where we recall that in any Fuchsian group Γ with cusps, if $G \in \Gamma$ satisfies $G \cdot \zeta = \zeta$ for some $\zeta \in \mathcal{P}_\Gamma$, then G is parabolic. Hence the map $z \mapsto A_k^{-1} P A_k(z)$ is an horizontal translation in $\mathbb{H}$. If s and s' are geodesics in $\mathbb{D}$ having ξ as common endpoint, then their pre-images in $\mathbb{H}$ under $\varphi \circ B \circ A_k$ are parallel vertical half lines whose distance does not depend on the choice of B in Equation (3.2). We have a well defined positive real number

$$\Delta(s, s', \xi) := \left|\mathrm{Re}\big(A_k^{-1} B^{-1} \varphi^{-1}(s)\big) - \mathrm{Re}\big(A_k^{-1} B^{-1} \varphi^{-1}(s')\big)\right|.$$

Fix a cuspidal word $W = (a_0, \dots, a_n)$ and the vertex ξ_W of $\Omega_\mathbb{D}$ associated to W in § 2.4. For $n \geq 1$ Equation (2.13) implies that the geodesics $s_{a_0}, F_{a_0}(s_{a_1}), \dots, F_{a_0,\dots,a_{n-1}}(s_{a_n})$ all have ξ_W as common endpoint. See Figure 3. Define the *geometric length* $|W| \geq 0$ of W as

$$|W| := \begin{cases} \Delta\big(s_{a_0}, F_{a_0,\dots,a_{n-1}}(s_{a_n}), \xi_W\big) & \text{if } n \geq 1 \\ 0 & \text{if } n = 0. \end{cases} \tag{3.3}$$

For $a \in \mathcal{A}$ set $G_a = \varphi^{-1} \circ F_a \circ \varphi$. Set $G_{a_0,\dots,a_n} := G_{a_0} \circ \dots \circ G_{a_n}$ for any word $(a_0, \dots, a_n)$ and $G_{W_0,\dots,W_r} = G_{a_0,\dots,a_n}$ if $(a_0, \dots, a_n) = W_0 * \dots * W_r$. Define the interval

$$[a_0, \dots, a_n]_\mathbb{H} := \varphi^{-1}\big([a_0, \dots, a_n]_\mathbb{D}\big) = G_{a_0,\dots,a_n}\big(\partial\mathbb{H} \setminus [\widehat{a_n}]_\mathbb{H}\big).$$

Set $[a_0, a_1, \dots]_\mathbb{H} := \varphi^{-1}\big([a_0, a_1, \dots]_\mathbb{D}\big)$, that is encode $\alpha \in \mathbb{R}$ by the same cutting sequence as $\varphi(\alpha) \in \mathbb{D}$. If $(a_n)_{n\in\mathbb{N}}$ has cuspidal decomposition $(W_r)_{r\in\mathbb{N}}$, Equation (2.14) becomes

$$\alpha = [W_0, W_1, \dots]_\mathbb{H} := [a_0, a_1, \dots]_\mathbb{H}. \tag{3.4}$$

For $r \in \mathbb{N}$ let W_r be the r-th cuspidal word. Set $\zeta_{W_r} := \varphi^{-1}(\xi_{W_r})$. The convergents of α are

$$\zeta_r := G_{W_0,\dots,W_{r-1}} \cdot \zeta_{W_r} \quad ; \quad r \in \mathbb{N}. \tag{3.5}$$

For $k = 1, \dots, p$ let $\mu_k > 0$ be such that the primitive parabolic element $P_k \in A_k \Gamma A_k^{-1}$ fixing ∞ acts by $P_k(z) = z + \mu_k$. Set $\mu := \max\{\mu_1, \dots, \mu_p\}$.

Theorem 3.1 (Main Theorem)**.** *For any $r \in \mathbb{N}$ with $|W_r| > 0$ we have*

$$\frac{1}{|W_r| + 2\mu} \leq D(G_{W_0,\dots,W_{r-1}} \cdot \zeta_{W_r})^2 \cdot |\alpha - G_{W_0,\dots,W_{r-1}} \cdot \zeta_{W_r}| \leq \frac{1}{|W_r|}. \tag{3.6}$$

Moreover there exists $\epsilon_0 > 0$ depending only on $\Omega_\mathbb{D}$ and on $\mathcal{S}$, such that for any $G \in \Gamma$ and $k = 1, \dots, p$ with $D(G \cdot z_k) \neq 0$ the condition

$$D(G \cdot z_k)^2 \cdot |\alpha - G \cdot z_k| < \epsilon_0$$

implies that there exists some $r \in \mathbb{N}$ such that

$$G \cdot z_k = G_{W_0,\dots,W_{r-1}} \cdot \zeta_{W_r} \quad \text{where} \quad |W_r| > 0. \tag{3.7}$$

Remark 3.2. *Equation* (3.6) *holds for any choice of $\mathcal{S}$ as in Equation* (1.5)*, and this follows because geometric length and denominators satisfy a form of equivariance under the choice of $\mathcal{S}$. Equation* (3.7) *shows that, for any choice of the subgroup Γ_0, all good enough approximations of a given α belong to the sequence of its convergents.*

3.2. **Reduced form of parabolic fixed points.** Fix $G \cdot z_k \in \mathcal{P}_\Gamma$. Recall Equation (3.1) and write elements of Γ_0 in the generators $\{G_a : a \in \mathcal{A}\}$. There exists an unique admissible word $b_0, \dots, b_m$ and a vertex ζ of $\Omega_{\mathbb{H}}$ which is not an endpoint of $e_{\widehat{b_m}}$ such that

$$G \cdot z_k = G_{b_0,\dots,b_m} \cdot \zeta.$$

The representation above is called the *reduced form* of the parabolic fixed point $G \cdot z_k$. In the next Lemmas 3.3 and 3.4, let $(b_0, \dots, b_m)$ be a non-trivial admissible word and let ζ_0 be a vertex of $\Omega_{\mathbb{H}}$ which is not an endpoint of $e_{\widehat{b_m}}$, so that $G_{b_0,\dots,b_m} \cdot \zeta_0$ is a parabolic fixed point written in its reduced form and different from ∞.

Lemma 3.3. *There exists a constant $\kappa_1 > 0$, depending only on $\Omega_{\mathbb{H}}$, such that*

$$\left|\zeta_0 - G^{-1}_{b_0,\dots,b_m} \cdot \infty\right| \geq \kappa_1,$$

that is the vertex ζ_0 and the pole of $G_{b_0,\dots,b_m}$ stay at distance uniformly bounded from below.

Proof. We have $G_{b_0,\dots,b_m}\left(\mathbb{R} \setminus [\widehat{b_m}]_{\mathbb{H}}\right) = [b_0, \dots, b_m]_{\mathbb{H}}$ By Equation (2.10). Since ∞ does not belong to the interior of $[b_0, \dots, b_m]_{\mathbb{H}}$ then the pole of $G_{b_0,\dots,b_m}$ belongs to the closure of $[\widehat{b_m}]_{\mathbb{H}}$. The Lemma follows because ζ_0 is a vertex of $\Omega_{\mathbb{H}}$ different from the endpoints of $e_{\widehat{b_m}}$. □

Lemma 3.4. *There exists a constant $\kappa_2 > 0$, depending only on $\Omega_{\mathbb{H}}$ and on $\mathcal{S}$, such that the following holds.*

(1) *If ζ_1 is a vertex of $\Omega_{\mathbb{H}}$ different from ζ_0, then*

$$D(G_{b_0,\dots,b_m} \cdot \zeta_0) \geq \kappa_2 \cdot D(G_{b_0,\dots,b_m} \cdot \zeta_1).$$

(2) *If b_{m+1} satisfies $b_{m+1} \neq \widehat{b_m}$ and ζ_2 is a vertex of $\Omega_{\mathbb{H}}$ with $G_{b_{m+1}} \cdot \zeta_2 \neq \zeta_0$, then*

$$D(G_{b_0,\dots,b_m} \cdot \zeta_0) \geq \kappa_2 \cdot D(G_{b_0,\dots,b_m,b_{m+1}} \cdot \zeta_2).$$

Proof. We prove Part (1). Set $G := G_{b_0,\dots,b_m}$, $\zeta := G \cdot \zeta_0$, and $\zeta' := G \cdot \zeta_1$. If $\zeta' = \infty$ then the statement is trivially true. If $D(G \cdot \zeta_1) \neq 0$, let $\zeta_0 = B_0 A_k \cdot \infty$ and $\zeta_1 = B_1 A_j \cdot \infty$ as in Equation (3.2). Referring to Equation (1.3), let c, d be the entries of G. Let a_0, c_0 and a_1, c_1 be the entries of $B_0 A_k$ and $B_1 A_j$ respectively. We prove an upper bound for

$$\frac{D(G_{b_0,\dots,b_m} \cdot \zeta_1)}{D(G_{b_0,\dots,b_m} \cdot \zeta_0)} = \left|\frac{ca_1 + dc_1}{ca_0 + dc_0}\right|.$$

We cannot have $c_0 = c_1 = 0$, because $\zeta_0 \neq \zeta_1$ and in particular ζ_0, ζ_1 cannot be both equal to ∞. Moreover $G \cdot \zeta_0$, $G \cdot \zeta_1$ are both different from ∞, thus condition $c = 0$ implies $c_0, c_1 \neq 0$. Hence for $c = 0$ Part (1) follows because the ratio above equals $|c_1/c_0|$, which varies in a finite set of values and is therefore bounded from above. If $c, c_0, c_1 \neq 0$ then

$$\left|\frac{ca_1 + dc_1}{ca_0 + dc_0}\right| = \left|\frac{c_1}{c_0}\right| \cdot \left|\frac{(a_1/c_1) - (-d/c)}{(a_0/c_0) - (-d/c)}\right| = \left|\frac{c_1}{c_0}\right| \cdot \left|\frac{\zeta_1 - (G^{-1} \cdot \infty)}{\zeta_0 - (G^{-1} \cdot \infty)}\right|.$$

In this case Part (1) follows because $|c_1/c_0|$ is bounded from above, and Lemma 3.3 gives a lower bound for the denominator of the second factor (the numerator is not bounded, but as it increases the ratio converges to 1). If $c, c_0 \neq 0$ and $c_1 = 0$ then Lemma 3.3 gives

$$\left|\frac{ca_1 + dc_1}{ca_0 + dc_0}\right| = \left|\frac{a_1}{c_0}\right| \cdot \left|\frac{1}{(a_0/c_0) - (-d/c)}\right| = \left|\frac{a_1}{c_0}\right| \cdot \left|\frac{1}{\zeta_0 - (G^{-1} \cdot \infty)}\right| \leq \left|\frac{a_1}{c_0 \cdot \kappa_1}\right|,$$

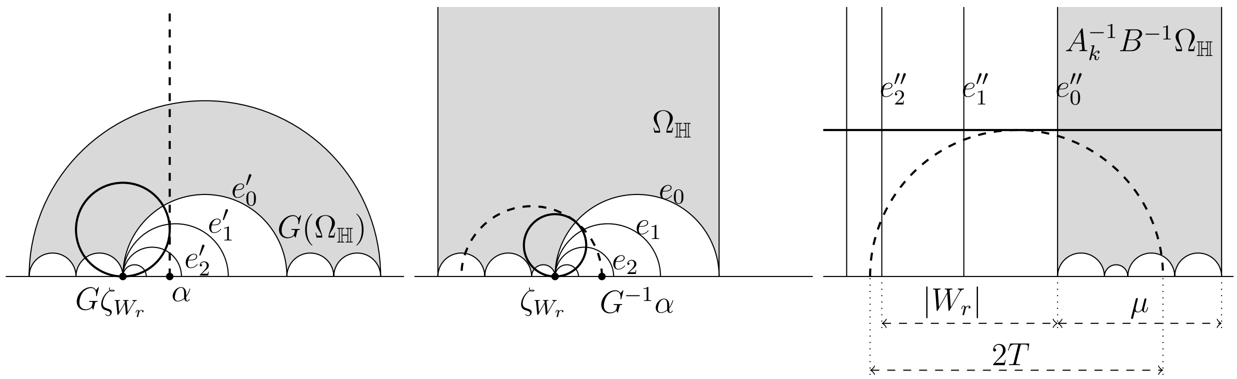

FIGURE 4. The r-th cuspidal word $W_r = (a_0, a_1, a_2)$ of α is the first cuspidal word of $G^{-1} \cdot \alpha$, where $G = G_{W_0,\dots,W_{r-1}}$. The vertex ζ_{W_r} of $\Omega_{\mathbb{H}}$ is common to the arcs $e_0 = e_{a_0}$, $e_1 := G_{a_0} e_{a_1}$ and $e_2 := G_{a_0 a_1} e_{a_2}$. The arcs $e'_i = G e_i$ share the vertex $G\zeta_{W_r}$. The point ζ_{W_r} is sent to ∞, and the arcs e_0, e_1, e_2 are sent to the parallel vertical arcs e''_0, e''_1, e''_2. We have $|W_r| = \big|\mathrm{Re}(e''_2) - \mathrm{Re}(e''_0)\big|$.

and Part (1) follows observing that a_1/c_0 varies in a finite set of values. Finally, if $c, c_1 \neq 0$ and $c_0 = 0$ then

$$\left|\frac{ca_1 + dc_1}{ca_0 + dc_0}\right| = \left|\frac{a_1}{a_0} - (-d/c)\frac{c_1}{a_0}\right| \leq \left|\frac{a_1}{a_0}\right| + |G^{-1} \cdot \infty| \left|\frac{c_1}{a_0}\right|.$$

In this case $\zeta_0 = \infty$, which is not an endpoint of $[\widehat{b_m}]$. Thus $[\widehat{b_m}]$ is contained in the compact interval of $\mathbb{R}$ delimited by the two parallel vertical segments of $\Omega_{\mathbb{H}}$. Hence $|G^{-1} \cdot \infty|$ is uniformly bounded, because the pole $G^{-1} \cdot \infty$ belongs to the closure of $[\widehat{b_m}]$ (see proof of Lemma 3.3). Part (1) follows in this case too, and the proof is complete. Part (2) follows similarly, replacing ζ_1 by $\zeta_* := G_{b_{m+1}} \cdot \zeta_2$ and observing that, since $G_{b_{m+1}}$ varies in the finite set $\{G_a : a \in \mathcal{A}\}$ then also the entries of $X \in \mathrm{SL}(2, \mathbb{R})$ with $G_{b_{m+1}} \cdot \zeta_2 = X \cdot \infty$ vary in a finite set. Moreover $\zeta_0 \neq \zeta_*$, and thus $G \cdot \zeta_0 \neq G \cdot \zeta_*$. $\square$

3.3. **Proof of Theorem 3.1.** By a standard separation property of parabolic fixed points (a proof is in § A in [6]), there exists a constant $S_0 > 0$, depending only on Γ and on $\mathcal{S}$, such that for any $G \cdot z_i$ and $F \cdot z_j$ in $\mathcal{P}_\Gamma$ with $G \cdot z_i \neq F \cdot z_j$ we have

$$|G \cdot z_i - F \cdot z_j| \geq \frac{S_0}{D(G \cdot z_i) D(F \cdot z_j)}. \tag{3.8}$$

Let $\alpha = [a_0, a_1, \dots]_{\mathbb{H}} = [W_0, W_1, \dots]_{\mathbb{H}}$ be the expansion of $\alpha \in \mathbb{R}$ as in Equation (3.4).

3.3.1. *Proof of Equation* (3.6). Fix $r \in \mathbb{N}$ with $|W_r| > 0$. Take $k \in \{1, \dots, p\}$ and $B \in \Gamma$ as in Equation (3.2), that is $\zeta_{W_r} = BA_k \cdot \infty$. As in Figure 4, let $T > 0$ be such that the horoball

$$B_T := G_{W_0,\dots,W_{r-1}} BA_k\big(\{z \in \mathbb{H} : \mathrm{Im}(z) > T\}\big)$$

is tangent at $G_{W_0,\dots,W_{r-1}} \cdot \zeta_{W_r}$ with radius $\rho(B_T) = |\alpha - G_{W_0,\dots,W_{r-1}} \cdot \zeta_{W_r}|$. Equation (1.6) and Equation (1.7) give

$$D(G_{W_0,\dots,W_{r-1}} \cdot \zeta_{W_r})^2 \cdot |\alpha - G_{W_0,\dots,W_{r-1}} \cdot \zeta_{W_r}| = c^2(G_{W_0,\dots,W_{r-1}} BA_k) \cdot \frac{\mathrm{Diam}(B_T)}{2} = \frac{1}{2T}.$$

The geodesic in $\mathbb{H}$ with endpoints $(G_{W_0,\dots,W_{r-1}}BA_k)^{-1}\cdot\infty$ and $(G_{W_0,\dots,W_{r-1}}BA_k)^{-1}\cdot\alpha$ is tangent to $\{z\in\mathbb{H}:\mathrm{Im}(z)>T\}$. Equation (3.6) follows because Equation (3.3) gives

$$|W_r|\leq 2T\leq |W_r|+2\mu.$$

3.3.2. *Proof of Equation* (3.7). Referring to § 3.2, let ζ_0 be the vertex of $\Omega_{\mathbb{H}}$ and $(b_0,\dots,b_m)$ be the admissible word such that the reduced form of the parabolic fixed point $G\cdot z_k$ is

$$G\cdot z_k=G_{b_0,\dots,b_m}\cdot\zeta_0,$$

where ζ_0 is not an endpoint of $e_{\widehat{b_m}}$ whenever $(b_0,\dots,b_m)$ is not the empty word. Assume $D(G\cdot z_k)^2|\alpha-G\cdot z_k|<\epsilon_0$, where the constant $\epsilon_0>0$ will be determined later.

Step (0). Assume that $(b_0,\dots,b_m)$ is the empty word, so that $\zeta_0=G\cdot z_k\neq\infty$. Consider the extra assumption $|W_0|>0$ and $\zeta_0=\zeta_{W_0}$ on pairs (α,ζ_0), where $\zeta_{W_0}=\varphi^{-1}(\xi_{W_0})$ and ξ_{W_0} is the vertex of $\Omega_{\mathbb{D}}$ associated to W_0 as in § 2.4. Define $\epsilon_0>0$ by

$$\epsilon_0:=\inf_{(\alpha,\zeta_0)}D(\zeta_0)^2\cdot|\alpha-\zeta_0|,$$

where the infimum is taken over all pairs (α,ζ_0) not satisfying the extra assumption. With such ϵ_0, the statement follows whenever $(b_0,\dots,b_m)$ is the empty word.

Step (1). Now assume that $(b_0,\dots,b_m)$ is not the empty word. Then $G\cdot z_k$ is an interior point of $[b_0,\dots,b_m]_{\mathbb{H}}$. Let ζ_1,ζ_2 be the endpoints of $[\widehat{b_m}]$, which are vertices of $\Omega_{\mathbb{H}}$ different from ζ_0. The endpoints of $[b_0,\dots,b_m]_{\mathbb{H}}$ are $\zeta_i':=G_{b_0,\dots,b_m}\cdot\zeta_i$ for $i=1,2$, according to Equation (2.10). Let $N\geq -1$ be maximal with $a_n=b_n$ for any $n=0,\dots,N$, where the last condition is empty for $N=-1$, and where $N\leq m$. Observe that condition $N\leq m-1$ implies $\alpha\notin[b_0,\dots,b_m]_{\mathbb{H}}$, and therefore

$$\begin{aligned}|\alpha-G\cdot z_k|&\geq\min_{i=1,2}|\zeta_i'-G\cdot z_k|=\min_{i=1,2}|G_{b_0,\dots,b_m}\cdot\zeta_i-G_{b_0,\dots,b_m}\cdot\zeta_0|\\&\geq\frac{S_0}{D(G_{b_0,\dots,b_m}\cdot\zeta_0)}\cdot\min_{i=1,2}\frac{1}{D(G_{b_0,\dots,b_m}\cdot\zeta_i)}\geq\frac{S_0\kappa_2}{D(G_{b_0,\dots,b_m}\cdot\zeta_0)^2},\end{aligned}$$

where the third inequality follows from Part (1) of Lemma 3.4 and the second from Equation (3.8). Therefore $N=m$, provided that $\epsilon_0<\kappa_2S_0$.

We proved $[a_0,\dots,a_m]_{\mathbb{H}}=[b_0,\dots,b_m]_{\mathbb{H}}$. Moreover $G\cdot z_k$ does not belong to the interior of $[a_0,\dots,a_m,a_{m+1}]_{\mathbb{H}}$, since the latter is a subinterval of $[b_0,\dots,b_m]_{\mathbb{H}}$ delimited by the image under $G_{b_0,\dots,b_m}$ of two consecutive vertices of $\Omega_{\mathbb{H}}$. The same argument as in the first part of Step (1), which is left to the reader, shows that $G\cdot z_k$ is an endpoint of $[a_0,\dots,a_m,a_{m+1}]_{\mathbb{H}}$.

Step (2). We show that $G\cdot z_k=G_{b_0,\dots,b_m}\cdot\zeta_0$ is an endpoint of $[a_0,\dots,a_{m+2}]_{\mathbb{H}}$. Otherwise $G\cdot z_k$ doesn't belong to the closure of $[a_0,\dots,a_{m+2}]_{\mathbb{H}}$. Since $\alpha\in[a_0,\dots,a_{m+2}]_{\mathbb{H}}$ then

$$\begin{aligned}|\alpha-G\cdot z_k|&\geq|G_{b_0,\dots,b_m,a_{m+1}}\cdot\zeta_3-G_{b_0,\dots,b_m}\cdot\zeta_0|\\&\geq\frac{S_0}{D(G_{b_0,\dots,b_m}\cdot\zeta_0)D(G_{b_0,\dots,b_m,a_{m+1}}\cdot\zeta_3)}\geq\frac{S_0\kappa_2}{D(G_{b_0,\dots,b_m}\cdot\zeta_0)^2},\end{aligned}$$

where $G_{b_0,\dots,b_m,a_{m+1}}\cdot\zeta_3$ is the endpoint of $[a_0,\dots,a_{m+2}]_{\mathbb{H}}$ which is closest to $G\cdot z_k$ and where ζ_3 is a vertex of $\Omega_{\mathbb{H}}$ which is not an endpoint of $e_{\widehat{a_{m+1}}}$. We use Equation (3.8) and Part (2) of Lemma 3.4. The inequality is absurd by condition $\epsilon_0<\kappa_2S_0$.

Step (3). Let r be minimal such that $(a_0,\dots,a_m)$ is an initial factor of $W_0*\dots*W_{r-1}$. If $(a_0,\dots,a_{m+2})$ is also an initial factor of $W_0*\dots*W_{r-1}$, then $G_{W_0,\dots,W_{r-1}}\cdot\xi_{W_{r-1}}$ is a common endpoint of the intervals $[a_0,\dots,a_m]_{\mathbb{H}}$, $[a_0,\dots,a_{m+1}]_{\mathbb{H}}$ and $[a_0,\dots,a_{m+2}]_{\mathbb{H}}$, according to Equation (2.13). Without loss of generality we have

$$G_{W_0,\dots,W_{r-1}}\cdot\xi_{W_{r-1}}=\inf[a_0,\dots,a_m]_{\mathbb{H}}=\inf[a_0,\dots,a_{m+1}]_{\mathbb{H}}=\inf[a_0,\dots,a_{m+2}]_{\mathbb{H}}.$$

The common endpoint is not $G\cdot z_k$, which belongs to the interior of $[a_0,\dots,a_m]_{\mathbb{H}}$. Thus Step (1) implies $G\cdot z_k=\sup[a_0,\dots,a_{m+1}]_{\mathbb{H}}$, which is absurd because $G\cdot z_k$ is an endpoint of $[a_0,\dots,a_{m+2}]_{\mathbb{H}}$ by Step (2). Hence $W_0*\dots*W_{r-1}$ is either equal to $(a_0,\dots,a_m)$ or to $(a_0,\dots,a_{m+1})$. Moreover (a_{m+1},a_{m+2}) is a cuspidal word, because $[a_0,\dots,a_{m+1}]_{\mathbb{H}}$ and $[a_0,\dots,a_{m+2}]_{\mathbb{H}}$ share the endpoint $G\cdot z_k$.

• In case $W_0*\dots*W_{r-1}=(a_0,\dots,a_m)$ the word (a_{m+1},a_{m+2}) is an initial factor of W_r, that is $|W_r|>0$ and $\zeta_0=\zeta_{W_r}$.

• In case $W_0*\dots*W_{r-1}=(a_0,\dots,a_{m+1})$ the word $W':=(a_{m+1})*W_r$ is also cuspidal (this is allowed by Remark 2.2). If $|W_r|=0$, that is $W_r=(a_{m+2})$, then $G\cdot z_k$ does not belong to the closure of $[a_0,\dots,a_{m+3}]_{\mathbb{H}}$ and we get an absurd by

$$|\alpha-G\cdot z_k|\geq|G_{b_0,\dots,b_m}\cdot\zeta_0-G_{b_0,\dots,b_m,a_{m+1},a_{m+2}}\cdot\zeta_3|\geq\frac{S_0\kappa_2}{D(G_{b_0,\dots,b_m}\cdot\zeta_0)^2},$$

where ζ_3 is a vertex of $\Omega_{\mathbb{H}}$ and $G_{b_0,\dots,b_m,a_{m+1},a_{m+2}}\cdot\zeta_3$ is the endpoint of $[a_0,\dots,a_{m+3}]_{\mathbb{H}}$ which is closest to $G\cdot z_k$. In the last inequality we reason as in Step (2), replacing κ_2 by a smaller constant and extending Part (2) of Lemma 3.4 one more step, in order to compare $D(G_{b_0,\dots,b_m}\cdot\zeta_0)$ and $D(G_{b_0,\dots,b_m,a_{m+1},a_{m+2}}\cdot\zeta_3)$. Since W' is cuspidal with $|W'|>0$ we have $\zeta_0=\zeta_{W'}$. But we have also $\zeta_{W'}=G_{a_{m+1}}\cdot\zeta_{W_r}$, which implies

$$G_{b_0,\dots,b_m}\cdot\zeta_0=G_{a_0,\dots,a_m}\cdot G_{a_{m+1}}\cdot\zeta_{W_r}=G_{W_0,\dots,W_{r-1}}\cdot\zeta_{W_r}.$$

In both cases Equation (3.7) follows. The proof of Theorem 3.1 is complete. □

References

[1] M. Artigiani, L. Marchese, C. Ulcigrai: *The Lagrange spectrum of a Veech surface has a Hall ray*, Groups, Geometry, and Dynamics, 10 (2016), 1287-1337.

[2] M. Artigiani, L. Marchese, C. Ulcigrai: *Persistent Hall rays for Lagrange Spectra at cusps of Riemann surfaces*, Ergodic Theory and Dynamical Systems, Vol. 40, Issue 8, 2020, 2017-2072.

[3] R. Bowen, C. Series: *Markov maps associated with fuchsian groups.* Publications Matématiques de l'IHES, tome 50 (1979), pages 153-170.

[4] S. Katok: *Fuchsian groups.* Chicago Lectures in Mathematics, University of Chicago Press, 1992.

[5] J. Lehner: *Diophantine approximation on Hecke groups.* Glasgow Math. J., Vol. 27, 1985, 117-127.

[6] L. Marchese: *Transfer operators and dimension of bad sets for non-uniform Fuchsian lattices.* ArXiv:1812.11921.

[7] H. Nakada: *On the Lenstra constant associated to the Rosen continued fractions.* J. Eur. Math. Soc., Vol. 12, 2010, 55-70.

[8] S. J. Patterson: *Diophantine approximation in Fuchsian groups.* Phil. Trans. Roy. Soc. London Ser. A, vol. 282, num. 1309, 1976, 527-563.

[9] D. Rosen: *A class of continued fraction associated to certain properly discontinuous groups.* Duke Math. J., Vol. 21, 1954, 549-562.

[10] D. Rosen, T. Schmidt: *Hecke groups and continued fractions.* Bull. Austral. Math. Soc., Vol. 46, 1992, 459-474.

[11] T. Schmidt, M. Sheingorn: *Riemann surfaces have hall rays at each cusp.* Illinois journal of Mathematics, Vol. 41, number 3, 1997, 378-397.

[12] P. Tukia: *On discrete groups of the unit disk and their isomorphisms.* Annales Academiae Scientiarum Fennicae, Series A I, 504 (1972), 1-45.

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