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# ON GOOD APPROXIMATIONS AND BOWEN-SERIES EXPANSION 

LUCA MARCHESE


#### Abstract

We consider the continued fraction expansion of real numbers under the action of a non-uniform lattice in $\operatorname{PSL}(2, \mathbb{R})$ and prove metric relations between the convergents and a natural geometric notion of good approximations.


## 1. Introduction

Let $\mathbb{H}:=\{z \in \mathbb{C}: \operatorname{Im}(z)>0\}$ be the upper half plane and for $p / q \in \mathbb{Q}$ let $H_{p / q} \subset \mathbb{H}$ be the circle of diameter $1 / q^{2}$ tangent at $p / q$. Set $H_{\infty}=\{z \in \mathbb{H}: \operatorname{Im}(z)>1\}$ and consider the family $\left\{H_{p / q}: p / q \in \mathbb{Q} \cup\{\infty\}\right\}$ of Ford circles, which are the orbit of $H_{\infty}$ under the projective action of the modular group $\operatorname{SL}(2, \mathbb{Z})$, that is the group of $2 \times 2$ matrices with coefficients $a, b, c, d$ in $\mathbb{Z}$ (notation refers to Equation (1.3) below). Any two circles are either disjoint or tangent, and Figure 1 shows that for any irrational $\alpha$ there exist infinitely many $p / q \in \mathbb{Q}$ with $\alpha \in \Pi\left(H_{p / q}\right)$, that is $|\alpha-p / q|<(1 / 2) q^{-2}$, where $\Pi(x+i y):=x$. This defines the sequence


Figure 1. Balls $G\left(H_{k}\right), k \in \mathbb{Z}$, tangent to $H_{p / q}=G\left(H_{\infty}\right)$, where $p / q=G \cdot \infty$.
of geometric good approximations of $\alpha$ as the sequence of $p_{n} / q_{n}$ in $\mathbb{Q}$ with $\alpha \in \Pi\left(B_{p_{n} / q_{n}}\right)$. The same sequence arises from the continued fraction expansion $\alpha=a_{0}+\left[a_{1}, a_{2}, \ldots\right]$ of $\alpha$, indeed the convergents $p_{n} / q_{n}:=a_{0}+\left[a_{1}, \ldots, a_{n}\right]$ satisfy:

$$
\begin{equation*}
|\alpha-p / q|<(1 / 2) q^{-2} \Rightarrow p / q=p_{n} / q_{n} \text { for some } n \geq 1 \tag{1.1}
\end{equation*}
$$

The first $n+1$ partial quotients $a_{1}, \ldots, a_{n+1}$ approximate $\alpha$ with error given by

$$
\begin{equation*}
\frac{1}{2+a_{n+1}} \leq q_{n}^{2} \cdot\left|\alpha-p_{n} / q_{n}\right| \leq \frac{1}{a_{n+1}} \text { for any } n \in \mathbb{N} \tag{1.2}
\end{equation*}
$$

Rosen continued fractions where introduced in [9], in relation to diophantine approximation for Hecke groups, proving in particular an extension of Equation (1.2), which was later

[^0]improved by [7]. Equation (1.1) was extended to Rosen continued fraction in [5], where the sharp constant replacing $1 / 2$ was obtained in [10]. In this note we consider diophantine approximation for a general non-uniform lattice Fuchsian group, in relation to the so-called Bowen-Series expansion of real numbers ([3]). Our Main Theorem 3.1 provides an extension of Equations (1.1) and (1.2) to this setting. This result is used in [6] to approximate the dimension of sets of badly approximable points by the dimension of dynamically defined regular Cantor sets. The study of the high part of Markov and Lagrange spectra is also a natural application, in the spirit of [11], [1] and [2]. In general, Theorem 3.1 applies to a large variety of problems in diophantine approximations, since it translates diophantine properties into ergodic properties of the Bowen-Series expansion.

Let $\operatorname{SL}(2, \mathbb{C})$ be the group of matrices

$$
G=\left(\begin{array}{ll}
a & b  \tag{1.3}\\
c & d
\end{array}\right)
$$

with $a, b, c, d \in \mathbb{C}$ and $a d-b c=1$, where any such $G$ acts on points $z \in \mathbb{C} \cup\{\infty\}$ by

$$
\begin{equation*}
G \cdot z:=\frac{a z+b}{c z+d} \tag{1.4}
\end{equation*}
$$

Denote $a=a(G), b=b(G), c=c(G)$ and $d=d(G)$ the coefficients of $G$ as in Equation (1.3). The group $\operatorname{SL}(2, \mathbb{R})$ of $G$ with coefficients $a, b, c, d$ in $\mathbb{R}$ acts by isometries on $\mathbb{H}$ via Equation (1.4), and inherits a topology from the identification with the set of $(a, b, c, d) \in \mathbb{R}^{4}$ with $a d-b c=1$. A Fuchsian group is a discrete subgroup $\Gamma<\operatorname{SL}(2, \mathbb{R})$. Referring to [4], we say that $\Gamma$ is a lattice if it has a Dirichlet region $\Omega \subset \mathbb{H}$ with finite hyperbolic area. If $\Omega$ is not compact, then the lattice $\Gamma$ is said non-uniform. In this case the intersection $\bar{\Omega} \cap \partial \mathbb{H}$ is a finite non-empty set, whose elements are called the vertices at infinity of $\Omega$. A point $z \in \mathbb{R} \cup\{\infty\}$ is a parabolic fixed point for $\Gamma$ if there exists $P \in \Gamma$ parabolic with $P(z)=z$. Let $\mathcal{P}_{\Gamma}$ be the set of parabolic fixed points of $\Gamma$, which is equal to the orbit under $\Gamma$ of the vertices at infinity of $\Omega$. The set $\mathcal{P}_{\Gamma}$ is dense in $\mathbb{R}$. Two points $z_{1}$ and $z_{2}$ in $\mathcal{P}_{\Gamma}$ are equivalent if $z_{2}=G\left(z_{1}\right)$ for some $G \in \Gamma$. Any non-uniform lattice $\Gamma$ has a finite number $p \geq 1$ of equivalence classes $\left[z_{1}\right], \ldots,\left[z_{p}\right]$ of parabolic fixed points, called the cusps of $\Gamma$.

Let $\Gamma$ be a non-uniform lattice with $p \geq 1$ cusps. Fix a list $\mathcal{S}=\left(A_{1}, \ldots, A_{p}\right)$ of elements $A_{k} \in \mathrm{SL}(2, \mathbb{R})$ such that the points

$$
\begin{equation*}
z_{k}=A_{k} \cdot \infty \quad \text { for } \quad k=1, \ldots, p \tag{1.5}
\end{equation*}
$$

form a complete set $\left\{z_{1}, \ldots, z_{p}\right\} \subset \mathcal{P}_{\Gamma}$ of inequivalent parabolic fixed points. A natural choice for $z_{1}, \ldots, z_{p}$ is a maximal set of non-equivalent vertices at infinity of a fundamental domain. Any element of $\mathcal{P}_{\Gamma}$ has the form $G \cdot z_{k}$ for some $G \in \Gamma$ and $k=1, \ldots, p$. We have horoballs

$$
B_{k}:=A_{k}(\{z \in \mathbb{H}: \operatorname{Im}(z)>1\}) \quad \text { with } \quad k=1, \ldots, p,
$$

each $B_{k}$ being tangent to $\mathbb{R} \cup\{\infty\}$ at $z_{k}$. We can have $A_{k}=\mathrm{Id}$, that is $z_{k}=\infty$ and $B_{k}=H_{\infty}$. Thus $G\left(B_{k}\right)$ is a ball tangent to the real line at $G \cdot z_{k}$ for any $G \in \Gamma$ with $G \cdot z_{k} \neq \infty$. These balls generalize Ford circles and we measure how their diameter shrinks to zero as $G$ varies in $\Gamma$ with the denominator

$$
D\left(G \cdot z_{k}\right):=\left\{\begin{array}{cl}
1 / \sqrt{\operatorname{Diam}\left(G\left(B_{k}\right)\right)} & \text { if } \quad G \cdot z_{k} \neq \infty \\
0 & \text { if } G \cdot z_{k}=\infty
\end{array}\right.
$$

Recall that for any $T>0$ and any $G \in \mathrm{SL}(2, \mathbb{R})$ with $c(G) \neq 0$ we have

$$
\begin{equation*}
\operatorname{Diam}(G(\{z \in \mathbb{H}: \operatorname{Im}(z)>T\}))=\frac{1}{T c^{2}(G)}, \tag{1.6}
\end{equation*}
$$

where we refer to the notation of Equation (1.3). Hence

$$
\begin{equation*}
D\left(G \cdot z_{k}\right)=\left|c\left(G A_{k}\right)\right| \quad \text { for any } \quad G \cdot z_{k} \in \mathcal{P}_{\Gamma} . \tag{1.7}
\end{equation*}
$$

In [8], Patterson proves that there exists a constant $M=M(\Gamma, \mathcal{S})>0$ such that for any $Q>0$ big enough and any $\alpha \in \mathbb{R}$ there exists $G \in \Gamma$ and $k \in\{1, \ldots, p\}$ with

$$
\left|\alpha-G \cdot z_{k}\right| \leq \frac{M}{D\left(G \cdot z_{k}\right) Q} \quad \text { and } \quad 0<D\left(G \cdot z_{k}\right) \leq Q
$$

For $\Gamma=\operatorname{SL}(2, \mathbb{Z}), \mathcal{S}=\{\operatorname{Id}\}$ and $M=1$ Patterson's Theorem gives the Classical Dirichlet Theorem. In general, for any $\alpha \in \mathbb{R}$ we obtain infinitely many $G \cdot z_{k} \in \mathcal{P}_{\Gamma}$ with

$$
\begin{equation*}
\left|\alpha-G \cdot z_{k}\right| \leq \frac{M}{D^{2}\left(G \cdot z_{k}\right)} \tag{1.8}
\end{equation*}
$$

The Bowen-Series expansion ([3]) provides a coding $\alpha=\left[W_{1}, W_{2}, \ldots\right]$ of a real number $\alpha$, where for $r \geq 1$ we call cuspidal words the symbols $W_{r}$, which belong to a countable alphabet $\mathcal{W}$ (definitions are in $\S 2$ and $\S 3$ ). Cuspidal words $W \in \mathcal{W}$, that were introduced in [1] and [2], label a subset of elements $\left\{G_{W}: W \in \mathcal{W}\right\}$ of $\Gamma$, which generalize the role played in the theory of classical continued fractions by the matrices

$$
\left(\begin{array}{cc}
1 & a_{2 k+1} \\
0 & 1
\end{array}\right) \quad \text { and } \quad\left(\begin{array}{cc}
1 & 0 \\
a_{2 k} & 1
\end{array}\right) \quad \text { with } \quad a_{2 k}, a_{2 k+1} \in \mathbb{N}^{*} \text { for any } k \in \mathbb{N} \text {. }
$$

The coding is a continuous bijection $\Sigma \rightarrow \mathbb{R}$, where $\Sigma \subset \mathcal{W}^{\mathbb{N}}$ is a subshift with aperiodic transition matrix (see [6]). For $r \geq 1$ the first $r$ symbols in the expansion of $\alpha=\left[W_{1}, W_{2}, \ldots\right]$ define $\zeta_{r}=\zeta_{r}\left(W_{1}, \ldots, W_{r}\right) \in \mathcal{P}_{\Gamma}$, see Equation (3.5). This extends the classical notion of convergents $p_{n} / q_{n}$ of $\alpha$. The main result of this note is Theorem 3.1 in $§ 3$. We give the following preliminary statement (see also Remark 3.2).

Main Theorem (Theorem 3.1). Fix $\alpha=\left[W_{1}, W_{2}, \ldots\right]$ which is not an element of $\mathcal{P}_{\Gamma}$. The convergents $\zeta_{r}=\zeta_{r}\left(W_{1}, \ldots, W_{r}\right)$ approximate $\alpha$ with error given by an analogue of Equation (1.2). Moreover there exists a constant $\epsilon_{0}>0$ such that any $G \cdot z_{k} \in \mathcal{P}_{\Gamma}$ satisfying Equation (1.8) with $M=\epsilon_{0}$ belongs to the sequence $\left(\zeta_{r}\right)_{r \geq 1}$.

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## 2. The Bowen-Series expansion

We follow $\S 3$ in [6], which is itself based on $\S 2.4$ in [1] and $\S 2$ in [2]. The original construction is in [3], where it is defined a Markov map, which is orbit equivalent to the action of a given finitely generated Fuchsian group of the first kind. In our setting such Markov map corresponds to an acceleration of the map in Equation (2.7) below. This § 2 describes the coding by cusidal words. The same description appears in [6], where it is
followed by the study of the combinatorial and metric properties of the subshift related to the coding. Consider the unit disc $\mathbb{D}:=\{z \in \mathbb{C}:|z|<1\}$ and the map

$$
\begin{equation*}
\varphi: \mathbb{H} \rightarrow \mathbb{D} \quad ; \quad \varphi(z):=\frac{z-i}{z+i} \tag{2.1}
\end{equation*}
$$

The conjugate of $\operatorname{SL}(2, \mathbb{R})$ under $\varphi$ is the group $\operatorname{SU}(1,1)$ of $F \in \mathrm{GL}(2, \mathbb{C})$ with

$$
F=\left(\begin{array}{cc}
\alpha & \bar{\beta}  \tag{2.2}\\
\beta & \bar{\alpha}
\end{array}\right) \quad \text { with } \quad|\alpha|^{2}-|\beta|^{2}=1 .
$$

Denote $\alpha=\alpha(F)$ and $\beta=\beta(F)$ the coefficients of $F$ as in Equation (2.2).
2.1. Isometric circles. Consider $F \in \mathrm{SU}(1,1)$ and $\alpha=\alpha(F), \beta=\beta(F)$ as in Equation (2.2). Assume $\beta \neq 0$ and let $\omega_{F}:=-\bar{\alpha} / \beta$ be the pole of $F$. The isometric circle $I_{F}$ of $F$ is the euclidean circle centered at $\omega_{F}$ with radius $\rho(F):=|\beta|^{-1}$, that is

$$
I_{F}:=\left\{\xi \in \mathbb{C}:\left|\xi-\omega_{F}\right|\right\} .
$$

We have $F\left(I_{F}\right)=I_{F^{-1}}$, where $\rho(F)=\rho\left(F^{-1}\right)$ and $\left|\omega_{F^{-1}}\right|=\left|\omega_{F}\right|$. See Theorem 3.3.2 in [4]. Moreover $I_{F} \cap \mathbb{D}$ is a geodesic of $\mathbb{D}$ for any $F \in \operatorname{SU}(1,1)$, by Theorem 3.3.3 in [4]. Denote $U_{F}$ the disc in $\mathbb{C}$ with $\partial U_{F}=I_{F}$, that is the interior of $I_{F}$.
2.2. Labelled ideal polygon. Let $\Gamma \subset \operatorname{SU}(1,1)$ be a non-uniform lattice. According to [12], there exist a free subgroup $\Gamma_{0}<\Gamma$ with finite index $\left[\Gamma_{0}: \Gamma\right]<+\infty$. See also $\S 2.2$ of [6]. In particular $\beta(F) \neq 0$ for any $F \in \Gamma_{0}$, referring to Equation (2.2), so that the isometric circle $I_{F}$ and the disc $U_{F}$ introduced in $\S 2.1$ are defined. The origin $0 \in \mathbb{D}$ is not a fixed point of any $F \in \Gamma_{0}$ and Theorem 3.3.5 in [4] implies that the set

$$
\begin{equation*}
\Omega_{0}:=\overline{\mathbb{D} \backslash \bigcup_{F \in \Gamma_{0}} U_{F}} \tag{2.3}
\end{equation*}
$$

is a Dirichlet region for $\Gamma_{0}$. Recall from [4] that $\Omega_{0}$ is an hyperbolic polygon with an even number $2 d$ of sides, denoted by the letter $s$, and with $2 d$ vertices, denoted by the letter $\xi$ (see also $\S 2.4$ of [6]). All vertices of $\Omega_{0}$ belong to $\partial \mathbb{D}$, because $\Gamma_{0}$ is free. Any side $s$ is a complete geodesic in $\mathbb{D}$ and for any such $s$ there exists an unique $F \in \Gamma$ such that $F(s)$ is another side of $\Omega_{0}$ with $F(s) \neq s$. The sides $s$ and $F(s)$ are thus paired. See Figure 2. The set of pairings generates $\Gamma_{0}$, according to Theorem 3.5.4 in [4]. For a convenient labelling, consider two finite alphabets $\mathcal{A}_{0}$ and $\widehat{\mathcal{A}_{0}}$, both with $d$ elements, and a map

$$
\iota: \mathcal{A}_{0} \cup \widehat{\mathcal{A}_{0}} \rightarrow \mathcal{A}_{0} \cup \widehat{\mathcal{A}_{0}} \quad \text { with } \quad \iota^{2}=\mathrm{Id} \quad \text { and } \quad \iota\left(\mathcal{A}_{0}\right)=\widehat{\mathcal{A}_{0}}
$$

that is an involution of $\mathcal{A}_{0} \cup \widehat{\mathcal{A}_{0}}$ which exchanges $\mathcal{A}_{0}$ with $\widehat{\mathcal{A}_{0}}$. Set $\mathcal{A}:=\mathcal{A}_{0} \cup \widehat{\mathcal{A}_{0}}$ and for any $a \in \mathcal{A}$, denote $\widehat{a}:=\iota(a)$.

- Label the sides of $\Omega_{0}$ by the letters in $\mathcal{A}$, so that for any $a \in \mathcal{A}$ the sides $s_{a}$ and $s_{\widehat{a}}$ are those which are paired by the action of $\Gamma_{0}$.
- For any pair of sides $s_{a}$ and $s_{\widehat{a}}$ as above, let $F_{a}$ be the unique element of $\Gamma_{0}$ such that

$$
\begin{equation*}
F_{a}\left(s_{\widehat{a}}\right)=s_{a} . \tag{2.4}
\end{equation*}
$$

- For any $a \in \mathcal{A}$ we have $F_{\widehat{a}}=F_{a}^{-1}$, and the latter form a set of generators for $\Gamma_{0}$.

In the following we denote $\Omega_{\mathbb{D}}:=\Omega_{0} \subset \mathbb{D}$ the labelled ideal polygon defined above and $\Omega_{\mathbb{H}}:=\varphi^{-1}\left(\Omega_{\mathbb{D}}\right) \subset \mathbb{H}$ its preimage under the map in Equation (2.1).


Figure 2. Ideal polygon labelled by $\mathcal{A}=\{a, b, c, d, \widehat{a}, \widehat{b}, \widehat{c}, \widehat{d}\}$.
2.3. The boundary map. Parametrize $\operatorname{arcs} J \subset \partial \mathbb{D}$ by $t \mapsto e^{-i t}$ with $t \in(x, y)$. Set $\inf J:=e^{-i x}$ and $\sup J:=e^{-i y}$. We say that $J$ is right open if $\inf J \in J$ and $\sup J \notin J$. Let $\Gamma_{0}<\Gamma$ be a finite index free subgroup and $\Omega_{\mathbb{D}}$ be an ideal polygon for $\Gamma_{0}$ labelled by $\mathcal{A}$, as in § 2.2.

For $a \in \mathcal{A}$ let $F_{a}$ be the map in Equation (2.4). Let $I_{F_{a}}$ be the isometric circle of $F_{a}$ and $U_{F_{a}}$ be its interior, as in $\S 2.1$. Recall that $s_{\widehat{a}}=I_{F_{a}} \cap \mathbb{D}$ and $s_{a}=I_{F_{\widehat{a}}} \cap \mathbb{D}$. Let $[a]_{\mathbb{D}}$ be the right open arc of $\partial \mathbb{D}$ cut by the side $s_{a}$, that is

$$
[a]_{\mathbb{D}}:=U_{F_{\widehat{a}}} \cap \partial \mathbb{D} .
$$

Set $\xi_{a}^{L}:=\inf [a]_{\mathbb{D}}$ and $\xi_{a}^{R}:=\sup [a]_{\mathbb{D}}$. Figure 2 shows examples of such notation. In order to take account of the cyclic order in $\partial \mathbb{D}$ of the $\operatorname{arcs}[a]_{\mathbb{D}}$, fix $a_{0} \in \mathcal{A}$ and define a map $o: \mathcal{A} \rightarrow \mathbb{Z} / 2 d \mathbb{Z}$ setting $o\left(a_{0}\right):=0$ and

$$
\begin{equation*}
o(b)=o(a)+1 \quad \bmod 2 d \quad \text { for } \quad a, b \in \mathcal{A} \quad \text { with } \quad \xi_{a}^{R}=\xi_{b}^{L} \tag{2.5}
\end{equation*}
$$

We have $F_{a}\left(I_{F_{a}}\right)=I_{F_{\widehat{a}}}$ for any $a \in \mathcal{A}$, thus $F_{a}$ sends the complement of $[\widehat{a}]_{\mathbb{D}}$ to $[a]_{\mathbb{D}}$, that is

$$
\begin{equation*}
F_{a}\left(\partial \mathbb{D} \backslash[\widehat{a}]_{\mathbb{D}}\right)=[a]_{\mathbb{D}} \tag{2.6}
\end{equation*}
$$

The Bowen-Series map is the map $\mathcal{B S}: \partial \mathbb{D} \rightarrow \partial \mathbb{D}$ defined by

$$
\begin{equation*}
\mathcal{B S}(\xi):=F_{a}^{-1}(\xi) \quad \text { iff } \quad \xi \in[a]_{\mathbb{D}} . \tag{2.7}
\end{equation*}
$$

The boundary expansion of a point $\xi \in \partial \mathbb{D}$ is the sequence $\left(a_{k}\right)_{k \in \mathbb{N}}$ of letters $a_{k} \in \mathcal{A}$ with

$$
\begin{equation*}
\mathcal{B S}^{k}(\xi) \in\left[a_{k}\right]_{\mathbb{D}} \quad \text { for any } k \in \mathbb{N} . \tag{2.8}
\end{equation*}
$$

By Equation (2.6), any such sequence satisfies the so-called no backtracking Condition:

$$
\begin{equation*}
a_{k+1} \neq \widehat{a_{k}} \text { for any } k \in \mathbb{N} . \tag{2.9}
\end{equation*}
$$

A finite word $\left(a_{0}, \ldots, a_{n}\right)$ satisfying Condition (2.9) corresponds to a factor of the map $\mathcal{B S}: \partial \mathbb{D} \rightarrow \partial \mathbb{D}$, that is a finite concatenation $F_{a_{n}}^{-1} \circ \cdots \circ F_{a_{0}}^{-1}$ arising from iterations of $\mathcal{B S}$. We call admissible word, or simply word, any finite or infinite word in the letters of $\mathcal{A}$ satisfying Condition (2.9). We use the notation

$$
F_{a_{0}, \ldots, a_{n}}:=F_{a_{0}} \circ \cdots \circ F_{a_{n}} \in \Gamma_{0} .
$$

Define the right open arc $\left[a_{0}, \ldots, a_{n}\right]_{\mathbb{D}}$ as the set of $\xi \in \partial \mathbb{D}$ such that $\mathcal{B} \mathcal{S}^{k}(\xi) \in\left[a_{k}\right]_{\mathbb{D}}$ for any $k=0, \ldots, n$, that is

$$
\begin{equation*}
\left[a_{0}, \ldots, a_{n}\right]_{\mathbb{D}}:=F_{a_{0}, \ldots, a_{n-1}}\left[a_{n}\right]_{\mathbb{D}}=F_{a_{0}, \ldots, a_{n}}\left(\partial \mathbb{D} \backslash\left[\widehat{a_{n}}\right]_{\mathbb{D}}\right) \tag{2.10}
\end{equation*}
$$

Two such arcs satisfy $\left[a_{0}, \ldots, a_{n}\right]_{\mathbb{D}} \subset\left[b_{0}, \ldots, b_{m}\right]_{\mathbb{D}}$ if and only if $m \geq n$ and $a_{k}=b_{k}$ for any $k=0, \ldots, n$. It is easy to see that $\left[a_{0}, \ldots, a_{n}\right]_{\mathbb{D}}$ shrinks to a point as $n \rightarrow \infty$. See Lemma 3.1 in [6] for a proof. A sequence $\left(a_{k}\right)_{k \in \mathbb{N}}$ satisfying Condition (2.9) corresponds to a point $\xi=\left[a_{0}, a_{1}, \ldots\right]_{\mathbb{D}}$ in $\partial \mathbb{D}$, where we use the notation

$$
\left[a_{0}, a_{1}, \ldots\right]_{\mathbb{D}}:=\bigcap_{n \in \mathbb{N}}\left[a_{0} \ldots, a_{n}\right]_{\mathbb{D}} .
$$

Conversely, if $\left(a_{k}\right)_{k \in \mathbb{N}}$ is the boundary expansion of $\xi \in \partial \mathbb{D}$, then $\xi=\left[a_{0}, a_{1}, \ldots\right]_{\mathbb{D}}$. The Bowen-Series map $\mathcal{B S}$ is the shift on the space of admissible infinite words.
2.4. Cuspidal words. Consider the map $o: \mathcal{A} \rightarrow \mathbb{Z} / 2 d \mathbb{Z}$ in Equation (2.5). The definitions in $\S 2.3$ easily imply Lemma 2.1 below. See Lemma 3.2 in [6] for a proof.
Lemma 2.1. Let $\left(a_{0}, \ldots, a_{n}\right)$ be a word satisfying Condition (2.9) with $n \geq 1$ and $a_{0}=a_{n}$. The map $F_{a_{0}, \ldots, a_{n-1}}$ is a parabolic element of $\Gamma_{0}$ fixing $\xi_{a_{0}}^{R}$ if and only if

$$
\begin{equation*}
o\left(a_{k+1}\right)=o\left(\widehat{a_{k}}\right)-1 \quad \text { for any } \quad k=0, \ldots, n-1 \tag{2.11}
\end{equation*}
$$

The map $F_{a_{0}, \ldots, a_{n-1}}$ is a parabolic element of $\Gamma_{0}$ fixing $\xi_{a_{0}}^{L}$ if and only if

$$
\begin{equation*}
o\left(a_{k+1}\right)=o\left(\widehat{a_{k}}\right)+1 \quad \text { for any } \quad k=0, \ldots, n-1 \tag{2.12}
\end{equation*}
$$

Let $W=\left(a_{0}, \ldots, a_{n}\right)$ be an admissible word. We say that $W$ is a cuspidal word if it is the initial factor of an admissible word $\left(a_{0}, \ldots, a_{m}\right)$ with $m \geq n$ such that $F_{a_{0}, \ldots, a_{m}}$ is a parabolic element of $\Gamma_{0}$ fixing a vertex of $\Omega_{\mathbb{D}}$.

- If $n \geq 1$ and Equation (2.11) is satisfied, we say that $W$ is a right cuspidal word. In this case we define its type by $\varepsilon(W):=R$ and we set $\xi_{W}:=\xi_{a_{0}}^{R}$.
- If $n \geq 1$ and Equation (2.12) is satisfied, we say that $W$ is a left cuspidal word. In this case we define its type by $\varepsilon(W):=L$ and we set $\xi_{W}:=\xi_{a_{0}}^{L}$.
- If $n=0$, that is $W=\left(a_{0}\right)$ has just one letter, the type $\varepsilon(W)$ is not defined. We set by convention $\xi_{W}:=\xi_{a_{0}}^{R}$.
If $W=\left(a_{0}, \ldots, a_{n}\right)$ is cuspidal with $n \geq 1$, Lemma 2.1 implies $\xi_{a_{k}}^{\varepsilon(W)}=F_{a_{k}} \cdot \xi_{a_{k+1}}^{\varepsilon(W)}$ for any $k=0, \ldots, n-1$ and it follows

$$
\begin{equation*}
\xi_{W}=\partial\left[a_{0}\right]_{\mathbb{D}} \cap \partial\left[a_{0}, a_{1}\right]_{\mathbb{D}} \cap \cdots \cap \partial\left[a_{0}, \ldots, a_{n}\right]_{\mathbb{D}} \tag{2.13}
\end{equation*}
$$

that is the $n+1$ arcs above share $\xi_{W}$ as common endpoint (see also $\S 2.4$ in [2] and $\S 4.3$ in [1]). A sequence $\left(a_{n}\right)_{n \in \mathbb{N}}$ is said cuspidal if any initial factor $\left(a_{0}, \ldots, a_{n}\right)$ with $n \in \mathbb{N}$ is a cuspidal word, and eventually cuspidal if there exists $k \in \mathbb{N}$ such that $\left(a_{n+k}\right)_{n \in \mathbb{N}}$ is a cuspidal sequence.
2.5. The cuspidal acceleration. If $W=\left(b_{0}, \ldots, b_{m}\right)$ and $W^{\prime}=\left(a_{0}, \ldots, a_{n}\right)$ are words with $a_{0} \neq \widehat{b_{m}}$, define the word $W * W^{\prime}:=\left(b_{0}, \ldots, b_{m}, a_{0}, \ldots, a_{n}\right)$. Let $\left(a_{n}\right)_{n \in \mathbb{N}}$ be a sequence satisfying Condition (2.9) and not eventually cuspidal.

Initial step: Set $n(0):=0$. Let $n(1) \in \mathbb{N}$ be the maximal integer $n(1) \geq 1$ such that $\left(a_{0}, \ldots, a_{n(1)-1}\right)$ is cuspidal, then set $W_{0}:=\left(a_{0}, \ldots, a_{n(1)-1}\right)$.


Figure 3. Geometric length $|W|$ of a right cuspidal word $W=\left(a_{0}, a_{1}, a_{2}, a_{3}\right)$. The arrows inside $\Omega_{\mathbb{D}}$ represent the action of $F_{a_{0}}, F_{a_{1}}, F_{a_{2}}$. The arcs $s_{0}:=s_{a_{0}}$, $s_{1}:=F_{a_{0}}\left(s_{a_{1}}\right), s_{2}:=F_{a_{0}, a_{1}}\left(s_{a_{2}}\right)$ and $s_{3}:=F_{a_{0}, a_{1}, a_{2}}\left(s_{a_{3}}\right)$ share the common vertex $\xi_{W}$, which is sent to $\infty$ under the map $A_{k}^{-1} B^{-1} \varphi^{-1}$. Thus the arcs $s_{0}, s_{1}, s_{2}, s_{3}$ in $\mathbb{D}$ are sent to parallel vertical $\operatorname{arcs} e_{i}:=\varphi^{-1}\left(s_{i}\right)$ in $\mathbb{H}$.

Recursive step: Fix $r \geq 1$ and assume that the instants $n(0)<\cdots<n(r)$ and the cuspidal words $W_{0}, \ldots, W_{r-1}$ are defined. Define $n(r+1) \geq n(r)+1$ as the maximal integer such that $\left[a_{n(r)}, \ldots, a_{n(r+1)-1}\right]$ is cuspidal, then set

$$
W_{r}:=\left(a_{n(r)}, \ldots, a_{n(r+1)-1}\right)
$$

The sequence of words $\left(W_{r}\right)_{r \in \mathbb{N}}$ is called the cuspidal decomposition of $\left(a_{n}\right)_{n \in \mathbb{N}}$. We have of course $a_{0}, a_{1}, a_{2} \cdots=W_{0} * W_{1} * \ldots$. For any $\xi=\left[a_{0}, a_{1}, \ldots\right]_{\mathbb{D}}$, if $\left(W_{r}\right)_{r \in \mathbb{N}}$ is the cuspidal decomposition of $\left(a_{n}\right)_{n \in \mathbb{N}}$, we write

$$
\begin{equation*}
\xi=\left[a_{0}, a_{1}, \ldots\right]_{\mathbb{D}}=\left[W_{0}, W_{1}, \ldots\right]_{\mathbb{D}} \tag{2.14}
\end{equation*}
$$

Remark 2.2. If $W_{r-1}:=\left(a_{n(r-1)}, \ldots, a_{n(r)-1}\right)$ and $W_{r}:=\left(a_{n(r)}, \ldots, a_{n(r+1)-1}\right)$ are two consecutive cuspidal words in the cuspidal decomposition of a sequence $\left(a_{n}\right)_{n \in \mathbb{N}}$ satisfying Condition (2.9), then the word $\left(a_{n(r)-1}, a_{n(r)}, \ldots, a_{n(r+1)-1}\right)$ can be cuspidal.

## 3. The main Theorem 3.1

The tools in $\S 2$ induce a boundary expansion on $\mathbb{R}$. Let $\Gamma_{0}<\Gamma$ be the free subgroup and $\Omega_{\mathbb{D}} \subset \mathbb{D}$ be the ideal polygon in $\S 2.2$. Recall that $\mathcal{P}_{\Gamma_{0}}=\Gamma_{0}\left(\Omega_{\mathbb{D}} \cap \partial \mathbb{D}\right)$ by Theorem 4.2 .5 in [4]. Since $\Gamma_{0}$ has finite index in $\Gamma$ then the two groups have the same set of parabolic fixed points, that is

$$
\begin{equation*}
\mathcal{P}_{\Gamma}=\Gamma_{0}\left(\Omega_{\mathbb{D}} \cap \partial \mathbb{D}\right) \tag{3.1}
\end{equation*}
$$

3.1. Geometric length of cuspidal words and main statement. Fix $\mathcal{S}=\left(A_{1}, \ldots, A_{p}\right)$ as in Equation (1.5). Let $\Omega_{\mathbb{H}}:=\varphi^{-1}\left(\Omega_{\mathbb{D}}\right) \subset \mathbb{H}$ be the pre-image of $\Omega_{\mathbb{D}}$ under the map in Equation (2.1). Any vertex $\xi$ of $\Omega_{\mathbb{D}}$ corresponds to an unique vertex $\zeta=\varphi^{-1}(\xi)$ of $\Omega_{\mathbb{H}}$. For any such vertex $\zeta$ consider $B \in \Gamma$ and $k \in\{1, \ldots, p\}$ with

$$
\begin{equation*}
\zeta=B A_{k} \cdot \infty \tag{3.2}
\end{equation*}
$$

Any side $s_{a}$ of $\Omega_{\mathbb{D}}$ corresponds to an unique side $e_{a}:=\varphi^{-1}\left(s_{a}\right)$ of $\Omega_{\mathbb{H}}$, where $a \in \mathcal{A}$. If $B A_{k} \cdot \infty=B^{\prime} A_{j} \cdot \infty$, then $j=k$. Moreover $B^{\prime}=B P$, where $P \in \Gamma$ is parabolic fixing
$A_{k} \cdot \infty$, where we recall that in any Fuchsian group $\Gamma$ with cusps, if $G \in \Gamma$ satisfies $G \cdot \zeta=\zeta$ for some $\zeta \in \mathcal{P}_{\Gamma}$, then $G$ is parabolic. Hence the map $z \mapsto A_{k}^{-1} P A_{k}(z)$ is an horizontal translation in $\mathbb{H}$. If $s$ and $s^{\prime}$ are geodesics in $\mathbb{D}$ having $\xi$ as common endpoint, then their pre-images in $\mathbb{H}$ under $\varphi \circ B \circ A_{k}$ are parallel vertical half lines whose distance does not depend on the choice of $B$ in Equation (3.2). We have a well defined positive real number

$$
\Delta\left(s, s^{\prime}, \xi\right):=\left|\operatorname{Re}\left(A_{k}^{-1} B^{-1} \varphi^{-1}(s)\right)-\operatorname{Re}\left(A_{k}^{-1} B^{-1} \varphi^{-1}\left(s^{\prime}\right)\right)\right| .
$$

Fix a cuspidal word $W=\left(a_{0}, \ldots, a_{n}\right)$ and the vertex $\xi_{W}$ of $\Omega_{\mathbb{D}}$ associated to $W$ in § 2.4. For $n \geq 1$ Equation (2.13) implies that the geodesics $s_{a_{0}}, F_{a_{0}}\left(s_{a_{1}}\right), \ldots, F_{a_{0}, \ldots, a_{n-1}}\left(s_{a_{n}}\right)$ all have $\xi_{W}$ as common endpoint. See Figure 3. Define the geometric length $|W| \geq 0$ of $W$ as

$$
|W|:= \begin{cases}\Delta\left(s_{a_{0}}, F_{a_{0}, \ldots, a_{n-1}}\left(s_{a_{n}}\right), \xi_{W}\right) & \text { if } n \geq 1  \tag{3.3}\\ 0 & \text { if } n=0\end{cases}
$$

For $a \in \mathcal{A}$ set $G_{a}=\varphi^{-1} \circ F_{a} \circ \varphi$. Set $G_{a_{0}, \ldots, a_{n}}:=G_{a_{0}} \circ \cdots \circ G_{a_{n}}$ for any word $\left(a_{0}, \ldots, a_{n}\right)$ and $G_{W_{0}, \ldots, W_{r}}=G_{a_{0}, \ldots, a_{n}}$ if $\left(a_{0}, \ldots, a_{n}\right)=W_{0} * \cdots * W_{r}$. Define the interval

$$
\left[a_{0}, \ldots, a_{n}\right]_{\mathbb{H}}:=\varphi^{-1}\left(\left[a_{0}, \ldots, a_{n}\right]_{\mathbb{D}}\right)=G_{a_{0}, \ldots, a_{n}}\left(\partial \mathbb{H} \backslash\left[\widehat{a_{n}}\right]_{\mathbb{H}}\right) .
$$

Set $\left[a_{0}, a_{1}, \ldots\right]_{\mathbb{H}}:=\varphi^{-1}\left(\left[a_{0}, a_{1}, \ldots\right]_{\mathbb{D}}\right)$, that is encode $\alpha \in \mathbb{R}$ by the same cutting sequence as $\varphi(\alpha) \in \mathbb{D}$. If $\left(a_{n}\right)_{n \in \mathbb{N}}$ has cuspidal decomposition $\left(W_{r}\right)_{r \in \mathbb{N}}$, Equation (2.14) becomes

$$
\begin{equation*}
\alpha=\left[W_{0}, W_{1}, \ldots\right]_{\mathbb{H}}:=\left[a_{0}, a_{1}, \ldots\right]_{\mathbb{H}} . \tag{3.4}
\end{equation*}
$$

For $r \in \mathbb{N}$ let $W_{r}$ be the $r$-th cuspidal word. Set $\zeta_{W_{r}}:=\varphi^{-1}\left(\xi_{W_{r}}\right)$. The convergents of $\alpha$ are

$$
\begin{equation*}
\zeta_{r}:=G_{W_{0}, \ldots, W_{r-1}} \cdot \zeta_{W_{r}} \quad ; \quad r \in \mathbb{N} \tag{3.5}
\end{equation*}
$$

For $k=1, \ldots, p$ let $\mu_{k}>0$ be such that the primitive parabolic element $P_{k} \in A_{k} \Gamma A_{k}^{-1}$ fixing $\infty$ acts by $P_{k}(z)=z+\mu_{k}$. Set $\mu:=\max \left\{\mu_{1}, \ldots, \mu_{p}\right\}$.

Theorem 3.1 (Main Theorem). For any $r \in \mathbb{N}$ with $\left|W_{r}\right|>0$ we have

$$
\begin{equation*}
\frac{1}{\left|W_{r}\right|+2 \mu} \leq D\left(G_{W_{0}, \ldots, W_{r-1}} \cdot \zeta_{W_{r}}\right)^{2} \cdot\left|\alpha-G_{W_{0}, \ldots, W_{r-1}} \cdot \zeta_{W_{r}}\right| \leq \frac{1}{\left|W_{r}\right|} \tag{3.6}
\end{equation*}
$$

Moreover there exists $\epsilon_{0}>0$ depending only on $\Omega_{\mathbb{D}}$ and on $\mathcal{S}$, such that for any $G \in \Gamma$ and $k=1, \ldots, p$ with $D\left(G \cdot z_{k}\right) \neq 0$ the condition

$$
D\left(G \cdot z_{k}\right)^{2} \cdot\left|\alpha-G \cdot z_{k}\right|<\epsilon_{0}
$$

implies that there exists some $r \in \mathbb{N}$ such that

$$
\begin{equation*}
G \cdot z_{k}=G_{W_{0}, \ldots, W_{r-1}} \cdot \zeta_{W_{r}} \quad \text { where } \quad\left|W_{r}\right|>0 \tag{3.7}
\end{equation*}
$$

Remark 3.2. Equation (3.6) holds for any choice of $\mathcal{S}$ as in Equation (1.5), and this follows because geometric length and denominators satisfy a form of equivariance under the choice of $\mathcal{S}$. Equation (3.7) shows that, for any choice of the subgroup $\Gamma_{0}$, all good enough approximations of a given $\alpha$ belong to the sequence of its convergents.
3.2. Reduced form of parabolic fixed points. Fix $G \cdot z_{k} \in \mathcal{P}_{\Gamma}$. Recall Equation (3.1) and write elements of $\Gamma_{0}$ in the generators $\left\{G_{a}: a \in \mathcal{A}\right\}$. There exists an unique admissible word $b_{0}, \ldots, b_{m}$ and a vertex $\zeta$ of $\Omega_{\mathbb{H}}$ which is not an endpoint of $e_{\widehat{b_{m}}}$ such that

$$
G \cdot z_{k}=G_{b_{0}, \ldots, b_{m}} \cdot \zeta
$$

The representation above is called the reduced form of the parabolic fixed point $G \cdot z_{k}$. In the next Lemmas 3.3 and 3.4, let $\left(b_{0}, \ldots, b_{m}\right)$ be a non-trivial admissible word and let $\zeta_{0}$ be a vertex of $\Omega_{\mathbb{H}}$ which is not an endpoint of $e_{\widehat{b_{m}}}$, so that $G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}$ is a parabolic fixed point written in its reduced form and different from $\infty$.

Lemma 3.3. There exists a constant $\kappa_{1}>0$, depending only on $\Omega_{\mathbb{H}}$, such that

$$
\left|\zeta_{0}-G_{b_{0}, \ldots, b_{m}}^{-1} \cdot \infty\right| \geq \kappa_{1},
$$

that is the vertex $\zeta_{0}$ and the pole of $G_{b_{0}, \ldots, b_{m}}$ stay at distance uniformly bounded from below.
Proof. We have $\left.G_{b_{0}, \ldots, b_{m}}\left(\mathbb{R} \backslash \widehat{b_{m}}\right]_{\mathbb{H}}\right)=\left[b_{0}, \ldots, b_{m}\right]_{\mathbb{H}}$ By Equation (2.10). Since $\infty$ does not belong to the interior of $\left[b_{0}, \ldots, b_{m}\right]_{\mathbb{H}}$ then the pole of $G_{b_{0}, \ldots, b_{m}}$ belongs to the closure of $\left[\widehat{b_{m}}\right]_{\mathbb{H}}$. The Lemma follows because $\zeta_{0}$ is a vertex of $\Omega_{\mathbb{H}}$ different from the endpoints of $e_{\widehat{b_{m}}}$.

Lemma 3.4. There exists a constant $\kappa_{2}>0$, depending only on $\Omega_{\mathbb{H}}$ and on $\mathcal{S}$, such that the following holds.
(1) If $\zeta_{1}$ is a vertex of $\Omega_{\mathbb{H}}$ different from $\zeta_{0}$, then

$$
D\left(G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}\right) \geq \kappa_{2} \cdot D\left(G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{1}\right)
$$

(2) If $b_{m+1}$ satisfies $b_{m+1} \neq \widehat{b_{m}}$ and $\zeta_{2}$ is a vertex of $\Omega_{\mathbb{H}}$ with $G_{b_{m+1}} \cdot \zeta_{2} \neq \zeta_{0}$, then

$$
D\left(G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}\right) \geq \kappa_{2} \cdot D\left(G_{b_{0}, \ldots, b_{m}, b_{m+1}} \cdot \zeta_{2}\right)
$$

Proof. We prove Part (1). Set $G:=G_{b_{0}, \ldots, b_{m}}, \zeta:=G \cdot \zeta_{0}$, and $\zeta^{\prime}:=G \cdot \zeta_{1}$. If $\zeta^{\prime}=\infty$ then the statement is trivially true. If $D\left(G \cdot \zeta_{1}\right) \neq 0$, let $\zeta_{0}=B_{0} A_{k} \cdot \infty$ and $\zeta_{1}=B_{1} A_{j} \cdot \infty$ as in Equation (3.2). Referring to Equation (1.3), let $c, d$ be the entries of $G$. Let $a_{0}, c_{0}$ and $a_{1}, c_{1}$ be the entries of $B_{0} A_{k}$ and $B_{1} A_{j}$ respectively. We prove an upper bound for

$$
\frac{D\left(G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{1}\right)}{D\left(G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}\right)}=\left|\frac{c a_{1}+d c_{1}}{c a_{0}+d c_{0}}\right|
$$

We cannot have $c_{0}=c_{1}=0$, because $\zeta_{0} \neq \zeta_{1}$ and in particular $\zeta_{0}, \zeta_{1}$ cannot be both equal to $\infty$. Moreover $G \cdot \zeta_{0}, G \cdot \zeta_{1}$ are both different from $\infty$, thus condition $c=0$ implies $c_{0}, c_{1} \neq 0$. Hence for $c=0$ Part (1) follows because the ratio above equals $\left|c_{1} / c_{0}\right|$, which varies in a finite set of values and is therefore bounded from above. If $c, c_{0}, c_{1} \neq 0$ then

$$
\left|\frac{c a_{1}+d c_{1}}{c a_{0}+d c_{0}}\right|=\left|\frac{c_{1}}{c_{0}}\right| \cdot\left|\frac{\left(a_{1} / c_{1}\right)-(-d / c)}{\left(a_{0} / c_{0}\right)-(-d / c)}\right|=\left|\frac{c_{1}}{c_{0}}\right| \cdot\left|\frac{\zeta_{1}-\left(G^{-1} \cdot \infty\right)}{\zeta_{0}-\left(G^{-1} \cdot \infty\right)}\right| .
$$

In this case Part (1) follows because $\left|c_{1} / c_{0}\right|$ is bounded from above, and Lemma 3.3 gives a lower bound for the denominator of the second factor (the numerator is not bounded, but as it increases the ratio converges to 1 ). If $c, c_{0} \neq 0$ and $c_{1}=0$ then Lemma 3.3 gives

$$
\left|\frac{c a_{1}+d c_{1}}{c a_{0}+d c_{0}}\right|=\left|\frac{a_{1}}{c_{0}}\right| \cdot\left|\frac{1}{\left(a_{0} / c_{0}\right)-(-d / c)}\right|=\left|\frac{a_{1}}{c_{0}}\right| \cdot\left|\frac{1}{\zeta_{0}-\left(G^{-1} \cdot \infty\right)}\right| \leq\left|\frac{a_{1}}{c_{0} \cdot \kappa_{1}}\right|,
$$



Figure 4. The $r$-th cuspidal word $W_{r}=\left(a_{0}, a_{1}, a_{2}\right)$ of $\alpha$ is the first cuspidal word of $G^{-1} \cdot \alpha$, where $G=G_{W_{0}, \ldots, W_{r-1}}$. The vertex $\zeta_{W_{r}}$ of $\Omega_{\mathbb{H}}$ is common to the $\operatorname{arcs} e_{0}=e_{a_{0}}, e_{1}:=G_{a_{0}} e_{a_{1}}$ and $e_{2}:=G_{a_{0} a_{1}} e_{a_{2}}$. The arcs $e_{i}^{\prime}=G e_{i}$ share the vertex $G \zeta_{W_{r}}$. The point $\zeta_{W_{r}}$ is sent to $\infty$, and the $\operatorname{arcs} e_{0}, e_{1}, e_{2}$ are sent to the parallel vertical $\operatorname{arcs} e_{0}^{\prime \prime}, e_{1}^{\prime \prime}, e_{2}^{\prime \prime}$. We have $\left|W_{r}\right|=\left|\operatorname{Re}\left(e_{2}^{\prime \prime}\right)-\operatorname{Re}\left(e_{0}^{\prime \prime}\right)\right|$.
and Part (1) follows observing that $a_{1} / c_{0}$ varies in a finite set of values. Finally, if $c, c_{1} \neq 0$ and $c_{0}=0$ then

$$
\left|\frac{c a_{1}+d c_{1}}{c a_{0}+d c_{0}}\right|=\left|\frac{a_{1}}{a_{0}}-(-d / c) \frac{c_{1}}{a_{0}}\right| \leq\left|\frac{a_{1}}{a_{0}}\right|+\left|G^{-1} \cdot \infty\right|\left|\frac{c_{1}}{a_{0}}\right| .
$$

In this case $\zeta_{0}=\infty$, which is not an endpoint of $\left[\widehat{b_{m}}\right]$. Thus $\left[\widehat{b_{m}}\right]$ is contained in the compact interval of $\mathbb{R}$ delimited by the two parallel vertical segments of $\Omega_{\mathbb{H}}$. Hence $\left|G^{-1} \cdot \infty\right|$ is uniformly bounded, because the pole $G^{-1} \cdot \infty$ belongs to the closure of $\left[\widehat{b_{m}}\right]$ (see proof of Lemma 3.3). Part (1) follows in this case too, and the proof is complete. Part (2) follows similarly, replacing $\zeta_{1}$ by $\zeta_{*}:=G_{b_{m+1}} \cdot \zeta_{2}$ and observing that, since $G_{b_{m+1}}$ varies in the finite set $\left\{G_{a}: a \in \mathcal{A}\right\}$ then also the entries of $X \in \operatorname{SL}(2, \mathbb{R})$ with $G_{b_{m+1}} \cdot \zeta_{2}=X \cdot \infty$ vary in a finite set. Moreover $\zeta_{0} \neq \zeta_{*}$, and thus $G \cdot \zeta_{0} \neq G \cdot \zeta_{*}$.
3.3. Proof of Theorem 3.1. By a standard separation property of parabolic fixed points (a proof is in $\S$ A in [6]), there exists a constant $S_{0}>0$, depending only on $\Gamma$ and on $\mathcal{S}$, such that for any $G \cdot z_{i}$ and $F \cdot z_{j}$ in $\mathcal{P}_{\Gamma}$ with $G \cdot z_{i} \neq F \cdot z_{j}$ we have

$$
\begin{equation*}
\left|G \cdot z_{i}-F \cdot z_{j}\right| \geq \frac{S_{0}}{D\left(G \cdot z_{i}\right) D\left(F \cdot z_{j}\right)} \tag{3.8}
\end{equation*}
$$

Let $\alpha=\left[a_{0}, a_{1}, \ldots\right]_{\mathbb{H}}=\left[W_{0}, W_{1}, \ldots\right]_{\mathbb{H}}$ be the expansion of $\alpha \in \mathbb{R}$ as in Equation (3.4).
3.3.1. Proof of Equation (3.6). Fix $r \in \mathbb{N}$ with $\left|W_{r}\right|>0$. Take $k \in\{1, \ldots, p\}$ and $B \in \Gamma$ as in Equation (3.2), that is $\zeta_{W_{r}}=B A_{k} \cdot \infty$. As in Figure 4 , let $T>0$ be such that the horoball

$$
B_{T}:=G_{W_{0}, \ldots, W_{r-1}} B A_{k}(\{z \in \mathbb{H}: \operatorname{Im}(z)>T\})
$$

is tangent at $G_{W_{0}, \ldots, W_{r-1}} \cdot \zeta_{W_{r}}$ with radius $\rho\left(B_{T}\right)=\left|\alpha-G_{W_{0}, \ldots, W_{r-1}} \cdot \zeta_{W_{r}}\right|$. Equation (1.6) and Equation (1.7) give

$$
D\left(G_{W_{0}, \ldots, W_{r-1}} \cdot \zeta_{W_{r}}\right)^{2} \cdot\left|\alpha-G_{W_{0}, \ldots, W_{r-1}} \cdot \zeta_{W_{r}}\right|=c^{2}\left(G_{W_{0}, \ldots, W_{r-1}} B A_{k}\right) \cdot \frac{\operatorname{Diam}\left(B_{T}\right)}{2}=\frac{1}{2 T} .
$$

The geodesic in $\mathbb{H}$ with endpoints $\left(G_{W_{0}, \ldots, W_{r-1}} B A_{k}\right)^{-1} \cdot \infty$ and $\left(G_{W_{0}, \ldots, W_{r-1}} B A_{k}\right)^{-1} \cdot \alpha$ is tangent to $\{z \in \mathbb{H}: \operatorname{Im}(z)>T\}$. Equation (3.6) follows because Equation (3.3) gives

$$
\left|W_{r}\right| \leq 2 T \leq\left|W_{r}\right|+2 \mu
$$

3.3.2. Proof of Equation (3.7). Referring to $\S 3.2$, let $\zeta_{0}$ be the vertex of $\Omega_{\mathbb{H}}$ and $\left(b_{0}, \ldots, b_{m}\right)$ be the admissible word such that the reduced form of the parabolic fixed point $G \cdot z_{k}$ is

$$
G \cdot z_{k}=G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}
$$

where $\zeta_{0}$ is not an endpoint of $e_{\widehat{b_{m}}}$ whenever $\left(b_{0}, \ldots, b_{m}\right)$ is not the empty word. Assume $D\left(G \cdot z_{k}\right)^{2}\left|\alpha-G \cdot z_{k}\right|<\epsilon_{0}$, where the constant $\epsilon_{0}>0$ will be determined later.

Step (0). Assume that $\left(b_{0}, \ldots, b_{m}\right)$ is the empty word, so that $\zeta_{0}=G \cdot z_{k} \neq \infty$. Consider the extra assumption $\left|W_{0}\right|>0$ and $\zeta_{0}=\zeta_{W_{0}}$ on pairs $\left(\alpha, \zeta_{0}\right)$, where $\zeta_{W_{0}}=\varphi^{-1}\left(\xi_{W_{0}}\right)$ and $\xi_{W_{0}}$ is the vertex of $\Omega_{\mathbb{D}}$ associated to $W_{0}$ as in $\S 2.4$. Define $\epsilon_{0}>0$ by

$$
\epsilon_{0}:=\inf _{\left(\alpha, \zeta_{0}\right)} D\left(\zeta_{0}\right)^{2} \cdot\left|\alpha-\zeta_{0}\right|,
$$

where the infimum is taken over all pairs $\left(\alpha, \zeta_{0}\right)$ not satisfying the extra assumption. With such $\epsilon_{0}$, the statement follows whenever $\left(b_{0}, \ldots, b_{m}\right)$ is the empty word.

Step (1). Now assume that $\left(b_{0}, \ldots, b_{m}\right)$ is not the empty word. Then $G \cdot z_{k}$ is an interior point of $\left[b_{0}, \ldots, b_{m}\right]_{\mathbb{H}}$. Let $\zeta_{1}, \zeta_{2}$ be the endpoints of $\left.\widehat{b_{m}}\right]$, which are vertices of $\Omega_{\mathbb{H}}$ different from $\zeta_{0}$. The endpoints of $\left[b_{0}, \ldots, b_{m}\right]_{\mathbb{H}}$ are $\zeta_{i}^{\prime}:=G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{i}$ for $i=1,2$, according to Equation (2.10). Let $N \geq-1$ be maximal with $a_{n}=b_{n}$ for any $n=0, \ldots, N$, where the last condition is empty for $N=-1$, and where $N \leq m$. Observe that condition $N \leq m-1$ implies $\alpha \notin\left[b_{0}, \ldots, b_{m}\right]_{\mathbb{H}}$, and therefore

$$
\begin{aligned}
\left|\alpha-G \cdot z_{k}\right| & \geq \min _{i=1,2}\left|\zeta_{i}^{\prime}-G \cdot z_{k}\right|=\min _{i=1,2}\left|G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{i}-G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}\right| \\
& \geq \frac{S_{0}}{D\left(G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}\right)} \cdot \min _{i=1,2} \frac{1}{D\left(G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{i}\right)} \geq \frac{S_{0} \kappa_{2}}{D\left(G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}\right)^{2}}
\end{aligned}
$$

where the third inequality follows from Part (1) of Lemma 3.4 and the second from Equation (3.8). Therefore $N=m$, provided that $\epsilon_{0}<\kappa_{2} S_{0}$.

We proved $\left[a_{0}, \ldots, a_{m}\right]_{\mathbb{H}}=\left[b_{0}, \ldots, b_{m}\right]_{\mathbb{H}}$. Moreover $G \cdot z_{k}$ does not belong to the interior of $\left[a_{0}, \ldots, a_{m}, a_{m+1}\right]_{\mathbb{H}}$, since the latter is a subinterval of $\left[b_{0}, \ldots, b_{m}\right]_{\mathbb{H}}$ delimited by the image under $G_{b_{0}, \ldots, b_{m}}$ of two consecutive vertices of $\Omega_{\mathbb{H}}$. The same argument as in the first part of Step (1), which is left to the reader, shows that $G \cdot z_{k}$ is an endpoint of $\left[a_{0}, \ldots, a_{m}, a_{m+1}\right]_{\mathbb{H}}$.

Step (2). We show that $G \cdot z_{k}=G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}$ is an endpoint of $\left[a_{0}, \ldots, a_{m+2}\right]_{\mathbb{H}}$. Otherwise $G \cdot z_{k}$ doesn't belong to the closure of $\left[a_{0}, \ldots, a_{m+2}\right]_{\mathbb{H}}$. Since $\alpha \in\left[a_{0}, \ldots, a_{m+2}\right]_{\mathbb{H}}$ then

$$
\begin{aligned}
\left|\alpha-G \cdot z_{k}\right| & \geq\left|G_{b_{0}, \ldots, b_{m}, a_{m+1}} \cdot \zeta_{3}-G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}\right| \\
& \geq \frac{S_{0}}{D\left(G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}\right) D\left(G_{b_{0}, \ldots, b_{m}, a_{m+1}} \cdot \zeta_{3}\right)} \geq \frac{S_{0} \kappa_{2}}{D\left(G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}\right)^{2}},
\end{aligned}
$$

where $G_{b_{0}, \ldots, b_{m}, a_{m+1}} \cdot \zeta_{3}$ is the endpoint of $\left[a_{0}, \ldots, a_{m+2}\right]_{\mathbb{H}}$ which is closest to $G \cdot z_{k}$ and where $\zeta_{3}$ is a vertex of $\Omega_{\mathbb{H}}$ which is not an endpoint of $e_{\widehat{a_{m+1}}}$. We use Equation (3.8) and Part (2) of Lemma 3.4. The inequality is absurd by condition $\epsilon_{0}<\kappa_{2} S_{0}$.

Step (3). Let $r$ be minimal such that $\left(a_{0}, \ldots, a_{m}\right)$ is an initial factor of $W_{0} * \cdots * W_{r-1}$. If $\left(a_{0}, \ldots, a_{m+2}\right)$ is also an initial factor of $W_{0} * \cdots * W_{r-1}$, then $G_{W_{0}, \ldots, W_{r-1}} \cdot \xi_{W_{r-1}}$ is a common endpoint of the intervals $\left[a_{0}, \ldots, a_{m}\right]_{\mathbb{H}},\left[a_{0}, \ldots, a_{m+1}\right]_{\mathbb{H}}$ and $\left[a_{0}, \ldots, a_{m+2}\right]_{\mathbb{H}}$, according to Equation (2.13). Without loss of generality we have

$$
G_{W_{0}, \ldots, W_{r-1}} \cdot \xi_{W_{r-1}}=\inf \left[a_{0}, \ldots, a_{m}\right]_{\mathbb{H}}=\inf \left[a_{0}, \ldots, a_{m+1}\right]_{\mathbb{H}}=\inf \left[a_{0}, \ldots, a_{m+2}\right]_{\mathbb{H}} .
$$

The common endpoint is not $G \cdot z_{k}$, which belongs to the interior of $\left[a_{0}, \ldots, a_{m}\right]_{\mathbb{H}}$. Thus Step (1) implies $G \cdot z_{k}=\sup \left[a_{0}, \ldots, a_{m+1}\right]_{\mathbb{H}}$, which is absurd because $G \cdot z_{k}$ is an endpoint of $\left[a_{0}, \ldots, a_{m+2}\right]_{\text {Hi }}$ by Step (2). Hence $W_{0} * \cdots * W_{r-1}$ is either equal to $\left(a_{0}, \ldots, a_{m}\right)$ or to $\left(a_{0}, \ldots, a_{m+1}\right)$. Moreover $\left(a_{m+1}, a_{m+2}\right)$ is a cuspidal word, because $\left[a_{0}, \ldots, a_{m+1}\right]_{\mathbb{H}}$ and $\left[a_{0}, \ldots, a_{m+2}\right]_{\mathbb{H}}$ share the endpoint $G \cdot z_{k}$.

- In case $W_{0} * \cdots * W_{r-1}=\left(a_{0}, \ldots, a_{m}\right)$ the word $\left(a_{m+1}, a_{m+2}\right)$ is an initial factor of $W_{r}$, that is $\left|W_{r}\right|>0$ and $\zeta_{0}=\zeta_{W_{r}}$.
- In case $W_{0} * \cdots * W_{r-1}=\left(a_{0}, \ldots, a_{m+1}\right)$ the word $W^{\prime}:=\left(a_{m+1}\right) * W_{r}$ is also cuspidal (this is allowed by Remark 2.2). If $\left|W_{r}\right|=0$, that is $W_{r}=\left(a_{m+2}\right)$, then $G \cdot z_{k}$ does not belong to the closure of $\left[a_{0}, \ldots, a_{m+3}\right]_{\mathbb{H}}$ and we get an absurd by

$$
\left|\alpha-G \cdot z_{k}\right| \geq\left|G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}-G_{b_{0}, \ldots, b_{m}, a_{m+1}, a_{m+2}} \cdot \zeta_{3}\right| \geq \frac{S_{0} \kappa_{2}}{D\left(G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}\right)^{2}},
$$

where $\zeta_{3}$ is a vertex of $\Omega_{\mathbb{H}}$ and $G_{b_{0}, \ldots, b_{m}, a_{m+1}, a_{m+2}} \cdot \zeta_{3}$ is the endpoint of $\left[a_{0}, \ldots, a_{m+3}\right]_{\mathbb{H}}$ which is closest to $G \cdot z_{k}$. In the last inequality we reason as in Step (2), replacing $\kappa_{2}$ by a smaller constant and extending Part (2) of Lemma 3.4 one more step, in order to compare $D\left(G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}\right)$ and $D\left(G_{b_{0}, \ldots, b_{m}, a_{m+1}, a_{m+2}} \cdot \zeta_{3}\right)$. Since $W^{\prime}$ is cuspidal with $\left|W^{\prime}\right|>0$ we have $\zeta_{0}=\zeta_{W^{\prime}}$. But we have also $\zeta_{W^{\prime}}=G_{a_{m+1}} \cdot \zeta_{W_{r}}$, which implies

$$
G_{b_{0}, \ldots, b_{m}} \cdot \zeta_{0}=G_{a_{0}, \ldots, a_{m}} \cdot G_{a_{m+1}} \cdot \zeta_{W_{r}}=G_{W_{0}, \ldots, W_{r-1}} \cdot \zeta_{W_{r}} .
$$

In both cases Equation (3.7) follows. The proof of Theorem 3.1 is complete.

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