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Physical projectors for multi-leg helicity amplitudes

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Abstract: We present a method for building physical projector operators for multi-leg helicity amplitudes. For any helicity configuration of the external particles, we define a physical projector which singles out the corresponding helicity amplitude. For processes with more than four external legs, these physical projectors depend on significantly fewer tensor structures and exhibit a remarkable simplicity compared with projector operators defined with traditional approaches. As an example, we present analytic formulas for a complete set of projectors for five-gluon scattering. These have been validated by reproducing known results for five-gluon amplitudes up to one-loop.

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1 Introduction

After the discovery of the Higgs boson, particle physics has entered a new, unprecedented phase, at least in the recent decades. In spite of the fact that only a small fraction of the expected full LHC data set has been analysed, we have already been able to confirm the Standard Model of Particle Physics as the correct theory of Nature with unprecedented precision and for energies that span an impressively large number of orders of magnitude. Given this state of affairs, the apparent absence of clear signs of new physics has pushed the particle physics community into a new era of precision physics. Indeed, by comparing precise theoretical predictions for suitable observables with equally precise experimental results, the discovery potential of the LHC can be pushed to energies beyond its direct reach, increasing our chances to spot elusive signs of new physics phenomena.

Among the ingredients required to produce precise theoretical predictions for complex observables at the LHC, the calculation of scattering amplitudes for multi-particle final state processes has an important place. In order to match the experimental precision of many measurements at the LHC, two-loop corrections for several processes with up to five external particles are required. While we have a quite robust understanding of how such calculations should be performed in perturbative quantum field theory, their technical complexity constitutes often a major bottleneck. Indeed, in spite of the many advancements which have already brought many previously impossible calculations within reach, our current technology to treat multi-loop and multi-leg processes has only recently obtained its first results for processes with more than four external legs and more work is needed to generalise these to other processes.

There are many reasons why these processes are difficult. In this paper, we focus in particular on one of them, which has to do with the way we usually approach the calculation of scattering amplitudes at loop orders higher than one. In fact, while at one loop the underlying simplicity of the scattering amplitudes has allowed to develop techniques and automated tools to deal with these calculations, their generalisation to higher-loop orders has revealed to be quite non-trivial. The methods of integrand reduction [1, 2] and generalized unitarity [3–6] have been extended beyond one loop [7–11] and used to obtain the first analytic results for two-loop five parton amplitudes in the planar limit [12–14]. Very recently, the first non-planar five-point two-loop amplitudes have also been published [15–19]. In spite of this, a lot of progress will be required before these ideas can be applied generally to any class of processes.

An alternative approach to compute multi-loop scattering amplitudes, which is in principle entirely general and can be applied to any process at any perturbative order, consists in identifying Lorentz-invariant form factors, which can, in turn, be extracted from the relevant Feynman diagrams through suitable projector operators. This method has proven to be very successful in the calculation of a large number of lower-point (i.e. up to four external particles) scattering amplitudes up to two and three loops in perturbative quantum field theory. In spite of being very general, its applicability to multi-leg processes has been hindered by the increasing complexity of the relevant projection operators when more than four external particles are considered. The idea behind this method is very simple. Given the scattering on n particles of different spin, one parametrises the corresponding scattering amplitude in terms of a combination of scalar *form factors* which multiply all possible tensor structures compatible with the symmetries of the process under consideration. Since the tensors form a basis, for each of these form factors a projection operator can be defined as a linear combination of the same tensors, whose coefficients are fixed requiring that the projector singles out the correct form factor. Such tensor structures are interpreted as generic d-dimensional objects and all manipulations are performed in conventional dimensional regularisation (CDR). Clearly, the larger the number of external particles grows, the more independent tensors have to be included, such that going from four to five external legs typically implies a jump in one order of magnitude in the number of tensors needed and, therefore, in the corresponding form factors. Moreover, deriving the projectors requires in general to solve a dense linear system of equations, with as many equations as the number of independent tensor structures. Solving this system becomes very soon impractical with conventional computer algebra systems. Even if the solution can often be easily found using alternative approaches (for example, finite fields and multivariate reconstruction [20–23]), the final result will, in general, be extremely cumbersome, making its practical utility unclear. For these reasons, while progress has recently been made in defining the required projectors for the case of five-gluon scattering [24], their use for generic five- and higher-point scattering amplitudes is commonly considered to be a very difficult endeavour.

In this paper, we will show that this is not necessarily the case. Indeed, while the tensor decomposition described above implies that external particles are taken to be d-dimensional, one of the things that modern techniques have taught us is that substantial simplifications happen when helicity amplitudes with only physical four-dimensional external states are considered. Starting from this insight, in this paper we will show that by fixing the helicities of the external states, one can define a set of *physical projectors* which single out at once the corresponding helicity amplitudes. In general, there will be as many

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physical projectors as many independent helicity amplitudes and each of these projectors will be expressed as a linear combination of the original tensors, with rational coefficients that depend on the kinematic invariants. As a matter of fact, for processe with more than four external legs, the number of physical projectors will typically be much smaller than the original number of d-dimensional ones. Moreover, in these cases, when expressed in terms of the original tensors, only a subset of them will contribute and their number will correspond exactly to the number of independent helicity amplitudes in the process under consideration. Finally, the corresponding coefficients will be orders of magnitude simpler than the ones of the original projectors. In order words, the approach described in this paper allows us to get rid at once of all spurious tensor structures which correspond to the extra $(d-4)$ unphysical degrees of freedom associated to the external states and to define extremely compact projector operators that single out directly the physical degrees of freedom from the corresponding scattering amplitudes. To demonstrate the effectiveness of the new projectors, we will apply them to the calculation of one-loop corrections to five-gluon scattering in QCD. Recently, an alternative approach, which also exploits the simplifications coming from four-dimensional external states, has been proposed in [25] and we will comment more later about differences and similarities to our method.

The rest of the paper is organised as follows. In section 2, we start by recalling how the standard d-dimensional projectors work and elucidate the shortcomings of the standard approach. Inspired by this, in section 3 we illustrate how to define physical projectors which overcome most of these issues. In section 4 we use these ideas in order to build a complete set of physical projectors for the scattering of five gluons in QCD. We then apply explicitly the newly derived projectors to the calculation of one-loop corrections to five-gluon scattering in QCD. We stress here that such calculation is usually deemed to be impractical already at one-loop order with the use of standard d-dimensional projectors. Our approach, instead, allows to complete the analytic calculations of the one-loop corrections in a few hours on an average laptop computer, by resorting only to standard computer algebra systems as FORM [26] and Reduze [27]. Finally, we draw our conclusions in section 5.

2 Shortcomings of the standard approach

Before discussing the general idea behind the definition of physical projectors, we remind the reader of the way standard d-dimensional projectors work. We will stress why they are so useful in the context of multiloop calculations and, at the same time, highlight the shortcomings of the traditional approach.

Typically, multiloop calculations start with the enumeration of the Feynman diagrams which contribute to the process considered at the corresponding perturbative order. Diagrams always take the form of a multiple integral over the momenta of the virtual particles running in the loops, whose integrand is given by a rational function in the scalar products among the loop momenta, the momenta of the external particles and all their polarization structures (polarization vectors, spin-chains, etc). By factoring out all loop-independent tensor structures, one is then left with a large combination of tensor integrals, which are

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notoriously very difficult to compute as long as the tensor indices are not contracted. On the contrary, many effective tools are available for the calculation of so-called *scalar Feynman integrals*, where all loop momenta are contracted either among themselves or with the momenta of the external particles. Indeed, large numbers of linear relations among these integrals can be derived, among which the most prominent role is played by the so-called *integration-by-part identities* (IBPs) [28–30]. By solving these identities, the large majority of these integrals can be expressed in terms of a much smaller subset of independent *master integrals*. Moreover, the very same IBPs allow one to derive differential equations satisfied by the master integrals [31–33], which are typically much simpler to solve compared to attempting their direct integration over the loop momenta. While these steps are in general not straightforward, a lot of progress has been recently made in their systematisation [34–40], and we will not address this aspect in this paper.

Instead, we will focus on the previous step, i.e. on the manipulations required to go from the tensor integrals stemming from the Feynman diagrams, to the corresponding scalar integrals for which the technology above can be applied. Different possible solutions to this problem exist and in what follows we will focus on one possible approach. This consists in deriving suitable *projection operators* which, once applied on the relevant Feynman diagrams, allow one to project out the required combinations of scalar Feynman integrals in terms of *scalar form factors* from the overall, non-perturbative, Lorentz and Dirac tensor structures. To fix the notation, let us consider the scattering of n spin-1 vector bosons, which we assume to be all outgoing for definiteness, i.e.

$$0 \to V_1(p_1) + \ldots + V_n(p_n). \tag{2.1}$$

While working with spin-1 particles will allow us to reduce the clutter in the notation, it should be clear that the inclusion of external particles with different spin (scalar, spinors, etc) would not change any of the conclusions of the following discussion.

We start by observing that Lorentz invariance alone requires that the scattering amplitude for (2.1) can be schematically written as

$$\mathcal{A}(p_1, \ldots, p_{n-1}) = \epsilon_1^{\mu_1} \ldots \epsilon_n^{\mu_n} \, \mathcal{T}_{\mu_1,\ldots,\mu_n}(p_1, \ldots, p_{n-1}) \,, \tag{2.2}$$

where $\epsilon_j^{\mu_j} = \epsilon_j^{\mu_j}(p_j)$ are the polarization vectors associated to the external bosons and $\mathcal{T}_{\mu_1,\ldots,\mu_n}(p_1, \ldots, p_n)$ is a rank-n Lorentz tensor. This tensor may, in turn, be decomposed into a tensor basis compatible with the symmetries of the underlying theory and gauge invariance

$$\mathcal{T}^{\mu_1,\ldots,\mu_n}(p_1, \ldots, p_{n-1}) = \sum_{j=1}^{M} \mathcal{F}_j \, T_j^{\mu_1,\ldots,\mu_n} \,, \tag{2.3}$$

where the $\mathcal{F}_j$ are scalar form factors. As it should be easy to realise, the number of independent tensors M increases extremely fast with the number of external legs. While their exact number depends on whether the external particles are massless or massive, one can easily go from a handful of tensors for 3 external bosons, to $\mathcal{O}(10)$ for 4, up to $\mathcal{O}(100)$ for 5 and so on.

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Each of the form factors $\mathcal{F}_j$ can then be extracted by applying a suitably defined projection operator $\mathcal{P}_j$ on the Feynman diagrams which contribute to the scattering amplitude in the desired theory and to the desired perturbative order. To derive the projectors we use the fact that the M tensors are a basis, and write each projector as a linear combination of the same tensors, contracted with the respective polarisation structures:

$$\mathcal{P}_j = \epsilon^*_{1\,\mu_1} \dots \epsilon^*_{n\,\mu_n} \sum_k c_{jk}\, T_k^{\mu_1,\dots,\mu_n}\,. \tag{2.4}$$

The coefficients $c_{jk} = c_{jk}(d; p_1, \dots, p_{n-1})$ are, in general, rational functions of the number of space-time dimensions d and of the scalar products among the external momenta p_j. They can be determined by applying each of the projectors on the decomposition in eq. (2.2) and imposing that

$$\sum_{\text{pol}} \mathcal{P}_j\, \mathcal{A}(p_1, \dots, p_{n-1}) = \mathcal{F}_j\,, \tag{2.5}$$

where the sum runs over the polarisations of the external particles. More explicitly, the c_{ij} can be computed by inverting the following matrix

$$c^{-1}_{ij} = \left(\sum_{\text{pol}} \epsilon^*_{1\,\mu_1} \dots \epsilon^*_{n\,\mu_n} \epsilon_{1\,\nu_1} \dots \epsilon_{n\,\nu_n} \right) T_i^{\mu_1,\dots,\mu_n} T_j^{\nu_1,\dots,\nu_n}. \tag{2.6}$$

Notice that, in all the equations so far, the polarization vectors are treated symbolically. After summing them over the external polarization, each $\epsilon^*_{i\,\mu_i}\epsilon_{i\,\nu_i}$ is replaced by the expression consistent with the gauge constraints that have been applied in defining the basis in eq. (2.3). In this way, the matrix elements defined in the previous equations are rational functions of the Mandelstam invariants and the space-time dimensions d.

If the matrix in eq. (2.6) can be inverted, all form factors can in principle be computed in terms of scalar Feynman integrals, for which the technology of IBPs and differential equations can be employed. As a next step, one usually starts from the amplitude in eq. (2.2) and fixes the polarisations of the external states, forcing them in $d = 4$ space-time dimensions. This allows one to define *helicity amplitudes*, which can be written as linear combinations of the M scalar form factors $\mathcal{F}_j$. We note that this corresponds to working in the 't Hooft-Veltman scheme (tHV), where external states are taken in 4 space-time dimensions, while virtual ones are taken in d continuous dimensions [41].

While this construction is clearly very general, it should be equally clear that finding the solution of eq. (2.5) (that allows one to define the projectors in the first place) can become highly non-trivial when a large number of tensor structures is involved. Moreover, even if a solution can be found, the projectors themselves can become very soon extremely cumbersome, making their practical use quite difficult. Finally, one might wonder if taking well engineered linear combinations of tensors (and therefore linear combinations of the original form factors) as a new basis of objects could simplify the system in eq. (2.5) and with it, its final solutions. Unfortunately, since virtually any linear combination could work equally well, there is no obvious criterion to select a basis of tensors over any other.

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To show how the complexity of this approach can easily get out of hand, let us consider the prototypical example of the scattering of five massless spin-1 particles in a parity-invariant theory. Among the others, this covers the case of five-gluon scattering in QCD. By generating all possible Lorentz structures, one is left with 1724 tensors. Imposing that the external gluons are transversely polarised, i.e. $\epsilon_j \cdot p_j = 0$ for $j = 1, \ldots, 5$, and imposing invariance under gauge transformations (or equivalently fixing the gauge of the external gluons) reduces their number to 142. We proceed then by writing the scattering amplitude as

$$\mathcal{A}(p_1, \ldots, p_4) = \epsilon_1^{\mu_1} \ldots \epsilon_5^{\mu_5} \sum_{j=1}^{142} \mathcal{F}_j \, T_j^{\mu_1,\ldots,\mu_5}(p_1, \ldots, p_4) \,. \tag{2.7}$$

Following the discussion around eqs. (2.4), (2.5), we can attempt to derive the corresponding 142 projectors

$$\mathcal{P}_j = \epsilon^*_{1\mu_1} \ldots \epsilon^*_{5\mu_5} \sum_{k=1}^{142} c_{jk}(d; p_1, \ldots, p_4) \, T_k^{\mu_1,\ldots,\mu_5} \,. \tag{2.8}$$

The corresponding system of equations for the coefficients c_{jk} is too complicated to be solved by a naive use of Mathematica or FORM and alternative methods must be considered. A possible strategy towards a solution has been outlined in [24]. Another possibility consists in using techniques based on algebraic manipulations over finite fields: the system can be solved numerically modulo prime numbers and the exact symbolic solution can then be reconstructed from multiple numerical evaluations (see e.g. refs. [22, 23, 42]). While these techniques allow us to get to a solution quite easily, it is enough to look at the resulting projectors to understand the limits of this method. Indeed, the 142×142 coefficients c_{jk} in eq. (2.8) occupy alone 1Gb of disk space.[1] Having in mind the complexity of the Feynman diagrams required, for example, to compute the scattering of five gluons at two loops in QCD, it appears evident that such an approach is deemed to fail. Moreover, it should as well be clear that the perspective of using the same approach for even larger numbers of external particles (for example in the six-gluon case) appears entirely unfeasible.

Motivated by these problems, in the next section we describe how most of these limitations can be lifted by defining suitable *physical projector operators* which single out directly the helicity amplitudes required for the calculation we are interested in. As we will see, this approach applied to the case of five-gluon scattering will solve at once many problems. First, the majority of the 142 tensor structures in eq. (2.7) will turn out to be redundant. Moreover, in comparison with the 1Gb of data required to specify the standard projectors, our new physical projector operators will end up being extremely compact and easy to use.

[1]The inversion has been performed using FiniteFlow [23]. The dimension of 1Gb refers to the GCD-simplified (but not factorized) analytic result written to a file. We stress that this calculation was only done as a test and it is not required when using the physical projectors we present in this paper. By comparison, the file attached to this paper as supplementary material contains a full set of physical projectors for the same process in about 750 Kb.

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3 Physical projectors for helicity amplitudes

In this section, we present the main result of this paper. We show that, by projecting directly onto the helicity amplitudes defined in tHV scheme,[2] we can build a set of *physical projectors* having compact analytic expressions and involving a substantially smaller number of tensor structures.

Let us consider once more the general decomposition for the scattering amplitude of n spin-1 bosons in eqs. (2.2), (2.3). Once more, particles of different spin can be accommodated by a straightforward generalisation of this discussion. Having an explicit representation for the general amplitude, we can imagine to consider four-dimensional external states and fix their helicities in all possible ways. We define the total number of helicity amplitudes to be h_λ. Clearly, in a case where not all external particles are different, many of the helicity configurations will not be independent and may be obtained from the independent ones by permutations of the external legs and complex conjugation. We ignore this detail for now. If the helicity of the boson j is λ_j, we write the scattering amplitude as $\mathcal{A}_{\lambda_1,\ldots,\lambda_n}(p_1,\ldots,p_n)$. In the case of n massless external spin-1 bosons, each particle can have two helicity states, such that there will be in total $h_\lambda = 2^n$ different helicity amplitudes. We stress that, while the helicity amplitudes are enough to reconstruct the full structure of the scattering amplitude, typically their number grows with the number of external particles much slower than the number of independent tensors. Indeed, for 5 massless external spin-1 bosons, there are only $h_\lambda = 32$ independent helicity configurations, in comparison with the $M = 142$ different tensor structures discussed in the previous section. For 6 external gluons, there are only $h_\lambda = 64$. Armed with this observation, we would like to define projectors operators which, instead of projecting on all "unphysical" form factors $\mathcal{F}_j$, project only onto the h_λ independent helicity amplitudes.

We first recall that, for all helicities λ_j, we can define explicit polarization states using the spinor-helicity formalism [44–47], in terms of massless spinors $|j\rangle = |p_j\rangle$ and $|j] = |p_j]$ with negative and positive helicity. As an example, polarization vectors ϵ^μ_λ for massless bosons can be defined as

$$\epsilon^\mu_+(p) = \frac{\langle\eta|\sigma^\mu|p]}{\sqrt{2}\,\langle\eta\, p\rangle}, \qquad \epsilon^\mu_-(p) = \frac{\langle p|\sigma^\mu|\eta]}{\sqrt{2}\,[p\,\eta]}, \tag{3.1}$$

where η is an arbitrary reference vector. Analogous formulas exist for polarization states of particles with different spin and massive particles as well. Moreover, when one deals with spinor products, it is often convenient to work with objects which are invariant under little group scaling

$$|j\rangle \to t_j\,|j\rangle\,, \quad |j] \to t_j^{-1}|j]\,. \tag{3.2}$$

It is always possible to define a rescaled amplitude $\overline{\mathcal{A}}_{\lambda_1,\ldots,\lambda_n}(p_1,\ldots,p_{n-1})$ which is invariant under little-group scaling by dividing it by a suitable prefactor $K_{\lambda_1,\ldots,\lambda_n}$ in the spinor

[2]Our approach actually applies to any dimensional regularization scheme where the external states are treated in four dimensions. In particular, the projectors built with our method are also valid in the Four-Dimensional-Helicity scheme [43], since the latter only differs from tHV because of a different treatment of the internal gluon states. In the remainder of this paper, we still only refer to the tHV scheme for simplicity.

products

$$\overline{\mathcal{A}}_{\lambda_1,\ldots,\lambda_n} = \frac{1}{K_{\lambda_1,\ldots,\lambda_n}}\,\mathcal{A}_{\lambda_1,\ldots,\lambda_n}. \tag{3.3}$$

While there is no unique choice for the prefactor $K_{\lambda_1,\ldots,\lambda_n}$, it can be easily built based on the external helicities and it is independent of the loop order. Explicit examples will be given in the next section. The rescaled amplitudes $\overline{\mathcal{A}}_{\lambda_1,\ldots,\lambda_n}$ are by construction invariant under the little group transformation (3.2) and thus independent of any spinor phase.

With these concepts in mind, we start from eqs. (2.2), (2.3) and fix the helicities of the external particles as

$$\overline{\mathcal{A}}_{\lambda_1,\ldots,\lambda_n}(p_1,\ldots,p_{n-1}) = \frac{1}{K_{\lambda_1,\ldots,\lambda_n}}\,\epsilon^{\mu_1}_{\lambda_1}\ldots\epsilon^{\mu_n}_{\lambda_n}\sum_{j=1}^{M}\mathcal{F}_j\,T_j^{\mu_1,\ldots,\mu_n}. \tag{3.4}$$

We stress that, while in the previous sections we treated the external states mostly symbolically (i.e. only in order to yield a sum over polarizations), here the external polarization states $\epsilon^{\mu_j}_{\lambda_j}$ are explicit polarizations built out of spinors. We can rewrite the previous equation as

$$\overline{\mathcal{A}}_{\lambda_1,\ldots,\lambda_n}(p_1,\ldots,p_{n-1}) = \sum_{j=1}^{M}\mathcal{F}_j\,R_j^{\lambda_1,\ldots,\lambda_n} \tag{3.5}$$

where we have defined

$$R_j^{\lambda_1,\ldots,\lambda_n} = \frac{1}{K_{\lambda_1,\ldots,\lambda_n}}\,\epsilon^{\mu_1}_{\lambda_1}\ldots\epsilon^{\mu_n}_{\lambda_n}\,T_j^{\mu_1,\ldots,\mu_n}. \tag{3.6}$$

Because the objects R_j defined in eq. (3.6) are also invariant under little group transformations, they can be parametrised in terms of $3n-10$ invariants x_j,

$$R_j^{\lambda_1,\ldots,\lambda_n} = R_j^{\lambda_1,\ldots,\lambda_n}(x_1,\ldots,x_{3n-10}). \tag{3.7}$$

These invariants, in turn, can always be chosen such that all scalar quantities involving spinors and polarization vectors are rational functions of the x_j. We point out that, for kinematics with external massive particles, the functions $R_j^{\lambda_1,\ldots,\lambda_n}$ will also depend on the external masses.

This allows us to formulate our central result. In fact, at this point, we simply promote eq. (3.5) to become a new *helicity projection operator* by the formal substitution $\mathcal{F}_j \to \mathcal{P}_j$, where $\mathcal{P}_j$ is the projector that singles out $\mathcal{F}_j$ as defined in eqs. (2.4), (2.5). We have defined in this way a set of as many helicity projectors as the number of independent helicity amplitudes

$$\mathcal{P}_{\lambda_1,\ldots,\lambda_n} = \sum_{j=1}^{M} R_j^{\lambda_1,\ldots,\lambda_n}\,\mathcal{P}_j\,. \tag{3.8}$$

By using eq. (2.4) and remembering that all scalar products $p_i\cdot p_j$ can be written as rational functions in the variables x_j, we immediately see that the new helicity projectors will be also written as a linear combination of the original tensors

$$\mathcal{P}_{\lambda_1,\ldots,\lambda_n}(p_1,\ldots,p_{n-1}) = \epsilon^{*}_{1\mu_1}\ldots\epsilon^{*}_{n\mu_n}\sum_{k=1}^{M}\mathcal{C}_k^{\lambda_1,\ldots,\lambda_n}T_k^{\mu_1,\ldots,\mu_n} \tag{3.9}$$

where the $\mathcal{C}_k = \mathcal{C}_k^{\lambda_1,\dots,\lambda_n}(d; x_1, \dots, x_{3n-10})$ will be rational functions in d and in the x_j. One should realise here that, while the projectors in eq. (3.9) are defined to project out the helicity amplitudes $\mathcal{A}_{\lambda_1,\dots,\lambda_n}$, the polarisation vectors ϵ_j that appear on the right-hand side of the previous equation do not have their polarisation fixed. On the contrary, as already explained above, they are applied on the Feynman diagrams by summing over their helicities as described in eq. (2.5). By construction, these projectors single out the (rescaled) helicity amplitudes

$$\overline{\mathcal{A}}_{\lambda_1,\dots,\lambda_n} = \sum_{\text{pol}} \mathcal{P}_{\lambda_1,\dots,\lambda_n} \mathcal{A}(p_1, \dots, p_{n-1}) \,. \tag{3.10}$$

As hinted to above, in general the sum in eq. (3.9) should run over all M independent tensors and the coefficients $\mathcal{C}_k^{\lambda_1,\dots,\lambda_n}$ will depend explicitly on the number of space-time dimensions d. Nevertheless, it turns out that for processes with more than four external legs, this is not the case. In fact, we observe that in order to compute helicity amplitudes we are allowed to only consider the projection of the tensor $\mathcal{T}_{\mu_1,\dots,\mu_n}$ defined in eq. (2.2) onto the four-dimensional physical space, where the external polarisations live. For processes with five or more external legs, this four-dimensional space is spanned by four independent external momenta. Hence, we may restrict the tensor basis to span the physical space defined by four independent external legs only. This can be effectively achieved in the decomposition of eq. (2.3), simply by removing all tensors containing the metric tensor $g^{\mu\nu}$. This can also be seen by observing that in this case we can decompose $g^{\mu\nu}$ into a four-dimensional part and a (-2ϵ)-dimensional part, as

$$g^{\mu\nu} = g^{\mu\nu}_{[4]} + g^{\mu\nu}_{[-2\epsilon]} = g^{\mu\nu}_{[-2\epsilon]} + \mathcal{O}(p_i^\mu p_j^\nu), \tag{3.11}$$

where the last equality states that the four dimensional metric tensor $g^{\mu\nu}_{[4]}$ is a linear combination of tensors $p_i^\mu p_j^\nu$ built out of the four independent external momenta. Hence, we are allowed to replace $g^{\mu\nu} \to g^{\mu\nu}_{[-2\epsilon]}$ in our general tensor decomposition. We then observe that tensors with $g^{\mu\nu}_{[-2\epsilon]}$ are trivially orthogonal to the other tensors and the inversion of the matrix in eq. (2.6) can be performed separately in the four-dimensional and in the (-2ϵ)-dimensional space. Moreover, all the coefficients $R_j^{\lambda_1,\dots,\lambda_n}$ multiplying tensors which depend on $g^{\mu\nu}_{[-2\epsilon]}$ also vanish by orthogonality. Putting everything together, this is effectively equivalent to removing the metric tensor $g^{\mu\nu}$ from the very beginning in the tensor decomposition. A corollary of this observation is the fact that the physical projectors for these processes are independent of the space-time dimension d (because such a dependence may only come from the metric tensor). We stress, again, that this is true only for processes with more than four external legs.

Let us see what this implies for a generic n-point amplitude with $n \geq 5$ vector bosons. As we just stated, we may build a physical tensor basis directly from tensor of the form $p_{j_1}^{\mu_1} \cdots p_{j_n}^{\mu_n}$, where all the p_{j_k} are drawn from a subset of four (independent) external momenta. In total, we have 4^n such combinations. For spin-1 bosons we can always impose transversality for each external particle, i.e. $\epsilon_j \cdot p_j = 0$, going down in this way to 3^n tensors. Moreover, if the bosons are massless, by fixing their gauge, e.g. with the cyclic choice $\epsilon_j \cdot p_{j+1} = 0$, with $p_{n+1} = p_1$, we are left with a total of 2^n independent tensors. This is consistent with the fact that, for the scattering of n massless spin-1 bosons, there are $h_\lambda = 2^n$ independent helicity amplitudes and we expect only 2^n tensors to be relevant for their reconstruction. Hence, the inversion needed for computing the physical projectors can be performed in a (significantly smaller) h_λ-dimensional tensor subspace.

We verified this by computing physical projectors in several five-point examples. In the case of five-gluon scattering (which will be discussed more in detail in the next section), as we saw in section 2, the general d-dimensional tensor structure requires 142 tensors after gauge-invariance and transversality conditions have been imposed on the external gluons, while only the $h_\lambda = 2^5 = 32$ structures of the form $p_{j_1}^{\mu_1} p_{j_2}^{\mu_2} p_{j_3}^{\mu_3} p_{j_4}^{\mu_4} p_{j_5}^{\mu_5}$ turn out to contribute to the helicity projectors. Similarly, we also studied the scattering of four gluons and a scalar, which is relevant for Higgs boson plus two jets production at hadron colliders, $gg \to Hgg$. In this case one has $h_\lambda = 2^4 = 16$ independent helicity amplitudes. On the other hand, the full d-dimensional tensor structure would require 43 different tensors after gauge-invariance and transversality conditions have been imposed on the external gluons. We verified explicitly that in order to project directly over helicity amplitudes we need, as expected, only the 16 tensor structures built out of the 4 independent external momenta $p_{j_1}^{\mu_1} p_{j_2}^{\mu_2} p_{j_3}^{\mu_3} p_{j_4}^{\mu_4}$.

To summarise, given these considerations, in order to reconstruct the helicity projectors defined in equation eq. (3.8) one never needs to go through the whole d-dimensional tensor structure. In practice, if the number of external legs is $n \geq 5$, we simply reinterpret all formulas above, i.e. eqs. (3.4), (3.5), (3.8), (3.9) with $M = h_\lambda$, as the number of the independent tensors in $d = 4$ dimensions built from the combinations $p_{j_1}^{\mu_1} \cdots p_{j_n}^{\mu_n}$. This allows us to simplify even further the derivation of the helicity projectors since, in eq. (2.6), only a $h_\lambda \times h_\lambda$ matrix has to be considered instead of a typically much larger $M \times M$ one. One may also perform the inversion in eq. (2.6), either in the full tensor space or in the physical subspace, numerically over finite fields and reconstruct the analytic physical projectors directly. With the latter approach, using FiniteFlow [23], the analytic reconstruction of the physical projectors becomes extremely efficient. For five-point processes, it typically takes a few seconds on a modern laptop.

We conclude this section with an observation about the choice of variables x_j, which we will then make more explicit in the next section with an example. As already stated, one can choose invariants which offer a rational parametrization of the spinor components, up to a little group and a Poincaré transformation. With this choice, all the functions $R_j^{\lambda_1,\dots,\lambda_n}$ and $\mathcal{C}_k^{\lambda_1,\dots,\lambda_n}$ above are rational. A notable example are momentum twistor variables [48–50]. Alternatively, one may choose to use Mandelstam invariants instead. In this case, one can still obtain a unique representation for $R_j^{\lambda_1,\dots,\lambda_n}$ and $\mathcal{C}_k^{\lambda_1,\dots,\lambda_n}$ by identifying a set of independent square roots and requiring the result to be multilinear in these square roots. The coefficients of each independent monomial in these square roots, which are rational functions of the Mandelstam invariants, can be treated independently of each other. As an example, with massless 5-point kinematics, we have only one square root which can be identified with the parity odd invariant $\mathrm{tr}_5 = \mathrm{tr}(\gamma_5\, p_1\, p_2\, p_3\, p_4)$. Every little-group invariant function R of the spinor variables can thus be written in a unique way as $R = R_+ + \mathrm{tr}_5\, R_-$ where the parity-even and odd components R_+ and R_- are rational functions of Mandelstam invariants. In practice, in order to obtain such a representation for our physical projectors, it is often convenient to first get a rational representation of the $R_j^{\lambda_1,\ldots,\lambda_n}$ in terms of momentum twistors variables and then convert it back to Mandelstam invariants, since this sidesteps the need of performing tedious spinor algebra. After that, the parity even and odd components of $\mathcal{C}_k^{\lambda_1,\ldots,\lambda_n}$ can be computed from the corresponding ones of $R_j^{\lambda_1,\ldots,\lambda_n}$, independently of each other. The same approach easily generalises to the presence of several independent square roots.

Before going on with an explicit example, it is interesting to compare our method to the one recently proposed in [25], which also exploits explicitly simplifications coming from taking external particles in $d = 4$ space-time dimensions. While this is conceptually similar to our approach and we expect the conclusions to be equivalent, there are some important differences. In comparison to [25], we do not need to perform an explicit decomposition of the external polarisation states in terms of four-dimensional external momenta in order to see the relevant simplifications in the tensor structure. Using our approach, our helicity projectors are *uniquely* written as linear combinations of standard, d-dimensional projector operators. This allow us to perform all manipulations in the standard tHV scheme, without having to make sure that our d-dimensional regularisation scheme is consistent. Finally, this different point of view allows us to see straight-away how the potential of the method can be fully exploited only starting from $n \geq 5$ external particles, where the simplifications to the tensor structure become more substantial. As described above, for these processes, we can immediately exclude all tensor structures which are not independent, by using the formal decomposition in eq. (3.11).

4 Physical projectors for five-gluon scattering

In order to show the potential of the method that we propose, in this section we apply it to the case of five-gluon scattering in QCD. In section 2, we have already pointed to the difficulties in applying standard d-dimensional projectors to the scattering of five gluons. In particular, we have shown that a generic tensor decomposition requires 142 independent structures, see eq. (2.7), and that the corresponding projector operators given in eq. (2.8) appear to be extremely cumbersome.

Let us then start off by considering the scattering of five massless gluons

$$0 \to g(p_1) + g(p_2) + g(p_3) + g(p_4) + g(p_5)\,,$$

with $p_5 = -p_1 - p_2 - p_3 - p_4$ and $p_j^2 = 0$ for $j = 1, \ldots, 5$. The amplitude depends on five independent kinematical invariants, which we pick to be s_{12}, s_{23}, s_{34}, s_{45} and s_{51}, where $s_{ij} = (p_i + p_j)^2$. The parity-odd invariant

$$\mathrm{tr}_5 = \mathrm{tr}(\gamma_5\, \not{p}_1\, \not{p}_2\, \not{p}_3\, \not{p}_4)$$

will also play an important role in the following discussion. As already outlined in eq. (2.7), the most general tensor decomposition of the scattering amplitude reads

$$\mathcal{A}(p_1,\ldots,p_4) = \epsilon_1^{\mu_1}\ldots\epsilon_5^{\mu_5}\sum_{j=1}^{142}\mathcal{F}_j\,T_j^{\mu_1,\ldots,\mu_5}(p_1,\ldots,p_4)\,, \tag{4.1}$$

where the tensors $T_j^{\mu_1,\ldots,\mu_5}(p_1,\ldots,p_4)$ are built out of the four independent momenta p_j^{μ}, $j=1,\ldots,4$ and the metric tensor $g^{\mu\nu}$. In order to be left with only 142 tensor structures we have used the fact that $\epsilon_j\cdot p_j = 0$ for $j=1,\ldots,5$ and we have also imposed a cyclic gauge choice on the external gluons as follows

$$\epsilon_1\cdot p_2 = \epsilon_2\cdot p_3 = \epsilon_3\cdot p_4 = \epsilon_4\cdot p_5 = \epsilon_5\cdot p_1 = 0\,. \tag{4.2}$$

While fixing the gauge explicitly is not necessary, it is useful to obtain tensors that are as compact as possible.

As argued in detail in the previous section, since the scattering amplitude depends on 4 independent momenta and we are interested in projecting directly on the physical helicity amplitudes, we can drop all tensors in (4.1) which depend explicitly on the metric tensor $g^{\mu\nu}$. In this way we are left, as expected, with the 32 tensors

$$\begin{aligned}
T_1^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_1^{\mu_3}\,p_1^{\mu_4}\,p_2^{\mu_5}\,p_3^{\mu_1}\,, & T_2^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_1^{\mu_3}\,p_2^{\mu_4}\,p_2^{\mu_5}\,p_3^{\mu_1}\,,\\
T_3^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_1^{\mu_3}\,p_1^{\mu_4}\,p_3^{\mu_1}\,p_3^{\mu_5}\,, & T_4^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_1^{\mu_3}\,p_2^{\mu_4}\,p_3^{\mu_1}\,p_3^{\mu_5}\,,\\
T_5^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_1^{\mu_4}\,p_2^{\mu_3}\,p_2^{\mu_5}\,p_3^{\mu_1}\,, & T_6^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_2^{\mu_3}\,p_2^{\mu_4}\,p_2^{\mu_5}\,p_3^{\mu_1}\,,\\
T_7^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_1^{\mu_4}\,p_2^{\mu_3}\,p_3^{\mu_1}\,p_3^{\mu_5}\,, & T_8^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_2^{\mu_3}\,p_2^{\mu_4}\,p_3^{\mu_1}\,p_3^{\mu_5}\,,\\
T_9^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_3}\,p_1^{\mu_4}\,p_2^{\mu_5}\,p_3^{\mu_1}\,p_4^{\mu_2}\,, & T_{10}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_3}\,p_2^{\mu_4}\,p_2^{\mu_5}\,p_3^{\mu_1}\,p_4^{\mu_2}\,,\\
T_{11}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_3}\,p_1^{\mu_4}\,p_3^{\mu_1}\,p_3^{\mu_5}\,p_4^{\mu_2}\,, & T_{12}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_3}\,p_2^{\mu_4}\,p_3^{\mu_1}\,p_3^{\mu_5}\,p_4^{\mu_2}\,,\\
T_{13}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_4}\,p_2^{\mu_3}\,p_2^{\mu_5}\,p_3^{\mu_1}\,p_4^{\mu_2}\,, & T_{14}^{\mu_1,\ldots,\mu_5} &= p_2^{\mu_3}\,p_2^{\mu_4}\,p_2^{\mu_5}\,p_3^{\mu_1}\,p_4^{\mu_2}\,,\\
T_{15}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_4}\,p_2^{\mu_3}\,p_3^{\mu_1}\,p_3^{\mu_5}\,p_4^{\mu_2}\,, & T_{16}^{\mu_1,\ldots,\mu_5} &= p_2^{\mu_3}\,p_2^{\mu_4}\,p_3^{\mu_1}\,p_3^{\mu_5}\,p_4^{\mu_2}\,,\\
T_{17}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_1^{\mu_3}\,p_1^{\mu_4}\,p_2^{\mu_5}\,p_4^{\mu_1}\,, & T_{18}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_1^{\mu_3}\,p_2^{\mu_4}\,p_2^{\mu_5}\,p_4^{\mu_1}\,,\\
T_{19}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_1^{\mu_3}\,p_1^{\mu_4}\,p_3^{\mu_5}\,p_4^{\mu_1}\,, & T_{20}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_1^{\mu_3}\,p_2^{\mu_4}\,p_3^{\mu_5}\,p_4^{\mu_1}\,,\\
T_{21}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_1^{\mu_4}\,p_2^{\mu_3}\,p_2^{\mu_5}\,p_4^{\mu_1}\,, & T_{22}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_2^{\mu_3}\,p_2^{\mu_4}\,p_2^{\mu_5}\,p_4^{\mu_1}\,,\\
T_{23}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_1^{\mu_4}\,p_2^{\mu_3}\,p_3^{\mu_5}\,p_4^{\mu_1}\,, & T_{24}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_2}\,p_2^{\mu_3}\,p_2^{\mu_4}\,p_3^{\mu_5}\,p_4^{\mu_1}\,,\\
T_{25}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_3}\,p_1^{\mu_4}\,p_2^{\mu_5}\,p_4^{\mu_1}\,p_4^{\mu_2}\,, & T_{26}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_3}\,p_2^{\mu_4}\,p_2^{\mu_5}\,p_4^{\mu_1}\,p_4^{\mu_2}\,,\\
T_{27}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_3}\,p_1^{\mu_4}\,p_3^{\mu_5}\,p_4^{\mu_1}\,p_4^{\mu_2}\,, & T_{28}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_3}\,p_2^{\mu_4}\,p_3^{\mu_5}\,p_4^{\mu_1}\,p_4^{\mu_2}\,,\\
T_{29}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_4}\,p_2^{\mu_3}\,p_2^{\mu_5}\,p_4^{\mu_1}\,p_4^{\mu_2}\,, & T_{30}^{\mu_1,\ldots,\mu_5} &= p_2^{\mu_3}\,p_2^{\mu_4}\,p_2^{\mu_5}\,p_4^{\mu_1}\,p_4^{\mu_2}\,,\\
T_{31}^{\mu_1,\ldots,\mu_5} &= p_1^{\mu_4}\,p_2^{\mu_3}\,p_3^{\mu_5}\,p_4^{\mu_1}\,p_4^{\mu_2}\,, & T_{32}^{\mu_1,\ldots,\mu_5} &= p_2^{\mu_3}\,p_2^{\mu_4}\,p_3^{\mu_5}\,p_4^{\mu_1}\,p_4^{\mu_2}\,.
\end{aligned} \tag{4.3}$$

With these tensors we can therefore rewrite (4.1) as

$$\mathcal{A}(p_1,\ldots,p_4) = \epsilon_1^{\mu_1}\ldots\epsilon_5^{\mu_5}\sum_{j=1}^{32}\mathcal{F}_j\,T_j^{\mu_1,\ldots,\mu_5}(p_1,\ldots,p_4) + \mathcal{O}(g^{\mu\nu}_{[-2\epsilon]})\,, \tag{4.4}$$

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where $\mathcal{O}(g^{\mu\nu}_{[-2\epsilon]})$ indicates tensor structures which live in the (-2ϵ)-dimensional space and do not contribute to helicity amplitudes.

Starting from this tensor, we put to zero all terms proportional to $\mathcal{O}(g^{\mu\nu}_{[-2\epsilon]})$ and fix the helicities of the five external gluons in all possible ways by using the spinor-helicity formalism. For every helicity configuration, we then define a rescaled amplitude $\overline{\mathcal{A}}_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5}$ which is invariant under little group transformations, see eq. (3.2). This can be achieved by dividing the corresponding amplitudes by a suitable prefactor $K_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5}$ for the $h_\lambda = 2^5 = 32$ different helicity configurations. For the helicity configurations which are zero at tree-level in QCD we choose

$$K_{+++++} = \frac{s_{12}^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\,, \qquad K_{-++++} = \frac{\left(\langle 12\rangle[23]\langle 31\rangle\right)^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \tag{4.5}$$

and cyclic permutations thereof. For the MHV amplitudes, instead, we can choose the tree-level Parke-Taylor amplitudes as rescaling factor, e.g.

$$K_{--+++} = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\,, \tag{4.6}$$

and similarly for the remaining 9 configurations. Scaling factors for the helicity configurations with three or more negative helicities can be obtained by complex conjugation of eqs. (4.5), (4.6).

Before deriving the helicity projectors, it is convenient to obtain a rational parametrisation of the spinor products $\langle ij\rangle$, $[ij]$ and of the external invariants s_{ij}, since this avoids the need of performing tedious spinor algebra. For the case of five massless external particles, we can use the parametrisation in terms of momentum twistors [48] provided in [51]. We define a momentum twistor Z_j for each momentum and write the parametrisation in matrix form as

$$Z = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1+x_2}{x_1 x_2} & \frac{1+x_3(1+x_2)}{x_1 x_2 x_3} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{x_4}{x_2} & 1 \\ 0 & 0 & 1 & 1 & \frac{x_4 - x_5}{x_4} \end{pmatrix}, \tag{4.7}$$

where the x_j are momentum twistor variables. The kinematic invariants can the be written as

$$\begin{aligned} &s_{12} = x_1\,, && s_{23} = x_1 x_4\,, && s_{34} = x_1(x_4 + x_3 x_4 - x_2 x_3 + x_2 x_3 x_5)/x_2 \\ &s_{45} = x_1 x_5\,, && s_{51} = x_1 x_3(x_2 - x_4 + x_5)\,. && \end{aligned} \tag{4.8}$$

Similarly, for the parity-odd invariant we find

$$\mathrm{tr}_5 = -x_1^2\left(x_3\left(x_5-1\right)x_2^2 + \left(2x_3+1\right)x_4 x_2 - \left(x_3+1\right)x_4\left(x_4 - x_5\right)\right)/x_2. \tag{4.9}$$

An explicit parametrisation of the spinor components in terms of these variables is given in eq. (5.10) of [22] (see also ref. [50] for a generalisation to other processes).

For each helicity configuration, we now proceed with defining the functions $R_j^{\lambda_1,\ldots,\lambda_5}$, see eq. (3.6), as rational functions of the momentum twistor variables. As pointed out at the end of section 3, we may now choose to continue using the variables x_j or alternatively switch back to Mandelstam invariants. In this example we pick the latter option. It is straightforward to invert the relations in eqs. (4.8), (4.9) and write

$$R_j^{\lambda_1,\ldots,\lambda_5} = R_{+,j}^{\lambda_1,\ldots,\lambda_5}(s_{ij}) + \mathrm{tr}_5\, R_{-,j}^{\lambda_1,\ldots,\lambda_5}(s_{ij}), \tag{4.10}$$

where $R_{\pm,j}^{\lambda_1,\ldots,\lambda_5}$ are rational functions of the Mandelstam invariants s_{ij}. Notice that this representation is unique.

Having defined the functions $R_j^{\lambda_1,\ldots,\lambda_5}$, we are now ready to reconstruct our physical projectors, defined as in eq. (3.8). We use FiniteFlow [22] to invert the 32×32 matrix and reconstruct directly the physical projectors as linear combinations of the 32 tensors in (4.3)

$$\mathcal{P}_{\lambda_1,\ldots,\lambda_5} = \epsilon^*_{1\,\mu_1}\ldots\epsilon^*_{5\,\mu_5}\sum_{k=1}^{32}\mathcal{C}_k^{\lambda_1,\ldots,\lambda_5}T_k^{\mu_1,\ldots,\mu_5}. \tag{4.11}$$

Similarly to the coefficients $R_j^{\lambda_1,\ldots,\lambda_5}$ above, we can write the $\mathcal{C}_k^{\lambda_1,\ldots,\lambda_5}$ in eq. (4.11) as

$$\mathcal{C}_k^{\lambda_1,\ldots,\lambda_5} = \mathcal{C}_{+,k}^{\lambda_1,\ldots,\lambda_5} + \mathrm{tr}_5\,\mathcal{C}_{-,k}^{\lambda_1,\ldots,\lambda_5} \tag{4.12}$$

where the parity even and odd parts $\mathcal{C}_{+,k}^{\lambda_1,\ldots,\lambda_5}$ and $\mathcal{C}_{-,k}^{\lambda_1,\ldots,\lambda_5}$ are rational functions of the Mandelstam invariants s_{ij} and are only determined by $R_{+,j}^{\lambda_1,\ldots,\lambda_5}$ and $R_{-,j}^{\lambda_1,\ldots,\lambda_5}$ respectively. Explicit expressions for the coefficients $\mathcal{C}_{\pm,k}^{\lambda_1,\ldots,\lambda_5}$ for a full set of helicity configurations are given in ancillary files. As exemplification, we write down explicitly the coefficients of the parity-even part of the projector on the all-plus helicity amplitude. By defining

$$\mathcal{C}_{+,k}^{+++++} = \frac{4s_{23}s_{34}s_{45}s_{51}}{\sqrt{2}\,s_{12}\Delta(p_1,p_2,p_4,p_4)^2}\,\overline{\mathcal{C}}_{+,k}^{+++++}$$

where $\Delta(p_1,p_2,p_4,p_4)$ is the Gram-determinant of the four momenta

$$\begin{aligned}&\Delta(p_1,p_2,p_4,p_4)\\&\quad=\left(-s_{23}s_{34}+s_{12}\left(s_{23}-s_{51}\right)+s_{45}\left(s_{34}+s_{51}\right)\right)^2+4s_{34}\left(s_{12}+s_{23}-s_{45}\right)s_{45}s_{51}\,,\end{aligned}$$

we find

$$\begin{aligned}
\overline{\mathcal{C}}_{+,1}^{+++++} &= \left(s_{12}+s_{23}-s_{34}-s_{45}\right)\left(s_{23}+s_{34}-s_{51}\right)^2,\\
\overline{\mathcal{C}}_{+,2}^{+++++} &= \left(s_{23}+s_{34}-s_{51}\right)\left(s_{23}^2-\left(s_{45}+s_{51}\right)s_{23}+s_{12}\left(s_{23}-s_{51}\right)+\left(s_{34}+s_{45}\right)s_{51}\right),\\
\overline{\mathcal{C}}_{+,3}^{+++++} &= \left(s_{23}+s_{34}-s_{51}\right)^2\left(s_{12}-s_{34}+s_{51}\right),\\
\overline{\mathcal{C}}_{+,4}^{+++++} &= \left(s_{23}+s_{34}-s_{51}\right)\left(s_{12}\left(s_{23}-s_{51}\right)+\left(s_{23}+s_{34}-s_{51}\right)s_{51}\right),\\
\overline{\mathcal{C}}_{+,5}^{+++++} &= \left(s_{12}+s_{23}-s_{45}\right)\left(s_{23}+s_{34}-s_{51}\right)\left(s_{23}-s_{45}-s_{51}\right),\\
\overline{\mathcal{C}}_{+,6}^{+++++} &= \left(s_{12}+s_{23}-s_{45}\right)\left(s_{23}+s_{34}-s_{51}\right)\left(s_{23}-s_{45}-s_{51}\right),
\end{aligned}$$

$$
\begin{aligned}
\overline{\mathcal{C}}^{+++++}_{+,7} &= -\left(s_{23}+s_{34}-s_{51}\right)\left(s_{12}-s_{34}+s_{51}\right)\left(-s_{23}+s_{45}+s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,8} &= \left(s_{23}+s_{34}-s_{51}\right)\left(s_{23}-s_{45}-s_{51}\right)\left(s_{12}+s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,9} &= \left(s_{12}^2+\left(s_{23}-s_{34}-s_{45}\right)s_{12}+s_{23}\left(s_{23}-s_{45}-s_{51}\right)\right)\left(s_{23}+s_{34}-s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,10} &= \left(s_{23}-s_{51}\right)s_{12}^2+\left(2s_{23}^2-2\left(s_{45}+s_{51}\right)s_{23}+\left(s_{34}+s_{45}\right)s_{51}\right)s_{12}+s_{23}\left(-s_{23}+s_{45}+s_{51}\right)^2,\\
\overline{\mathcal{C}}^{+++++}_{+,11} &= s_{12}\left(s_{23}+s_{34}-s_{51}\right)\left(s_{12}-s_{34}+s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,12} &= s_{12}\left(s_{23}^2-s_{45}s_{23}+s_{12}\left(s_{23}-s_{51}\right)+\left(s_{34}-s_{51}\right)s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,13} &= \left(s_{12}+s_{23}-s_{45}\right)\left(s_{23}+s_{34}-s_{51}\right)\left(s_{23}-s_{45}-s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,14} &= \left(s_{12}+s_{23}-s_{45}\right)\left(s_{23}-s_{45}-s_{51}\right)\left(s_{12}+s_{23}-s_{45}-s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,15} &= 0,\\
\overline{\mathcal{C}}^{+++++}_{+,16} &= s_{12}\left(s_{12}+s_{23}-s_{45}\right)\left(s_{23}-s_{45}-s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,17} &= s_{23}\left(s_{12}+s_{23}-s_{45}\right)\left(s_{23}+s_{34}-s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,18} &= s_{23}\left(s_{12}+s_{23}-s_{45}\right)\left(s_{23}+s_{34}-s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,19} &= s_{23}\left(s_{23}+s_{34}-s_{51}\right)\left(s_{12}-s_{34}+s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,20} &= s_{23}\left(s_{23}+s_{34}-s_{51}\right)\left(s_{12}+s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,21} &= -\left(s_{12}+s_{23}-s_{45}\right)\left(-s_{23}^2+\left(-s_{34}+s_{45}+s_{51}\right)s_{23}+s_{34}\left(s_{12}-s_{34}+s_{51}\right)\right),\\
\overline{\mathcal{C}}^{+++++}_{+,22} &= \left(s_{23}+s_{34}\right)\left(s_{12}+s_{23}-s_{45}\right)\left(s_{23}-s_{45}-s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,23} &= -\left(s_{12}-s_{34}+s_{51}\right)\left(-s_{23}^2+\left(s_{45}+s_{51}\right)s_{23}+s_{12}s_{34}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,24} &= \left(s_{23}-s_{45}-s_{51}\right)\left(s_{12}\left(s_{23}+s_{34}\right)+s_{23}s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,25} &= s_{23}\left(s_{12}+s_{23}-s_{45}\right)\left(s_{23}+s_{34}-s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,26} &= s_{23}\left(s_{12}+s_{23}-s_{45}\right)\left(s_{12}+s_{23}-s_{45}-s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,27} &= 0,\\
\overline{\mathcal{C}}^{+++++}_{+,28} &= s_{12}s_{23}\left(s_{12}+s_{23}-s_{45}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,29} &= -\left(s_{12}+s_{23}-s_{45}\right)^2\left(s_{12}-s_{23}-s_{34}+s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,30} &= \left(s_{12}+s_{23}-s_{45}\right)^2\left(s_{23}-s_{45}-s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,31} &= -s_{12}\left(s_{12}+s_{23}-s_{45}\right)\left(s_{12}-s_{34}+s_{51}\right),\\
\overline{\mathcal{C}}^{+++++}_{+,32} &= s_{12}\left(s_{12}+s_{23}-s_{45}\right)\left(s_{23}-s_{45}-s_{51}\right).
\end{aligned}
\tag{4.13}
$$

As a check of the consistency of our approach, we can obtain the same result starting from a full d-dimensional tensor decomposition. In particular, we could ignore the fact that $g^{\mu\nu}$ is not linearly independent and decide to start from the full d-dimensional tensor in eq. (4.1). If we do so, we can formally write the physical helicity projectors as linear combinations of the original 142 d-dimensional projectors. We then use FiniteFlow [22] to invert the corresponding 142×142 matrix in eq. (2.6) numerically and use this to reconstruct only the physical projectors directly in terms of the original tensor structures $T_j^{\mu_1,\ldots,\mu_5}$, as we did in eq. (3.9). The analytic reconstruction takes a couple of minutes on a modern laptop. As a result, as expected, we find that all the coefficients which multiply the tensors which depend on $g^{\mu\nu}$ turn out to be zero and we can recover the very same result discussed above. Clearly, by removing the (-2ϵ)-dimensional tensors from the beginning, all manipulations are much simpler and reconstruction procedure runs through only in a few seconds.

4.1 Five-gluon scattering at one-loop in QCD

As a validation of the helicity projectors newly derived, we have used them to compute the one-loop corrections to five-gluon scattering in QCD. While this calculation is rather simple using modern one-loop techniques based on either generalised unitarity or integrand reduction, see for example ref. [52], it would clearly constitute a challenge using standard projection-based techniques.

Following common practice in these calculations, we decompose the tree-level and one-loop five-gluon helicity amplitudes in terms of coloured-ordered primitive amplitudes. It is well known that all one-loop primitive amplitudes can be obtained from the coefficient of the colour structure $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5})$ of the following helicity configurations

$$\mathcal{A}_{+++++}\,, \quad \mathcal{A}_{-++++}\,, \quad \mathcal{A}_{--+++}\,, \quad \mathcal{A}_{-+-++}\,.$$

In order to compute these amplitudes, we first generate all relevant Feynman diagrams with QGRAF [53] and sort them selecting only the ones corresponding to the relevant colour-ordered amplitudes. We then proceed by applying on each diagrams the projectors defined in eq. (4.11). In practice, we prefer to compute for each diagram the 32 contractions with the 32 tensors in eq. (4.3) independently. More explicitly, for every Feynman diagram $\mathcal{D}_j$, we extract the gluon polarisation vectors

$$\mathcal{D}_j = (\epsilon_{1\,\mu_1}\dots\epsilon_{5\,\mu_5})D_j^{\mu_1,\dots,\mu_5}\,,$$

and compute the quantity

$$\mathcal{D}_{jk} = \sum_{\mathrm{pol}} \left(\epsilon^*_{1\,\mu_1}\dots\epsilon^*_{5\,\mu_5}\epsilon_{1\,\nu_1}\dots\epsilon_{5\,\nu_5}\right) T_k^{\mu_1,\dots,\mu_5}\, D_j^{\nu_1,\dots,\nu_5}\,. \tag{4.14}$$

Due to the transversality and gauge constraints that we imposed on the tensor structures, our polarisations sums are given by

$$\sum_{\mathrm{pol}} \epsilon^*_{1,\mu_1}\,\epsilon_{1\,\nu_1} = -g_{\mu_1\nu_1} + \frac{p_{1,\mu_1}p_{2,\nu_1}+p_{2,\mu_1}p_{1,\nu_1}}{p_1\cdot p_2}\,, \tag{4.15}$$

and cyclic permutations thereof. Once all $\mathcal{D}_{jk}$ have been computed, the relevant helicity amplitudes can be computed by summing all Feynman diagrams and assembling them together as in eq. (4.11). While all these manipulations could be performed efficiently using FiniteFlow [23], the simplicity of the tensor structures and of the helicity projectors allow us to perform them using FORM [26] and Reduze [27] in few hours on a laptop. In our calculation we have included the full dependence on the number of colours N_c and the number of fermions N_f and we have verified explicitly that our unrenormalised helicity amplitudes agree with known results, even before substituting the explicit analytical results for the master integrals [54].

5 Conclusions

We presented an efficient method for building physical projector operators for helicity amplitudes, which is suitable for applications to multi-leg processes. While it is common belief that a projector-based approach to compute multi-loop multi-leg scattering amplitudes in perturbative QFT becomes soon impractical due to the proliferation of the number of tensor structures and of the complexity of the corresponding projectors required, in this paper we have shown that this is not necessarily the case. In particular, we have demonstrated that if one aims to build projection operators that reconstruct only physical helicity amplitudes, huge simplifications take place due the large redundancy of the generic d-dimensional tensor structure. It turns out that, when considering the scattering of $n \geq 5$ particles of arbitrary spin, the number of different helicity amplitudes h_λ provides a higher bound for the number of different tensor structures that are required in order to reconstruct them. Hence, in these cases, one can obtain a full set of independent helicity amplitudes from the contraction of the amplitude with no more than h_λ tensor structures. Moreover, the corresponding projection operators turn out to be substantially simpler.

Starting from $n = 5$ external legs, this method yields additional drastic simplifications compared to traditional projector-based approaches. Indeed, in this case the entire four-dimensional space can be spanned by the four independent external momenta and all tensor structures which involve the metric tensor $g^{\mu\nu}$ turn out to be redundant. We have demonstrated this explicitly by studying the tensor decomposition for five-gluon scattering in QCD and comparing the standard d-dimensional approach with our method. We have found that, while in the standard approach there are 142 independent tensor structures and thus 142 rather cumbersome projection operators, their number drops to 32 when projecting directly on the independent helicity amplitudes. As expected, 32 is also the number of different helicity configurations $2^5 = 32$. We derived explicitly the helicity projection operators, which are attached to the arXiv submission of this paper, and we validated them by computing the tree-level and one-loop corrections to five-gluon scattering in QCD. The simplifications obtained in this way were so substantial that the whole calculation could be completed in few hours on a laptop computer with a straightforward application of standard computer algebra systems. Similar simplifications were observed when computing projector operators for other five-point processes, such as the scattering of four gluons and a (massive) scalar.

While recent progress in integrand reduction and generalised unitarity has considerably improved the possibilities of computing multi-leg helicity amplitudes, enhancing the spectrum of techniques which can tackle these processes is definitely useful for further progress. Given the generality, the relative easy-of-use and the familiarity of projector-based approaches compared to alternative techniques, we believe that making them applicable to more complex processes will prove beneficial to future calculations. We also stress that, despite being commonly seen as an alternative, integrand reduction may in principle be applied in conjunction with projection operators.

While this constitutes a very interesting development in view of the calculations of two-loop corrections to other processes which involve five particles in the final state, we note that one should expect even more substantial simplifications when considering even more particles in the final state. Clearly, we are well aware of the fact that these calculations are extremely complicated irrespective of the approach used. Nevertheless, we believe that this paper provides an important contribution to substantially simplify one of the computationally most demanding steps required.

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