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On the almost sure convergence of sums

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ON THE ALMOST SURE CONVERGENCE OF SUMS

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ABSTRACT. Two counterexamples, addressing questions raised in [1] and [3], are provided. Both counterexamples are related to chaoses. Let $F_n = Y_n + Z_n$, where Y_n and Z_n belong to different chaoses of uniformly bounded degree. It may be that $F_n \xrightarrow{a.s.} 0$, $F_n \xrightarrow{L_{2+\delta}} 0$ and $E\{\sup_n |F_n|^\delta\} < \infty$, for some $\delta > 0$, and yet Y_n fails to converge to 0 a.s.

1. INTRODUCTION

Throughout, $(X_n : n \in \mathbb{N})$ is a sequence of real independent random variables such that $E(X_n) = 0$ and $E(X_n^2) = 1$. For $p \in \mathbb{N}$, we denote by $\mathcal{F}_p$ the collection of those functions $f : \mathbb{N}^p \to \mathbb{R}$ such that f is symmetric (i.e., invariant under permutations of its arguments), f vanishes on the diagonal (i.e., $f(j_1, \ldots, j_p) = 0$ whenever $j_r = j_s$ for some $r \neq s$), and

$$\sum_{j_1,\ldots,j_p \in \mathbb{N}} f(j_1, \ldots, j_p)^2 < \infty.$$

Moreover, for $p \in \mathbb{N}$ and $f \in \mathcal{F}_p$, we let

$$Q_p(f) = \sum_{j_1,\ldots,j_p \in \mathbb{N}} f(j_1, \ldots, j_p) \, X_{j_1} \ldots X_{j_p}.$$

Here, the series converges (a.s. and in L_2) because of the martingale convergence theorem. In fact,

$$M_n = \sum_{j_1,\ldots,j_p=1}^{n} f(j_1, \ldots, j_p) \, X_{j_1} \ldots X_{j_p}$$

is a martingale with respect to the filtration $\sigma(X_1, \ldots, X_n)$ and

$$\sup_n E\{M_n^2\} = \sum_{j_1,\ldots,j_p \in \mathbb{N}} f(j_1, \ldots, j_p)^2.$$

Therefore, $Q_p(f) = \lim_n M_n$ where the equality is a.s. and in the L_2-sense.

Multilinear forms in independent random variables, such as $Q_p(f)$, play a role in various frameworks, including multiple stochastic integration, harmonic analysis, Boolean functions, geometry of Banach spaces, random graphs, concentration of measures, Malliavin calculus and the Malliavin-Stein method; see [1] and references therein.

Recently, the following problem has come to the fore. Suppose that

$$F_n \xrightarrow{a.s.} F,$$

where F_n and F are real random variables of the form

$$F_n = a_n + \sum_{p=1}^{q} Q_p(f_{n,p}) \quad \text{and} \quad F = a + \sum_{p=1}^{q} Q_p(f_p)$$

for some $q \in \mathbb{N}$, f_p, $f_{n,p} \in \mathcal{F}_p$ and a, $a_n \in \mathbb{R}$. Can one deduce that

(1) $\qquad a = \lim_n a_n \quad \text{and} \quad Q_p(f_p) \overset{a.s.}{=} \lim_n Q_p(f_{n,p}) \text{ for all } p = 1, \dots, q$?

As possibly expected, the answer is no. To give conditions for (1), hence, is a quite natural problem.

For instance, by a result of Poly and Zheng [3, Theorem 1.3], condition (1) holds whenever

$$\sup_n E\{|X_n|^{2+\delta}\} < \infty \qquad \text{for some } \delta > 0.$$

Subsequently, this result has been improved by [1, Cor. 2.9], where condition (1) is shown to be true whenever $(X_n^2 : n \in \mathbb{N})$ is uniformly integrable.

Next, for our purposes, we also need to recall the following result.

Theorem 1. (Theorem 1.1 of [3]). *Let F_n and F be real random variables and $S_q = \bigoplus_{p=0}^{q} \mathcal{H}_p$, where $q \in \mathbb{N}$ and $\mathcal{H}_p$ is the p-th Gaussian Wiener chaos (associated to a given isonormal Gaussian process). If $F_n \overset{a.s.}{\longrightarrow} F$ and $F_n \in S_q$ for all n, then*

$$F \in S_q \quad \text{and} \quad J_p(F_n) \overset{a.s.}{\longrightarrow} J_p(F) \qquad \text{for } p = 0, 1, \dots, q$$

where J_p is the projection operator onto $\mathcal{H}_p$.

In connection with the above results, the following questions are raised in [3].

(i) Does condition (1) hold if each X_n is a two point variable ?

(ii) Let $T_q = \bigoplus_{p=0}^{q} \mathcal{I}_p$, where $\mathcal{I}_p$ is the p-th Poisson-Wiener chaos; see point (*) below and [2, Chap. 18]. Is it possible to replace S_q with T_q in Theorem 1 ? Precisely, if $F_n \overset{a.s.}{\longrightarrow} F$ and $F, F_n \in T_q$ for all n, can one deduce that $J_p(F_n) \overset{a.s.}{\longrightarrow} J_p(F)$ for all $p = 0, 1, \dots, q$?

Both (i) and (ii) have been answered (in the negative) by Adamczak; see Examples 2.11 and 3.1 of [1].

It is worth noting that, under the assumptions of Theorem 1, one also obtains $F_n \overset{L_2}{\longrightarrow} F$ (thanks to the hypercontractivity of the Ornstein-Uhlenbeck semigroup). In addition, by Theorem 3.2 of [1], question (ii) has a positive answer provided the sequence (F_n) is L_2-bounded and

(2) $$E\{\sup_n |F_n|\} < \infty.$$

Whether or not condition (2) can be dropped, however, is an open problem; see [1, Example 3.3]. Hence, questions (i)-(ii) could be restated as follows.

(j) Does condition (1) hold if each X_n is a two point variable and $F_n \overset{L_2}{\longrightarrow} F$?

(jj) Is it possible to remove condition (2) if $F_n \overset{L_2}{\longrightarrow} F$? Precisely, if $F_n \overset{a.s.}{\longrightarrow} F$, $F_n \overset{L_2}{\longrightarrow} F$ and $F, F_n \in T_q$ for all n, can one deduce that $J_p(F_n) \overset{a.s.}{\longrightarrow} J_p(F)$ for all $p = 0, 1, \dots, q$?

The only purpose of this note is to answer (in the negative) questions (j)-(jj). Our counterexamples are actually simple (the main tool is the Borel-Cantelli lemma) and help to unify the current state of the art on the subject. The second counterexample suggests that, not only condition (2) can not be removed, but to weaken it seems to be quite hard.

2. THE COUNTEREXAMPLES

Example 2. (Question (j) has a negative answer). Let (Y_n) be a sequence of independent random variables such that $Y_n \in \{-1, 1\}$ and $p_n = P(Y_n = 1) \in (0, 1)$. Define

$$X_n = \frac{Y_n - 2p_n + 1}{2\sqrt{p_n(1 - p_n)}} = \sqrt{\frac{1 - p_n}{p_n}} 1_{\{Y_n = 1\}} - \sqrt{\frac{p_n}{1 - p_n}} 1_{\{Y_n = -1\}},$$

$$F = 0 \quad \text{and} \quad F_n = p_{2n+1} X_{2n} + \sqrt{p_{2n+1}(1 - p_{2n+1})} X_{2n} X_{2n+1}.$$

Such X_n are independent with $E(X_n) = 0$ and $E(X_n^2) = 1$, and we have to choose p_n in such a way that $F_n \xrightarrow{L_2} 0$, $F_n \xrightarrow{a.s.} 0$, but $p_{2n+1} X_{2n}$ fails to converge to 0 a.s. Take, for instance,

$$p_{2n} = 1/n \quad \text{and} \quad p_{2n+1} = n^{-1/\sqrt{\log n}} \quad \text{for } n \geq 2.$$

Then:

- $E(F_n^2) = p_{2n+1} \to 0$.
- Let $A = \{Y_{2n} = Y_{2n+1} = 1 \text{ for infinitely many } n\}$. Since

$$\sum_n P(Y_{2n} = Y_{2n+1} = 1) = \sum_n n^{-1-1/\sqrt{\log n}} < \infty,$$

  then $P(A) = 0$. Moreover, on the set A^c, one obtains

$$F_n = F_n 1_{\{Y_{2n+1} = 1\}} = F_n 1_{\{Y_{2n} = -1, Y_{2n+1} = 1\}} = -\sqrt{1/(n-1)}$$

  for each n large enough. (The first equality is because $F_n \neq 0$ if and only if $Y_{2n+1} = 1$). Hence, $F_n \xrightarrow{a.s.} 0$.
- Let $B = \{Y_{2n} = 1 \text{ for infinitely many } n\}$. The Borel-Cantelli lemma yields $P(B) = 1$. Moreover, on the set B, one obtains

$$p_{2n+1} X_{2n} = p_{2n+1} \sqrt{\frac{1 - p_{2n}}{p_{2n}}} = \sqrt{n-1} \, n^{-1/\sqrt{\log n}}$$

  for infinitely many n. Therefore, $p_{2n+1} X_{2n}$ fails to converge to 0 a.s. (since $\sqrt{n-1} \, n^{-1/\sqrt{\log n}} \to \infty$).

Example 2 is straightforward. The next example is slightly more elaborate.

Example 3. (Question (jj) has a negative answer). Let (Y_n) be a sequence of independent Poisson random variables. Define $\lambda_n = E(Y_n)$ and

$$X_n = \frac{Y_n - \lambda_n}{\sqrt{\lambda_n}}, \quad F = 0, \quad F_n = \lambda_{2n+1} X_{2n} + \sqrt{\lambda_{2n+1}} X_{2n} X_{2n+1}.$$

As shown below, $F_n \in T_2$ and $J_1(F_n) = \lambda_{2n+1} X_{2n}$ for all n. Therefore, to answer (jj) in the negative, it suffices to choose λ_n in such a way that

$$F_n \xrightarrow{L_2} 0, \quad F_n \xrightarrow{a.s.} 0, \quad \lambda_{2n+1} X_{2n} \text{ fails to converge to 0 a.s.}$$

We will prove a little bit more. In fact, we will also obtain

$$F_n \xrightarrow{L_{2+\delta}} 0 \quad \text{and} \quad E\big\{\sup_n |F_n|^\delta\big\} < \infty \quad \text{for some } \delta > 0.$$

To begin with, we collect some useful facts in a lemma.

Lemma 4. *If Y has a Poisson distribution with parameter $\lambda > 0$, then*

$$P(Y > j) \le \lambda^{j+1}/(j+1)! \qquad \text{for } j = 0, 1, \dots$$

In particular, $P(Y > 0) \le \lambda$. Moreover, if $\lambda \in (0, 1]$, then

$$E\big\{|Y - \lambda|^{5/2}\big\} \le \sqrt{8}\,\lambda \quad \text{and} \quad E(Y^{5/2}) \le \sqrt{15}\,\lambda.$$

Proof. By Taylor formula, there is $\gamma \in (0, \lambda)$ such that

$$P(Y > j) = e^{-\lambda} \sum_{k>j} \lambda^k/k! = e^{-\lambda}\Big\{ e^\lambda - \sum_{k=0}^j \lambda^k/k! \Big\} = e^{-\lambda} e^\gamma \lambda^{j+1}/(j+1)!$$

Hence, $P(Y > j) \le \lambda^{j+1}/(j+1)!$ follows from $\gamma < \lambda$. Next, suppose $\lambda \in (0, 1]$ and note that $E\big\{|Y - \lambda|\big\} \le 2\,\lambda$ and $E\big\{(Y - \lambda)^4\big\} = 3\lambda^2 + \lambda \le 4\lambda$. Therefore,

$$E\big\{|Y - \lambda|^{5/2}\big\} = E\big\{|Y - \lambda|^2 |Y - \lambda|^{1/2}\big\} \le \sqrt{E\big\{(Y - \lambda)^4\big\} E\big\{|Y - \lambda|\big\}} \le \sqrt{8}\,\lambda.$$

Similarly, since $E(Y^4) = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda \le 15\lambda$,

$$E(Y^{5/2}) = E\big\{Y^2 Y^{1/2}\big\} \le \sqrt{E(Y^4)\,E(Y)} \le \sqrt{15}\,\lambda.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next, we note that $F_n = X_{2n} Y_{2n+1}$ and we let

$$\lambda_{2n} = n^{-3/4} \quad \text{and} \quad \lambda_{2n+1} = n^{-5/16}.$$

- First, by Lemma 4, one obtains $F_n \xrightarrow{L_{5/2}} 0$. In fact,

$$E(|F_n|^{5/2}) = E\big\{|X_{2n} Y_{2n+1}|^{5/2}\big\} = \frac{E\big\{|Y_{2n} - \lambda_{2n}|^{5/2}\big\}}{\lambda_{2n}^{5/4}} E(Y_{2n+1}^{5/2})$$

$$\le \sqrt{120}\,\frac{\lambda_{2n+1}}{\lambda_{2n}^{1/4}} = \sqrt{120}\,n^{-1/8} \longrightarrow 0.$$

- We next prove $F_n \xrightarrow{a.s.} 0$. Fix $\epsilon > 0$ and define

$$A = \big\{Y_{2n+1} > \epsilon\,n^{3/8} \text{ for infinitely many } n\big\} \quad \text{and}$$

$$H = \big\{Y_{2n} \ne 0, \, Y_{2n+1} \ne 0 \text{ for infinitely many } n\big\}.$$

On $(A \cup H)^c = A^c \cap H^c$, for each n large enough, one obtains $Y_{2n+1} = 0$ or $Y_{2n} = 0$ and $Y_{2n+1} \le \epsilon\,n^{3/8}$. In the first case, $F_n = X_{2n} Y_{2n+1} = 0$, while in the second

$$|F_n| = |X_{2n} Y_{2n+1}| = \sqrt{\lambda_{2n}}\,Y_{2n+1} = n^{-3/8}\,Y_{2n+1} \le \epsilon.$$

Hence, to prove $F_n \xrightarrow{a.s.} 0$, it suffices to see that $P(A) = P(H) = 0$. In turn, this follows from

$$\epsilon^2 \sum_n P\big(Y_{2n+1} > \epsilon\, n^{3/8}\big) \le \sum_n \frac{E\big(Y_{2n+1}^2\big)}{n^{3/4}} = \sum_n \frac{\lambda_{2n+1} + \lambda_{2n+1}^2}{n^{3/4}}$$

$$\le 2 \sum_n \frac{\lambda_{2n+1}}{n^{3/4}} = 2 \sum_n n^{-17/16} < \infty$$

and

$$\sum_n P\big(Y_{2n} \ne 0,\, Y_{2n+1} \ne 0\big) \le \sum_n \lambda_{2n}\lambda_{2n+1} = \sum_n n^{-17/16} < \infty.$$

- To show that $\lambda_{2n+1}X_{2n}$ does not converge to 0 a.s., we argue as in Example 2. Let $B = \big\{Y_{2n} = 1 \text{ for infinitely many } n\big\}$. The Borel-Cantelli lemma yields $P(B) = 1$. On the other hand, on the set B, one obtains

$$\lambda_{2n+1}\, X_{2n} = \lambda_{2n+1}\, \frac{1 - \lambda_{2n}}{\sqrt{\lambda_{2n}}} = n^{1/16} - n^{-11/16}$$

for infinitely many n. Therefore, $\lambda_{2n+1}X_{2n}$ fails to converge to 0 a.s.

- Finally, we prove $E(M^\delta) < \infty$ for each $\delta \in (0, 1/24)$, where

$$M = \sup_n |F_n|.$$

Since $E(M^\delta) = \int_0^\infty P\big(M > t^{1/\delta}\big)\, dt$, it suffices to show that

$$(3) \qquad\qquad P(M > t) = \mathrm{O}(t^{-1/24}) \qquad \text{as } t \to \infty.$$

For each $t > 0$, Lemma 4 yields

$$P(M > t) \le \sum_n P\big(|F_n| > t\big) = \sum_n P\big(|X_{2n}\, Y_{2n+1}| > t\big)$$

$$\le \sum_n \Big\{ P\big(0 < Y_{2n+1} \le \sqrt{t},\, |X_{2n}| > \sqrt{t}\big) + P\big(Y_{2n+1} > \sqrt{t}\big) \Big\}$$

$$\le \sum_n P\big(Y_{2n+1} > 0\big) P\big(|X_{2n}| > \sqrt{t}\big) + \sum_n \frac{\lambda_{2n+1}^{[\sqrt{t}]+1}}{([\sqrt{t}]+1)!}$$

Define $a = \sum_n \lambda_{2n+1}^4 = \sum_n n^{-5/4}$. If $t \ge 9$, then $[\sqrt{t}] + 1 \ge 4$ and $\sqrt{\lambda_{2n}\, t} > \lambda_{2n}$. Hence,

$$P(M > t) \le \sum_n \lambda_{2n+1} P\big(|Y_{2n} - \lambda_{2n}| > \sqrt{\lambda_{2n}\, t}\big) + \sum_n \frac{\lambda_{2n+1}^4}{[\sqrt{t}]^2}$$

$$\le \sum_n \lambda_{2n+1} P\big(Y_{2n} > \sqrt{\lambda_{2n}\, t}\big) + \frac{a}{(\sqrt{t}-1)^2}.$$

If $n \le [t^{2/3}]$, then $\lambda_{2n} = n^{-3/4} \ge t^{-1/2}$ and

$$P\big(Y_{2n} > \sqrt{\lambda_{2n}\, t}\big) \le P\big(Y_{2n} > t^{1/4}\big) \le t^{-1/4} E(Y_{2n}) = t^{-1/4}\, \lambda_{2n}.$$

Otherwise, if $n > [t^{2/3}]$, Lemma 4 yields

$$P\big(Y_{2n} > \sqrt{\lambda_{2n}\, t}\big) \le P(Y_{2n} > 0) \le \lambda_{2n}.$$

Therefore, letting $b = \sum_n \lambda_{2n+1} \lambda_{2n} = \sum_n n^{-17/16}$, one obtains

$$\sum_n \lambda_{2n+1} P\big(Y_{2n} > \sqrt{\lambda_{2n}\, t}\big) \leq t^{-1/4} \sum_{n=1}^{[t^{2/3}]} \lambda_{2n+1} \lambda_{2n} + \sum_{n>[t^{2/3}]} \lambda_{2n+1} \lambda_{2n}$$

$$\leq b\, t^{-1/4} + \sum_{n>[t^{2/3}]} n^{-17/16}$$

$$\leq b\, t^{-1/4} + 16\, [t^{2/3}]^{-1/16}$$

$$\leq b\, t^{-1/4} + 16 \big\{t^{2/3} - 1\big\}^{-1/16}.$$

To summarize,

$$P(M > t) \leq b\, t^{-1/4} + 16 \big\{t^{2/3} - 1\big\}^{-1/16} + \frac{a}{(\sqrt{t} - 1)^2} \qquad \text{for each } t \geq 9,$$

and this proves condition (3).

To close the paper, it remains to see that $F_n \in T_2$ and $J_1(F_n) = \lambda_{2n+1} X_{2n}$.

(*) The Poisson-Wiener chaos. Let $(\mathcal{X}, \mathcal{E}, \mu)$ be a σ-finite measure space and N a Poisson process with intensity μ on some probability space $(\Omega, \mathcal{A}, P)$. Then,

$$L_2(\Omega, \sigma(N), P) = \bigoplus_{p=0}^{\infty} \mathcal{I}_p \qquad \text{with } \mathcal{I}_p \perp \mathcal{I}_q \text{ for } p \neq q.$$

The closed subspace $\mathcal{I}_p$, called the *p-th Poisson-Wiener chaos*, is the collection of all random variables of the form $I_p(f)$, where I_p is the multiple Poisson integral of order p and f ranges over the symmetric elements of $L_2(\mathcal{X}^p, \mathcal{E}^p, \mu^p)$. We refer to [2, Chapter 12] for the general definition of I_p. Here, it suffices to note that, if $A = A_1 \times \ldots \times A_p$ with $A_i \in \mathcal{E}$, $\mu(A_i) < \infty$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$I_p\Big(\sum_\pi 1_A \circ \pi\Big) = p! \prod_{i=1}^{p} \big\{N(A_i) - \mu(A_i)\big\}$$

where the sum is over all permutations π of $\mathcal{X}^p$. (A permutation of $\mathcal{X}^p$ is meant as a function $\pi : \mathcal{X}^p \to \mathcal{X}^p$ of the form $\pi(x_1, \ldots, x_p) = (x_{j_1}, \ldots, x_{j_p})$ where $(j_1, \ldots, j_p)$ is a fixed permutation of $(1, \ldots, p)$).

Next, let $\mathcal{X} = [0, \infty)$, $\mathcal{E}$ the Borel σ-field, μ the Lebesgue measure, and

$$Y_n = N(A_n) \qquad \text{where } A_n = \Big(\sum_{i=1}^{n-1} \lambda_i, \sum_{i=1}^{n} \lambda_i\Big].$$

Then, (Y_n) is an independent sequence, each Y_n has a Poisson distribution with parameter λ_n, and

$$\sqrt{\lambda_{2n+1}}\, X_{2n}\, X_{2n+1} = I_2(f_n) \in \mathcal{I}_2 \qquad \text{where}$$

$$f_n(x, y) = \frac{1}{2\sqrt{\lambda_{2n}}} \big\{1_{A_{2n}}(x) 1_{A_{2n+1}}(y) + 1_{A_{2n}}(y) 1_{A_{2n+1}}(x)\big\}.$$

Similarly,

$$\lambda_{2n+1} X_{2n} = I_1\Big(\lambda_{2n}^{-1/2} \lambda_{2n+1} 1_{A_{2n}}\Big) \in \mathcal{I}_1.$$

Hence,

$$F_n = \lambda_{2n+1}\, X_{2n} + \sqrt{\lambda_{2n+1}}\, X_{2n}\, X_{2n+1} \in T_2 \quad \text{and} \quad J_1(F_n) = \lambda_{2n+1}\, X_{2n}.$$

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