Widths of highly excited resonances in multidimensional molecular predissociation

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Vania Sordoni died on October 10*th,* 2019*.*
May this paper contribute to keep her memory alive.

Abstract. We investigate the simple resonances of a 2 by 2 matrix of n-dimensional semiclassical Schrödinger operators that interact through a first order differential operator. We assume that one of the two (analytic) potentials admits a well with non empty interior, while the other one is non trapping and creates a barrier between the well and infinity. Under a condition on the resonant state inside the well, we find an optimal lower bound on the width of the resonance. The method of proof relies on Carleman estimates, microlocal propagation of the microsupport, and a refined study of a non involutive double characteristic problem in the framework of Sjöstrand's analytic microlocal theory.

1. Introduction.

The mathematical study of molecular predissociation goes back to [**Kl**], where a general framework is set in order to study this physical phenomenon. This framework mainly consists of semiclassical matrix Schrödinger operators, where the semiclassical parameter h is the square root of the ratio of electronic to nuclear masses, and where potential wells interact with unbounded classically allowed regions, giving rise to molecular resonances.

In the case of two interacting electronic levels, the standard Agmon estimates method also provides a general upper bound for the widths of these resonances. However, only very few results exist concerning a lower bound of such quantities (let us recall that the physical importance of the resonance widths lies in the fact that their inverses represent the life-time of the corresponding metastable molecule).

To our knowledge, there are two cases only where such a lower bound is obtained. The first is when the resonance is close to the ground state of the bonding potential: see [**GrMa**] where a complete semiclassical asymptotic of the width is obtained. The second one is when the resonance is highly excited and the dimension of the nuclei-space is one: see [**As**]. In both cases, the lower bound is optimal, in the sense that it is of the same order of magnitude as the upper bound given by Agmon estimates.

The purpose of this paper is to extend Ashida's result [**As**] to the case of multidimensional nuclei-space.

The geometrical structure of this situation is very similar to that of shape resonances, in the sense that one potential forms an energy well surrounded by a potential barrier (in which both potentials are involved), and immersed inside a sea corresponding to the second potential. And actually, both the strategy of the proof and the result (namely, the link between the width if the resonance and the size of the resonant state inside the well) are similar to that of [**DaMa**] where excited shape resonances are studied.

Roughly speaking, the starting point is the explicit link between the width of the resonance and the size of the resonant state inside the sea, and the strategy consists in reasoning by contradiction and in propagating the smallness of the resonant state from the inside of the sea up to the boundary of the well. In the case of shape resonances, this means first crossing the shore (that is, the boundary of the sea), then propagating inside the barrier, and finally arriving at the boundary of the well. In the case of molecular predissociation, an extra difficulty appears: the crossing of the so called "crest", that is the place where the two potentials coincide. In contrast with the other types of propagation (that involve the principal symbol only), the crossing of the crest is very sensitive to the lower order interaction terms. Indeed, such a propagation would not take place without these terms, and the mathematical problem in itself has many to do with old results on propagation for scalar operators with non-involutive double-characteristics (see, e.g., [**LaLa**], [**PeSo**]). The difference is that here, after conjugation by the convenient weight, the principal symbols of the two operators that are involved become complex-valued. Then, in order to prove the propagation, it becomes necessary to use a FBI transform (in the sense of [**Sj**]), associated with a complex canonical transform, such that *both* operators take a simpler form.

Concerning the two other types of propagation (across the boundaries of the sea and of the well), their proofs mainly follow those of [**DaMa**], [**Ma1**], with additional technical difficulties due to the fact that we deal with a matrix-operator (while those considered in [**DaMa**], [**Ma1**] are scalar).

The paper is organized as follows: In Section 2, we describe our analytical and geometrical assumptions, and we recall some basic facts about the Agmon distance. Section 3 is devoted to the statement of our main results. In Section 4 we give some properties of the resonant state, and we establish the link between the resonance width and the size of the resonant state inside the sea. In Section 5 we specify this link through a contradiction argument, by transferring the smallness of the resonance width to the smallness of the resonant state inside the sea. Then this smallness is first propagated across the shore in Section 6, then across the crest in Section 7, and finally up to the well in Section 8. A converse result is proved in Section 9, and Section 10 concerns the possible examples of application.

2.   Assumptions.

We consider the semiclassical 2×2 matrix Schrödinger operator,

$$P = \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix} \tag{2.1}$$

with,

$$P_j = P_j(x, hD_x) := -h^2\Delta + V_j(x) \quad (j = 1, 2),$$

where $x = (x_1, \ldots, x_n)$ is the current variable in $\mathbb{R}^n$ $(n \geq 1)$, $h > 0$ denotes the semiclassical parameter, $W = W(x, hD_x)$ is a first-order semiclassical differential operator, and W^* stands for the formal adjoint of W.

This kind of operator appears in the Born–Oppenheimer approximation of molecules, after reduction to an effective Hamiltonian (see [**KMSW**], [**MaSo**]). In that case, the quantity h^2 stands for the inverse of the mass of the nuclei.

ASSUMPTION 1.   The potentials V_1 and V_2 are smooth and bounded on $\mathbb{R}^n$, and satisfy,

$$\begin{aligned} &\text{The set } U := \{V_1 \leq 0\} \text{ is compact and connected,} \\ &\nabla V_1 \neq 0 \text{ on the boundary } \partial U \text{ of } U, \text{ and } \liminf_{|x| \to \infty} V_1 > 0; \end{aligned} \tag{2.2}$$

$$\begin{aligned} &V_2|_U > 0 \text{ and } E = 0 \text{ is a non-trapping energy for } V_2; \\ &V_2 \text{ has a strictly negative limit as } |x| \to \infty. \end{aligned} \tag{2.3}$$
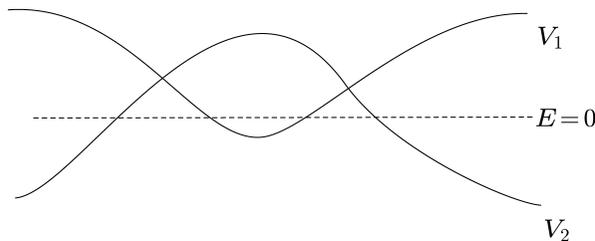
We define the island $\widehat{\mathcal{I}}$ as the bounded open connected component of $\{V_2 > 0\}$ containing U, and the sea as the unbounded closed set,

$$\mathcal{M} := \mathbb{R}^n \backslash \widehat{\mathcal{I}}. \tag{2.4}$$

With (2.2)–(2.3), the well U for V_1 is included in the island.

The fact that 0 is a non-trapping energy for V_2 means that, for any $(x, \xi) \in p_2^{-1}(0)$, one has $|\exp tH_{p_2}(x, \xi)| \to +\infty$ as $t \to \infty$, where $p_2(x, \xi) := \xi^2 + V_2(x)$ is the symbol of P_2, and $H_{p_2} := (\nabla_\xi p_2, -\nabla_x p_2)$ is the Hamilton field of p_2.

Such conditions (2.2)–(2.3) correspond to the situation of *molecular predissociation* as described in [**Kl**].

We plan to study the resonances of P near the energy level $E = 0$. Since our methods will strongly rely on analytic microlocal analysis, we also add the following assumption of analyticity:

Assumption 2.   The potentials V_1 and V_2, together with the coefficients of W, extend to bounded holomorphic functions near a complex sector of the form, $\mathcal{S}_\delta := \{x \in \mathbb{C}^n \, ; \, |\text{Im}\, x| \leq \delta|\text{Re}\, x|\}$, with $\delta > 0$. Moreover V_2 tends to its limit at ∞ in this sector and $\text{Re}\, V_1$ stays away from 0 outside a compact set of this sector.

Remark 2.1.   As a matter of fact, it would have been possible for us to consider more general kinds of interactions, such as first-order pseudodifferential operators with analytic symbols (see, e.g., [**GrMa**]). However, this would have just led to heavier notations and technical complications, without changing the key-ideas of the proof.

Now, as in [**GrMa**], we define the cirque Ω_0 as,

$$\Omega_0 = \{x \in \mathbb{R}^n ; V_1(x) < V_2(x)\}. \tag{2.5}$$

(Hence, the well is in the cirque and the cirque is in the island.)

Thereafter, the boundary $\partial\Omega_0$ of Ω_0 will be called the crest, and the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ will be called the shore.

We also consider the Lithner–Agmon distance d associated to the pseudo-metric

$$(\min(V_1, V_2))_+ dx^2.$$

Such a metric is considered in [**Kl**], [**GrMa**], [**Pe**]. There are three places where this metric is not a standard smooth one.

At first, inside the well U and the sea $\mathbb{R}^n \backslash \widehat{\mathcal{I}}$, where the metric degenerates completely.

Secondly on the crest $\partial\Omega_0$ (that is, at the points where $V_1 = V_2$). This case has been considered in Pettersson [**Pe**]. At such points, if one assumes that $\nabla V_1 \neq \nabla V_2$, then any geodesic that crosses transversally the hypersurface $\{V_1 = V_2\}$ is C^1 (and so is φ near the crossing point).

Finally there are the boundaries of the island $\partial\widehat{\mathcal{I}}$ and of the well ∂U, where $\min(V_1, V_2)$ vanishes. This situation was considered in [**HeSj2**], [**Ma1**], and we will follow them in further constructions.

We denote by φ the Lithner–Agmon distance to the well U,

$$\varphi(x) := d(U, x), \tag{2.6}$$

and we set,

$$S_0 := d(U, \mathcal{M}). \tag{2.7}$$

We also consider the set,

$$G := \left\{ \overline{\gamma \cap (\widehat{I} \backslash U)} \, ; \, \gamma = \text{minimal geodesics between } U \text{ and } \mathcal{M} \right\},$$

which coincides with the set of minimal geodesics between U and $\mathcal{M}$ that have only their end-points in $U \cup \mathcal{M}$. In particular, if $\gamma \in G$, then $\gamma \cap \mathcal{M}$ is a point of type 1 in the terminology of [**HeSj2**].

Let us also recall that the assumption that 0 is a non trapping energy for V_2 implies that $\nabla V_2 \neq 0$ on $\partial\widehat{\mathcal{I}}$, and therefore that $\partial\widehat{\mathcal{I}}$ is a smooth hypersurface.

3. Main results.

Under the previous assumptions we plan to study the resonances of the operator P given in (2.1).

In order to define the resonances we consider the distortion given as follows: Let $F(x) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $F(x) = 0$ for $|x| \leq R_0$, $F(x) = x$ for $|x|$ large enough. For $\theta > 0$ small enough, we define the distorted operator P_θ as the value at $\nu = i\theta$ of the extension to the complex of the operator $U_\nu P U_\nu^{-1}$, defined for ν real small enough, and analytic in ν, where we have set

$$U_\nu \phi(x) = \det(1 + \nu dF(x))^{1/2} \phi(x + \nu F(x)). \tag{3.1}$$

By using the Weyl perturbation theorem, one can see that there exists $\varepsilon_0 > 0$ such that for any $\theta > 0$ small enough, the spectrum of P_θ is discrete in $[-\varepsilon_0, \varepsilon_0] - i[0, \varepsilon_0\theta]$. The eigenvalues of P_θ are called the resonances of P [**HeMa**], [**HeSj2**], [**Hu**], [**Kl**].

In this paper we are interested in the imaginary part of these resonances.

For some fixed $\delta > 0$ arbitrarily small, let $\widetilde{V}_2$ be a C^∞ function that coincides with V_2 on $\{\mathrm{dist}(x, \mathcal{M}) \geq \delta\}$, and such that $\inf_{\mathbb{R}^n} \widetilde{V}_2 > 0$. By adapting the techniques used in [**HeSj2**, Section 9], it is not difficult to see that the resonances of P near 0 are exponentially close to the eigenvalues of the operator,

$$\widetilde{P} := \begin{pmatrix} -h^2\Delta + V_1 & hW \\ hW^* & -h^2\Delta + \widetilde{V}_2 \end{pmatrix}, \tag{3.2}$$

where the precise meaning is the following one: Let $I(h)$ be a closed interval containing 0, and $a(h) > 0$ such that $a(h) \to 0$ as $h \to 0_+$, and, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ satisfying,

$$a(h) \geq \frac{1}{C_\varepsilon} e^{-\varepsilon/h}; \tag{3.3}$$

$$\sigma(\widetilde{P}) \cap ((I(h) + [-2a(h), 2a(h)]) \backslash I(h)) = \emptyset, \tag{3.4}$$

for all $h > 0$ small enough. Then, there exists a constant $\varepsilon_1 > 0$ and a bijection,

$$\widetilde{\beta} : \sigma(\widetilde{P}) \cap I(h) \to \mathrm{Res}(P) \cap \Gamma(h),$$

(where we have set, $\Gamma(h) := (I(h) + [-a(h), a(h)]) + i[-\varepsilon_1, 0])$, such that, for any $\varepsilon > 0$, one has,

$$\widetilde{\beta}(\lambda) - \lambda = \mathcal{O}(e^{-(2S_0 - \varepsilon)/h}), \tag{3.5}$$

uniformly as $h \to 0_+$.

In particular, since the eigenvalues of $\widetilde{P}$ are real, one obtains that, for any $\varepsilon > 0$, the resonances ρ in $\Gamma(h)$ satisfy,

$$|\mathrm{Im}\,\rho| = \mathcal{O}(e^{-(2S_0 - \varepsilon)/h}). \tag{3.6}$$

Observe that, since $-h^2\Delta + \widetilde{V}_2$ is elliptic, the eigenvalues of $\widetilde{P}$ near 0 are actually close (up to $\mathcal{O}(h^2)$) to those of $-h^2\Delta + V_1$. In particular, using Weyl estimates, we see that the number of eigenvalues of $\widetilde{P}$ inside any small enough fix interval around 0 is $\mathcal{O}(h^{-n})$. Thus, possibly by excluding some particular values of h (e.g. by taking h along a convenient sequence tending to 0), it is in principle possible to construct many such intervals $I(h)$ satisfying (3.3)–(3.4) (see also [**HeSj1**, Section 2] and [**HeSj2**, Section 9]).

From now on, we consider the particular case where $I(h)$ consists of a unique value, that is, we assume,

ASSUMPTION 3. There exists $E(h) \in \mathbb{R}$ such that,

$$
\begin{aligned}
&E(h) \in \sigma_{disc}(\widetilde{P}); \\
&E(h) \to 0 \text{ as } h \to 0_+; \\
&\sigma(\widetilde{P}) \cap [E(h) - 2a(h), E(h) + 2a(h)] = \{E(h)\},
\end{aligned}
\tag{3.7}
$$

where $a(h)$ satisfies (3.3).

Applying (3.5), we denote by $\rho = \rho(h)$ the unique resonance of P satisfying $\rho - E(h) = \mathcal{O}(e^{-(2S_0-\varepsilon)/h})$ for all $\varepsilon > 0$. We also denote by u_0 the normalized eigenstate of $\widetilde{P}$ associated with $E(h)$.

REMARK 3.1. By standard results on the tunneling effect (see, e.g., [**HeSj1**]), it can be shown that the eigenvalues of $\widetilde{P}$ coincide, up to exponentially small errors, with those of the Dirichlet realization $P_\mathcal{D}$ of P on any domain $\mathcal{D}$ contained in $\widehat{I}$ and whose interior contains U. As a consequence, in Assumption 3 one can equivalently replace $\widetilde{P}$ with any of such $P_\mathcal{D}$'s.

As in [**DaMa**], [**Ma1**], in order to obtain a lower bound on the width $|\text{Im}\,\rho|$ of ρ, we need to add a further assumption on the size of u_0 near some geometric subset of ∂U.

ASSUMPTION 4. For any $\varepsilon > 0$ and for any neighborhood $\mathcal{W}$ of the set $\bigcup_{\gamma \in G}(\gamma \cap \partial U)$, there exists $C = C(\varepsilon, \mathcal{W}) > 0$ such that, for all $h > 0$ small enough, one has,

$$
\|u_0\|_{L^2(\mathcal{W}) \oplus L^2(\mathcal{W})} \geq \frac{1}{C} e^{-\varepsilon/h}.
$$

REMARK 3.2. Actually, introducing the microsupport $MS(u_0)$ of u_0 as, e.g., in [**Ma2**] (with obvious changes due to the fact that u_0 has two components), in our situation we have $MS(u_0) \cap \{x \in \partial U\} \subset \{\xi = 0\}$, and Assumption 4 can be rephrased into,

$$
MS(u_0) \cap \bigcup_{\gamma \in G}(\gamma \cap \partial U) \times \{0\} \neq \emptyset.
$$

REMARK 3.3. By standard results on the propagation of the microsupport (see, e.g., [**Ma2**]), one can see that Assumption 4 is always satisfied when $n = 1$. When $n \geq 2$, a sufficient condition is that for any neighborhood W of $\bigcup_{\gamma \in G}(\gamma \cap \partial U) \times \{0\}$, the set

$\bigcup_{t\in\mathbb{R}} \exp tH_{p_1}(W)$ is a neighborhood of $p_1^{-1}(0) \cap \{x \in U\}$, where $p_1(x,\xi) := \xi^2 + V_1(x)$ and $H_{p_1} := (\partial_\xi p_1, -\partial_x p_1)$ is the Hamilton field of p_1.

Now, we denote by $G_1 \subset G$ be the set of minimal geodesics γ such that $MS(u) \cap (\gamma \cap \partial U) \times \{0\} \neq \emptyset$. (By Assumption 4, one has $G_1 \neq \emptyset$.)

We also denote by $w_0(x,\xi)$ the principal symbol of W and by $w_0^*(x,\xi)$ the principal symbol of W^*.

We further assume,

ASSUMPTION 5.   There exists $\gamma \in G_1$ such that,

- γ intersects $\partial\Omega_0$ at a finite number of points $x^{(1)}, \ldots, x^{(N)}$ (ordered from the closest to $\partial\mathcal{M}$ up to the closest to U);

- All intersections between γ and $\partial\Omega_0$ are transversal;

- $\nabla V_1 \neq \nabla V_2$ on $\gamma \cap \partial\Omega_0$.

ASSUMPTION 6.   For all $j \in \{1, \ldots, N\}$, one has $w_0^*(x^{(j)}, i\nabla\varphi(x^{(j)})) \neq 0$ if j is odd, and $w_0(x^{(j)}, i\nabla\varphi(x^{(j)})) \neq 0$ if j is even.

Observe that N is necessarily odd. Moreover, the last property is nothing but a condition of ellipticity on the operators of interaction, similar to that appearing in [**As**], [**GrMa**].

Our main result is,

THEOREM 3.4.   *Under Assumptions* 1 *to* 6, *one has,*

$$\lim_{h\to 0_+} h \ln|\mathrm{Im}\, \rho(h)| = -2S_0.$$

REMARK 3.5.   In view of (3.6), this will be implied by the fact that, for any $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that,

$$|\mathrm{Im}\, \rho(h)| \geq \frac{1}{C} e^{-(2S_0+\varepsilon)/h},$$

for all $h > 0$ small enough.

Concerning the necessity of Assumption 4, we also have the following converse result:

THEOREM 3.6.   *Suppose that Assumptions* 1 *to* 3 *are satisfied, but Assumption* 4 *is not. Assume further that for all* $\gamma \in G$, γ *satisfies the properties listed in Assumption* 5. *Then, there exists* $\delta > 0$ *such that,*

$$|\mathrm{Im}\, \rho(h)| \leq e^{-2(S_0+\delta)/h}.$$

4.   Preliminaries.

In this section, we recall some basic facts on the resonant state $u = (u_1, u_2)$ of P associated with ρ, and normalized in such a way that,

$$\|u\|_{L^2(\widehat{\mathcal{I}})} = 1. \tag{4.1}$$

(From now on, we always write L^2 instead of $L^2 \oplus L^2$ in order to lighten the notations, and similarly for the Sobolev spaces.)

Typically, the properties we are going to recall can be deduced from the same arguments used in the scalar case (see, e.g., [**HeSj1**], [**HeSj2**], [**HiSi**]), and we refer the reader interested in more details to, e.g., [**GrMa**], [**Kl**].

First of all, by Lithner–Agmon estimates (as, e.g., in [**Kl**]), we have that, for any $\varepsilon > 0$ and any bounded set $\mathcal{B} \subset \mathbb{R}^n$,

$$\|e^{\varphi/h}u\|_{H^1(\mathcal{B})} = \mathcal{O}(e^{\varepsilon/h}), \tag{4.2}$$

where φ is defined in (2.6). In addition, with the same techniques used in [**HeSj2**], it can be seen that near the well U, u is exponentially close to u_0, in the sense that there exists $\delta > 0$ and a neighborhood Ω_U of $\overline{U}$ such that,

$$\|u - u_0\|_{H^1(\Omega_U)} = \mathcal{O}(e^{-\delta/h}).$$

In particular, the property of u_0 in Assumption 4 extends to u, too.

As in [**DaMa**], [**HeSj2**], we consider the set (called the set of "points of type 1" in [**HeSj2**]),

$$\mathcal{T}_1 := \bigcup_{\gamma \in G} (\gamma \cap \partial\mathcal{M}).$$

We also set $p_2(x,\xi) = \xi^2 + V_2(x)$ and, by the same arguments as in [**HeSj2**, Section 9] (but adapted to our case of a system), we see that if a bounded subset $\mathcal{B}$ of $\mathcal{M}$ stays away from the x-projection of the set,

$$\widetilde{\mathcal{T}}_1 := \bigcup_{t \in \mathbb{R}} \exp tH_{p_2}(\mathcal{T}_1 \times \{0\}),$$

then there exists $\delta > 0$ such that,

$$\|u\|_{H^1(\mathcal{B})} = \mathcal{O}(e^{-(S_0+\delta)/h}). \tag{4.3}$$

Now, let Ω_1 be a smooth bounded open domain containing the closure of the island $\widehat{\mathcal{I}}$, and write the interaction W as,

$$W = r_0(x) + hr_1(x) \cdot \nabla_x, \tag{4.4}$$

where r_0 is complex-valued, and r_1 is (complex) vector-valued. By the Stokes formula on Ω_1, we have,

$$(\operatorname{Im}\rho)\|u\|^2_{L^2(\Omega_1)} = -h^2 \operatorname{Im} \int_{\partial\Omega_1} \frac{\partial u}{\partial \nu} \cdot \overline{u}ds + h^2 \operatorname{Im} \int_{\partial\Omega_1} (r_1 \cdot \nu)u_2\overline{u_1}ds, \tag{4.5}$$

where ds is the surface measure on $\partial\Omega_1$, and ν stands for the outward pointing unit

normal to Ω_1. (Note that, if W had been a more general pseudodifferential operator, the previous formula would not have been valid anymore, but could have been replaced by another one involving the multiplication by a cut-off function instead of the restriction to Ω_1.)

Using (4.2)–(4.3) and (4.1), we easily deduce the existence of some $\delta > 0$ such that,

$$\operatorname{Im} \rho = -h^2 \operatorname{Im} \int_{\partial\Omega_1 \cap \mathcal{B}} \frac{\partial u}{\partial \nu} \cdot \overline{u} ds + h^2 \operatorname{Im} \int_{\partial\Omega_1 \cap \mathcal{B}} (r_1 \cdot \nu) u_2 \overline{u_1} ds + \mathcal{O}(e^{-(2S_0 + \delta)/h}), \quad (4.6)$$

where $\mathcal{B}$ is an arbitrarily small neighborhood of the x-projection $\Pi_x \widetilde{\mathcal{T}_1}$ of $\widetilde{\mathcal{T}_1}$.

Now, we fix some arbitrary $z_1 \in \Pi_x \mathcal{T}_1$, and we denote by Z_1 a small enough neighborhood of z_1 in ∂M. For $t_0 > 0$ sufficiently small, we set,

$$\Lambda_\pm := \bigcup_{0 < \pm t < 2t_0} \exp t H_{p_2}(Z_1 \times \{0\}).$$

Then $\Lambda_\pm \subset \{p_2 = 0\}$ and, since $Z_1 \times \{0\}$ is isotropic, $\Lambda_\pm$ is Lagrangian (see, e.g., [**Ma2**]). Since $\nabla V_2(z_1) \neq 0$ and $H_{p_2} = (2\xi, -\nabla V_2)$, it is also easy to check that both Λ_+ and Λ_- project bijectively on the base. Since in addition p_2 is an even function of ξ, we finally obtain (e.g., as in [**DaMa**, Section 5]) the existence of a real-analytic function ψ, defined on the x-projection of $\Lambda_\pm$, such that,

$$\begin{aligned} \Lambda_\pm \; : \; & \xi = \pm\nabla\psi(x); \\ & (\nabla\psi(x))^2 + V_2(x) = 0. \end{aligned} \quad (4.7)$$

Setting $z_0 := \Pi_x(\exp t_0 H_{p_2}(z_1, 0))$, and still denoting by ψ an holomorphic extension of ψ to a complex neighborhood of z_0, one can prove as in [**DaMa**, Proposition 5.1],

PROPOSITION 4.1. *For any $\varepsilon_1 > 0$, one has,*

$$e^{-i\psi/h + S_0/h} u \in H_{\varepsilon_1 |\operatorname{Im} x|, z_0},$$

where $H_{\varepsilon_1 |\operatorname{Im} x|, z_0}$ is the Sjöstrand's space consisting of h-dependent holomorphic functions $v = v(x; h)$ defined in a complex neighborhood of z_0, such that, for all $\varepsilon > 0$,

$$v(x, h) = \mathcal{O}(e^{(\varepsilon_1 |\operatorname{Im} x| + \varepsilon)/h}),$$

uniformly for $x \in \mathbb{C}^n$ close enough to z_0 and $h > 0$ small enough.

REMARK 4.2. Obviously, the result of this proposition can be re-written as,

$$u \in H_{-\operatorname{Im} \psi - S_0 + \varepsilon_1 |\operatorname{Im} x|, z_0}.$$

PROOF. The proof is very similar to that of [**DaMa**, Proposition 5.1], with the only difference that here we have to deal with a matrix-operator, instead of a scalar one. For the sake of completeness, we outline the main steps. At first, for x close enough to z_0, we write $v(x) := e^{-i\psi(x)/h + S_0/h} u(x)$ as the oscillating integral,

$$v(x) = (2\pi)^{-n} \int e^{i|\xi|\theta(x,y,\xi/|\xi|)} a\left(x-y, \frac{\xi}{|\xi|}\right) v(y)\chi(y)dyd\xi, \tag{4.8}$$

where χ is a cut-off function around z_0, $\theta(x,y,\tau) := (x-y)\tau + i(x-y)^2/2$, and $a(x,\tau) := 1 + ix\tau/2$. Thanks to the standard ellipticity of P, for $|\xi| \geq Ch^{-1}$ (with $C > 0$ a large enough constant), one can construct a $\mathbb{C}^2$-valued analytic symbol $b = b(x,y,\tau,\xi,h) \sim \sum_{k\geq 0} b_k(x,y,\tau,h)|\xi|^{-k}$ such that,

$$e^{-i|\xi|\theta(x,y,\tau)}Q(y,hD_y)\left(e^{i|\xi|\theta(x,y,\tau)}b\right) = a(x-y,\tau) + \mathcal{O}(e^{-\delta|\xi|}),$$

with $\delta > 0$, and where $Q(y,hD_y) := {}^t(e^{-i\psi(y)/h}P(y,hD_y)e^{i\psi(y)/h} - \rho)$. Inserting this estimate into (4.8) and using that, for all $\varepsilon > 0$, one has $v = \mathcal{O}(e^{\varepsilon/h})$ uniformly on the real near z_0, together with the fact that $Qv = 0$, we obtain,

$$v(x) = (2\pi)^{-n} \int_{|\xi|\leq C/h} e^{i|\xi|\theta(x,y,\xi/|\xi|)} a\left(x-y, \frac{\xi}{|\xi|}\right) v(y)\chi(y)dyd\xi + \mathcal{O}(e^{-\delta'/h}), \tag{4.9}$$

for some $\delta' > 0$.

Then, splitting the remaining integral into $\int_{\{|\xi|\leq \varepsilon_1/h\}}$ and $\int_{\{\varepsilon_1/h\leq|\xi|\leq C/h\}}$, we immediately observe that the first term $\int_{\{|\xi|\leq \varepsilon_1/h\}}$ is $\mathcal{O}(e^{(\varepsilon_1|\mathrm{Im}\,x|+\varepsilon)/h})$ for all $\varepsilon > 0$. Finally, using that u is outgoing and the results of [**HeSj2**, Section 9], we see that $MS(e^{S_0/h}u) \cap \Lambda_- = \emptyset$. Therefore, by propagation of MS, and since also $MS(e^{S_0/h}u) \cap (\partial M \times \mathbb{R}^n) \subset \{\xi = 0\}$, we deduce that, above a neighborhood of z_0, one necessarily has $MS(e^{S_0/h}u) \subset \Lambda_+$, and thus $MS(v) \subset \{\xi = 0\}$. But, after the change of variables $\xi \mapsto \xi/h$, this exactly means that the term $\int_{\{\varepsilon_1/h\leq|\xi|\leq C/h\}}$ is exponentially small as $h \to 0_+$. $\qquad\square$

Thanks to Proposition 4.1, we can enter the framework of analytic pseudodifferential calculus of [**Sj**, Sections 4–5]. In particular, working in the Sjöstrand's space $H_{-\mathrm{Im}\,\psi-S_0+\varepsilon_1|\mathrm{Im}\,x|,z_0}$, we can represent Pu as,

$$Pu(x) = \frac{1}{(2\pi h)^n} \int_{\Gamma(x)} e^{i(x-y)\xi/h-[(x-\alpha_x)^2+(y-\alpha_x)^2]/2h} p(\alpha_x,\xi;h)v(y)dyd\xi d\alpha_x, \tag{4.10}$$

where p is a matrix-valued analytic symbol that satisfies,

$$p(\alpha_x,\xi;h) = \begin{pmatrix} \xi^2 + V_1(\alpha_x) & 0 \\ 0 & \xi^2 + V_2(\alpha_x) \end{pmatrix} + \mathcal{O}(h),$$

and $\Gamma(x)$ is the (bounded, singular) contour of $\mathbb{C}^{3n}$ defined by,

$$\Gamma(x) : \begin{cases} \xi = \nabla\psi(\alpha_x) + 2i\varepsilon_1 \dfrac{\overline{x-y}}{|x-y|}\,; \\ |x-y| \leq r,\ y \in \mathbb{C}^n\ (r \text{ small enough with respect to } \varepsilon_1)\,; \\ |x-\alpha_x| \leq r,\ \alpha_x \in \mathbb{R}^n. \end{cases}$$

Now, as in [**DaMa**], we take local coordinates $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ centered at z_1, in such a way that $dV_2(z_1) \cdot x = -cx_n$ with $c > 0$. Then, taking advantage of the fact that hD_{x_n} and $h^2 D_{x_n}^2$ can be represented as in (4.10) with p substituted with ξ_n and ξ_n^2, respectively, we see that $p(\alpha_x, \xi; h)$ can actually be written as,

$$p(\alpha_x, \xi; h) = \begin{pmatrix} \xi_n^2 + a_1 & hb_1 + hb_2\xi_n \\ hb_3 + hb_4\xi_n & \xi_n^2 + a_2 \end{pmatrix},$$

where the a_j's and the b_j's are functions of $(\alpha_x, \xi'; h)$ only (not of ξ_n), and satisfy,

$$a_j(\alpha_x, \xi'; h) = (\xi')^2 + V_j(\alpha_x) + \mathcal{O}(h) \qquad (j = 1, 2);$$
$$b_k(\alpha_x, \xi'; h) = \mathcal{O}(1) \quad \text{on } \Gamma(x) \qquad (k = 1, 2, 3, 4).$$

Now, on $\Gamma(x)$, we have that ξ remains close to $\nabla\psi(\alpha_x)$, which in turn remains close to $(0, \sqrt{c\alpha_{x_n}})$, with α_{x_n} close to $\delta_0 := z_{0,n} > 0$. In particular, a_1 remains close to $V_1(z_0) > 0$, and a_2 remains close to $V_2(z_0) < 0$.

Then, re-writing $p(\alpha_x, \xi; h)$ as,

$$p(\alpha_x, \xi; h) = \left(\xi_n \mathbf{I}_2 + \frac{h}{2}\mathbf{J}\right)^2 + A$$

with,

$$\mathbf{J} := \begin{pmatrix} 0 & b_2 \\ b_4 & 0 \end{pmatrix};$$
$$A := \begin{pmatrix} a_1 & hb_1 \\ hb_3 & a_2 \end{pmatrix} - \frac{h^2}{4}\mathbf{J}^2,$$

and using the analytic symbolic calculus of [**Sj**], we see that, as an operator on $H_{-\operatorname{Im} \psi - S_0 + \varepsilon_1 |\operatorname{Im} x|, z_0}$, $P - \rho$ can be factorized into,

$$P - \rho = (hD_{x_n} \mathbf{I}_2 + B_+)(hD_{x_n} \mathbf{I}_2 + B_-),$$

where the symbol $\beta_\pm$ of $B_\pm$ does not depend on ξ_n, and satisfies,

$$\beta_\pm(\alpha_x, \xi'; h) = \pm \begin{pmatrix} i\sqrt{a_1 - \rho} & 0 \\ 0 & \sqrt{\rho - a_2} \end{pmatrix} + \mathcal{O}(h).$$

(Here, $\sqrt{\cdot}$ stand for the principal square-root of complex numbers with positive real part.)

In particular, $\xi_n \mathbf{I}_2 + \beta_+$ remains elliptic on $\Gamma(x)$, and thus, by applying an analytic parametrix of $hD_{x_n} \mathbf{I}_2 + B_+$, the equation $Pu = \rho u$ becomes,

$$(hD_{x_n} + B_-)u = 0 \quad \text{in } H_{-\operatorname{Im} \psi - S_0 + \varepsilon_1 |\operatorname{Im} x|, z_0}. \tag{4.11}$$

In the same way, $P - \rho$ can also be factorized into,

$$P - \rho = (hD_{x_n} \mathbf{I}_2 + \widetilde{B}_+)(hD_{x_n} \mathbf{I}_2 + \widetilde{B}_-),$$

where the symbol $\widetilde{\beta}_\pm$ of $\widetilde{B}_\pm$ satisfies,

$$\widetilde{\beta}_\pm(\alpha_x, \xi'; h) = \pm \begin{pmatrix} -i\sqrt{a_1 - \rho} & 0 \\ 0 & \sqrt{\rho - a_2} \end{pmatrix} + \mathcal{O}(h).$$

Since $\xi_n \mathbf{I}_2 + \widetilde{\beta}_+$ remains elliptic, too, on $\Gamma(x)$, this also leads to,

$$(hD_{x_n} + \widetilde{B}_-)u = 0 \quad \text{in } H_{-\operatorname{Im}\psi - S_0 + \varepsilon_1 |\operatorname{Im} x|, z_0}. \tag{4.12}$$

At that point, we can proceed as in [**DaMa**], [**Ma2**] (or, also, [**HeSj2**, Section 10]), and, by taking a realization on the real domain of B_- and $\widetilde{B}_-$ near z_0 (see [**Sj**]), we conclude to the existence of four analytic pseudodifferential operators Q_j and $\widetilde{Q}_j$ ($j = 1, 2$) acting on $L^2(\{x_n = \delta_0\})$, such that, near z_0,

$$\begin{aligned}
h\frac{\partial u_1}{\partial x_n}\bigg|_{x_n = \delta_0} &= Q_1 u_1 + \mathcal{O}(h)\|\chi u_2\|_{\{x_n = \delta_0\}} = \widetilde{Q}_1 u_1 + \mathcal{O}(h)\|\chi u_2\|_{\{x_n = \delta_0\}}; \\
h\frac{\partial u_2}{\partial x_n}\bigg|_{x_n = \delta_0} &= Q_2 u_2 + \mathcal{O}(h)\|\chi u_1\|_{\{x_n = \delta_0\}} = \widetilde{Q}_2 u_2 + \mathcal{O}(h)\|\chi u_1\|_{\{x_n = \delta_0\}},
\end{aligned} \tag{4.13}$$

where χ is some cut-off function that localizes near z_0, and the symbols q_j and $\widetilde{q}_j$ of Q_j and $\widetilde{Q}_j$ satisfy,

$$\begin{aligned}
q_1(x', \xi'; h) &= -\sqrt{(\xi')^2 + V_1(x', \delta_0) - \rho} + \mathcal{O}(h); \\
\widetilde{q}_1(x', \xi'; h) &= \sqrt{(\xi')^2 + V_1(x', \delta_0) - \rho} + \mathcal{O}(h); \\
q_2(x', \xi'; h) &= i\sqrt{\rho - (\xi')^2 - V_2(x', \delta_0)} + \mathcal{O}(h); \\
\widetilde{q}_2(x', \xi'; h) &= i\sqrt{\rho - (\xi')^2 - V_2(x', \delta_0)} + \mathcal{O}(h).
\end{aligned} \tag{4.14}$$

More precisely (see [**Sj**, Section 5]), these operators (say, Q_j) are of the type,

$$\begin{aligned}
Q_j v(x'; h) = \frac{1}{(2\pi h)^{3(n-1)/2}} \int_{\mathbb{R}^{3(n-1)}} & e^{i(x' - y')\xi'/h - [(x' - \alpha_{x'})^2 + (y' - \alpha_{x'})^2]/h} \\
& \times q_j(x', \xi'; h)\chi_0(x', y', \alpha_{x'}, \xi')v(y')dy'd\alpha_{x'}d\xi',
\end{aligned} \tag{4.15}$$

where $\chi_0 \in C_0^\infty(\mathbb{R}^{4(n-1)})$ is a cut-off function on a small enough neighborhood of the point $(z_0', z_0', z_0', \nabla_{x'}\psi(z_0))$.

(Note that, even for such realizations on the real of analytic pseudodifferential operators, ξ' remains close to 0.)

Here we observe the important fact that $Q_1 + \widetilde{Q}_1$ is $\mathcal{O}(h)$, a consequence of which is that (4.13) actually implies,

$$\begin{aligned}
h\frac{\partial u_1}{\partial x_n}\bigg|_{x_n = \delta_0} &= \mathcal{O}(h)\|\chi u_1\|_{\{x_n = \delta_0\}} + \mathcal{O}(h)\|\chi u_2\|_{\{x_n = \delta_0\}}; \\
h\frac{\partial u_2}{\partial x_n}\bigg|_{x_n = \delta_0} &= Q_2 u_2 + \mathcal{O}(h)\|\chi u_1\|_{\{x_n = \delta_0\}}.
\end{aligned} \tag{4.16}$$

Now, denoting by C_2 the pseudodifferential operators of the same form as Q_2, with symbol $c_2 := [\text{Re}\,\rho - (\xi')^2 - V_2(x',\delta_0)]^{1/4}$, the symbolic calculus of [**Sj**] gives us,

$$Q_2 = iC_2^* C_2 + \mathcal{O}(h).$$

Turning back to formula (4.6), we can treat in that way all the points of $\partial\Omega_1 \cap \mathcal{B}$, and, using a convenient partition of unity $(\chi_j^2)_{1 \le j \le N}$ on this set, we can write,

$$
\begin{aligned}
\text{Im}\,\rho &= -h^2 \sum_j \text{Im} \left\langle \chi_j \frac{\partial u}{\partial \nu}, \chi_j u \right\rangle_{L^2(\partial\Omega_1)} + \mathcal{O}(h^2)\|u\|_{L^2(\partial\Omega_1)}^2 + \mathcal{O}(e^{-(2S_0+\delta)/h}) \\
&= -h \sum_j \text{Im}\,\langle i\chi_j C_{2,j} u_2, C_{2,j}\chi_j u_2 \rangle_{L^2(\partial\Omega_1)} + \mathcal{O}(h^2)\|u\|_{L^2(\partial\Omega_1)}^2 + \mathcal{O}(e^{-(2S_0+\delta)/h}) \\
&= -h \sum_j \|C_{2,j}\chi_j u_2\|_{L^2(\partial\Omega_1)}^2 + \mathcal{O}(h^2)\|u\|_{L^2(\partial\Omega_1)}^2 + \mathcal{O}(e^{-(2S_0+\delta)/h}),
\end{aligned}
$$

where the $C_{2,j}$'s $(j = 1,\dots,N)$ are the pseudodifferential operators constructed as before on the support of χ_j, and where we have used (4.16) and the fact that the commutator $[\chi_j, C_{2,j}]$ is $\mathcal{O}(h)$.

Using the ellipticity of the symbol of $C_{2,j}$, we deduce the existence of a constant $C_0 > 0$ such that,

$$|\text{Im}\,\rho| \ge \frac{h}{C_0} \sum_j \|\chi_j u_2\|_{L^2(\partial\Omega_1)}^2 - C_0 h^2 \|u\|_{L^2(\partial\Omega_1)}^2 - C_0 e^{-(2S_0+\delta)/h},$$

and thus, since $\sum_j \chi_j^2 = 1$ on $\partial\Omega_1 \cap \mathcal{B}$, while $u = \mathcal{O}(e^{-(S_0+\delta')/h})$ (with $\delta' > 0$) on $\partial\Omega_1 \backslash \mathcal{B}$, we have proved,

PROPOSITION 4.3.   *There exist two positive constants C_0 and δ, such that,*

$$|\text{Im}\,\rho| \ge \frac{h}{C_0} \|u_2\|_{L^2(\partial\Omega_1)}^2 - C_0 h^2 \|u_1\|_{L^2(\partial\Omega_1)}^2 - C_0 e^{-(2S_0+\delta)/h},$$

for all $h > 0$ small enough.

5.   Reductio ad absurdum.

From now on, we proceed by contradiction, assuming the existence of some constant $\delta_1 > 0$ such that,

$$|\text{Im}\,\rho| = \mathcal{O}(e^{-(2S_0+\delta_1)/h}), \tag{5.1}$$

uniformly as $h \to 0_+$ (possibly along some sequence).

By Proposition 4.3, this implies,

$$\|u_2\|_{L^2(\partial\Omega_1)}^2 = \mathcal{O}(h\|u_1\|_{L^2(\partial\Omega_1)}^2 + e^{-(2S_0+\delta')/h}), \tag{5.2}$$

for some positive constant δ. Here, let us observe that the dependence with respect

to Ω_1 of the constants C_0 and δ that appear in Proposition 4.3 is only related to the ellipticity of $\xi_n + \sqrt{(\xi')^2 + V_2(x)}$ on $\{(x, \nabla\psi(x)); x \in \partial\Omega_1\}$. In particular, they can be taken uniformly as long as the distance between $\partial\Omega_1$ and $\{V_2 = 0\}$ remains larger than some fixed positive constant. Therefore, choosing a convenient 1-parameter family of such open sets Ω_1 (for instance $\Omega_{1,t} := \{V_2(x) = -t\}$ with $t \in [t_0, t_1]$, $0 < t_0 < t_1$ small enough), and integrating with respect to this parameter, we obtain from (5.2),

$$\|u_2\|^2_{L^2(\mathcal{A})} = \mathcal{O}(h\|u_1\|^2_{L^2(\mathcal{A})} + e^{-(2S_0+\delta')/h}), \tag{5.3}$$

where now $\mathcal{A}$ is a topological annulus surrounding $\widehat{\mathcal{I}}$, e.g. of the type,

$$\mathcal{A} = \{-t_1 < V_2(x) < -t_0, x \in \Omega_1'\}, \tag{5.4}$$

where Ω_1' stands for some fixed small enough neighborhood of $\widehat{\mathcal{I}}$. Then, using the equation $Pu = \rho u$, we have,

$$\langle(-h^2\Delta + V_1 - \rho)u_1, u_1\rangle_{L^2(\mathcal{A})} = -h\langle Wu_2, u_1\rangle_{L^2(\mathcal{A})},$$

and thus, by Stokes formula,

$$\|h\nabla u_1\|^2_{\mathcal{A}} + \langle(V_1 - \rho)u_1, u_1\rangle_{\mathcal{A}} - h^2\langle\partial_\nu u_1, u_1\rangle_{\partial\mathcal{A}}$$
$$= \mathcal{O}(h)\left(\|u_1\|^2_{\mathcal{A}} + \|u_2\|^2_{\mathcal{A}} + h\|u_1\|^2_{\partial\mathcal{A}} + h\|u_2\|^2_{\partial\mathcal{A}} + \|h\nabla u_1\|^2_{\mathcal{A}}\right),$$

where ∂_ν is the outward pointing normal derivative on $\partial\mathcal{A}$. Taking the real part, and using that Re $(V_1 - \rho)$ is positive near $\mathcal{A}$, together with the fact that, thanks to (4.16), one has $\langle\partial_\nu u_1, u_1\rangle_{\partial\mathcal{A}} = \mathcal{O}(\|u\|^2_{\partial\mathcal{A}})$, we deduce (for h sufficiently small),

$$\|h\nabla u_1\|^2_{\mathcal{A}} + \|u_1\|^2_{\mathcal{A}} = \mathcal{O}(h\|u_2\|^2_{\mathcal{A}} + h^2\|u\|^2_{\partial\mathcal{A}}).$$

Now, using both (5.3) and (5.2) (with $\partial\mathcal{A}$ instead of $\partial\Omega_1$), this gives us,

$$\|h\nabla u_1\|^2_{\mathcal{A}} + \|u_1\|^2_{\mathcal{A}} = \mathcal{O}(h^2\|u_1\|^2_{\mathcal{A}} + h^2\|u_1\|^2_{\partial\mathcal{A}} + e^{-(2S_0+\delta')/h}),$$

that is, for h sufficiently small,

$$h^2\|\nabla u_1\|^2_{\mathcal{A}} + \|u_1\|^2_{\mathcal{A}} = \mathcal{O}(h^2\|u_1\|^2_{\partial\mathcal{A}} + e^{-(2S_0+\delta')/h}). \tag{5.5}$$

At this point, we can use the standard Sobolev estimate,

$$\|u_1\|^2_{\partial\mathcal{A}} = \mathcal{O}(\|u_1\|^2_{\mathcal{A}} + \|u_1\|_{\mathcal{A}}\|\nabla u_1\|_{\mathcal{A}}),$$

that implies,

$$h^2\|u_1\|^2_{\partial\mathcal{A}} = \mathcal{O}(h\|u_1\|^2_{\mathcal{A}} + h^3\|\nabla u_1\|^2_{\mathcal{A}}),$$

and thus, once inserted into (5.5), permits us to conclude that, for h small enough,

$$\|h\nabla u_1\|^2_{\mathcal{A}} + \|u_1\|^2_{\mathcal{A}} = \mathcal{O}(e^{-(2S_0+\delta')/h}).$$

Summing up, and gathering with (5.3), we have proved,

PROPOSITION 5.1. *Under Assumption* (5.1), *and with* $\mathcal{A}$ *given in* (5.4), *there exists* $\delta_1 > 0$ *such that,*

$$\|u\|_{L^2(\mathcal{A})}^2 = \mathcal{O}(e^{-(S_0+\delta_1)/h}).$$

The purpose of the next sections will be to propagate this smallness of u across the shore $\partial\mathcal{M}$, the crest $\partial\Omega_0$, and up to the interior of the well U, in such a way that one finally gets a contradiction with Assumption 4.

6. Propagation across the shore $\partial\mathcal{M}$.

In order to propagate the smallness of u across $\partial\mathcal{M}$, we adopt the same strategy as in [**DaMa**], that is, we start by establishing global Carleman estimates around $\partial\mathcal{M}$.

We fix some $\mu_0 > 0$ sufficiently small, and, for $\mu \in (0, \mu_0]$, we consider the neighborhood $\mathcal{N}_\mu$ of $\partial\mathcal{M}$ given by,

$$\mathcal{N}_\mu := \{-\mu_0 \leq V_2(x) \leq \mu, \ x \in \Omega_1\}, \tag{6.1}$$

(where, as before, Ω_1 is some fixed small enough neighborhood of $\widehat{\mathcal{I}}$).

We also set,

$$\Sigma := \mathcal{N} \cap \{V_2 = \mu\};$$
$$\Sigma_0 := \mathcal{N} \cap \{V_2 = -\mu_0\}.$$

By the same geometrical considerations as in [**DaMa**, Section 6], we see that, on Σ, the function φ satisfies,

$$\varphi\mid_{\Sigma_\mu} \geq S_0 - c_0\mu^{3/2},$$

where the constant $c_0 > 0$ does not depend on μ. Therefore, using (4.2) and Sobolev estimates, we deduce that, for any $\mu, \varepsilon > 0$ small enough, we have,

$$u = \mathcal{O}(e^{-(S_0-c_0\mu^{3/2}-\varepsilon)/h}) \quad \text{uniformly on } \mathcal{N}_\mu, \tag{6.2}$$

and the same holds for all the derivatives of u.

In addition, since Σ_0 stays away from $\widehat{\mathcal{I}}$, by Proposition 5.1 (plus standard Sobolev estimates), we know the existence of some $\delta_0 > 0$ constant such that,

$$\|u\|_{H^2(\Sigma_0)} = \mathcal{O}(e^{-(S_0+\delta_0)/h}). \tag{6.3}$$

We plan to extend this estimate up to Σ. We set,

$$v(x) := e^{\alpha(\mu-V_2(x))/h}u(x), \tag{6.4}$$

where $\alpha > 0$ satisfies,

$$2\alpha\mu_0 \le \delta_0.$$

Then, by (6.2), for any $\varepsilon > 0$, v satisfies,

$$\|v\|_{H^2(\Sigma)} = \mathcal{O}(e^{-(S_0 - c_0\mu^{3/2} - \varepsilon)/h}), \tag{6.5}$$

and, by (6.3),

$$\|v\|_{H^2(\Sigma_0)} = \mathcal{O}(e^{-S_0/h}). \tag{6.6}$$

Moreover, v is solution to,

$$(A + iB)v = 0,$$

with,

$$A := P - \operatorname{Re}\rho - \alpha^2(\nabla V_2)^2;$$

$$B := -2h\alpha(\nabla V_2) \cdot D_x + ih\alpha\Delta V_2 - \operatorname{Im}\rho - ih\alpha\begin{pmatrix} 0 & -r_1 \cdot \nabla V_2 \\ \overline{r_1} \cdot \nabla V_2 & 0 \end{pmatrix},$$

where we have used (4.4). In particular, A and B are formally selfadjoint, and we have,

$$0 = \|(A + iB)v\|^2_{L^2(\mathcal{N}_\mu)} = \|Av\|^2_{L^2(\mathcal{N}_\mu)} + \|Bv\|^2_{L^2(\mathcal{N}_\mu)} + 2\operatorname{Im}\langle Av, Bv\rangle_{L^2(\mathcal{N}_\mu)}. \tag{6.7}$$

As in [**DaMa**], the key-point is the following Carleman estimate:

LEMMA 6.1.   *If α and μ_0 are chosen sufficiently small, there exists a constant $C > 0$ and, for all $\varepsilon > 0$, a constant $C_\varepsilon > 0$, such that,*

$$\operatorname{Im}\langle Av, Bv\rangle_{L^2(\mathcal{N}_\mu)} \ge \frac{h}{C}\|v_2\|^2_{L^2(\mathcal{N}_\mu)} - Ch\|v_1\|^2_{L^2(\mathcal{N}_\mu)} - Ch\|Av\|^2_{L^2(\mathcal{N}_\mu)}$$
$$- C_\varepsilon e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h},$$

uniformly for $h > 0$ small enough.

PROOF.   Using Green's formula, together with (6.5)–(6.6) and the fact that $\partial\mathcal{N}_\mu = \Sigma_0 \cup \Sigma$, we obtain,

$$\operatorname{Im}\langle Av, Bv\rangle_{L^2(\mathcal{N}_\mu)} = \frac{i}{2}\langle[A, B]v, v\rangle_{L^2(\mathcal{N}_\mu)} + \mathcal{O}(e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h}), \tag{6.8}$$

uniformly with respect to h, and with ε, μ arbitrarily small. Moreover, setting

$$\mathbf{V} := \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}, \mathbf{W} := \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix}, \mathbf{R} := \begin{pmatrix} 0 & -r_1 \cdot \nabla V_2 \\ \overline{r_1} \cdot \nabla V_2 & 0 \end{pmatrix},$$

$$A_0 := -h^2\Delta + \mathbf{V} - \operatorname{Re}\rho - \alpha^2(\nabla V_2)^2 = A - h\mathbf{W},$$

$$B_0 := -2h\alpha(\nabla V_2) \cdot D_x + ih\alpha\Delta V_2 - \operatorname{Im}\rho = B + ih\alpha\mathbf{R},$$

we have,

$$\frac{i}{2}[A, B] = \frac{i}{2}[A_0, B_0] + h[\mathbf{W}, B] - ih\alpha[A_0, \mathbf{R}], \qquad (6.9)$$

and, as in [**DaMa**, Section 6], a straightforward computation leads to,

$$\frac{i}{2}[A_0, B_0] = \alpha h^3 \left(Q_2 + 2Q_1 + \frac{1}{2}(\Delta^2 V_2) \right) + \alpha h \nabla V_2 \cdot \nabla \mathbf{V} - \alpha^3 h Q_0 \qquad (6.10)$$

with,

$$Q_2 := 2 \sum_{j,k} (\partial_j \partial_k V_2) \partial_j \partial_k;$$
$$Q_1 := \nabla(\Delta V_2) \cdot \nabla;$$
$$Q_0 := (\nabla V_2) \cdot (\nabla(\Delta V_2)^2).$$

Here, we first observe,

$$\langle (\nabla V_2 \cdot \nabla \mathbf{V})v, v \rangle_{\mathcal{N}_\mu} = \langle (\nabla V_2 \cdot \nabla V_1)v_1, v_1 \rangle + \|(\nabla V_2)v_2\|_{\mathcal{N}_\mu}^2$$

and therefore, since ∇V_2 never vanishes on $\partial \mathcal{M}$, there exists a constant $C_0 > 0$ (depending only on the geometry of V_1 and V_2 near $\partial \mathcal{M}$), such that,

$$\langle (\nabla V_2 \cdot \nabla \mathbf{V})v, v \rangle_{\mathcal{N}_\mu} \geq \frac{1}{C_0} \|v_2\|_{\mathcal{N}_\mu}^2 - C_0 \|v_1\|_{\mathcal{N}_\mu}^2. \qquad (6.11)$$

Then, by doing an integration by parts and by using (6.5)–(6.6) we first see,

$$\|h\nabla v\|_{\mathcal{N}_\mu}^2 = \mathcal{O}(\|Av\|_{\mathcal{N}_\mu}^2 + \|v\|_{\mathcal{N}_\mu}^2 + e^{-2(S_0 - c_0 \mu^{3/2} - \varepsilon)/h}), \qquad (6.12)$$

and then,

$$h^2 \langle Q_2 v, v \rangle_{\mathcal{N}_\mu} = \mathcal{O}(\|h\nabla v\|_{\mathcal{N}_\mu}^2 + h^2 \|v\|_{\mathcal{N}_\mu}^2 + e^{-2(S_0 - c_0 \mu^{3/2} - \varepsilon)/h})$$
$$= \mathcal{O}(|\langle -h^2 \Delta v, v \rangle_{\mathcal{N}_\mu}| + h^2 \|v\|_{\mathcal{N}_\mu}^2 + e^{-2(S_0 - c_0 \mu^{3/2} - \varepsilon)/h}).$$

Therefore, for any constant $C \geq 1$ arbitrarily large, and $h > 0$ sufficiently small,

$$h^2 |\langle Q_2 v, v \rangle_{\mathcal{N}_\mu}| \leq C \|h^2 \Delta v\|_{\mathcal{N}_\mu}^2 + \frac{2}{C} \|v\|_{\mathcal{N}_\mu}^2 + C_\varepsilon e^{-2(S_0 - c_0 \mu^{3/2} - \varepsilon)/h}$$

(where C_ε depends on ε only, which in turn is arbitrarily small). We deduce (with some new constant $C' > 0$),

$$h^2 |\langle Q_2 v, v \rangle_{\mathcal{N}_\mu}| \leq 2C \|Av\|_{\mathcal{N}_\mu}^2 + 2C \|(\mathbf{V} - \text{Re}\,\rho - \alpha^2 (\nabla V_2)^2)v\|_{\mathcal{N}_\mu}^2 + C'h\|v\|_{\mathcal{N}_\mu}^2$$
$$+ C'h\|h\nabla v\|_{\mathcal{N}_\mu}^2 + \frac{2}{C} \|v\|_{\mathcal{N}_\mu}^2 + C_\varepsilon e^{-2(S_0 - c_0 \mu^{3/2} - \varepsilon)/h}. \qquad (6.13)$$

Now, for h small enough, on $\mathcal{N}_\mu$ we have $|V_2 - \text{Re}\,\rho - \alpha^2 (\nabla V_2)^2| \leq 2\mu_0 + \alpha^2 |\nabla V_2|^2$, and thus,

$$\|(\mathbf{V} - \operatorname{Re}\rho - \alpha^2(\nabla V_2)^2)v\|^2_{\mathcal{N}_\mu} \le (2\mu_0 + c\alpha^2)^2\|v_2\|^2_{\mathcal{N}_\mu} + C''\|v_1\|^2_{\mathcal{N}_\mu}. \tag{6.14}$$

As a consequence, by first choosing C sufficiently large, then μ_0 and α sufficiently small, we can make true the inequality

$$\frac{2}{C} + 2C(2\mu_0 + c\alpha^2)^2 \le \frac{1}{4C_0}, \tag{6.15}$$

where C_0 is the constant appearing in (6.11).

Inserting (6.14)–(6.15) into (6.13), and using (6.12), for h small enough we obtain (possibly by increasing C''),

$$h^2|\langle Q_2 v, v\rangle_{\mathcal{N}_\mu}| \le C''\|Av\|^2_{\mathcal{N}_\mu} + \frac{1}{2C_0}\|v_2\|^2_{\mathcal{N}_\mu} + C''\|v_1\|^2_{\mathcal{N}_\mu} + C_\varepsilon e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h}. \tag{6.16}$$

Since Q_1 is a vector-field, by similar (but rougher) arguments, we also have,

$$\begin{aligned}
h|\langle Q_1 v, v\rangle_{\mathcal{N}_\mu}| &= \mathcal{O}(\|h\nabla v\|_{\mathcal{N}_\mu}\|v\|_{\mathcal{N}_\mu}) = \mathcal{O}(\|h\nabla v\|^2_{\mathcal{N}_\mu} + \|v\|^2_{\mathcal{N}_\mu}) \\
&= \mathcal{O}(\|Av\|^2_{\mathcal{N}_\mu} + \|v\|^2_{\mathcal{N}_\mu} + e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h})
\end{aligned} \tag{6.17}$$

and, of course,

$$|\langle Q_0 v, v\rangle_{\mathcal{N}_\mu}| = \mathcal{O}(\|v\|^2_{\mathcal{N}_\mu}). \tag{6.18}$$

Using (6.16)–(6.18) together with (6.10) and (6.11), we obtain,

$$\begin{aligned}
\operatorname{Re}\frac{i}{2}\langle[A_0, B_0]v, v\rangle_{\mathcal{N}_\mu} &\ge \frac{\alpha h}{3C_0}\|v_2\|^2_{\mathcal{N}_\mu} - C_1\alpha h(\|Av\|^2_{\mathcal{N}_\mu} + \|v_1\|^2_{\mathcal{N}_\mu}) \\
&\quad - C_\varepsilon e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h},
\end{aligned} \tag{6.19}$$

with $C_1 \ge 1$ independent of h, α and μ, all of them small enough.

In addition, we also have,

$$[\mathbf{W}, B] = [\mathbf{W}, B_0] - i\alpha h[\mathbf{W}, \mathbf{R}],$$

and thus, since B_0 is a scalar first-order semiclassical differential operator, and $\mathbf{R}$ is 0-th order,

$$[\mathbf{W}, B] = \alpha h F_1(x, hD_x)$$

where $F_1(x, hD_x)$ is a matrix of first-order semiclassical differential operators with bounded coefficients. As a consequence, by the same arguments as before, we have,

$$\langle h[\mathbf{W}, B]v, v\rangle = \mathcal{O}(h^2\|Av\|^2_{\mathcal{N}_\mu} + h^2\|v\|^2_{\mathcal{N}_\mu} + e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h}). \tag{6.20}$$

Finally, since A_0 is a scalar second-order semiclassical differential operator, we see that $h^{-1}[A_0, \mathbf{R}]$ is a matrix of first-order semiclassical differential operators with bounded coefficients, too, and thus,

$$\langle i\alpha h[A_0, \mathbf{R}]v, v\rangle = \mathcal{O}(h^2 \|Av\|^2_{\mathcal{N}_\mu} + h^2 \|v\|^2_{\mathcal{N}_\mu} + e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h}). \tag{6.21}$$

Inserting (6.19)–(6.21) into (6.9), and using (6.8), the result follows. □

For h small enough, Lemma 6.1 and (6.7) imply,

$$\|Av\|^2_{\mathcal{N}_\mu} + \|Bv\|^2_{\mathcal{N}_\mu} + h\|v_2\|^2_{\mathcal{N}_\mu} = \mathcal{O}(h\|v_1\|^2_{\mathcal{N}_\mu} + e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h}). \tag{6.22}$$

Now, we prove,

LEMMA 6.2.   *For any $C \geq 1$, there exists $C_1 = C_1(C) > 0$ such that, for all $h > 0$ small enough, one has,*

$$\|Av\|^2_{\mathcal{N}_\mu} \geq \frac{h}{C_1}\|h\nabla v\|^2_{\mathcal{N}_\mu} + \frac{1}{C_1}\|v_1\|^2_{\mathcal{N}_\mu} - \frac{h}{C}\|v_2\|^2_{\mathcal{N}_\mu} - C_1 e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h}.$$

PROOF.   We have,

$$\|Av\|^2_{\mathcal{N}_\mu} = \|A_0 v\|^2_{\mathcal{N}_\mu} + \|h\mathbf{W}v\|^2_{\mathcal{N}_\mu} + 2h\mathrm{Im}\,\langle [A_0, \mathbf{W}]v, v\rangle_{\mathcal{N}_\mu}, \tag{6.23}$$

and thus, setting $A_j := -h^2\Delta + V_j - \mathrm{Re}\,\rho - \alpha^2(\nabla V_2)^2$ $(j = 1, 2)$,

$$\|Av\|^2_{\mathcal{N}_\mu} \geq \|A_1 v_1\|^2_{\mathcal{N}_\mu} + \|A_2 v_2\|^2_{\mathcal{N}_\mu} + 2h\mathrm{Im}\,\langle [A_0, \mathbf{W}]v, v\rangle_{\mathcal{N}_\mu}. \tag{6.24}$$

Moreover, by an integration by parts,

$$\langle A_1 v_1, v_1\rangle_{\mathcal{N}_\mu} = \|h\nabla v_1\|^2_{\mathcal{N}_\mu} + \langle (V_1 - \mathrm{Re}\,\rho - \alpha^2(\nabla V_2)^2)v_1, v_1\rangle_{\mathcal{N}_\mu} + \mathcal{O}(e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h}),$$

and, by Assumption 1, one has $\min V_1|_{\partial\mathcal{M}} > 0$. Therefore, by Cauchy–Schwarz inequality, for α and h sufficiently small, we obtain the existence of a constant $C_1 > 0$ such that,

$$\|A_1 v_1\|_{\mathcal{N}_\mu} \cdot \|v_1\|_{\mathcal{N}_\mu} \geq \|h\nabla v_1\|^2_{\mathcal{N}_\mu} + \frac{1}{C_1}\|v_1\|^2_{\mathcal{N}_\mu} + \mathcal{O}(e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h}).$$

Since also $\|A_1 v_1\|_{\mathcal{N}_\mu} \cdot \|v_1\|_{\mathcal{N}_\mu} \leq 2C_1\|A_1 v_1\|^2_{\mathcal{N}_\mu} + (1/(2C_1))\|v_1\|^2_{\mathcal{N}_\mu}$, we deduce,

$$\|A_1 v_1\|^2_{\mathcal{N}_\mu} \geq \frac{1}{2C_1}\|h\nabla v_1\|^2_{\mathcal{N}_\mu} + \frac{1}{4C_1^2}\|v_1\|^2_{\mathcal{N}_\mu} + \mathcal{O}(e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h}). \tag{6.25}$$

On the other hand, setting $\widetilde{V}_2 := V_2 - \mathrm{Re}\,\rho - \alpha^2(\nabla V_2)^2$, we have,

$$\begin{aligned}\|A_2 v_2\|^2_{\mathcal{N}_\mu} &= \|h^2\Delta v_2\|^2_{\mathcal{N}_\mu} + \|\widetilde{V}_2 v_2\|^2_{\mathcal{N}_\mu} + 2\mathrm{Im}\,\langle [-h^2\Delta, \widetilde{V}_2]v_2, v_2\rangle_{\mathcal{N}_\mu} \\ &= \|h^2\Delta v_2\|^2_{\mathcal{N}_\mu} + \|\widetilde{V}_2 v_2\|^2_{\mathcal{N}_\mu} + \mathcal{O}(h\|h\nabla v_2\|_{\mathcal{N}_\mu}\|v_2\|_{\mathcal{N}_\mu} + h^2\|v_2\|^2_{\mathcal{N}_\mu}),\end{aligned} \tag{6.26}$$

and, still with an integration by parts and Cauchy–Schwarz inequality,

$$\|h^2\Delta v_2\|_{\mathcal{N}_\mu}\|v_2\|_{\mathcal{N}_\mu} \geq \langle -h^2\Delta v_2, v_2\rangle_{\mathcal{N}_\mu} = \|h\nabla v_2\|^2_{\mathcal{N}_\mu} + \mathcal{O}(e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h}).$$

Then, writing,

$$\|h^2 \Delta v_2\|_{\mathcal{N}_\mu} \|v_2\|_{\mathcal{N}_\mu} \leq \frac{1}{Ch} \|h^2 \Delta v_2\|_{\mathcal{N}_\mu}^2 + Ch \|v_2\|_{\mathcal{N}_\mu}^2,$$

where $C \geq 1$ is arbitrary, we deduce,

$$\|h^2 \Delta v_2\|_{\mathcal{N}_\mu}^2 \geq Ch \|h\nabla v_2\|_{\mathcal{N}_\mu}^2 - C^2 h^2 \|v_2\|_{\mathcal{N}_\mu}^2 + \mathcal{O}(e^{-2(S_0 - c_0 \mu^{3/2} - \varepsilon)/h}). \qquad (6.27)$$

Inserting (6.27) into (6.26), we obtain the existence of a constant $C_2 > 0$ such that, for any $C \geq C_2$,

$$\|A_2 v_2\|_{\mathcal{N}_\mu}^2 \geq Ch \|h\nabla v_2\|_{\mathcal{N}_\mu}^2 - 2C^2 h^2 \|v_2\|_{\mathcal{N}_\mu}^2 - C_2 h \|h\nabla v_2\|_{\mathcal{N}_\mu} \|v_2\|_{\mathcal{N}_\mu}$$
$$+ \mathcal{O}(e^{-2(S_0 - c_0 \mu^{3/2} - \varepsilon)/h}). \qquad (6.28)$$

In particular,

$$\|A_2 v_2\|_{\mathcal{N}_\mu}^2 \geq (C - C_2 \sqrt{C})h \|h\nabla v_2\|_{\mathcal{N}_\mu}^2 - \left(2C^2 h^2 + \frac{C_2 h}{\sqrt{C}}\right) \|v_2\|_{\mathcal{N}_\mu}^2 + \mathcal{O}(e^{-2(S_0 - c_0 \mu^{3/2} - \varepsilon)/h}),$$

and thus, for C sufficiently large,

$$\|A_2 v_2\|_{\mathcal{N}_\mu}^2 \geq \sqrt{C}h \|h\nabla v_2\|_{\mathcal{N}_\mu}^2 - \left(2C^2 h^2 + \frac{C_2 h}{\sqrt{C}}\right) \|v_2\|_{\mathcal{N}_\mu}^2 + \mathcal{O}(e^{-2(S_0 - c_0 \mu^{3/2} - \varepsilon)/h}). \quad (6.29)$$

Finally, concerning the term $2h\mathrm{Im}\,\langle [A_0, \mathbf{W}]v, v\rangle_{\mathcal{N}_\mu}$ appearing in (6.24), since A_0 is scalar, we have,

$$h\langle [A_0, \mathbf{W}]v, v\rangle_{\mathcal{N}_\mu} = \mathcal{O}(h^2)\left(\sum_{|\alpha| \leq 2} |\langle (hD_x)^\alpha v, v\rangle_{\mathcal{N}_\mu}|\right),$$

and thus, with an integration by parts, and still using (6.5)–(6.6),

$$h\langle [A_0, \mathbf{W}]v, v\rangle_{\mathcal{N}_\mu} = \mathcal{O}(h^2)(\|h\nabla v\|_{\mathcal{N}_\mu}^2 + \|v\|_{\mathcal{N}_\mu}^2) + \mathcal{O}(e^{-2(S_0 - c_0 \mu^{3/2} - \varepsilon)/h}). \qquad (6.30)$$

Inserting (6.25), (6.29), and (6.30) into (6.24), we obtain the existence of a constant $C_3 > 0$ such that, for any $C \geq 1$ sufficiently large, and for h sufficiently small (depending on C), one has,

$$\|Av\|_{\mathcal{N}_\mu}^2 \geq \frac{1}{C_3} \|h\nabla v_1\|_{\mathcal{N}_\mu}^2 + \frac{1}{C_3} \|v_1\|_{\mathcal{N}_\mu}^2 + Ch \|h\nabla v_2\|_{\mathcal{N}_\mu}^2 - \frac{h}{C} \|v_2\|_{\mathcal{N}_\mu}^2$$
$$- C_3 h^2 \|h\nabla v\|_{\mathcal{N}_\mu}^2 - C_3 h^2 \|v\|_{\mathcal{N}_\mu}^2 - C_3 e^{-2(S_0 - c_0 \mu^{3/2} - \varepsilon)/h}.$$

In particular, Lemma 6.2 follows. $\qquad \square$

Gathering Lemma 6.2 and (6.22), we obtain,

$$\frac{h}{C_1} \|h\nabla v\|_{\mathcal{N}_\mu}^2 + \frac{1}{C_1} \|v_1\|_{\mathcal{N}_\mu}^2 + h\left(1 - \frac{1}{C}\right) \|v_2\|_{\mathcal{N}_\mu}^2 = \mathcal{O}(h\|v_1\|_{\mathcal{N}_\mu}^2 + e^{-2(S_0 - c_0 \mu^{3/2} - \varepsilon)/h}),$$

where $C \geq 1$ is arbitrarily (and sufficiently) large, and $\varepsilon > 0$ is arbitrary. Choosing $C \geq 2$, we conclude that, for any $\varepsilon > 0$, for any $\mu > 0$ sufficiently small, there exists $h_0 = h_0(\varepsilon, \mu) > 0$ such that, for $h \in (0, h_0]$, one has,

$$\|h\nabla v\|^2_{\mathcal{N}_\mu} + \|v\|^2_{\mathcal{N}_\mu} = \mathcal{O}(e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h}). \tag{6.31}$$

In particular,

$$\|h\nabla v\|^2_{\mathcal{N}_{\mu/2}} + \|v\|^2_{\mathcal{N}_{\mu/2}} = \mathcal{O}(e^{-2(S_0 - c_0\mu^{3/2} - \varepsilon)/h}). \tag{6.32}$$

Now, recall from (6.4) that, on $\mathcal{N}_{\mu/2}$, on has $|v| \geq e^{(1/2)\alpha\mu/h}|u|$. Therefore, (6.32) implies,

$$\|h\nabla u\|^2_{\mathcal{N}_{\mu/2}} + \|u\|^2_{\mathcal{N}_{\mu/2}} = \mathcal{O}(e^{-(2S_0 + \alpha\mu - 2c_0\mu^{3/2} - 2\varepsilon)/h}).$$

Here, $\alpha > 0$ has been previously fixed sufficiently small, while μ can still be taken arbitrarily small. Therefore, by choosing $\mu > 0$ in such a way that $4c_0\mu^{3/2} \leq \alpha\mu$ (that is $\mu \leq \alpha^2/(16c_0^2)$), and by taking $\varepsilon \leq c_0\mu^{3/2}$, we have proved,

PROPOSITION 6.3.   *There exists $\delta_2 > 0$ and a neighborhood $\mathcal{N}$ of $\partial\mathcal{M}$, such that,*

$$\|u\|_{L^2(\mathcal{N})} + \|h\nabla u\|_{L^2(\mathcal{N})} = \mathcal{O}(e^{-(S_0 + \delta_2)/h}).$$

7.   Propagation across the crest $\partial\Omega_0$.

7.1.   Preliminaries.

In this section, we work near the minimal d-geodesic $\gamma \in G_1$ given by Assumption 5, and we will travel it, starting from $\partial\mathcal{M}$ up to ∂U. We denote by $x_0 \in \partial\mathcal{M}$ its starting point, by $y_0 \in \partial U$ its ending point, and by $x^{(1)}, \ldots, x^{(N)}$ the sequence of points that constitute $\gamma \cap \partial\Omega_0$, ordered from the closest to $\partial\mathcal{M}$ up to the closest to U. We also denote by $\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(N+1)}$ the portions of $\gamma \backslash (\partial\Omega_0 \cup \partial\mathcal{M} \cup \partial U)$ that are in-between $\partial\mathcal{M}$ and $x^{(1)}$, $x^{(1)}$ and $x^{(2)}, \ldots, x^{(N)}$ and U, respectively, in such a way that we have,

$$\gamma = \{x_0\} \cup \gamma^{(1)} \cup \{x^{(1)}\} \cup \gamma^{(2)} \cup \cdots \cup \{x^{(N)}\} \cup \gamma^{(N+1)} \cup \{y_0\},$$

where the unions are all disjoints.

Since γ is a minimal d-geodesic, we know that the function φ is analytic in a neighborhood of $\gamma^{(1)}$ (that of course may shrink as we get closer to the end points of $\gamma^{(1)}$), and satisfies $|\nabla\varphi|^2 = V_2$ there. Moreover, by standard arguments of Riemannian geometry, $\gamma^{(1)}$ can be parametrized on some interval $(0, t_1)$ in such a way that, for all $t \in (0, t_1)$, one has,

$$\dot{\gamma}^{(1)}(t) = -\nabla\varphi(\gamma^{(1)}(t)). \tag{7.1}$$

Now, we set,

$$v := e^{\varphi/h} u.$$

By Proposition 6.3, we know that there exists $\delta_2 > 0$ such that $u = \mathcal{O}(e^{-(S_0+\delta_2)/h})$ near x_0 (and analogous estimates for the derivatives of u). Hence (since $\varphi(x_0) = S_0$ and $\varphi \leq S_0$ everywhere), we obtain that v is exponentially small near $\gamma^{(1)}(t)$ for all t close enough to 0.

On the other hand, using the equation $Pu = \rho u$, on a neighborhood of $\gamma^{(1)}$ we obtain,

$$\left(-h^2\Delta + 2h(\nabla\varphi)\cdot\nabla - |\nabla\varphi|^2 + h(\Delta\varphi) + \mathbf{V} + h\mathbf{W} - \rho\right)v = 0,$$

that we prefer to write as,

$$\left(\frac{i}{2}h^2\Delta + h(\nabla\varphi)\cdot D_x + \frac{i}{2}V_2 - \frac{ih}{2}(\Delta\varphi) - \frac{i}{2}\mathbf{V} - \frac{ih}{2}\mathbf{W} + \frac{i\rho}{2}\right)v = 0,$$

that is,

$$Q(x, hD_x; h)v = 0, \tag{7.2}$$

where $Q(x, hD_x; h)$ is a semiclassical differential matrix-operator, with diagonal principal symbol $q_0 = \mathrm{diag}(q_1, q_2)$ given by,

$$q_1(x, \xi) = (\nabla\varphi)\cdot\xi - \frac{i}{2}\xi^2 + \frac{i}{2}(V_2 - V_1)$$

$$q_2(x, \xi) = (\nabla\varphi)\cdot\xi - \frac{i}{2}\xi^2.$$

Since $V_1 > V_2$ near $\gamma^{(1)}$, we see that q_1 is elliptic there, while, thanks to (7.1), the curve $t \mapsto (\gamma^{(1)}(t), 0)$ is a (real) integral curve of the Hamilton flow $H_{q_2} = (\nabla_\xi q_2, -\nabla_x q_2)$ of q_2. As a consequence, Equation (7.2) can actually be locally reduced to a scalar one (e.g. by using a parametrix of Q_1), and we can apply standard results on the micro-support (see, e.g., [**Ma2**, Chapter 4] or [**Ma1**, Section 5]). In particular, by Hanges' theorem of propagation, and since v is exponentially small near $\gamma^{(1)}(t)$ for t small, we deduce (denoting by $MS(v)$ the micro-support of v),

$$\gamma^{(1)} \times \{0\} \cap MS(v) = \emptyset. \tag{7.3}$$

But since $q_2^{-1}(0) \subset \{\xi = 0\}$, we also know that $MS(v) \subset \{\xi = 0\}$, and thus, (7.3) implies,

$$\gamma^{(1)} \times \mathbb{R}^n \cap MS(v) = \emptyset.$$

Still by standard results of analytic microlocal analysis (see, e.g. [**Ma2**], [**Sj**]), we conclude that v is exponentially small on a whole neighborhood of $\gamma^{(1)}$ as $h \to 0_+$, that is, we have proved,

PROPOSITION 7.1.   *For any $x \in \gamma^{(1)}$, there exists $\delta = \delta(x) > 0$ such that $ue^{\varphi/h} = \mathcal{O}(e^{-\delta/h})$ on a neighborhood of x, uniformly for $h > 0$ sufficiently small.*

The next step will consist in crossing $\partial\Omega_0$ at the ending point $x^{(1)}$ of $\gamma^{(1)}$.

By [**Pe**], Theorem 2.14, we know that there exists a neighborhood $\mathcal{V}_1$ of $x^{(1)}$ and two positive real-analytic functions φ_1, φ_2 on $\mathcal{V}_1$, such that,

$$
\begin{aligned}
&\varphi_1 = \varphi \text{ on } \mathcal{V}_1 \cap \{V_1 < V_2\}; \\
&\varphi_2 = \varphi \text{ on } \mathcal{V}_1 \cap \{V_2 < V_1\}; \\
&|\nabla\varphi_j(x)|^2 = V_j(x) \quad (j = 1, 2); \\
&\varphi_1 = \varphi_2 \text{ and } \nabla\varphi_1 = \nabla\varphi_2 \text{ on } \mathcal{V}_1 \cap \partial\Omega_0; \\
&\varphi_1(x) - \varphi_2(x) \sim d(x, \partial\Omega_0)^2.
\end{aligned}
\tag{7.4}
$$

Actually, φ_1 is nothing but the analytic continuation of $\varphi(x)$ from $\mathcal{V}_1 \cap \{V_1 < V_2\}$ obtained by using the metric $V_1(x)dx^2$ near $x^{(1)}$ (instead of $\min(V_1, V_2)dx^2$), while φ_2 is the (analytic) phase function of the Lagrangian manifold obtained as the flow-out of $\{(x, \nabla\varphi_1(x))\,;\, x \in \mathcal{V}_1 \cap \partial\Omega_0\}$ under the Hamilton flow of $q_2(x, \xi) := \xi^2 - V_2(x)$.

Now, we set,

$$
v := e^{\varphi_2/h}u,
$$

and

$$
\mathcal{V}_1^{\pm} := \mathcal{V}_1 \cap \{\pm V_1 > \pm V_2\}.
$$

By (4.2) and standard Sobolev estimates, we know that, for all $\varepsilon > 0$, w is $\mathcal{O}(e^{(\varphi_2 - \varphi + \varepsilon)/h})$ together with all its derivatives. In particular, since, by (7.4), $\varphi = \varphi_1 > \varphi_2$ on $\mathcal{V}_1^-$, we obtain that v is locally exponentially small there. Moreover, by Proposition 7.1, w is also locally exponentially small in $\mathcal{V}_1^+$ (on which $\varphi = \varphi_2$). Thus, the microsupport of v satisfies,

$$
MS(v) \cap \{x \in \mathcal{V}_1\} \subset \{V_1 = V_2\}.
\tag{7.5}
$$

On the other hand, v is solution to $Qv = \rho v$ with,

$$
Q := e^{\varphi_2/h}Pe^{-\varphi_2/h},
$$

and the semiclassical principal symbol of Q is given by,

$$
\begin{aligned}
q_0(x, \xi) &= \begin{pmatrix} (\xi + i\nabla\varphi_2)^2 + V_1 & 0 \\ 0 & (\xi + i\nabla\varphi_2)^2 + V_2 \end{pmatrix} \\
&= \begin{pmatrix} \xi^2 + V_1 - V_2 + 2i\xi \cdot \nabla\varphi_2 & 0 \\ 0 & \xi^2 + 2i\xi \cdot \nabla\varphi_2 \end{pmatrix}.
\end{aligned}
$$

In particular, the characteristic set $\mathrm{Char}(Q) := (\det q_0)^{-1}(0)$ of Q satisfies,

$$
\mathrm{Char}(Q) \subset \{\xi = 0\} \cup \{\xi^2 = V_2(x) - V_1(x)\},
\tag{7.6}
$$

and since we know that $MS(v) \subset \mathrm{Char}(Q)$ (see, e.g., [**Ma2**]), we conclude from (7.5)–(7.6) that we have,

$$MS(v) \cap \{x \in \mathcal{V}_1\} \ \subset \ \{V_1(x) = V_2(x)\,,\, \xi = 0\} = \partial\Omega_0 \times \{0\}. \tag{7.7}$$

Therefore, in order to prove that v is exponentially small near $x^{(1)}$, it remains to show,

PROPOSITION 7.2.   *One has,*

$$(x^{(1)}, 0) \notin MS(v).$$

REMARK 7.3.   This result can be seen as a propagation of the microsupport for a (matrix-) operator with non-involutive double characteristics, and has connections with the results proved in [**LaLa**], [**PeSo**].

7.2.   Proof of Proposition 7.2.

To begin with, let us write in a more precise way the expression of Q. We have,

$$Q = \begin{pmatrix} (hD_x + i\nabla\varphi_2)^2 + V_1 & hr_0(x) + ihr_1(x) \cdot (hD_x + i\nabla\varphi_2) \\ h\bar{r}_0(x) - ih(hD_x + i\nabla\varphi_2) \cdot \bar{r}_1(x) & (hD_x + i\nabla\varphi_2)^2 + V_2 \end{pmatrix},$$

and thus, the equation $Qv = \rho v$ can be written as,

$$\begin{cases} (Q_1 - \rho)v_1 = hA_1 v_2\,; \\ (Q_2 - \rho)v_2 = hA_2 v_1, \end{cases} \tag{7.8}$$

with,

$$Q_1 := -h^2\Delta + V_1 - V_2 + 2i\nabla\varphi_2 \cdot hD_x + h\Delta\varphi_2\,;$$
$$Q_2 := -h^2\Delta + 2i\nabla\varphi_2 \cdot hD_x + h\Delta\varphi_2,$$

and,

$$A_1 := -r_0(x) + ir_1(x) \cdot (hD_x + i\nabla\varphi_2)\,;$$
$$A_2 := -\bar{r}_0(x) + i\bar{r}_1(x) \cdot (hD_x + i\nabla\varphi_2) + h\,\mathrm{div}\,\bar{r}_1(x).$$

(In particular, by Assumption 6, A_2 is elliptic at $(x^{(1)}, 0)$.)

We plan to transform $Q_2 - \rho$ into ihD_{z_n} microlocally near $(x^{(1)}, 0)$ by the use of a convenient Fourier–Bros–Iagolnitzer transform in the same spirit as in [**Sj**, Section 7]. For this purpose, we first take local (real-analytic) coordinates $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ centered at $x^{(1)}$ in such a way that the hypersurface $\partial\Omega_0 = \{V_1 = V_2\}$ becomes $\{x_n = 0\}$ (and $\{V_1 > V_2\}$ becomes $\{x_n > 0\}$). Then, using Assumption 5, we see that $\nabla\varphi_2$ is transversal to $\{V_1 = V_2\}$, and thus (since it also points-out towards $\{V_1 > V_2\}$) another change of analytic coordinates transforms the vector-field $2\nabla\varphi_2(x) \cdot \nabla_x$ into ∂_{x_n}. Therefore, in the new local coordinates (that we still denote by (x', x_n)), the operators Q_1 and Q_2 become,

$$Q_1 = \Lambda(x, hD_x) + ihD_{x_n} + x_n g(x)\,;$$
$$Q_2 = \Lambda(x, hD_x) + ihD_{x_n},$$

where $\Lambda(x, hD_x)$ is a second-order semiclassical differential operator with non-negative real-valued principal symbol $\Lambda_0(x, \xi)$ homogeneous with respect to ξ, and where the function g is real-analytic and positive at $x = 0$.

We first get rid of ρ in the system (7.8). For this purpose, we consider the function $\sigma = \sigma(x)$ solution to,

$$\begin{cases} i\dfrac{\partial \sigma}{\partial x_n} + \Lambda_0(x, \nabla_x \sigma) = \rho\,; \\ \sigma\,|_{x_n=0} = 0, \end{cases} \tag{7.9}$$

and we set,

$$w := e^{-i\sigma/h} v.$$

Since $\sigma = \mathcal{O}(\rho)$ and $\rho = \rho(h) \to 0$ as $h \to 0$, for any $\varepsilon > 0$ we still have $w = \mathcal{O}(e^{\varepsilon/h})$, and w is solution to the system,

$$\begin{cases} \widetilde{Q}_1 w_1 = h\widetilde{A}_1 w_2\,; \\ \widetilde{Q}_2 w_2 = h\widetilde{A}_2 w_1, \end{cases} \tag{7.10}$$

with,

$$\widetilde{Q}_1 := \widetilde{\Lambda}(x, hD_x) + ihD_{x_n} + x_n g(x)\,;$$
$$\widetilde{Q}_2 := \widetilde{\Lambda}(x, hD_x) + ihD_{x_n},$$

where $\widetilde{\Lambda}(x, hD_x)$ is a second-order semiclassical differential operator, with principal symbol,

$$\widetilde{\Lambda}_0(x, \xi) = \Lambda_0(x, \xi) + (\nabla_\xi \Lambda_0)(x, \xi) \cdot \nabla_x \sigma(x), \tag{7.11}$$

and with $\widetilde{A}_2$ elliptic at $(0, 0)$.

In particular, we have,

$$\widetilde{\Lambda}_0(x, \xi) = \mathcal{O}(|\xi|^2 + |\rho \xi|); \quad \nabla_\xi \widetilde{\Lambda}_0(x, \xi) = \mathcal{O}(|\xi| + |\rho|). \tag{7.12}$$

At this point, it may be useful to make a few considerations, in order to explain that the reasons for which the computations below (that may seem somehow obscure) do work, are actually rather natural. A way to understand this consists in considering the toy-model system,

$$\begin{cases} (ihD_{x_n} + x_n)w_1 = hw_2\,; \\ ihD_{x_n} w_2 = hw_1. \end{cases}$$

Substituting $w_1 = iD_{x_n} w_2$ in the first equation, we obtain $Qw_2 = 0$ where the Weyl-symbol q of Q satisfies,

$$q(x, \xi; h) = -\xi_n^2 + ix_n\xi_n - \frac{h}{2} + \mathcal{O}(h^2).$$

If in addition we conjugate Q with $e^{-2\delta x_n^2/h}$ with $\delta > 0$ small enough, then we obtain a differential operator with principal symbol $\widetilde{q}$ satisfying,

$$\mathrm{Re}\, \widetilde{q}(x, \xi; h) \le -\xi_n^2 - \delta x_n^2 - \frac{h}{4}. \tag{7.13}$$

Therefore, if the solution is exponentially small for $x_n \ne 0$, it becomes natural to expect it to be (microlocally) exponentially small at $(0,0)$, too.

In several dimensions and in the analytic setting, however, the technical procedure is more complicated, and in order to eliminate the term $\widetilde{\Lambda}(x, hD_x)$, it will be necessary to use a Fourier integral operator with complex phase-function.

The microlocal transformation of $\widetilde{Q}_2$ will also take into account the operator $\widetilde{Q}_1$, in the sense that we will do it in such a way that the operator of multiplication by $x_n g(x)$ keeps a nice form. In order to do so, we consider the holomorphic function $\psi_0 = \psi_0(x, z')$, solution to the system,

$$\begin{cases} \dfrac{\partial \psi_0}{\partial x_n} + i\widetilde{\Lambda}_0(x, \nabla_x \psi_0) = i\mu x_n \,; \\ \psi_0\,|_{x_n=0} = i\mu(z' - x')^2/2. \end{cases} \tag{7.14}$$

(Here, $\mu > 0$ is a constant that will be fixed small enough later on.)

Then, we consider the holomorphic function $\psi = \psi(x, z)$ defined on a complex neighborhood of $(0,0)$, solution to the eikonal system,

$$\begin{cases} \dfrac{\partial \psi}{\partial z_n} = -i\widetilde{\Lambda}_0(x, \nabla_x \psi) - \dfrac{\partial \psi}{\partial x_n} \,; \\ \psi\,|_{z_n=0} = \psi_0. \end{cases} \tag{7.15}$$

We prove,

Lemma 7.4.   *One has,*

$$\psi(z, x) = i\mu\frac{(z - x)^2}{2} + x_n\alpha(x', z') + \mathcal{O}\left(|(x_n, z_n)|^2(|x' - z'| + |\rho|) + |(x_n, z_n)|^3\right);$$

$$\nabla_{z'}\psi(z, x) = i\mu(z' - x') + x_n\nabla_{z'}\alpha(x', z') + \mathcal{O}(|(x_n, z_n)|^2);$$

$$\partial_{z_n}\psi(z, x) = i\mu(z_n - x_n) + \mathcal{O}(|z_n|(|x' - z'| + |\rho|) + |(x_n, z_n)|^2),$$

uniformly in a neighborhood of $(0,0)$*, where* $\alpha = \alpha(x', z')$ *is the unique complex solution near* 0 *to the equation,*

$$\alpha + i\Lambda_0(x', 0; i\mu(x' - z'), \alpha) = 0. \tag{7.16}$$

Proof.   From (7.14) and the definition of α, we immediately obtain,

$$\left.\frac{\partial \psi_0}{\partial x_n}\right|_{x_n=0} = \alpha,$$

and, using (7.12), we see that,

$$
\begin{aligned}
&\alpha = \mathcal{O}(|x' - z'|^2 + |\rho(x' - z')|)\,;\\
&|\nabla_{x'}\alpha| + |\nabla_{z'}\alpha| = \mathcal{O}(|x' - z'| + |\rho|).
\end{aligned}
\tag{7.17}
$$

Then, differentiating (7.14) with respect to x_n, we deduce,

$$\left.\frac{\partial^2 \psi_0}{\partial x_n^2}\right|_{x_n=0} = i\mu + \mathcal{O}(|x' - z'| + |\rho|),$$

and thus, by Taylor's formula,

$$
\begin{aligned}
&\psi_0(x, z') = i\mu\frac{(z' - x')^2}{2} + x_n\alpha(x', z') + i\mu\frac{x_n^2}{2} + \mathcal{O}((|x' - z'| + |\rho|)x_n^2 + |x_n|^3)\,;\\
&\nabla_x\psi_0(x, z') = \mathcal{O}(|x' - z'| + |x_n|).
\end{aligned}
\tag{7.18}
$$

On the other hand, using (7.15), we obtain,

$$\left.\frac{\partial \psi}{\partial z_n}\right|_{z_n=0} = -i\widetilde{\Lambda}_0(x, \nabla_x\psi_0) - \frac{\partial \psi_0}{\partial x_n} = -i\mu x_n.$$

In particular, $\nabla_{z'}\partial_{z_n}\psi\,|_{z_n=0} = 0$, and, differentiating (7.15),

$$
\begin{aligned}
\left.\frac{\partial^2 \psi}{\partial z_n^2}\right|_{z_n=0} &= -i(\nabla_\xi\widetilde{\Lambda}_0)(x, \nabla_x\psi_0) \cdot \nabla_x\left.\frac{\partial \psi}{\partial z_n}\right|_{z_n=0} - \partial_{x_n}\left.\frac{\partial \psi_0}{\partial z_n}\right|_{z_n=0}\\
&= i\mu + \mathcal{O}(|\nabla_x\psi_0| + |\rho|) = i\mu + \mathcal{O}(|x' - z'| + |x_n| + |\rho|).
\end{aligned}
$$

The result follows by Taylor's formula. □

Now, by [**Sj**, Theorem 9.3], we can find an analytic symbol $a(x, z; h)$ defined in a complex neighborhood of $(0, 0)$, such that,

$$e^{-i\psi(x,z)/h}\left(ihD_{z_n} - {}^tQ_2(x, hD_x)\right)\left(a(x, z; h)e^{i\psi(x,z)/h}\right) = \mathcal{O}(e^{-2\delta_0/h}),\tag{7.19}$$

where ${}^tQ_2(x, hD_x)$ stands for the formal transposed of Q_2.

Then, fixing a cut-off function $\chi_0 = \chi_0(x)$ on a small enough neighborhood of 0, we set,

$$Tw(z; h) := \int_{\mathbb{R}^n} e^{i\psi(x,z)/h}a(x, z; h)\chi_0(x)w(x)dx.\tag{7.20}$$

By Lemma 7.4 and (7.17), we have,

$$\nabla_x\psi(0, 0) = 0; \quad \mathrm{Im}\,\nabla_x^2\psi(0, 0) > 0; \quad \det\nabla_z\nabla_x\psi(0, 0) \neq 0,$$

and thus, by the general theory of [**Sj**] (and its obvious generalization for the study of the microsupport near the nul section of $T^*\mathbb{R}^n$: see, e.g., [**Ma2**]), T is a Fourier–Bros–Iagolnitzer transform that can be used to characterize the microsupport of h-dependent functions near $(0,0)$: Setting,

$$\Phi(z) := \sup_{x \in \mathbb{R}^n} (-\mathrm{Im}\, \psi(x,z)),$$

and introducing the local (complex) canonical transformation,

$$\kappa\,:\,(x, -\nabla_x \psi) \mapsto (z, \nabla_z \psi),$$

then, for any $w = w(x;h)$ (say, in $L^\infty(\mathcal{V}_0)$ where $\mathcal{V}_0$ is a neighborhood of 0 containing the support of χ_0), such that $\|w\|_{L^\infty(\mathcal{V}_0)} = \mathcal{O}(e^{\varepsilon/h})$ for all $\varepsilon > 0$ and h small enough, and for $(x_0, \xi_0) \in \mathbb{R}^{2n}$ close enough to $(0,0)$, we have the equivalence,

$$(x_0, \xi_0) \notin MS(w) \Leftrightarrow \begin{cases} \text{There exists } \delta > 0 \text{ and a complex neighborhood} \\ \omega_0 \text{ of the } z\text{-projection of } \kappa(x_0,\xi_0), \text{ such that} \\ \|e^{-\Phi/h} Tw\|_{L^2(\omega_0)} = \mathcal{O}(e^{-\delta/h}) \text{ uniformly for} \\ h \text{ small enough.} \end{cases} \tag{7.21}$$

Moreover, Tw belongs to the Sjöstrand space $H_{\Phi,0}$ of h-dependent holomorphic functions $v = v(z;h)$ near $z = 0$, such that, for all $\varepsilon > 0$, $v = \mathcal{O}(e^{(\Phi+\varepsilon)/h})$ uniformly as $h \to 0_+$.

Now, by construction (in particular (7.19)), near $z = 0$ we have,

$$T\widetilde{Q}_2 w = ih D_{z_n} Tw + \mathcal{O}(e^{(\Phi-\delta_0)/h}). \tag{7.22}$$

On the other hand, by [**Sj**, Proposition 7.4], we also have,

$$T(x_n g(x) w) = L Tw, \tag{7.23}$$

where $L = L(z, hD_z)$ is a pseudodifferential operator in the complex domain (in the sense of [**Sj**]), with principal symbol $l_0(z, \zeta)$ such that,

$$l_0(\kappa(x,\xi)) = x_n g(x). \tag{7.24}$$

We show,

Lemma 7.5. *One has,*

$$l_0(z, \zeta) = \left(z_n + \frac{i}{\mu}\zeta_n\right) g\left(z' + \frac{i}{\mu}\zeta'\right) + \mathcal{O}\left(|(z_n, \zeta_n)|(|\zeta'| + |\rho|) + |(z_n,\zeta_n)|^2\right).$$

Proof. By definition, if $(z, \zeta) = \kappa(x, \xi)$, then $\xi = -\nabla_x \psi(x, z)$ and $\zeta = \nabla_z \psi(x, z)$, and thus, by Lemma 7.4,

$$\zeta_n = i\mu(z_n - x_n) + \mathcal{O}\left(|z_n|(|x' - z'| + |\rho|) + |(x_n, z_n)|^2\right);$$
$$\zeta' = i\mu(z' - x') + x_n \nabla_{z'}\alpha(x', z') + \mathcal{O}\left(|(x_n, z_n)|^2\right).$$

In view of (7.17), we first deduce,

$$x_n = \mathcal{O}\left(|(z_n, \zeta_n)|\right);$$
$$x' = \mathcal{O}\left(|(z', \zeta')| + |\rho| \cdot |(z_n, \zeta_n)| + |(z_n, \zeta_n)|^2\right),$$

then,

$$x_n = z_n + \frac{i}{\mu}\zeta_n + \mathcal{O}\left(|z_n|(|x' - z'| + |\rho|) + |(z_n, \zeta_n)|^2\right);$$
$$x' = z' + \frac{i}{\mu}\zeta' + \mathcal{O}\left(|(z_n, \zeta_n)|(|x' - z'| + |\rho|) + |(z_n, \zeta_n)|^2\right),$$

and finally,

$$x_n = z_n + \frac{i}{\mu}\zeta_n + \mathcal{O}\left(|(z_n, \zeta_n)|(|\zeta'| + |\rho|) + |(z_n, \zeta_n)|^2\right);$$
$$x' = z' + \frac{i}{\mu}\zeta' + \mathcal{O}\left(|(z_n, \zeta_n)|(|\zeta'| + |\rho|) + |(z_n, \zeta_n)|^2\right).$$

Inserting this into (7.24), the result follows. $\qquad\square$

Now, we set,

$$\widehat{w} = (\widehat{w}_1, \widehat{w}_2) := Tw.$$

By (7.22) and (7.23), the system (7.10) becomes

$$\begin{cases} (ihD_{z_n} + L)\widehat{w}_1 = h\widehat{A}_1\widehat{w}_2\,; \\ ihD_{z_n}\widehat{w}_2 = h\widehat{A}_2\widehat{w}_1, \end{cases} \tag{7.25}$$

where $\widehat{A}_1, \widehat{A}_2$ are 0-th order pseudodifferential operators in the complex domain, with $\widehat{A}_2$ elliptic at $(0,0)$.

We also need a general result on the pseudodifferential operators in the complex domain. Let $A(z, hD_z)$ be such a operator, acting on some Sjöstrand's space H_{ϕ, z_0} with $\phi \in C^2$. Then (see [**Sj**, Chapter 4]), A can be written as,

$$Au(z) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(z)} e^{i(z-y)\zeta/h} a(z, \zeta; h) u(y; h) dy\, d\zeta,$$

where a is an analytic symbol defined near $(z_0, (2/i)\nabla\phi(z_0))$, and $\Gamma(z)$ is the complex contour of $\mathbb{C}^{2n}$ given by,

$$\Gamma(z) : \begin{cases} \zeta = \dfrac{2}{i}\nabla\phi(z) + iR(\overline{z - y})\,; \\ y \in \mathbb{C}^n,\ |z - y| \leq r, \end{cases} \tag{7.26}$$

where $R > 0$ is chosen sufficiently large with respect to the Hessian of ϕ, and $r > 0$ is chosen sufficiently small in such a way that u is well defined in the ball centered at

z_0 with radius $2Rr$ (in which case the previous formula defines $Au(z)$ for z in the ball centered at z_0 with radius Rr).

In all these formulas, $\nabla := (\nabla_{\mathrm{Re}\ z} - i\nabla_{\mathrm{Im}\ z})/2$ stands for the holomorphic gradient. We have,

LEMMA 7.6.

$$Au(z) = a\left(z, \frac{2}{i}\nabla\phi(z)\right)u(z) + (\nabla_\zeta a)\left(z, \frac{2}{i}\nabla\phi(z)\right)\left(hD_z - \frac{2}{i}\nabla\phi(z)\right)u(z) + v(z),$$

with,

$$\|e^{-\phi/h}v\|_{L^2(|z-z_0|<Rr)} \leq Ch\|e^{-\phi/h}u\|_{L^2(|z-z_0|<2Rr)},$$

where the positive constant C can be taken independent of R and r, as long as Rr remains small enough.

PROOF.   For $\zeta = (2/i)\nabla\phi(z) + iR(\overline{x-y})$, we write,

$$\begin{aligned}
a(z,\zeta) &= a\left(z, \frac{2}{i}\nabla\phi(z)\right) + iR(\nabla_\zeta a)\left(z, \frac{2}{i}\nabla\phi(z)\right)\cdot(\overline{x-y}) + \mathcal{O}(|z-y|^2) \\
&= a\left(z, \frac{2}{i}\nabla\phi(z)\right) + (\nabla_\zeta a)\left(z, \frac{2}{i}\nabla\phi(z)\right)\cdot\left(\zeta - \frac{2}{i}\nabla\phi(z)\right) + \mathcal{O}(|z-y|^2),
\end{aligned}$$

where the $\mathcal{O}(|z-y|^2)$ only depends on the behaviour of a near z_0. Then, we observe that, since $\Gamma(z)$ is a good contour (in the sense of [**Sj**]), along $\Gamma(z)$ we have,

$$\left|e^{\phi(z)/h}e^{i(z-x)\zeta/h}e^{-\phi(y)/h}\right| = \mathcal{O}(e^{-c|z-y|^2/h})$$

with $c > 0$ constant, and the result follows by an application of the Schur lemma (see, e.g., [**Ma2**]) and the fact that,

$$h^{-n}\int_{x\in\mathbb{C}^n}|x|^2e^{-c|x|^2/h}L(dx) = \mathcal{O}(h).$$

(Here, $L(dx)$ stands for the Lebesgue measure on $\mathbb{C}^n$.) $\square$

In order to perform exponential weighted estimates with Tw, we have to specify better the function Φ. We show,

LEMMA 7.7.

$$\Phi(z) = \mu\frac{(\mathrm{Im}\ z)^2}{2} + \mathcal{O}\left(|\rho|^4 + |\mathrm{Im}\ z'|^4 + |z_n|(|\mathrm{Im}\ z'|^2 + |\rho|^2) + |z_n|^3\right).$$

PROOF.   By definition, $\Phi(z) = -\mathrm{Im}\ \psi(x_c(z),z)$, where $x_c(z)$ is the (real) critical point of $x \mapsto -\mathrm{Im}\ \psi(x,z)$. By computations similar to those of Lemma 7.4, we see,

$$\nabla_{x'}\psi(x,z) = i\mu(x'-z') + x_n\nabla_{x'}\alpha(x',z') + \mathcal{O}(|(x_n,z_n)|^2);$$
$$\partial_{x_n}\psi(x,z) = \alpha(x',z') + i\mu(x_n-z_n) + \mathcal{O}(|(x_n,z_n)|^2).$$

Hence, using (7.17), the critical point $x_c(z) = (x'_c, x_n^c)$ of Im ψ satisfies,

$$x'_c - \mathrm{Re}\, z' = \mathcal{O}\left(|x_n^c|(|x'_c - z'| + |\rho|) + |(x_n^c, z_n)|^2\right);$$
$$x_n^c - \mathrm{Re}\, z_n = \mathcal{O}\left(|x'_c - z'|(|x'_c - z'| + |\rho|) + |(x_n^c, z_n)|^2\right).$$

We first deduce,

$$x'_c - \mathrm{Re}\, z' = \mathcal{O}\left(|x_n^c|(|\mathrm{Im}\, z'| + |\rho|) + |(x_n^c, z_n)|^2\right);$$
$$x_n^c = \mathcal{O}\left(|z_n| + |x'_c - z'|(|x'_c - z'| + |\rho|)\right),$$

and then,

$$
\begin{aligned}
x'_c - \mathrm{Re}\, z' &= \mathcal{O}\left(|z_n|(|\mathrm{Im}\, z'| + |\rho|) + |\mathrm{Im}\, z'|(|\mathrm{Im}\, z'|^2 + |\rho|^2) + |z_n|^2\right); \\
x'_c - z' &= \mathcal{O}\left(|\mathrm{Im}\, z'| + |\rho z_n| + |z_n|^2\right); \\
x_n^c - \mathrm{Re}\, z_n &= \mathcal{O}\left(|\mathrm{Im}\, z'|^2 + |\rho|\mathrm{Im}\, z'| + |\rho|^2|z_n| + |z_n|^2\right); \\
x_n^c &= \mathcal{O}\left(|z_n| + |\mathrm{Im}\, z'|^2 + |\rho|\mathrm{Im}\, z'|\right).
\end{aligned}
\tag{7.27}
$$

By Lemma 7.4, we also have,

$$
\begin{aligned}
\Phi(z) = {} & \frac{\mu}{2}(\mathrm{Im}\, z)^2 + \mathcal{O}\left(|x_c - \mathrm{Re}\, z|^2 + |x_n^c\alpha(x'_c, z')|\right) \\
& + \mathcal{O}\left(|(x_n^c, z_n)|^2(|x'_c - z'| + |\rho|) + |(x_n^c, z_n)|^3\right)
\end{aligned}
$$

and thus, using (7.27) and (7.17), we find,

$$
\begin{aligned}
\Phi(z) = {} & \mu\frac{(\mathrm{Im}\, z)^2}{2} + \mathcal{O}\left(|\mathrm{Im}\, z'|^4 + |\rho|^2|\mathrm{Im}\, z'|^2 + |\mathrm{Im}\, z'|^3|\rho|\right) \\
& + \mathcal{O}\left(|z_n|(|\mathrm{Im}\, z'|^2 + |\rho\mathrm{Im}\, z'|) + |z_n|^2(|\mathrm{Im}\, z'| + |\rho|) + |z_n|^3\right),
\end{aligned}
$$

and the result follows by Cauchy–Schwarz inequality. $\square$

In particular, in a small enough neighborhood of 0, we deduce from Lemma 7.7,

$$\Phi(z) \le \mu\frac{(\mathrm{Im}\, z)^2}{2} + \frac{1}{2}|\mathrm{Im}\, z'|^2 + C|z_n|^3 + |\rho|^2,$$

where $C > 0$ is a constant. For $\delta > 0$ small enough, we set,

$$\Phi_\delta(z) := \mu\frac{(\mathrm{Im}\, z_n)^2}{2} + 2|z'|^2 + C|z_n|^3 + |\rho|^2 - \delta(\mathrm{Re}\, z_n)^2. \tag{7.28}$$

In particular Φ_δ is C^2, and if μ has been chosen ≤ 1, we have,

$$\Phi(z) \le \Phi_\delta(z) - |z'|^2 + \delta(\mathrm{Re}\, z_n)^2. \tag{7.29}$$

Now, we fix $\delta_0 > 0$ sufficiently small, and we set,

$$\Omega := \{z \in \mathbb{C}^n \,;\, |z_n| < \delta_0, \, |z'| < \delta_0\}, \tag{7.30}$$

and we observe,

• On $\{|z_n| = \delta_0\} \cap \Omega$: Analysing $\kappa^{-1}(z, \zeta)$, we see that these points correspond to (x, ξ) such that $|(x_n, \xi_n)| \geq \delta_0/2$, and thus, by (7.7), there exists $\delta_1 > 0$ such that, for such points,

$$\widehat{w}(z; h) = \mathcal{O}(e^{(\Phi(z) - 2\delta_1)/h}).$$

Therefore, by (7.29),

$$\widehat{w}(z; h) = \mathcal{O}(e^{(\Phi_\delta(z) - 2\delta_1 + \delta\delta_0^2)/h}),$$

and thus, if we choose $\delta \leq \delta_1 \delta_0^{-2}$,

$$\widehat{w}(z; h) = \mathcal{O}(e^{(\Phi_\delta(z) - \delta_1)/h}). \tag{7.31}$$

• On $\{|z'| = \delta_0\} \cap \Omega$: For any $\varepsilon > 0$, we have,

$$\widehat{w}(z; h) = \mathcal{O}(e^{(\Phi(z) + \varepsilon)/h}).$$

Hence, by (7.29), and taking ε sufficiently small (and $\delta < 1/2$), at these points we obtain,

$$\widehat{w}(z; h) = \mathcal{O}(e^{(\Phi_\delta(z) + \varepsilon - \delta_0^2 + \delta\delta_0^2)/h}) = \mathcal{O}(e^{(\Phi_\delta(z) - (1/2)\delta_0^2)/h}).$$

Summing up, we have proved the existence of a constant $\delta_2 > 0$ such that,

$$e^{-\Phi_\delta/h}\widehat{w} = \mathcal{O}(e^{-\delta_2/h}) \quad \text{on } \partial\Omega. \tag{7.32}$$

Now, the idea is to perform estimates on $\widehat{w}$ in the space,

$$L^2_{\Phi_\delta}(\Omega) := L^2(\Omega \,;\, e^{-2\Phi_\delta(z)/h} L(dz)).$$

Before that, we go back to (7.25), and, taking advantage of the ellipticity of $\widehat{A}_2$, we re-write it as,

$$\begin{cases} \widehat{w}_1 = \dfrac{1}{h}\widehat{B}_2 \, ihD_{z_n}\widehat{w}_2 \,; \\ \widehat{A}_2(-hD_{z_n} + iL)\widehat{B}_2 \, hD_{z_n}\widehat{w}_2 = h^2\widehat{A}_2\widehat{A}_1\widehat{w}_2, \end{cases} \tag{7.33}$$

where $\widehat{B}_2$ is a parametrix of $\widehat{A}_2$ near $(0,0)$, and where the identities hold in $H_{\Phi,0}$.

We set,

$$\widehat{Q} := \widehat{A}_2(-hD_{z_n} + iL)\widehat{B}_2 \, hD_{z_n}, \tag{7.34}$$

and we plan to investigate the quantity,

$$J := \langle \widehat{Q}\widehat{w}_2, \widehat{w}_2 \rangle_{L^2_{\Phi_\delta}(\Omega)} = \int_\Omega e^{-2\Phi_\delta(z)/h} Q\widehat{w}_2(z)\, \overline{\widehat{w}_2(z)} L(dz). \qquad (7.35)$$

At first, we write,

$$\widehat{Q} = (-hD_{z_n} + iL)\, hD_{z_n} + [\widehat{A}_2, -hD_{z_n} + iL]\widehat{B}_2\, hD_{z_n},$$

and we approximate the pseudodifferential operators $[\widehat{A}_2, -hD_{z_n} + iL]\widehat{B}_2$ and L by using Lemma 7.6 with $\phi = \Phi_\delta$. We obtain,

$$\widehat{Q}\widehat{w}_2 = \widehat{Q}_0\widehat{w}_2 + v,$$

where $\widehat{Q}_0$ is the differential operator,

$$\widehat{Q}_0 := \left(-hD_{z_n} + il_0\left(z, \frac{2}{i}\nabla\Phi_\delta\right) + i(\nabla_\zeta l_0)\left(z, \frac{2}{i}\nabla\Phi_\delta\right)\left(hD_z - \frac{2}{i}\nabla\Phi_\delta\right)\right) hD_{z_n}, \quad (7.36)$$

and with v satisfying,

$$\|e^{-\Phi_\delta/h}v\|_{L^2(\Omega)} \le C_0 h\|e^{-\Phi_\delta/h}hD_{z_n}\widehat{w}_2\|_{L^2(\Omega')}$$

where Ω' is an arbitrarily small neighborhood of $\overline{\Omega}$, and where $C_0 > 0$ does not depend on the (sufficiently small) size of Ω. In the same way as for (7.32), we see that,

$$\|e^{-\Phi_\delta/h}hD_{z_n}\widehat{w}_2\|_{L^2(\Omega'\setminus\Omega)} = \mathcal{O}(e^{-\delta_2/h}),$$

and thus, we actually have,

$$\|e^{-\Phi_\delta/h}v\|_{L^2(\Omega)} \le C_0 h\|e^{-\Phi_\delta/h}hD_{z_n}\widehat{w}_2\|_{L^2(\Omega)} + Ce^{-\delta_2/h}, \qquad (7.37)$$

where $C > 0$ and δ_2 may depend on the size of Ω, but not C_0. Therefore, we can write,

$$J = \langle \widehat{Q}_0\widehat{w}_2, \widehat{w}_2 \rangle_{L^2_{\Phi_\delta}(\Omega)} + J_1, \qquad (7.38)$$

with,

$$|J_1| \le \left(C_0 h\|e^{-\Phi_\delta/h}hD_{z_n}\widehat{w}_2\|_{L^2(\Omega)} + Ce^{-\delta_2/h} \right) \|e^{-\Phi_\delta/h}\widehat{w}_2\|_{L^2(\Omega)}. \qquad (7.39)$$

We first study $J_0 := \langle \widehat{Q}_0\widehat{w}_2, \widehat{w}_2 \rangle_{L^2_{\Phi_\delta}(\Omega)}$. Since $\overline{\widehat{w}_2(z)}$ is anti-holomorphic, we have $hD_z\overline{\widehat{w}_2(z)} = 0$ and thus, an integration by parts leads us to,

$$J_0 = \int_\Omega {}^t\widehat{Q}_0(e^{-2\Phi_\delta(z)/h})\widehat{w}_2(z)\overline{\widehat{w}_2(z)}L(dz) + \mathcal{O}(e^{-\delta_2/h}), \qquad (7.40)$$

where ${}^t\widehat{Q}_0$ is the formal adjoint of $\widehat{Q}_0$, and is given by,

$${}^t\widehat{Q}_0 = -hD_{z_n}\left(hD_{z_n} + il_1(z) - i\left(hD_z + \frac{2}{i}\nabla\Phi_\delta \right)\cdot l_2(z) \right),$$

where we have set,

$$l_1(z) := l_0\left(z, \frac{2}{i}\nabla\Phi_\delta\right); \quad l_2(z) := (\nabla_\zeta l_0)\left(z, \frac{2}{i}\nabla\Phi_\delta\right).$$

Using that $hD_z(e^{-2\Phi_\delta(z)/h}) = -(2/i)\nabla\Phi_\delta$, and setting,

$$l_3(z) := \sum_{j=1}^n i\frac{\partial}{\partial z_j}\left(\frac{\partial l_0}{\partial \zeta_j}\left(z, \frac{2}{i}\nabla\Phi_\delta\right)\right),$$

we find,

$$
\begin{aligned}
{}^t\widehat{Q}_0(e^{-2\Phi_\delta(z)/h}) &= -hD_{z_n}\left(\left(-\frac{2}{i}\partial_{z_n}\Phi_\delta + il_1(z) + ihl_3(z)\right)e^{-2\Phi_\delta(z)/h}\right) \\
&= \left(4(\partial_{z_n}\Phi_\delta)^2 + 2l_1(z)\partial_{z_n}\Phi_\delta + hl_4(z)\right)e^{-2\Phi_\delta(z)/h},
\end{aligned}
$$

with,

$$l_4 := -2\partial_{z_n}^2\Phi_\delta - \partial_{z_n}l_1 + 2l_3\partial_{z_n}\Phi_\delta - h\partial_{z_n}l_3.$$

Then, going back to (7.28), we compute,

$$
\begin{aligned}
\partial_{z_n}\Phi_\delta &= -\delta\operatorname{Re} z_n - \frac{i\mu}{2}\operatorname{Im} z_n + \mathcal{O}(|z_n|^2); \\
\partial_{z_n}^2\Phi_\delta &= -\frac{\delta}{2} - \frac{\mu}{4} + \mathcal{O}(|z_n|); \\
\nabla_{z'}\Phi_\delta &= 4\overline{z'},
\end{aligned}
\tag{7.41}
$$

and thus, by Lemma 7.5,

$$
\begin{aligned}
l_1(z) &= (1-\delta)g(0)\operatorname{Re} z_n + \mathcal{O}\left(|z_n|(|z'| + |\rho|) + |z_n|^2\right); \\
l_4(z) &= \delta + \frac{\mu}{2} - \frac{(1-\delta)g(0)}{2} + \mathcal{O}(|z| + |\rho| + h).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\operatorname{Re}&\left(4(\partial_{z_n}\Phi_\delta)^2 + 2l_1(z)\partial_{z_n}\Phi_\delta + hl_4(z)\right) \\
&= -\mu^2|\operatorname{Im} z_n|^2 - 2\delta(1-3\delta)g(0)|\operatorname{Re} z_n|^2 - \frac{(1-\delta)g(0) - \mu - 2\delta}{2}h \\
&\quad + \mathcal{O}\left(|z_n|^3 + |z_n|^2(|z'| + |\rho|) + h(|z| + |\rho| + h)\right).
\end{aligned}
\tag{7.42}
$$

Now, we see on (7.42) that if we choose $\mu < g(0)$, δ small enough, and the diameter of Ω small enough, then there exists a constant $C_1 > 0$ (independent of the diameter of Ω) such that, on Ω,

$$\operatorname{Re}\left(4(\partial_{z_n}\Phi_\delta)^2 + 2l_1(z)\partial_{z_n}\Phi_\delta + hl_4(z)\right) \le -\frac{1}{C_1}(|z_n|^2 + h). \tag{7.43}$$

Of course, this estimate is the analogue of (7.13), obtained for the toy-model.

Going back to (7.38), we deduce from (7.43),

$$\text{Re } J_0 \leq -\frac{h}{C_1}\|\widehat{w}_2\|^2_{L^2_{\Phi_\delta}(\Omega)} + Ce^{-\delta_2/h}. \tag{7.44}$$

Now, concerning J_1, we write,

$$\|hD_{z_n}\widehat{w}_2\|^2_{L^2_{\Phi_\delta}(\Omega)} = \int_\Omega -hD_{z_n}(e^{-2\Phi_\delta/h})\,\widehat{w}_2\,\overline{hD_{z_n}\widehat{w}_2}\,L(dz) + \mathcal{O}(e^{-\delta_2/h})$$
$$\leq 2\|(\partial_{z_n}\Phi_\delta)\widehat{w}_2\|_{L^2_{\Phi_\delta}(\Omega)}\|hD_{z_n}\widehat{w}_2\|_{L^2_{\Phi_\delta}(\Omega)} + Ce^{-\delta_2/h},$$

and therefore, by (7.41),

$$\|hD_{z_n}\widehat{w}_2\|_{L^2_{\Phi_\delta}(\Omega)} \leq C_2\|z_n\widehat{w}_2\|_{L^2_{\Phi_\delta}(\Omega)} + C'e^{-\delta'_2/h},$$

with C_2, C', δ'_2 positive, and C_2 independent of the (sufficiently small) size of Ω.

Inserting into (7.39), and recalling (7.30), we deduce,

$$|J_1| \leq C_3\delta_0 h\|\widehat{w}_2\|^2_{L^2_{\Phi_\delta}(\Omega)} + C''e^{-\delta''_2/h} \tag{7.45}$$

with C_3, C'', δ''_2 positive, and C_3 independent of δ_0. Putting together (7.44) and (7.45), and choosing δ_0 sufficiently small, we finally obtain the existence of a (new) constant $C > 0$ such that

$$\langle \widehat{Q}\widehat{w}_2, \widehat{w}_2\rangle_{L^2_{\Phi_\delta}(\Omega)} \leq -\frac{h}{C}\|\widehat{w}_2\|^2_{L^2_{\Phi_\delta}(\Omega)} + Ce^{-\delta_2/h}. \tag{7.46}$$

Finally, going back to (7.33), we deduce,

$$h\|\widehat{w}_2\|^2_{L^2_{\Phi_\delta}(\Omega)} = \mathcal{O}(h^2)\|\widehat{w}_2\|^2_{L^2_{\Phi_\delta}(\Omega)} + \mathcal{O}(e^{-\delta'/h})$$

with $\delta' > 0$ constant, and thus, for h sufficiently small,

$$\|\widehat{w}_2\|^2_{L^2_{\Phi_\delta}(\Omega)} = \mathcal{O}(e^{-\delta'/h}).$$

Since $\Phi_\delta(0) = |\rho(h)|^2 \to 0$ as $h \to 0$, by taking $\Omega' \subset \Omega$ a sufficiently small complex neighborhood of 0, we obtain

$$\|\widehat{w}_2\|_{L^2(\Omega')} = \mathcal{O}(e^{-\delta'/4h}),$$

and by (7.21) (and the fact that $\Phi(0) \leq |\rho|^2$), this implies that $(0,0) \notin MS(w_2)$. Using the first equation of (7.33), we also deduce that $(0,0) \notin MS(w_1)$, and thus, finally, that $(0,0) \notin MS(w)$. Hence, Proposition 7.2 is proved. $\qquad\square$

8.   Propagation up to the well U.

Thanks to Proposition 7.2, we know that $e^{\varphi_2/h}u$ is exponentially small near $x^{(1)}$, and since $\varphi_2(x^{(1)}) = \varphi_1(x^{(1)}) = \varphi(x^{(1)})$, we conclude that the same is true for $e^{\varphi/h}u$. Then, proceeding as for Proposition 7.1, this information can be propagated along $\gamma^{(2)}$, up to a small neighborhood of $x^{(2)}$. But in $x^{(2)}$, the situation is completely similar to that in $x^{(1)}$, with the only difference that the roles of the two components u_1 and u_2 are exchanges. But thanks to Assumption 6, the arguments of the previous section can be first repeated identically, and we conclude that $e^{\varphi/h}u$ is exponentially small near $x^{(2)}$. Iterating N times this procedure, we arrive to the existence of $\delta > 0$ such that,

$$e^{\varphi/h}u = \mathcal{O}(e^{-\delta/h}) \text{ uniformly near } x^{(N)}. \tag{8.1}$$

Again, we can propagate this information along $\gamma^{(N+1)}$ up to a small neighborhood of $y_0 \in \partial U$, and it remains to show that, in that case, u is exponentially small near y_0 (since $\gamma \in G_1$, this will be in contradiction with Assumption 4, and the proof of Theorem 3.4 will be completed).

We adopt the strategy used in [**Ma2**, Theorem 1.1] (see also [**DaMa**, Section 7]), with some modifications due to the fact that we deal with a matrix operator, and with some technical improvements (where it has been possible) and corrections (where it has been necessary).

At first, proceeding as in [**Ma2**, Section 3], we slightly modify the definition of all the geometric quantities by changing the potentials V_1 and V_2 into $V_1 - \text{Re}\,\rho$ and $V_2 - \text{Re}\,\rho$. Since $\text{Re}\,\rho \to 0$ as $h \to 0$, this does not modify the geometric situation (see [**Ma2**, End of Section 1]), and we still denote by U, d, γ, y_0 the corresponding quantities after this change. From now on, the quantity $E := \text{Re}\,\rho$ is considered as an extra small parameter, with respect to which all the estimates will be uniform.

We also fix $y_1, z_1 \in \gamma^{(N+1)}$ such that $d(y_1, y_0) < d(z_1, y_0)$ and, for (y, z) sufficiently close to (y_1, z_1), we consider the function,

$$F(y, z) := d(y, z) - d(z, U).$$

Then F is analytic, and it is solution to,

$$(\nabla_y F(y, z))^2 = V_1(y) - E.$$

Moreover, F can be analytically continued with respect to y along γ in the direction of y_0. At y_0, the map $y \mapsto F(z, y)$ develops a singularity, typically in $\text{dist}(y, \mathcal{C}_z)^{3/2}$, where "dist" stands for the Euclidean distance, and $\mathcal{C}_z$ (the caustic set) is an analytic hypersurface tangent to ∂U at some point $y_0(z)$. However, using a technique form [**HeSj2**, Section 10], it is possible to go round this singularity in the complex domain and to extend analytically $F(y, z)$ up to $\{V_1(y) < E\}$, avoiding $\mathcal{C}_z$. Depending on the way we turn around $\mathcal{C}_z$, we obtain two possible (complex-valued) extensions $F_\pm(y, z)$, that satisfy,

$$\begin{aligned}
&(\nabla_y F_\pm(y, z))^2 = V_1(y) - E; \\
&F_-(y, z) = \overline{F_+(y, z)}; \quad \text{Im}\, F_+(y, z) \sim \text{dist}(y, \mathcal{C}_z)^{3/2}.
\end{aligned} \tag{8.2}$$

Furthermore, $\mathcal{C}_z$ has a contact of order exactly 2 with ∂U, at the unique point $y_0(y)$ that is connected to z through a minimal geodesic $\gamma_z \subset \mathbb{R}^n \backslash U$ (see [**Ma2**, Lemmas 3.1 and 3.2]). Thanks to this, if we fix some small real-analytic hypersurface $\Gamma \subset \overset{\circ}{U}$ transversal to the x-projection of the bicharacteristic $\bigcup_{t \in \mathbb{R}} \exp t H_{p_1}(y_0, 0)$, together with a real-analytic hypersurface $\Sigma \subset \mathbb{R}^n \backslash U$ transversal to γ at z_1, then, one can prove the existence of a constant $C > 0$ such that, for all $(y, z) \in \Gamma \times \Sigma$,

$$\operatorname{Re} F_\pm(y, z) \geq \frac{1}{C} |y - w(z)|^2, \tag{8.3}$$

where $w(z)$ is the point of intersection between Γ and the x-projection α_z of the bicharacteristic $\bigcup_{t \in \mathbb{R}} \exp t H_{p_1}(y_0(z), 0)$ (see [**Ma2**, Formula (3.5)]).

In addition, if (y', z') stands for local coordinates on $\Gamma \times \Sigma$, we also have (see [**Ma2**, Section 3]),

$$\nabla_{y'} \operatorname{Re} F_\pm(w(z), z) = 0; \quad \det \nabla_{y'} \nabla_{z'} \operatorname{Re} F_\pm(z_1, w(z_1)) \neq 0. \tag{8.4}$$

Since P_2 is elliptic near $\partial U \times \{0\}$, and $F_\pm$ satisfies (8.2), it can be used to construct asymptotic solutions $v_\pm = v_\pm(y, z; h)$ to the equation $(P(y, hD_y) - E)v_\pm \sim 0$. More precisely, for y away from $\mathcal{C}_z$, $v_\pm$ is constructed of the form $a_\pm(y, z; h)e^{-F_\pm(y,z)/h}$, with $a_\pm$ a (vector-valued) classical analytic symbol, while, for y close to $\mathcal{C}_z$, $v_\pm$ can be represented as a Airy-type integral (see [**Ma2**, Section 3]), and we have,

$$e^{\operatorname{Re} F_\pm(y,z)/h}(P(y, hD_y) - E)v_\pm(y, z) = \mathcal{O}(e^{-\delta/h}), \tag{8.5}$$

where δ is some positive constant, and where z may vary in a complex neighborhood of z_1, and y in a (real) neighborhood Ω' of y_0, that may be taken tubular around $\gamma \cup \alpha_{z_1}$, with boundary $\Gamma \cup \Sigma' \cup \mathcal{T}$, where $\mathcal{T} \cap (\gamma \cup \alpha_{z_1}) = \emptyset$ and $\Sigma' \subset \mathbb{R}^n \backslash U$.

In addition, since P_2 is elliptic there, the symbol appearing in (8.5) is of the form,

$$a_\pm = \begin{pmatrix} a_1^\pm \\ h a_2^\pm \end{pmatrix},$$

where $a_1^\pm$ and $a_2^\pm$ are 0-th order classical analytic symbols, with $a_1^\pm$ elliptic.

The next step consists in applying the Green formula to the (exponentially small) quantity $\langle (P - E)u, v_\pm(y, \cdot) \rangle_{L^2(\Omega')} - \langle u, (P - E)v_\pm(y, \cdot) \rangle_{L^2(\Omega')}$. Using the same notations as in (4.4)–(4.5), and setting $v_\pm = (v_1^\pm, v_2^\pm)$, this gives,

$$\int_{\partial \Omega'} \left(\frac{\partial u}{\partial \nu} \overline{v_\pm} - u \frac{\partial \overline{v_\pm}}{\partial \nu} + (r_1 \cdot \nu) \left(u_1 \overline{v_2^\pm} - u_2 \overline{v_1^\pm} \right) \right) ds = \mathcal{O}(e^{-\delta'/h}),$$

with $\delta' > 0$ constant.

Then, using the fact that $e^{\varphi/h}u$ is exponentially small on Σ', together with the properties of $F_\pm$ (in particular the fact that $\operatorname{Re} F_\pm(y, z_1) + d(y, U)$ remains non negative, and is positive for z away from $\gamma \cup \alpha_{z_1}$: see [**Ma2**, Formula (4.2)]), we obtain,

$$\int_\Gamma \left(\frac{\partial u}{\partial \nu} \overline{v_\pm} - u \frac{\partial \overline{v_\pm}}{\partial \nu} + (r_1 \cdot \nu) \left(u_1 \overline{v_2^\mp} - u_2 \overline{v_1^\mp} \right) \right) ds = \mathcal{O}(e^{-\delta'/h}). \tag{8.6}$$

Then, using the ellipticity of $a_1^\pm$, we can write,

$$
\begin{aligned}
v_2^\pm &= h \frac{a_2^\pm}{a_1^\pm} v_1^\pm =: h f_\pm v_1^\pm; \\
h \frac{\partial v_1^\pm}{\partial \nu} &= - \left(a_1^\pm \frac{\partial F_\pm}{\partial \nu} + h \frac{\partial a_1^\pm}{\partial \nu} \right) e^{-F_\pm/h} =: g_1^\pm v_1^\pm; \\
h \frac{\partial v_2^\pm}{\partial \nu} &= h f_\pm g_1^\pm v_1^\pm + h^2 \frac{\partial f}{\partial \nu} v_1^\pm := h g_2^\pm v_1^\pm,
\end{aligned}
\tag{8.7}
$$

where $f_\pm$, $g_j^\pm$ are 0-th order classical symbols, and $g_1^\pm = -\partial F_\pm/\partial \nu + \mathcal{O}(h)$. Inserting (8.7) into (8.6) and multiplying by h, we find,

$$\int_\Gamma (G_\pm u) \overline{v_1^\mp} ds = \mathcal{O}(e^{-\delta'/h}), \tag{8.8}$$

with,

$$G_\pm u := h \frac{\partial u_1}{\partial \nu} + h^2 \overline{f_\pm} \frac{\partial u_2}{\partial \nu} - \overline{g_1^\pm} u_1 - h \overline{g_2^\mp} u_2 + h(r_1 \cdot \nu)(h \overline{f^\pm} u_1 - u_2). \tag{8.9}$$

Now, we specify a little bit better our choice of Γ. We set,

$$f_2(y) := \operatorname{Im} F_+(y, z_1).$$

Then, it can be seen that, inside U, the curve α_{z_1} actually coincides with $\widetilde{\gamma} := \bigcup_{t>0} \exp t \nabla f_2(y_0)$ (indeed, using (8.2), we see that $\nabla_y \operatorname{Re} F_\pm(y, z_1) = 0$ along $\widetilde{\gamma}$, and, setting $x(t) := \exp 2t \nabla f_2(y_0)$ and $\xi(t) := (1/2)\dot{x}(t)$, we deduce that $(x(t), \xi(t))$ is solution to the Hamilton system associated with p_1).

Now, we make depend the hypersurface Γ of a parameter t close enough to some fixed $t_0 > 0$, and we choose it in such a way that $\Gamma = \Gamma_t$ is orthogonal to $\widetilde{\gamma}$ at $y_t := \exp t \nabla f_2(y_0)$.

Observing that the constant δ' appearing in (8.8) can be taken independent of t, we multiply this expression by $e^{-(s-t)^2/h}$ (where s is some extra complex parameter), and integrate with respect to t around t_0. We obtain,

$$\int_\omega (G_\pm u) \theta^\pm dy' dt = \mathcal{O}(e^{((\operatorname{Im} s)^2 - \delta')/h}), \tag{8.10}$$

where $\theta_\pm := J \overline{v_1^\mp} e^{-(s-t)^2/2h}$, J being the Jacobian coming from the change of coordinates $y \mapsto (y', t)$ (with $y' =$ Euclidean coordinates orthogonal to $\widetilde{\gamma}$ at y_{t_0}), and ω is a small volume around y_{t_0}.

By construction, $\theta_\pm$ is of the form,

$$\theta_\pm(y', t; z', \tau) = b^\pm e^{i \psi_\pm(y', t; z', s)/h},$$

where b is an elliptic analytic classical symbol, $\psi_\pm := iF_\pm + (i/2)(s-t)^2$, and z' stands for local coordinates on the complexified $\Sigma_\mathbb{C}$ of Σ.

In particular, using (8.3) and the fact that $\nabla_y \text{Re } F_\pm(y_{t_0}, z_1) = 0$ and (thanks to our choice of Γ_t) $\nabla_{y'} f_2(y_{t_0}) = 0$, we see that, for any $\tau_0 \in \mathbb{R}$, $\psi_\pm$ satisfies,

$$
\begin{aligned}
&\nabla_{(y',t)} \psi_\pm(0, t_0; z_1', t_0 - i\tau_0) = -(0, \tau_0); \\
&\text{Im } \nabla^2_{(y',t)} \psi_\pm(0, t_0; z_1', t_0 - i\tau_0) > 0; \\
&\det \nabla_{(y',t)} \nabla_{z',\tau} \psi_\pm(0, t_0; z_1', t_0 - i\tau_0) \neq 0; \\
&\Phi_\pm(z', \tau) := \sup_{(y',t) \text{ real}} \left(-\text{Im } \psi_\pm(0, t_0; z_1', t_0 - i\tau_0) \right) = \tau_0^2.
\end{aligned}
\tag{8.11}
$$

Therefore, according to the general theory of [**Sj**] (see also [**HeSj3**, Appendix a]), the estimate (8.10) expresses the fact that, for any $\tau_0 \in \mathbb{R}$, we have,

$$
(y_{t_0}; 0, \tau_0) \notin MS(G_\pm u). \tag{8.12}
$$

Now, since P_2 is elliptic on $\partial U \times \{0\}$, we can construct a microlocal parametrix of $P_2 - \rho$ near $(y_{t_0}, 0)$, and from the system $(P - \rho)u = 0$ we deduce,

$$
u_2 = hQu_1 \tag{8.13}
$$

where Q a 0-th order analytic pseudodifferential operator. Therefore, we see on (8.9) that $G_\pm u$ can be microlocally re-written as,

$$
G_\pm u \sim (1 + hA_\pm)h\frac{\partial u_1}{\partial \nu} - (\overline{g_1^\mp} + hB_\pm)u_1,
$$

where $A_\pm$ and $B_\pm$ are 0-th order analytic pseudodifferential operators (here, the symbol "$\sim$" means that the identity is valid in a microlocal sense only, and we refer to [**HeSj3**, Appendix] for details on this notion).

Since elliptic operators do not change the microsupport, by taking a parametrix of $(1 + hA_\pm)$ we see that (8.12) is equivalent to,

$$
(y_{t_0}; 0, \tau_0) \notin MS\left(h\frac{\partial u_1}{\partial \nu} - (\overline{g_1^\mp} + h\widetilde{B}_\pm)u_1 \right), \tag{8.14}
$$

still with $\widetilde{B}_\pm$ a 0-th order analytic pseudodifferential operator. Taking the difference between the two functions in (8.14) (that is, the one with the $+$ index and the one with the $-$ index), we deduce,

$$
(y_{t_0}; 0, \tau_0) \notin MS\left((\overline{g_1^+ - g_1^-} + h\widetilde{B}_+ - h\widetilde{B}_-)u_1 \right). \tag{8.15}
$$

Now, we observe on (8.7) that the symbols g_1^+ and g_1^- are such that $\text{Im } g_1^\pm$ is elliptic (this is because $\partial_\nu f_2(y_{t_0}) \neq 0$), and $\text{Im } g_1^- = -\text{Im } g_1^+ + \mathcal{O}(h)$. In particular, $g_1^+ - g_1^-$ is elliptic, and (8.15) implies that, for all $\tau_0 \in \mathbb{R}$, we have,

$$
(y_{t_0}; 0, \tau_0) \notin MS(u_1). \tag{8.16}
$$

Since, for all $\varepsilon > 0$, u_1 is $\mathcal{O}(e^{\varepsilon/h})$ uniformly near y_0, and u_1 is solution to $(P_1 - \rho - h^2 W^*)u_1 = 0$ there, by standard results on the propagation of the micro-support, we know that $MS(u_1)$ is invariant under the Hamilton flow of p_1. In particular, taking $\tau_0 = \sqrt{E - V_1(y_{t_0})}$ (so that $(y_{t_0}; 0, \tau_0)$ is in $\{p_1 = E\}$ and it can be joined to $(y_0, 0)$ by a bicharacteristic of p_1), we conclude that $(y_0, 0) \notin MS(u_1)$. By (8.13) this also implies that $(y_0, 0) \notin MS(u_2)$, and therefore,

$$(y_0, 0) \notin MS(u). \tag{8.17}$$

Recall that here, y_0 is the point of $\partial U = \partial U(E) := \{V_1 = E\}$ where the modified geodesic $\gamma = \gamma(E)$ reaches $U(E)$, and that, in (8.17), E is considered as an extra (h-independent) parameter. However, since all the previous estimates are uniform with respect to E small enough, and since $MS(u)$ is a closed subset of $\mathbb{R}^{2n}$, we can particularize to $E = E(h) := \operatorname{Re} \rho(h)$, and finally obtain (this time with the original quantities),

$$(\gamma \cap \partial U) \times \{0\} \ \cap \ MS(u) = \emptyset.$$

But, by Remark 3.2 (and the fact that u and u_0 are exponentially close to each other near y_0), this is in contradiction with Assumption 4, and therefore Theorem 3.4 is proved.

9. Proof of Theorem 3.6.

The fact that Assumption 4 is not satisfied means that, for any $\gamma \in G$, there exists $\delta_0 > 0$ and a neighbourhood $\mathcal{W}$ of $\gamma \cap \partial U$, such that

$$\|u\|_{L^2(\mathcal{W}) \oplus L^2(\mathcal{W})} = \mathcal{O}(e^{-\delta_0/h}). \tag{9.1}$$

In addition, since ∂U is compact, the value of ε_0 can be taken independent of γ.

By (4.6), it is enough to prove that $v := e^{\varphi/h} u$ remains exponentially small first along γ, and then along $\Pi_x \exp t H_{p_2}((\gamma \cap \partial \mathcal{M}) \times \{0\})$ (t small enough).

At first, using (9.1), we see that for $t > 0$ small enough (that is, for $\gamma(t)$ is sufficiently close to $\gamma \cap U$), we have $(\gamma(t), 0) \notin MS(v)$. Then, by the same argument of propagation used for (7.3), we see that this property remains valid for all $t \in [0, t_1)$, where t_1 is the first time for which $\gamma(t)$ reaches the crest $\partial \Omega_0$. Hence, using $MS(v) \cap [(\gamma \cap \Omega_0) \times \mathbb{R}^n] \subset \{\xi = 0\}$, we deduce the existence, for any $t_1' < t_1$, of $\delta_1 = \delta_1(t_1') > 0$ and a neighbourhood $\mathcal{W}_1$ of $\gamma(t_1')$ such that,

$$\|e^{\varphi/h} u\|_{L^2(\mathcal{W}_1) \oplus L^2(\mathcal{W}_1)} = \mathcal{O}(e^{-\delta_1/h}). \tag{9.2}$$

Fixing $t_1' < t_1$ sufficiently close to t_1, and taking local Euclidean coordinates (x', x_n) centered at $\gamma(t_1)$ such that $\dot{\gamma}(t_1)$ is in $\{x' = 0, x_n > 0\}$, we consider the weight-function,

$$\psi(x) := \varepsilon_1 (1 - \varepsilon_2^{-2} |x'|^2) \varphi_0 + (1 - \varepsilon_3) \varphi(x), \tag{9.3}$$

where $\varphi_0 := \varphi(\gamma(t_1))$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ are small enough. Denoting by (x_1', x_n^1) the corresponding coordinates of $\gamma(t_1')$, we have $x_n^1 < 0$, $x_n^1 \sim t_1' - t_1$, $x_1' = \mathcal{O}((t_1 - t_1')^2)$, and $\varphi(0, x_1') < \varphi(0)$.

Then, we consider the open set,

$$\omega := \{|x'| < \varepsilon_2 \,;\, |x_n| < |x_n^1|\},$$

and, setting $\varphi_+ := \min\{\varphi(x)\,;\, x_n = -x_n^1\,,\, x \in \omega\} > \varphi_0$, we observe,

- On $|x'| = \varepsilon_2$: We have $\psi(x) = (1 - \varepsilon_3)\varphi(x) < \varphi(x)$;

- On $x_n = x_n^1$: $\psi(x) \leq \varepsilon_1\varphi_0 + (1 - \varepsilon_3)\varphi(x) \leq \varphi(x) + \delta_1$ if we have chosen $\varepsilon_1 \leq \delta_1/\varphi_0$;

- On $x_n = -x_n^1$: $\psi(x) \leq \varepsilon_1\varphi_0 + (1-\varepsilon_3)\varphi(x) < \varphi(x)$ if we have chosen $\varepsilon_1 < \varepsilon_3(\varphi_+/\varphi_0)$;

- On ω_ε: $|\nabla\psi(x)|^2 = (1 - \varepsilon_3)^2|\nabla\varphi(x)|^2 + \mathcal{O}(|\varepsilon_1/\varepsilon_2|^2) \leq (1 - \varepsilon_3)|\nabla\varphi(x)|^2$ if we have chosen $\varepsilon_3 << 1$ and $\varepsilon_1 << \varepsilon_2$;

- $\psi(0) = (1 + \varepsilon_1 - \varepsilon_3)\varphi_0 > \varphi_0$ if we have chosen $\varepsilon_1 > \varepsilon_3$.

In particular, taking $\varepsilon_1 \in (\varepsilon_3, \varepsilon_3(\varphi_+/\varphi_0))$ and $\varepsilon_3 << \varepsilon_2 << 1$, on $\partial\omega$ we have $e^{\psi/h}u = \mathcal{O}(1)$. Thus, performing Agmon estimates on ω (by using Green's formula), we easily conclude,

$$\|h\nabla(e^{\psi/h}u)\|_{L^2(\omega)} + \|e^{\psi/h}u\|_{L^2(\omega)} = \mathcal{O}(1).$$

Since $\psi(0) > \varphi(\gamma(t_1))$, we deduce the existence of $\delta_2 > 0$ such that $e^{\varphi/h}u = \mathcal{O}(e^{-\delta_2/h})$ near $\gamma(t_1)$.

Again, as before we can propagate this information along γ up to the next point where γ crosses $\partial\Omega_0$, where the same argument applies. Iterating the procedure, we reach in this way any point of gamma, except its final end-point $\{x_0\} := \gamma \cap \partial\mathcal{M}$.

In particular, if we fix $\widetilde{x}_0 \in \gamma\backslash\{x_0\}$ close enough to x_0, then there exists $\delta > 0$ such that $e^{(\varphi+2\delta)/h}u$ is bounded near $\widetilde{x}_0$.

In a spirit similar to that of (9.3), for $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ small enough, we set,

$$\widetilde{\varphi}(x) = \min\left(\varphi(x) + \varepsilon_1\,;\, S_0 + d(x, \mathcal{M})\right),$$

and we define,

$$\widetilde{\psi}(x) := \varepsilon_2(1 - \varepsilon_3^{-2}|x'|^2)S_0 + (1 - \varepsilon_2)\widetilde{\varphi}(x),$$

where, this time, (x', x_n) are Euclidean coordinates centered at x_0, with $\{x_n = 0\} = T_{x_0}\partial\mathcal{M}$ and $\widetilde{x}_0 \in \{x_n < 0\}$.

We have,

- On $\{|x'| = 2\varepsilon_3\}$: $\widetilde{\psi}(x) = -3\varepsilon_2 S_0 + (1 - \varepsilon_2)\widetilde{\varphi}(x) \leq \varphi(x) + \varepsilon_1 - 3\varepsilon_2 S_0$, and thus, $\widetilde{\psi}(x) \leq \varphi(x)$ if $\varepsilon_1 \leq 3\varepsilon_2 S_0$;

- On $\{x_n = \widetilde{x}_n^0\}$ (where $\widetilde{x}_n^0 < 0$ is the last coordinate of $\widetilde{x}_0$): $\widetilde{\psi}(x) \leq \varepsilon_2 S_0 + \varphi(x) + \varepsilon_1 \leq \varphi(x) + \delta$ if we have taken $\varepsilon_2 S_0 + \varepsilon_1 \leq \delta$;

- On $\{x_n = |\widetilde{x}_n^0|\}$: $\widetilde{\psi}(x) \leq S_0$;

- On $\{|x'| < 2\varepsilon_3\,,\ |x_n| < |\widetilde{x}_n^0|\}$: $|\nabla\widetilde{\psi}(x)|^2 = (1-\varepsilon_2)^2|\nabla\widetilde{\varphi}(x)|^2 + \mathcal{O}(|\varepsilon_2/\varepsilon_3|^2) \leq (1-\varepsilon_2)|\nabla\widetilde{\varphi}(x)|^2$ if we have taken $\varepsilon_2 << \varepsilon_3$;

- On $\{\varphi(x) + \varepsilon_1 = S_0 + d(x, \mathcal{M})\} \cap \{|x'| \leq \varepsilon_3\}$: Since $d(x, \mathcal{M} \geq S_0 - \varphi(x)$, on this set we have $\varphi(x) \geq S_0 - \varepsilon_1/2$, and thus $\widetilde{\varphi}(x) \geq S_0 + \varepsilon_1/2$. Therefore $\widetilde{\psi}(x) \geq (1-\varepsilon_2)(S_0 + \varepsilon_1/2)$, and then, if we have taken $\varepsilon_1 > 2\varepsilon_2 S_0/(1-\varepsilon_2)$, we obtain $\widetilde{\psi}(x) > S_0$ there.

Hence, using the fact that $|\nabla\widetilde{\varphi}(x)|^2 \leq V_2(x)$ almost everywhere, and performing as before Agmon estimates on $\widetilde{\omega} := \{|x'| < 2\varepsilon_3\,,\ |x_n| < |\widetilde{x}_n^0|\}$, we obtain,

$$\|h\nabla(e^{\widetilde{\psi}/h}u)\|_{L^2(\widetilde{\omega})} + \|e^{\widetilde{\psi}/h}u\|_{L^2(\widetilde{\omega})} = \mathcal{O}(e^{\varepsilon/h}).$$

For all $\varepsilon > 0$, and thus, thanks to the last property of $\widetilde{\psi}$, denoting by ω_1 a small enough neighborhood of $\{\varphi(x) + \varepsilon_1 = S_0 + d(x, \mathcal{M})\} \cap \{|x'| \leq \varepsilon_3\} \cap \widetilde{\omega}$,

$$\|u\|_{H^1(\omega_1)} = \mathcal{O}(e^{-(S_0+\delta_1)/h}),$$

for some constant $\delta_1 > 0$.

This can be done along any $\gamma \in G$, and since we already know that $e^{S_0/h}u$ is exponentially small near $\partial\mathcal{M}\backslash\bigcup_{\gamma \in G}(\gamma \cap \partial\mathcal{M})$, we conclude that there exists a neighborhood $\mathcal{V}$ of $\partial\mathcal{M}$ such that, for any $\mu > 0$ small enough, $e^{S_0/h}u$ (together with all its derivatives) is exponentially small on $\mathcal{V} \cap \{V_2 = -\mu\}$.

Then, setting $\Omega_2 := \widehat{I} \cap \{V_2 < -\mu\}$ and applying the Stokes formula on Ω_2, we obtain,

$$(\operatorname{Im}\rho)\|u\|^2_{L^2(\Omega_2)} = -h^2\operatorname{Im}\int_{\partial\Omega_2}\frac{\partial u}{\partial\nu}\cdot\overline{u}ds + h^2\operatorname{Im}\int_{\partial\Omega_2}(r_1\cdot\nu)u_2\overline{u_1}ds, \qquad (9.4)$$

and the result follows.

10. About examples.

In contrast with the case of highly excited shape resonances (see [**DaMa**]) where examples are not easy to construct, here, because of the presence of two potentials, the task is simpler.

A first type of examples can be made by considering a rotationally invariant potential V_1. In that case, it is well known that the eigenvalues of P_1 corresponding to rotationally invariant eigenfunctions are separated by a gap of order h^2 from the rest of the spectrum. Since the eigenvalues of $\widetilde{P}$ (defined in (3.2)) coincide with those of P_1 up to some $\mathcal{O}(\|W\|^2 h^2)$, if the interaction W is sufficiently small, the same gap will hold for the corresponding eigenvalues of $\widetilde{P}$, so that Assumption 3 will be satisfied for them. In addition, one can see that the corresponding eigenfunctions of $\widetilde{P}$ are close, up to $\mathcal{O}(h^2)$, to those of P_1. On the other hand, the normalized rotationally invariant eigenfunctions of P_1 can be constructed by means of one dimensional (WKB or so) methods (see, e.g., [**Ya**]), and near the boundary of the well U (that corresponds to turning points), it can be seen that they are of size at least 1. Therefore, the same will be true for the corresponding

eigenfunctions of $\widetilde{P}$, and Assumption 4 will be satisfied, too. Concerning Assumption 5, it strongly depends on the choice of the potential V_2, and one can for instance take it in such a way that the crest of the cirque realizes its (Euclidean) distance with the well at only one point, with sufficient non degeneracy in order to satisfy Assumption 5.

Another kind of examples may be given by taking V_1 with separated variables. For instance, in dimension 2, one may take $V_1(x, y) = v_1(x) + v_2(y)$, where $v_2^{-1}(0)$ is a bounded interval with non empty interior, while $v_1^{-1}(0) = \{0\}$. In that case, as in [**Ma2**, Section 7], one can construct sequences of values of h for which some eigenvalues of P_1 that are close to 0 are separated by a gap of order h^3 from the rest of its spectrum, and the corresponding eigenfunctions are concentrated on $x = 0$. If in addition the interaction W vanishes at $x = 0$, then, the arguments of the previous paragraph can be repeated, and give other examples where Theorem 3.4 applies, for instance if V_2 is chosen in such a way that the minimal geodesics between the well and the sea starts on $x = 0$. If, on the contrary, V_2 is chosen in such a way that any minimal geodesic γ between the well and the sea starts away from $x = 0$, then the eigenfunction is exponentially small on $\gamma \cap U$, and Assumption 4 is not satisfied anymore. In that case, Theorem 3.6 applies and shows that the width of the corresponding resonance is exponentially smaller that $e^{-2S_0/h}$.

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