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Abtract

This study investigates the contributions made by Gaspard Monge and the students of his School to the stereotomy of vaulted systems in France between the eighteenth and nineteenth centuries. The complexity of the apparatuses and the generality of the proposed solutions express the extent of the contributions that descriptive geometry made to the applications that preceded it. First among these stereotomy, which, though in decline from an operational point of view, was considered fundamental in the schools that were then being founded. The ellipsoidal vault, the helicoidal apparatus and the *arrière voussure de Marseille* are expressions of the relationship between the operability of stereotomy and the theoretical speculations of descriptive geometry, which operates through the synthetic language of drawing. These applications make explicit a *modus operandi*, capable of resolving the problems of defining, representing and expressing the geometric properties of figures using the synthetic methods of descriptive geometry.

Introduction

The stereotomy of vaulted systems is an expression of a large repertoire of forms, widely practiced and described in the numerous treatises on stereotomy published from the second half of the sixteenth century until the end of the eighteenth century. During the nineteenth century and part of the twentieth, stereotomy became an application of descriptive geometry and thus was considered as such in the treatises, particularly those written by the students of the Monge School. In the years of the Industrial Revolution, the process of transforming stereotomy from an art into a science initiated by Amedée François Frèzier reached its conclusion in Gaspard Monge's descriptive geometry (Salvatore 2011). It was a structural change for stereotomy. This discipline, which resolved diversely complex apparatuses case by case finds, in descriptive geometry, a finality of generalization. Therefore, it was transformed from a useful tool aimed at solving practical constructive problems into a speculative science operating with the form in terms of maximum generalization. The generalizing character of the constructions at the basis of modern stereotomy is a tangible expression of the objectives of descriptive geometry (Calvo-López 2011). Theorized by Monge in those years, these objectives concern, as is known, the knowledge of the theories of form and their communication by means of graphical representation.

Stereotomy becomes a highly formative tool for those who intend to work with shapes, but especially for those who intend to operate with them through the tools of synthetic geometry, namely through drawing. Therefore, nineteenth-century stereotomy must be understood as an experimental laboratory in which form was treated via a synthetic method in terms of pure scientific speculation. It was based on the construction of graphic models founded on an exhaustive knowledge of geometrical theories, oriented to the generalization of the solutions. The stereotomic apparatuses that are investigated in this present contribution, namely the ellipsoidal vault, the helicoidal apparatus and the *arrière voussure de Marseille* are an expression of this assumption, evidence of which is found precisely in the algorithm that leads to their construction in space, explained here through continuous digital representation.

The critical re-reading of these constructions in a digital environment demonstrates a *modus operandi* that identifies in the model – first mentally, then graphically, today digitally – the ideal instrument for the study, derivation and validation of the properties of form. The graphic construction of these apparatuses is a fundamental operation towards fully understanding their geometric meaning through the language of descriptive geometry. Just as the quality of a musical composition is difficult to appreciate simply by reading its score, it is likewise difficult to acquire the awareness of a form's geometrical *raison d'etre* merely through verbal description: hence just as music must be expressed by voices or instruments, so descriptive geometry must be represented through drawing. In this regard, digital drawing makes a particularly significant contribution. In fact, if on the one hand it facilitates visualization by overcoming the limits of a drawing on a sheet of paper, on the other hand it requires a greater geometric awareness, due to the rigor and accuracy needed to operate.

We believe that the synthetic approach is the element of continuity that links traditional descriptive geometry to contemporary descriptive geometry, and that today this approach finds, in the mathematical digital representation, new impulse and new lymph. From this point of view, these studies on stereotomy are still relevant in terms of content and methodology and should be considered a valid contribution for those who intend to work with form using the synthetic method of drawing.

2. Stereotomy at the Monge School

The 1794 edition of the Journal de l'Écoles polytechniques presented a synthesis of the lectures given by Gaspard Monge at the École, collected into a single work four years later (Monge 1794: 6; Hachette 1828: VIII-X). In the introduction to his *Traité de Géométrie Descriptive*, Jean Nicolas Pierre Hachette, who was an adjunct professor in the École at the time, argues that the collection of these lessons can be considered the first treatise in which descriptive geometry is treated independently of its applications (Hachette 1828: X).

This observation, which flows between the lines of a rich introduction written by one of Monge's most brilliant disciples, summarizes the innovative extent of the contribution that Monge had given to the comprehensive knowledge that, by the end of the eighteenth century, could be considered well established.[1]

In the years during which Monge taught at the École Royale du Génie de Mézière,[2] descriptive geometry still did not exist. Nevertheless, its applications are taught, such as, for example, perspective, *défilement* and shadow theory, already practiced in the previous centuries.

Among these, stereotomy stands out as having assumed a prominent role at the school (Sakarovitch 1998: 220-227). Widely practiced until the first half of the eighteenth century, because of the numerous works in cut stone done mostly in the field of fortifications, this discipline began to decline in the last years of the Ancient régime, mainly due to the progressive abandonment of stone from the construction world. Although the operational needs that nourished these studies were lacking, stereotomy had been one of the privileged teachings at the École de Mézière from the very beginning. Chastillion, one of the founders of the school, who was convinced of the pedagogical and educational value of this discipline, attributed to it the merit of teaching how to visualize forms in space.[3] This heuristic value of stereotomy is expressed in art. 9 of the statutes of the École de Mézière, presumably drafted by Chastillion himself, and even more clearly expressed in his essay entitled *Traité des ombres dans le dessin géométral*[4]:

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[1] For further information on Gaspard Monge's role in descriptive geometry, see Vito Cardone's studies (Cardone 2017).

[2] The École du Génie de Mézière was founded in 1748; Monge taught stereotomy here from 1760. This School was followed in the coming years by the École Normale and the École Polytechnique, founded in 1794.

[3] About the teaching at the École Royale du Génie de Mézière, see Sakarovitch (1995: 208-210).

[4] This treatise (*mémoire*) was published by Theodore Olivier in *Applications de géométrie descriptive aux ombres, à la perspective, à la gnomonique et aux engranages*. In the first two notes Olivier specifies that this *mémoire* is part of a manuscripts collection of the library of the École d'application de l'artillierie et du Génie de Metz. This comes from the ancient École du Génie which was transferred from Mezière to Metz in 1793. Olivier hypothesizes that this essay, anonymous, was wrote between 1775 and 1780 for the education of the young officers of the *Génie* (Olivier 1847: 5-24).

On n'a rien trouvé de plus propre pour leur [les ingégneurs] procurer cette connaissance parfaite du dessin que leur faire suivre des cours de coupe des pierres et de bois; … Indipéndamment des avantages qui résultent de cette étude, relativement aux constructions dont les officiers du génie ont la direction, on conçoit facilement que, quand on fait développer toutes les faces et connaître tous les angles plans ou solides d'une pierre quelconque employée dans une voûte, une trompe, etc., ou d'une pièce de charpente employée dans un comble, un dôme, un escalier, etc. … que, quand on sait bien former la répresentation de toutes ces choses pour les faire entendre aux autres, on est état de les représenter comme si elles étaient déjà exécutées, et d'en combiner les différentes constructions pour les rendre autant parfaites qu'elles pouvent l'être (Olivier 1747: 6).[5]

Stereotomy, in fact, dealt with the entirety of the problems that affected, in a transversal way, all the other applications practiced until that time, which converged towards a well-defined and shared idea of the representation of space. These problems were solved, in the Mézière school, through overturning operations and auxiliary planes in double orthogonal projection, according to a code already consolidated and used in stereotomic practice.

To understand the scenario of those years we must imagine the mosaic tiles of a puzzle, separate, but ready to be assembled. The combination of these tiles resolved the two fundamental problems of representation at that time, which would be translated by Monge into the two fundamental objectives of descriptive geometry, namely: the synthetic control of the properties of shape and its representation in the plane through drawing (Monge 1798: 5). Before the theorization of descriptive geometry, these objectives were satisfied in stereotomic practice and, for this reason, its teaching was held in high esteem (Sakarovitch 2005). Moreover, the stereotomic procedures employed representation in plan and elevation, required to describe the objective characteristics of represented forms. Therefore, it is not surprising that, in the years of the Industrial Revolution, during which the engineering schools in question were founded in France (i.e., École de Mézière, the École Normale and the École Polytechnique) there was a need to relaunch these forms of representation. The goal was to find a synthetic, efficient and shared language of the communication of shape, oriented towards the industrial manufacture of the product, a language that found unequivocal definition in Monge's new descriptive geometry.[6]

According to Bruno Belhoste this essey, antecedent to 1764, must be attributed to Chastillon (Belhoste 1990: 11; Sakarovitch 1998: 85).

[5] There was nothing more clean for their [engineers] to provide this perfect knowledge of drawing than to have them take courses in stone and wood cutting; … regardless of the advantages which result from this study, relative to the constructions under the control of the engineering officers, it is easy to understand that, when we develop all the faces and know all the plane or solid angles of any stone used in a vault, *trompe*, etc., or of a frame used in an attic, a dome, a staircase, etc. … that, when we know how to form the representation of all these things to make them understand to others, we are able to represent them as if they were already executed, and to combine the different constructions to make them as perfect as they can be (translation by the authors).

[6] The École du Génie de Mézières, together with the École des Ponts et Chaussées, was active until 1803. In the early years of the Republic, which followed the French Revolution, the École Normale was established, where Monge taught for four years. In 1794 the École Polytechnique, first called the École centrale des travaux publiques, was founded, and it gradually replaced the first two. In the Polytechnique, the teaching of descriptive geometry appeared for the first time. Monge taught there for a short time, entrusting Hachette with responsibility for teaching this new science in 1795 (Sakarovitch 1998: 220-227).

In synthesis, Monge's great merit consists in having approached the question in terms of maximum abstraction, conferring to a 'geometrical descriptive' practice used in those years the dignity of science. He was responsible for the rationalization of the graphic processes of representation in plan and elevation, little theorized but widely practiced over several centuries.[7] In particular, his school was responsible for the elaboration of the theory of surfaces as we know it today, resolved through a synthetic method.

Théodore Olivier clarifies this idea in the introduction to his *Cours de Géométrie descriptive* written in 1843, arguing that descriptive geometry must be considered both as an art and a science. Olivier writes:

> *Pendant longtemp et depuis très-longtemps, l'art des projections était connu des stéréometres, et ainsi des appareilleurs pour la coupe des pierres et de charpentiers; mais c'est vraiment depuis Monge que la géométrie descriptive a été reconnue être une science, et c'est aux travaux de Monge qu'on le doit; car c'est lui qui le premier a démontré que, dans ce que l'on appelait l'art des projections, résidait réellement une méthode scientifique qui permettait de rechercher et de démontrer certaines vérités géométriques, et ainsi toutes celles relatives à la forme de l'espace figuré* (Olivier 1843: VI).[8]

Monge was among the main founders of the École Polytecnique. Here he taught *Géometrie déscriptive* and the *Analyse appliquée à la géometrie* (Descriptive Geometry and Analysis Applied to Geometry). He held the synthetic language of geometry in as high esteem as algebra, and was strongly convinced of the effectiveness of studying them together, an idea that permeates all of his work (Loria 1921: 108-109).[9] In this regard, Olivier reports Monge's words:

> *Si je refaisais mon ouvrage, qui a pour titre L'analyse appliquées à la géométrie, … j'écrirais en deux colonnes: dans la première je donnerais les démonstrations par l'analyse; dans la seconde, je donnerais les démonstrations par la géométrie descriptive, en d'autre terms, par la méthode des projections; et l'on serait peut-être … en lisant cet ouvrege, de voir que l'avantage serait presque toujours du côté de la seconde colonne, pour la clarté du raisonnement, la simplicité de la démonstration, et la facilité de l'application des théorèmes trouvés aux divers travaux des ingegneurs* (Olivier 1843: VI).[10]

In this process of the scientification of the art of projections, stereotomy became part of descriptive geometry, and its applications, considered pedagogically instructive, continued to be taught. They

[7] The first printed treatise on stereotomy was by Philibert de l'Orme in 1567. This work testifies to a wise use of the representation in plan and elevation. However, evidence of the use of this method can be seen in the previously published treatises on perspective, first of all that written by Piero della Francesca at the end of the fifteenth century.

[8] From a long time and for a very long time, the art of projection was known to stereometers, and thus to stone-cutters and carpenters; but it is really since Monge that descriptive geometry has been recognized as a science, and it is to Monge's works that we owe it; because it was he who first demonstrated that, in what was called the art of projections, really resided a scientific method which made it possible to seek and demonstrate certain geometric truths, and thus all those relating to the form of figurative space (translation by the authors).

[9] Loria says that, starting from Monge, physical models made with ropes, wood or metal, were used to illustrate the most complex geometric figures (Loria 1921: 122).

[10] If I redid my work, which has the title Analysis applied to geometry, ... I would write in two columns: in the first I would give the demonstrations by analysis; in the second, I would give the demonstrations by descriptive geometry, in other words, by the method of projections; and you would see perhaps... by reading this opening, that the advantage would almost always be on the side of the second column, for the clarity of the reasoning, the simplicity of the demonstration, and the ease of application of the theorems found in the various works of the engineers (translation by the authors).

would contribute to enrich the pages of the treatises on geometry up to the first decades of the twentieth century. The subjects treated are the classic ones of the stereotomic repertoire, namely vaults and stairs.[11] Just as occurs traditionally, spherical domes, ellipsoidal vaults, bias vaults, *trompes*, etc., recur among the vaulted systems, as can be seen in the consistent production of descriptive geometry and stereotomy treatises written by the students of the Monge School such as Jean Nicolas Pierre Hachette, Charles-François-Antoine Leroy, Jean-Paul Douliot, Joseph Adhemar, and Emile Le Jeune, to name just a few.

Among the numerous cases treated, some are particularly significant (Rabasa 2011: 719-725). These are apparatuses capable of resolving and generalizing particularly difficult conditions through scientific theories derived from what was then the new descriptive geometry.

3. The Ellipsoidal vault and the Lines of Curvature

The goal of stereotomy is the design of the apparatus, namely the definition of the most convenient way to break down a work into a set of ashlars. The choice of the morphology of the apparatus derives from different issues of a static or aesthetic order, but also from practical considerations related to the execution and cost of the work.

The principles of stereotomic design are explained in several treatises dedicated to the art of stone-cutting and can be summarized in a few key points. These points concern in particular the dihedral angles formed by contiguous faces of the same ashlar, which should not be overly acute in order not to be susceptible to breakage. But they also concerned the correct execution of the joining surfaces, which had to be accurately performed to allow contiguous ashlars to optimally adhere and evenly distribute the thrusts throughout the entire surface, not merely in some points. For this exigency of accuracy, flat surfaces were favored, then developable ones, and finally ruled surfaces. Developable surfaces, like ruled surfaces, could be reproduced on site through straight rulers; the developable surfaces in particular could be developed on the plane and reproduced in cardboard or metal models (the *panneaux*) to be unrolled on site to check their correct execution. Generations of geometers have provided various contributions aimed at satisfying these indications by optimizing the apparatuses and their manufacturing processes. Monge is the author of one of the most significant theoretical contributions in this regard. In fact, he was responsible for the theorization of the lines of curvature of a surface, lines that, according to Monge himself, would have solved the designing problems of the ashlars as a whole.

Lines of curvature belong to a surface and cover it without gaps. They have the property of having the direction of the main curvatures of the surface at each point and, therefore, are constantly orthogonal to each other. Imagine a point on a surface and the normal leading to the surface at that point. This normal is a straight line supporting a sheaf of planes which section the surface according to a system of curves. Each of these curves will have, at the point, a different curvature value. Among the infinite curves of section, two will have the maximum and minimum curvature value and will be called main curvatures. The tangents to these two curves in the contact point indicate the main directions of curvature which are orthogonal to each other, as had already been demonstrated by Euler in the second half of the eighteenth century (Fig. 1) (Euler 1767: 119-143).[12]

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[11] The repertoire of the subjects studied is generally referable to that published by Jean Baptiste de la Rue in his *Traité de la coupe des pierres* of 1728, a particularly substantial manual work. The numerous cases described in the treatise were quite well known at the time, because they were proposed as exercises in the stereotomy courses of the École de Mézière.

[12] The main curvatures will be taken up by Gauss for the theorization of the "Gaussian curvature" (product of the main curvatures) currently used today in mathematical NURBS modellers. The curvature at an individual point of a surface

Fig. 1 The principal lines of curvature of an ellipsoid

The possibility of building a network of orthogonal curves on a surface hinted at the possibility of solving, in terms of maximum generalization, most of the problems posed by the design of the ashlars. Monge synthesizes these problems in his lectures on descriptive geometry (Monge 1795; Monge 1798: 120-127), exhorting accuracy in the execution of the surfaces, both those of facing and joining, hoping for their mutual perpendicularity and, consequently, that of the respective edges, urging operation with developable joining surfaces. Finally, he hopes that the edges of the segments can lead to the character of the surface to which they belong.

The lines of curvature could satisfy all these conditions. In fact, they allowed the construction of orthogonal edges and permitted operating with developable joining surfaces, since the set of the normals to a line of curvature constitutes a developable surface, which is orthogonal to the same surface constructed on the second family of lines of curvature (Fig. 2).

Fig. 2 The developable surfaces normal to the ellipsoid surface

Hachette and later Adhemar recount that graphic constructions for the tracing of these lines had not been elaborated at the time, but that the problem was marginal, since they were unknowingly already used in the stereotomy of stone, because of a natural instinct in the construction of the *appareillage*. In the cones and cylinders used, for example, for barrel vaults and *trompes*, the generatrixes and sections normal to the internal axis to these surfaces are precisely the lines of curvature in question; likewise, in surfaces of revolution, such lines correspond to the network of meridians and parallels of the surface (Hachette 1822: 289-290; Adhemar 1856: 365). The problem therefore concerned some particular cases, such as that of the ellipsoidal vault, with which Monge deals for the first time in a paper published in 1795 in the *II cahier of the Journal de l'Ecole Polytechnique* (Monge 1795: 145-165) (Fig. 3).

Fig. 3 Monge's graphical construction of the ellipsoidal apparatus

The construction of the lines of curvature of an ellipsoid was not a trivial question for the time, especially if resolved with a synthetic method, namely through drawing. The lines of curvature of an ellipsoid are skew curves symmetrical with respect to the principal planes of the surface. Monge observed that these curves are projected on the main planes of the surface according to second-degree curves: an ellipse for the family that follows the maximum curvature directions, a hyperbola for the family that follows the minimum curvature directions. Because of the aforementioned symmetry, the curves of the two families share the center and the main axes and both tend to two significant points that lie on the common focal axis without ever reaching them. The normals on the plane of the principal axes of these conics, conducted from these two points, meet the surface of the ellipsoid at four notable points, which Monge named "umbilicuses".[13] In order to represent the ellipses and the hyperbola in plan and elevation in question an auxiliary ellipse and hyperbola are used, having the same center of the two conical series that must be built. To represent these auxiliary conics it is necessary to determine their axes **OP** and **OR**. For this purpose, the foci $\mathbf{F_1}$, $\mathbf{F_2}$, $\mathbf{F_3}$ of the ellipse, principal sections of the ellipsoid, are built. For the construction of the first axis, the lengths $\mathbf{OF_2'} = \mathbf{OF_2}$ and $\mathbf{OF_1'} = \mathbf{OF_1}$ are considered, then the straight line $\mathbf{AF_2'}$ and the parallel $\mathbf{F_1'P}$ are constructed, thus determining the length of the first axis. To establish that of the second axis, the length $\mathbf{OF'_3}$ =

can be of three types: positive, negative and zero. This property makes it possible to classify surfaces into different types: surfaces with negative curvature (such as ruled surfaces); surfaces with positive curvature (such as the ellipsoid); surfaces with zero curvature (such as developable surfaces); finally surfaces with mixed curvature (such as the torus).

[13] The umbilicuses are remarkable points of a surface where the main curvatures are indeterminate. A surface composed entirely of umbilicuses is the sphere; in every point of this surface the main curvatures are indeterminate.

$\mathbf{OF_3}$ is considered, then the straight line $\mathbf{BF_3'}$ and finally the parallel $\mathbf{F_1R}$. Once the auxiliary conics are known, any point **H** is constructed on the auxiliary hyperbola and the coordinates are given on the vault axes. These coordinates provide the length of the two half-axes of the ellipse, which in the first projection represents the first series of lines of curvature. In the same way, the coordinates of a point L on the auxiliary ellipse are constructed to obtain the half-axes of a hyperbola projection of the second curvature. The construction of the lines of curvature in elevation works in the same way. We observe that lines of curvature that project in the first projection according to an ellipse appear as hyperbolas in the second projection and vice versa (Fig. 3). Bringing this construction to a definition of a stereotomic apparatus, it is necessary to define in advance, in plan and elevation, a division into ashlars, and then draw the lines of curvature in correspondence with these partitions.

The construction, laborious but particularly interesting, became recurrent in the descriptive geometry and stereotomy treatises that followed, such as those of Hachette, Leroy and Adhemar. Innovative contributions are instead due to Jacques Binet, who demonstrated how the lines of curvature of a second-degree surface derive from the intersection of the given surface with a pair of quadrics of a different type, confocal to this one, and how these three surfaces are perpendicular to each other. The question was generalized a few years later by Charles Dupin, who showed how the surfaces of a tri-orthogonal family are sectioned according to their lines of curvature. Confocal quadrics are surfaces that enjoy this property, and are in particular a generic ellipsoid, a one-sheeted elliptical hyperboloid, and a two-sheeted elliptical hyperboloid (Hilbert and Cohn-Vossen 1972: 28-36, 238-251). The use of confocal quadrics allows a rigorous representation of the lines of curvature of an ellipsoid, today accurately reproducible through continuous digital representation (Fallavollita and Salvatore 2012: 65-71) (Fig. 4).

Fig. 4 The confocal quadric surfaces

The theory of lines of curvature applied to the case of an ellipsoid remained a speculative exercise rather than an operating practice throughout the nineteenth century.[14] Monge's strong belief in the universal character with which these remarkable classes of lines would solve stereotomy problems underwent a crisis in the face of specific practical problems that arose around the middle of the nineteenth century relating in particular to the construction of bias vaults.

4. The Helicoidal Apparatus

In the tradition of stereotomy, the question of bias vaults has always been particularly thorny.[15] At the beginning of the nineteenth century this once again became very topical, becoming one of the rare stereotomy applications practiced at the time. With the spread of railway lines throughout the territory, the problem of crossing became central, as Adhemar recounts in his *Traité de la coupe des Pierres* of 1840. Since sudden changes in track direction were not possible, the crossing of roads or canals was solved by designing bias bridges, consisting of barrel vaults formed of quadric, round or elliptical cylinders. The case of bias bridges is particularly interesting because it poses certain problems that Monge's theory of lines of curvature could not resolve. In the case of cylinders, in fact, the lines of curvature are generatixes and straight sections of the surface (perpendicular to the internal main axis). If the vault presents a strong obliquity, these lines cannot be used as joint edges, because the head ashlars would present strongly acute angles and would not be contrasted at the sides, where

[14] Monge imagined creating the vault of the halls hosting the legislative assemblies as an application of the theory of lines of curvature on the surface of an ellipsoid, in iron and glass (Monge 1795: 162-163).

[15] The theme of bias vaults recurs in all stereotomy treatises from de l'Orme onwards. Desargues would make it the subject of a dedicated treatment in his *Brouillon projet* dedicated to stonecutting.

force is greater. This would produce vacuum thrusts, which must necessarily be avoided to prevent collapse.

The solution that greatly resolves the issue is the helicoidal apparatus, developed in England, where the problem was quite present.[16] In France, the helicoidal apparatus was treated by Adhemar, who published it in his treatise, together with other notably interesting solutions given by Hachette some years earlier, relating in particular to small crossings, such as road passages under railway sections (Adhemar 1856: 404-407). Hachette offers two especially interesting solutions. The first resolves the apparatus by using an elliptical cylindrical vault sectioned according to circumferences on the face arch. In the case in question, the beds of the ashlars are flat surfaces that belong to the planes passing through the normal to the wall that must be crossed. In this apparatus, the joint edges of the vault are elliptical arches and the face arch ashlars have a slight obliquity, which accentuates as the depth of the passage to be crossed increases (Fig. 5). The second solution to which Adhemar's text refers once again covers a small bias passage. In this case the surface of the vault is a ruled one, namely a cylindroid, which belongs to the two semi-circular face arches, and to the normal of the wall that must be crossed. This particular type of vault presents an imperfection at the key, a sort of deformation, which makes it unusable from a practical point of view, limiting its interest to pedagogical exercises (Fig. 5).

Fig. 5 Models of the apparatuses for small crossing by Adhemar's treatise

The question of bias bridges, as mentioned above, is in general resolved by a helicoidal apparatus. This apparatus makes it possible to have equal internal ashlars, maintaining their orthogonality on the fronts, thus obtaining a more uniform distribution of loads. In this case, the joint edges of the ashlars belong to two systems of helices, having the same pitch and axis, coinciding with the axis of the cylindrical intrados surface. The joining surfaces are straight ruled helicoids with director plane, therefore orthogonal to the surface of the cylinder, belonging to the aforementioned helices (Fig. 6). The extrados surface of the vault is another cylinder, which also has the same axis as the first. The internal ashlars have four lateral faces formed by straight helicoids and two cylindrical intrados and extrados faces. Their joint edges are helices, except for the four orthogonal to the intrados, which are instead straight lines. The helicoidal faces, thus formed, are minimal surfaces with respect to the edges that delimit them. We recall, in fact, that the right helicoid is the only minimum surface consisting of straight lines (excluding the plane). The ashlars thus obtained, except those on the face and those connecting to the pier, can be moved with a helical movement in the longitudinal or transverse direction, overlapping the contiguous ashlars without waste.

Fig. 6 The helicoidal surfaces of junction

The solution found generalized the question in such a way that the apparatus had widespread diffusion and was present in descriptive geometry treatises in the years to come. The reproduction of the graphical procedure for the construction of the following apparatus is taken from the treatise written by Gino Fano in 1935, demonstrating the didactic value that, even in those years, was attributed to this type of construction (Fano 1935: 448-461).

The construction of the apparatus, in plan, avails itself of the properties of the developable surfaces that allow us to operate on the plan development of the surface. In the case in question, the intrados surface is a round cylinder and therefore, in its development, the front lines, oblique with respect to

[16] On the question of bias vaults and England's contributions to the solution of solving the problem, see (Sakarovitch 1995).

the axis, became sinusoids. Figure 7 shows the impost parallelogram of the skew vault. The obliquity angle of the vault is given, on the first projection plane, because by the axis of the vault and the segment that represents the first projection of the front arch. The planes in front, being generic flat sections of a round cylinder, are ellipses that project themselves as circumferences on the second projection plane. Therefore, the intrados surface develops in the plane according to a quadrilateral formed by two straight and parallel sides (**AD** and **BC**) and by two sinusoids. Once the development is complete, the number of the ashlars of the arch in front is established. The developed joint edges are straight lines passing through these partitions and perpendicular to the **AB** chords, while the discontinuous joint edges are parallel to the **AB** chords. In order for the ashlars to exactly divide the surface, the continuous joint edge coming out of **A** must pass through a division point of the **CD** chord. This means that the two systems of lines will not be perfectly perpendicular but the ashlars will still be equal to each other. For this purpose, a straight line perpendicular to **CD** from point **A** is detached. This straight line intersects the **CD** chord at a point $\mathbf{A_1}$. A point among those of the partition of the **CD** chord, particularly close to $\mathbf{A_1}$, is chosen, such as point **4** in the figure. The continuous joint edges will be parallel to the **A4** straight line. Although this passage introduces an approximation in the construction, it allows us to keep the continuous junction helices as perpendicular as possible to the arches in front. The discontinuous joint edges will be parallel to the **CD** chord and must pass through points **G**, **H** and **I**, intersection of the continuous joint edges with the impost edges **AD** and **BC**. Once the intrados drawing is obtained, it will be sufficient to envelop it on the surface in order to obtain the objective joint edges.

Fig. 7 The helicoidal apparatus and its net

The construction leads to the definition of three different types of ashlars: the internal ashlars, equal to each other; the head ashlars that have the visible front face flat; the impost bearings that rest on the piers. The internal ashlars, as already mentioned, are composed of helicoidal junction surfaces and cylindrical intrados and extrados surfaces. The head ashlars differ from the internal ones only for the flat front of facing. Finally, the impost ones, which are different from each other, are welded to the bearings, to ensure at the same time a consistent housing with the masonry, but especially to avoid the formation of an acute angle to the impost of the vault (Fig. 08).[17]

Fig. 8 The three different ashlars

5. The *Arrière Voussure de Marseille*

The *arrière voussure de Marseille* is the last case we present. This, perhaps more than the others, makes evident the contributions of descriptive geometric theories to the solution of operational problems of stereotomy.

The *arrière voussure* is a small vault used especially in architecture, necessary to span door and window openings in a wall thickness. This apparatus consists of a barrel vault, the *porte droite* and a second vault, the *arriére-voussure*, intended to cover the door opening.[18] It is an apparatus necessary to facilitate the entry of carriages and prevent the wooden doors, which close the opening, from striking against the wall.[19] The *arriére-voussure*, which therefore constitutes the last part of the passage, is formed by three ruled surfaces placed in tangential continuity between them, whose

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[17] For further information on the design of impost bearings, see the studies on ruled surfaces and the stereotomy of stone (Fallavollita and Salvatore 2012).

[18] There are several models of *arrière voussure*, but this in particular raises a descriptive geometric question that finds its scientific rigor in the theory of ruled surfaces.

[19] For further details on the vault and the related theorems, see (Fallavollita 2008).

common lines are two circumferential arches and a horizontal straight line, the axis of the vault. The peculiarity of the opening lies in the tangential condition existing between these three ruled surfaces that cover the splay of the vault. The geometric explanation is given by Hachette, who dedicates a chapter to this apparatus in the appendix of his *Traité de géométrie descriptive* (1828: 315-318) (Fig. 9).

Fig. 9 The construction of the *Coupe de Arriere Voussure de Marseille*

In the first surface of the *arriére-voussure*, the position of the straight generatrix is determined by three conditions (Fig. 10):

- it must belong to the circular arch of the connection groove with the adjacent barrel vault, the *porte droite*;

- it must belong to the circumference arch that delimits the top of the splay;

- finally it must belong to the horizontal axis of the vault, orthogonal to the wall surface.

Fig. 10 The model of the *Coupe de Arriere Voussure de Marseille* from the treatise of Hachette (1828: A.Pl.3)

The *arriére-voussure* and the groove have the same joint plane as the *porte droite*; since the flat junction surfaces of the segments of this opening pass through the horizontal axis of the door, these cut the intrados surface according to straight lines, the edges of the ashlars that compose the vault.

Suppose that the top arch of the splay, which serves as a directrix for the straight line generating the first surface of the *arriére-voussure*, is given. This arch, whatever its radius, will be cut from the planes that delimit the splay, at two points. The planes passing through these points and the axis of the door section the groove circumference in two other points. These four points determine the position and length of the limit lines of the first of the three ruled surfaces that compose the *arrière voussure*, the central one. Two additional ruled surfaces cover the remaining space on the sides. These admit, as directrix curves, the circumference of the groove and the axis of the vault, but they also lean on a third circumference that has a radius equal to that of the groove, placed on the splayed planes of the door. The radius of these arches must not be less than that of the door panels, which must be accommodated. In order for tangential continuity to exist between the central part of the *arrière voussure* and the lateral wings, these must share the same tangent plane along the contact generatrix. Two ruled surfaces that share a generatrix and admit three common tangent planes in three different points of same are tangent to each other in all points of the shared generatrix.[20]

The solution to this problem of continuity is an application of the properties of the ruled surfaces derived by Hachette starting from the Chasles theorem. According to this theorem:

> *I piani tangenti a una rigata sghemba nei punti di una generatrice non singolare formano un fascio avente per asse questa generatrice, e proiettivo alla punteggiata dei punti di contatto.*[21] (Fano 1935: 355).

If the surface is a developable one, the tangent plane along a generatrix g will always have the same position, in other words, it will be tangential along the whole generatrix of the surface.

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[20] The other two tangent planes are along the generatrix line at the points of intersection with the second and third directrix.

[21] The tangent planes to a ruled surface at the points of a non-singular generatrix form a sheaf having this generatrix as its axis, and projective to the dotted of the contact points (translation by the authors).

Chasles theorem led Hachette to discover and define the main properties of ruled surfaces[22] (Hachette 1828: XIII-XIV). Given a generic ruled **R**, we consider any generatrix **g** on it. There are an infinite number of quadric surfaces tangential to **R** along the entire generatrix **g**. Each of them has as generatrixes of the opposite system all those straight lines tangential to the R-ruled that rest on **g**. Therefore, any three of these are needed to determine a hyperboloid Q tangent.

> *Deux sufaces réglées qui on une droite commune et trois plans tangens communs en trois points de cette droite, sont tangents l'une à l'autre dans tous les points de la droite commune.*[23] (Hachette 1828: 96).

It is therefore possible to also obtain a tangential hyperbolic paraboloid (Fig. 11). To construct this paraboloid, three planes are taken tangential to the ruled surface along a generatrix (the horizontal line in Fig. 11). These touch the surface at three distinct points. It is sufficient to cut the three tangent planes with three planes parallel to a director plane. The straight sections thus identify three generatrices parallel to the director plane, which identify a single hyperbolic paraboloid tangent to the given ruled surface (a hyperboloid of revolution in the figure).

Fig. 11 Two ruled surfaces having a non-singular generatrix in common meet, in general, in no more than two points of this generatrix. If they touch in three points of the same generatrix, they will be tangent along the entire generatrix. The three tangent planes are identified by the straight line in common and by the three straight lines of the paraboloid

In conclusion, two ruled surfaces having a non-singular generatrix in common meet, in general, at no more than two points of this generatrix. If they touch in three points of the same generatrix, they will be tangential along the entire generatrix, as happens in the case of the *arrière voussure*.

6. Conclusions

The applications presented in the sections above show the contributions made by descriptive geometry, in terms of theoretical generalization, to the solution of stereotomic problems. These contributions show, on the one hand, the lively academic speculation of steoreotomy and, on the other, its decline as applied art. The proposed cases are examples of a *modus operandi* that investigates shape in space, showing the power of theory through drawing.

With this in mind, we asked ourselves whether this approach, which permeates the history of descriptive geometry up to its apogee, is still valid today. Therefore, we wondered if, beyond the interest in the history of this science, these studies are still relevant, considering that, from a purely applicative point of view, they were no longer used even in Monge's time, when stereotomy was a discipline aimed at pure academic speculation. This question appears very topical today, since we live in a time when descriptive geometry itself entered into crisis following the advent of the computer. Despite the efforts of some scholars of the Italian School to revive this science, in the international debate descriptive geometry is often considered a dead science. Taught today as a basic subject in the early years in some engineering and architecture courses, it seems unable to offer new research scenarios. However, we believe that this science is still alive and that the "crisis" arises from a misunderstanding that has reduced it, for several years by now, to a set of graphic codes capable of transferring a three-dimensional space to the two-dimensional reality of a drawing plane. If this is,

[22] The properties of the ruled surfaces were studied and discovered by Hachette and published in his treatise of 1828. Years later, Gino Fano, grouped them under the heading "consequences of the Chasles theorem", because of the theorem that still carries today its name, although it was Hachette who derived its properties (Fano 1935: 355-357).

[23] Two ruled surfaces which have a common line and three common tangent planes in three points of this line, are tangent to each other in all the points of the common line (translation by the authors).

and was, the sole purpose of descriptive geometry, then those scholars who decree the death, or at least the obsolescence, of this science are right. Instead, if descriptive geometry has as its main objective the study of forms, their properties and their spatial relations, as another group of scholars are convinced, then this science can acquire renewed vigor today, thanks to digital representation.

A small demonstration of the fact that digital representation is only a more effective tool than pencil and compass is also deduced from these studies on stereotomy. To construct these apparatuses, such as the ellipsoidal vault, the helicoidal apparatus or the *arrière voussure* vault, it is essential to have a thorough knowledge of the geometric properties of the represented shapes and be able to translate them into spatial constructions. The study and construction of these geometries must utilize the synthetic method that allow them to be visualized and, in the virtual world, almost touched. Today, it seems simpler to draw an ellipsoidal vault than it was in Monge's day. This presumed simplicity is in part an illusion and lies mostly in the power of digital visualization. It is easier to see and understand shapes in space because we can observe them as if they were three-dimensional entities actually present. This illusory advantage led us and still leads us to believe that we can easily control and understand complex shapes. This is deceptive, since anyone who tries to tackle the apparatuses described in these few paragraphs will realize how important and complex it is to fully understand their geometrical reasons for being.

Acknowledgments

All images are by the authors unless otherwise noted.

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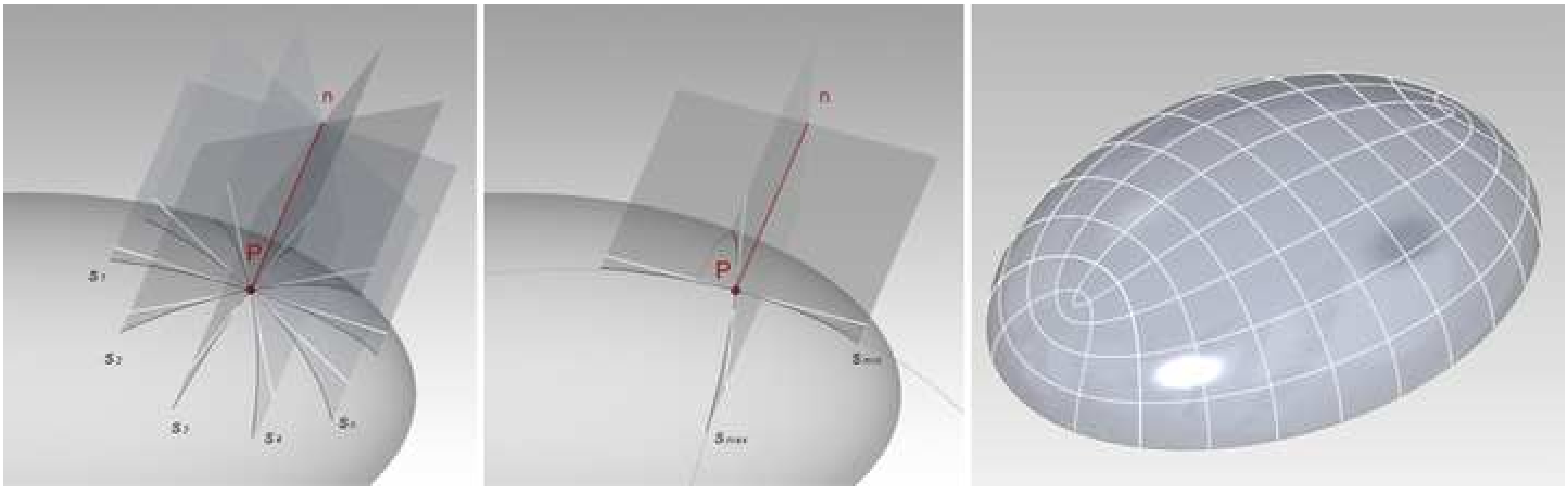
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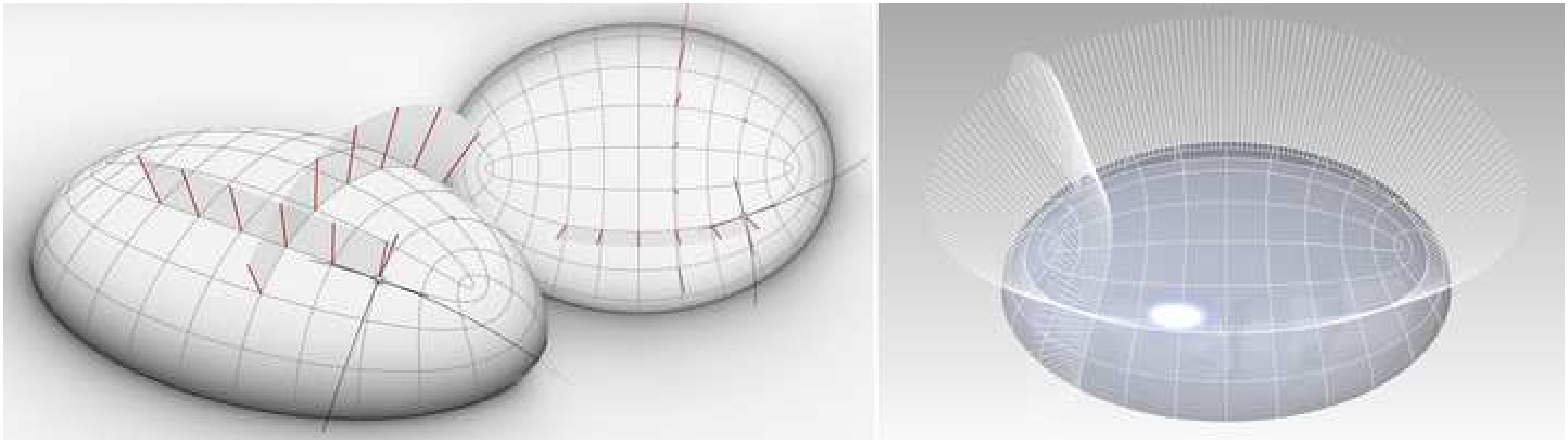

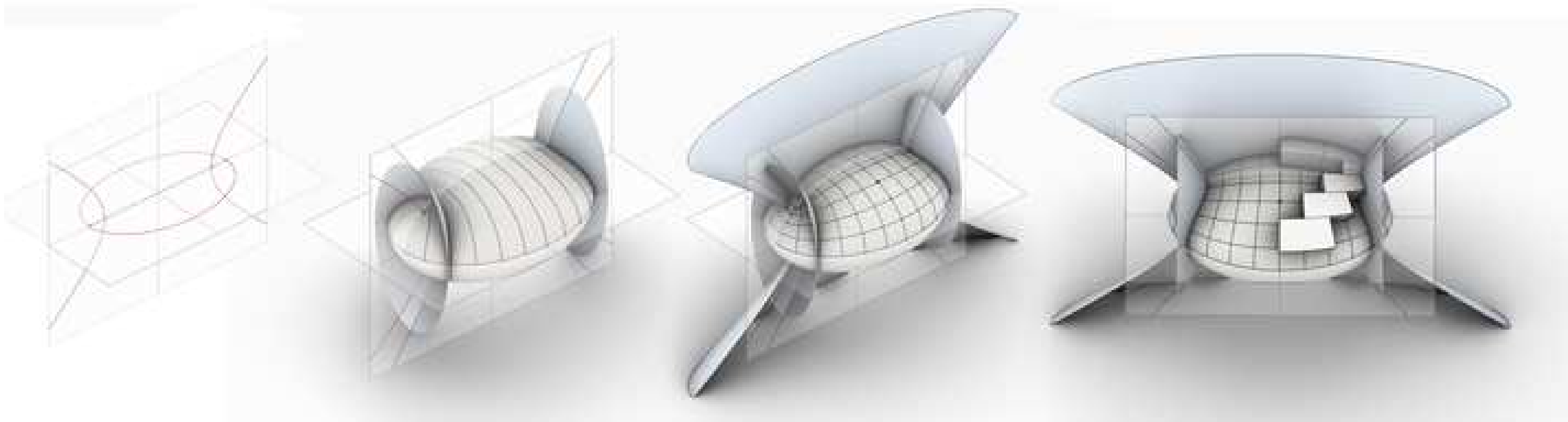

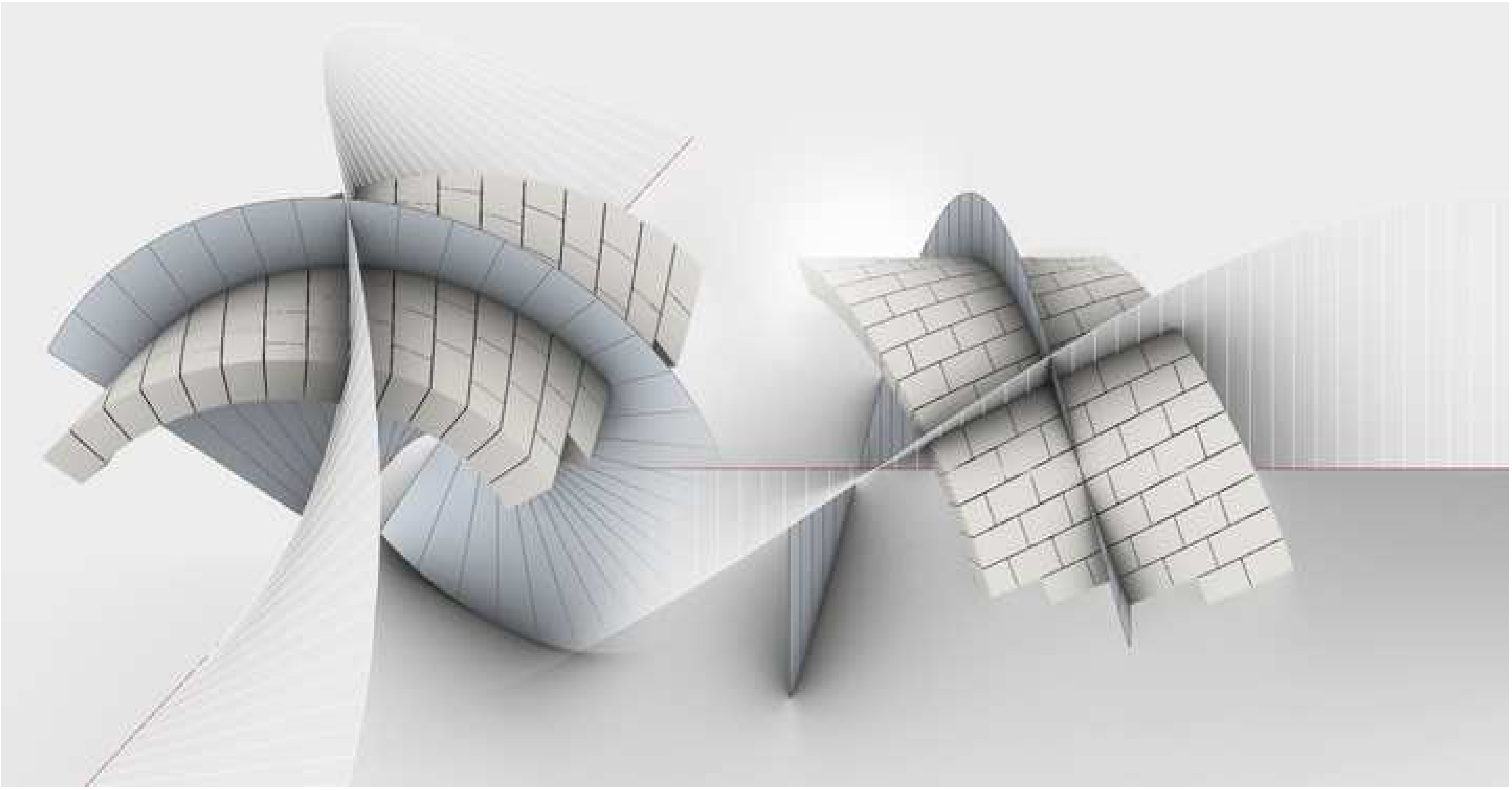

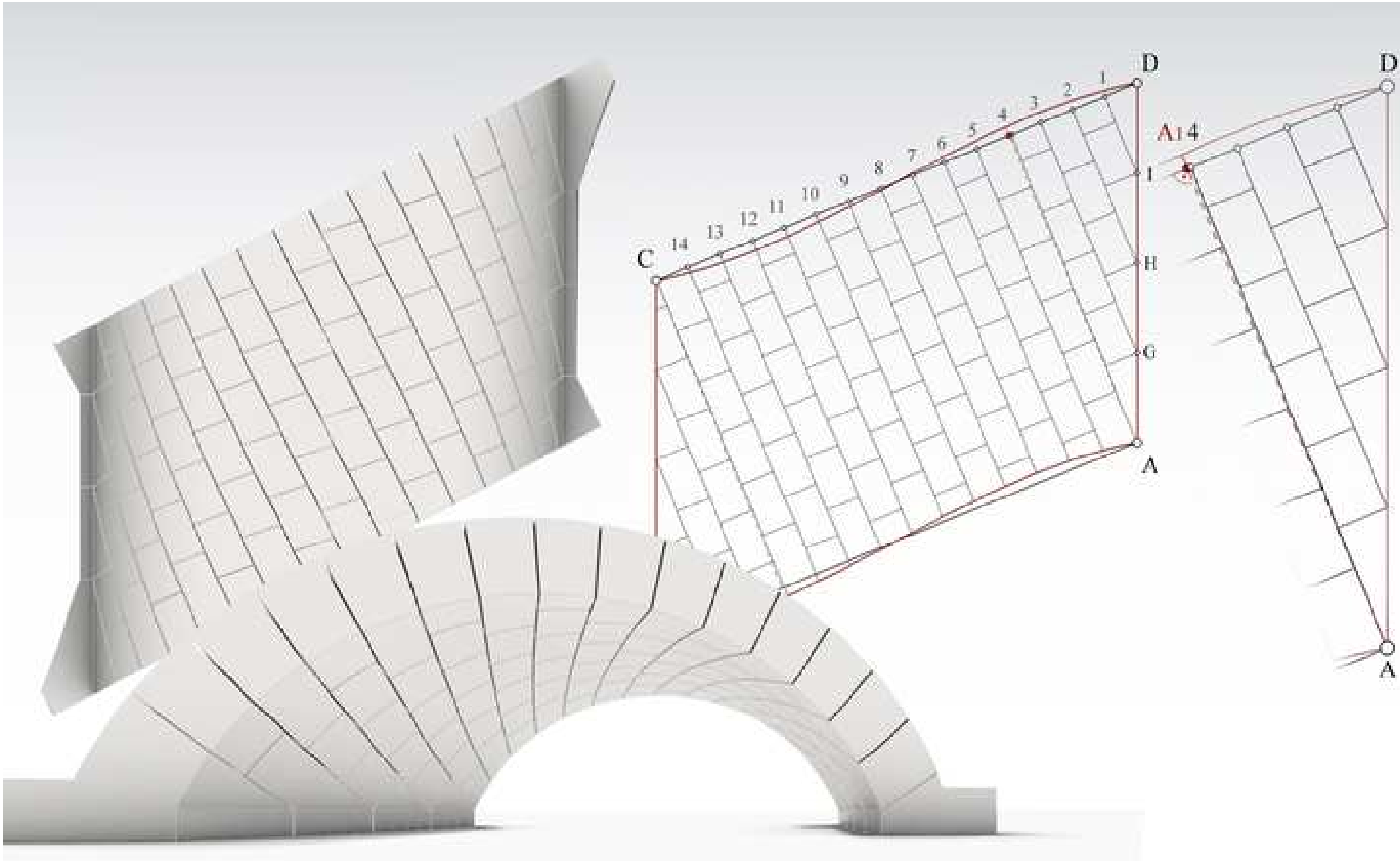
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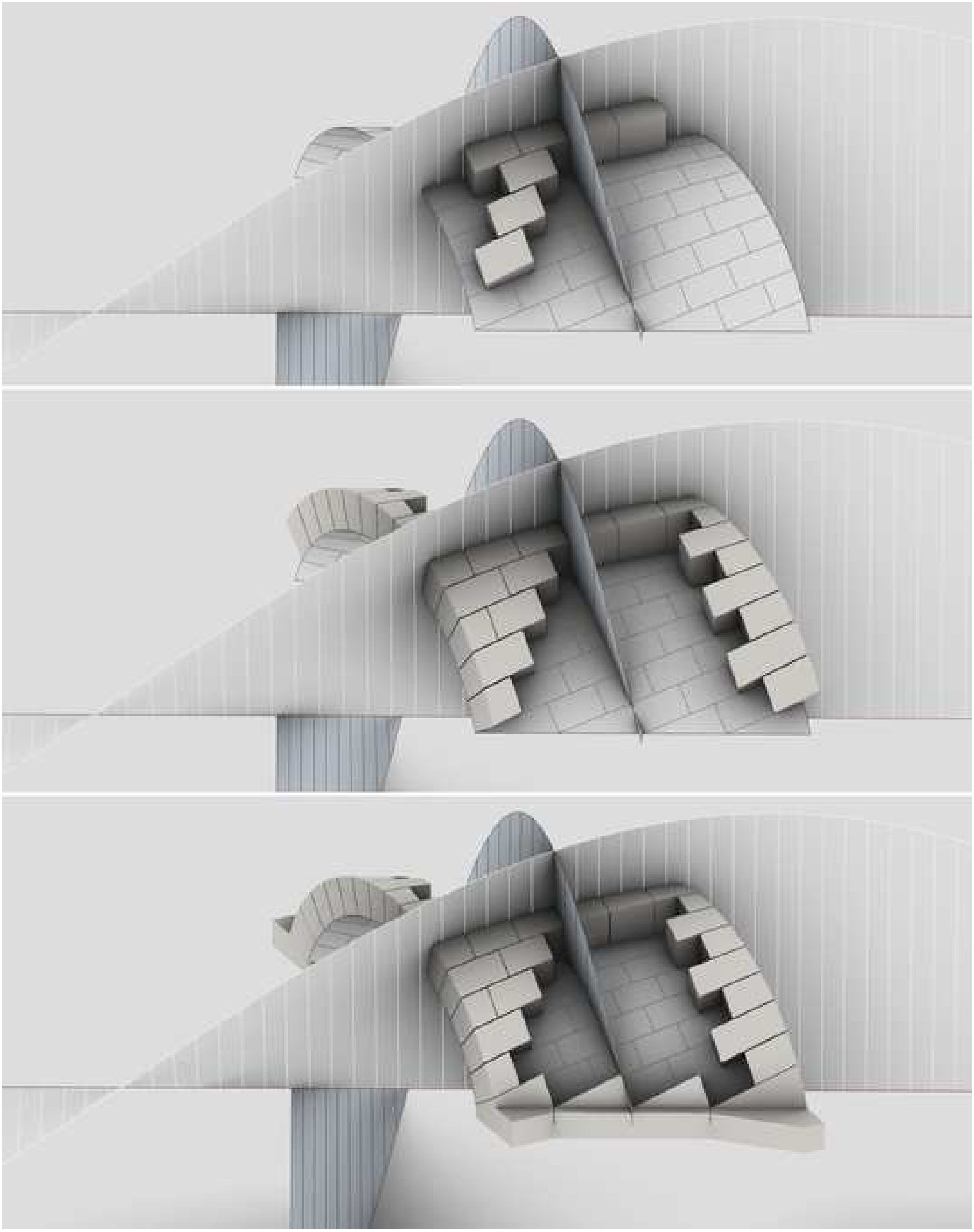

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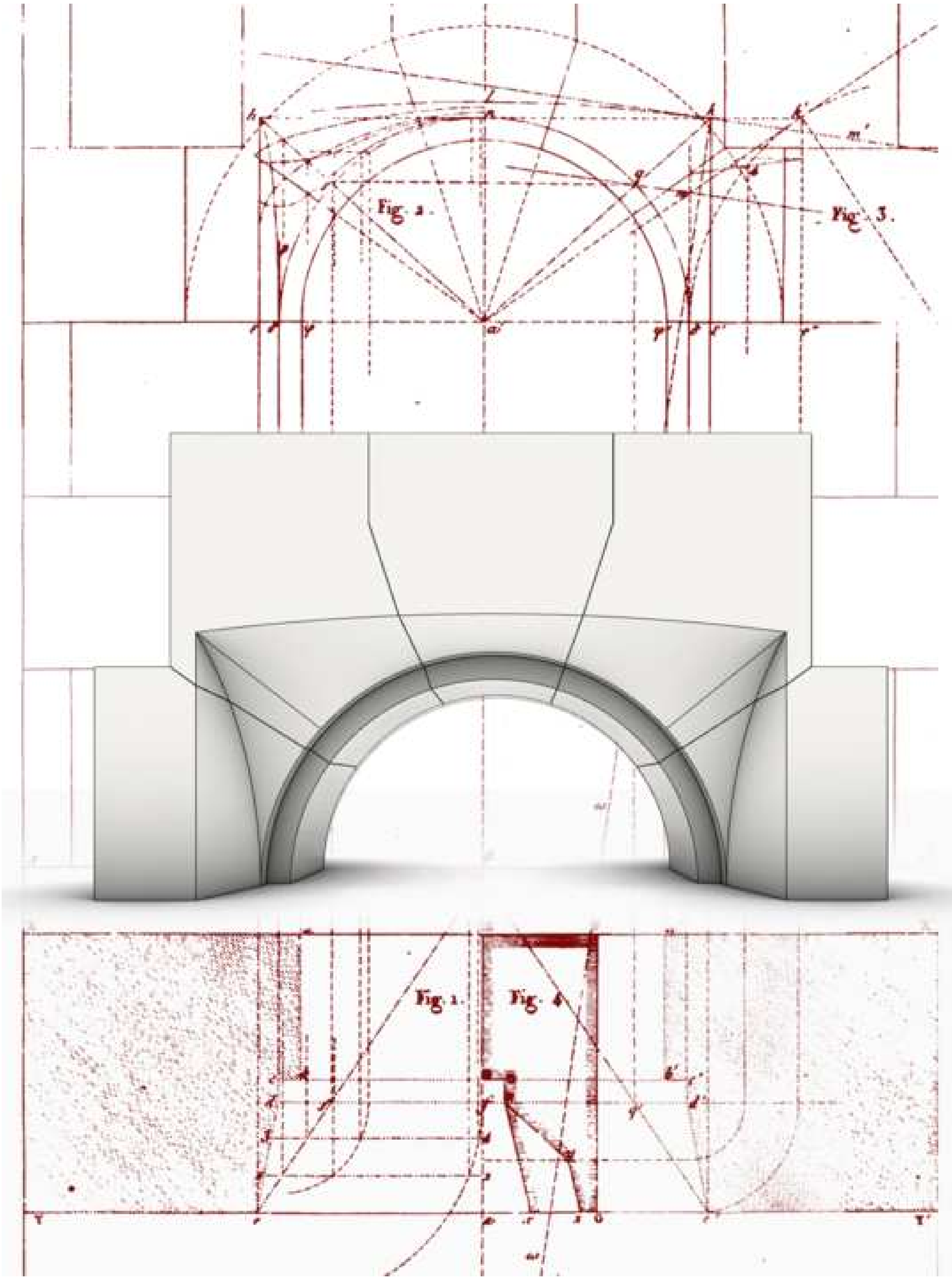

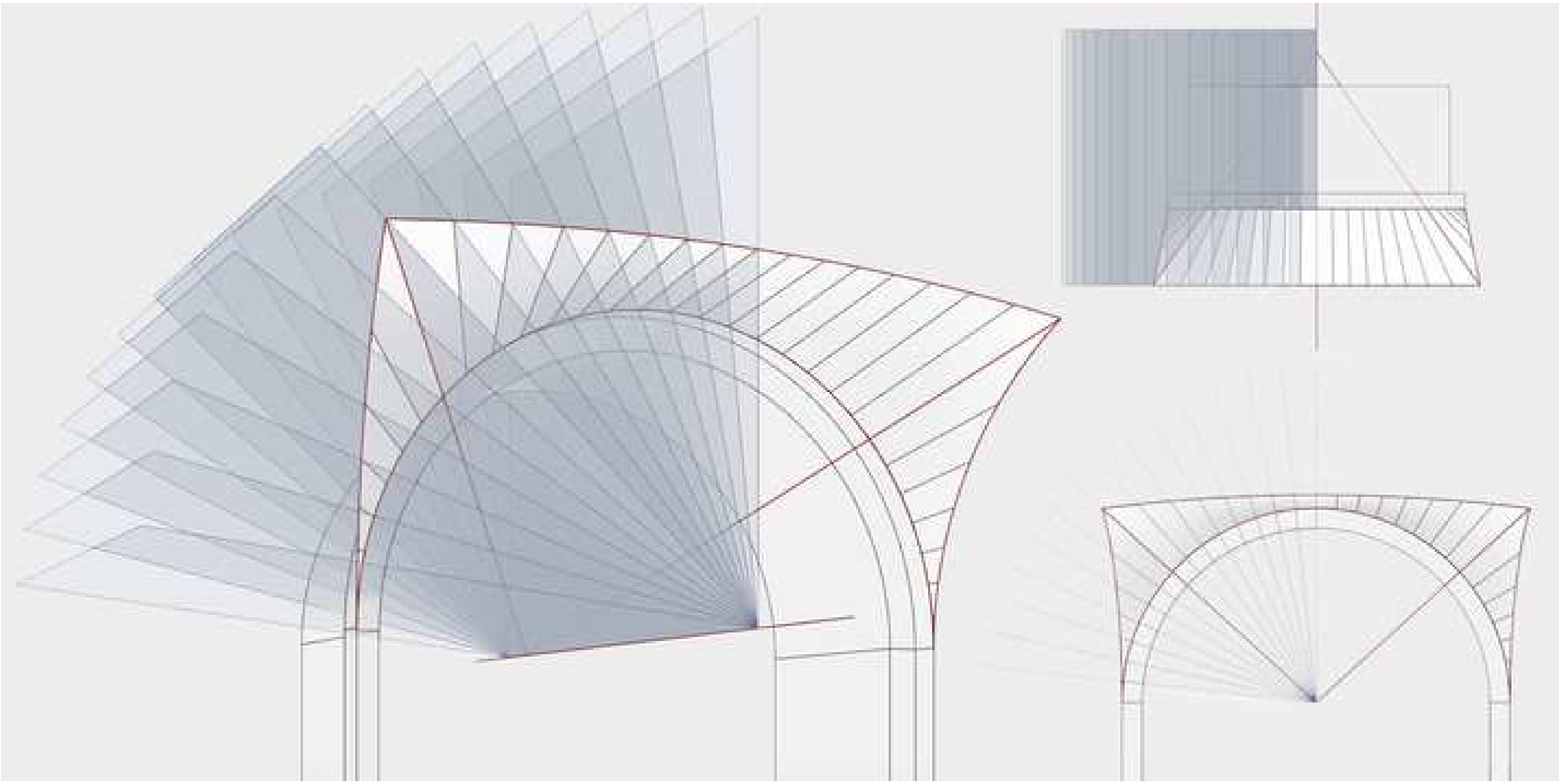

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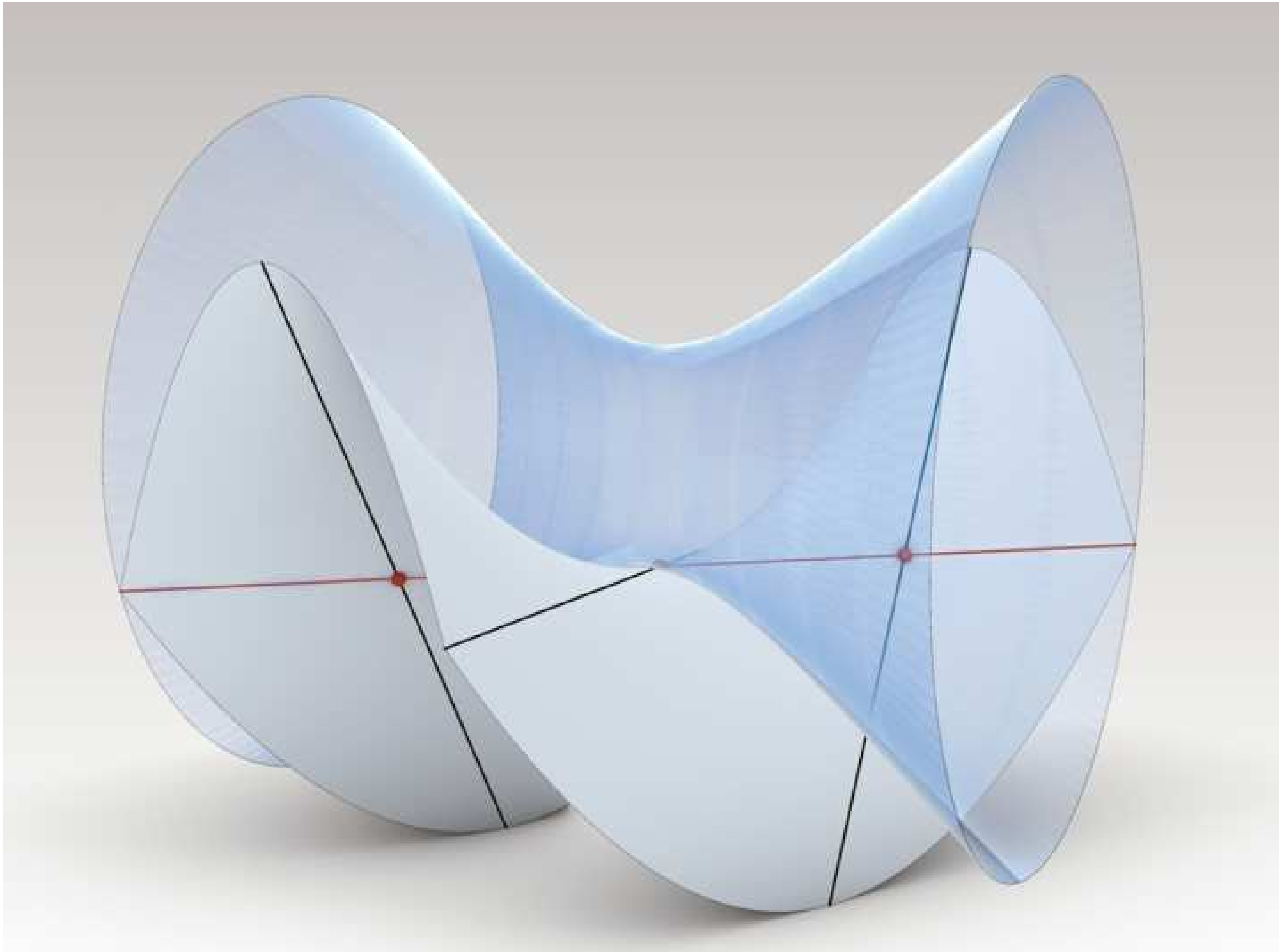

Fig 3 new

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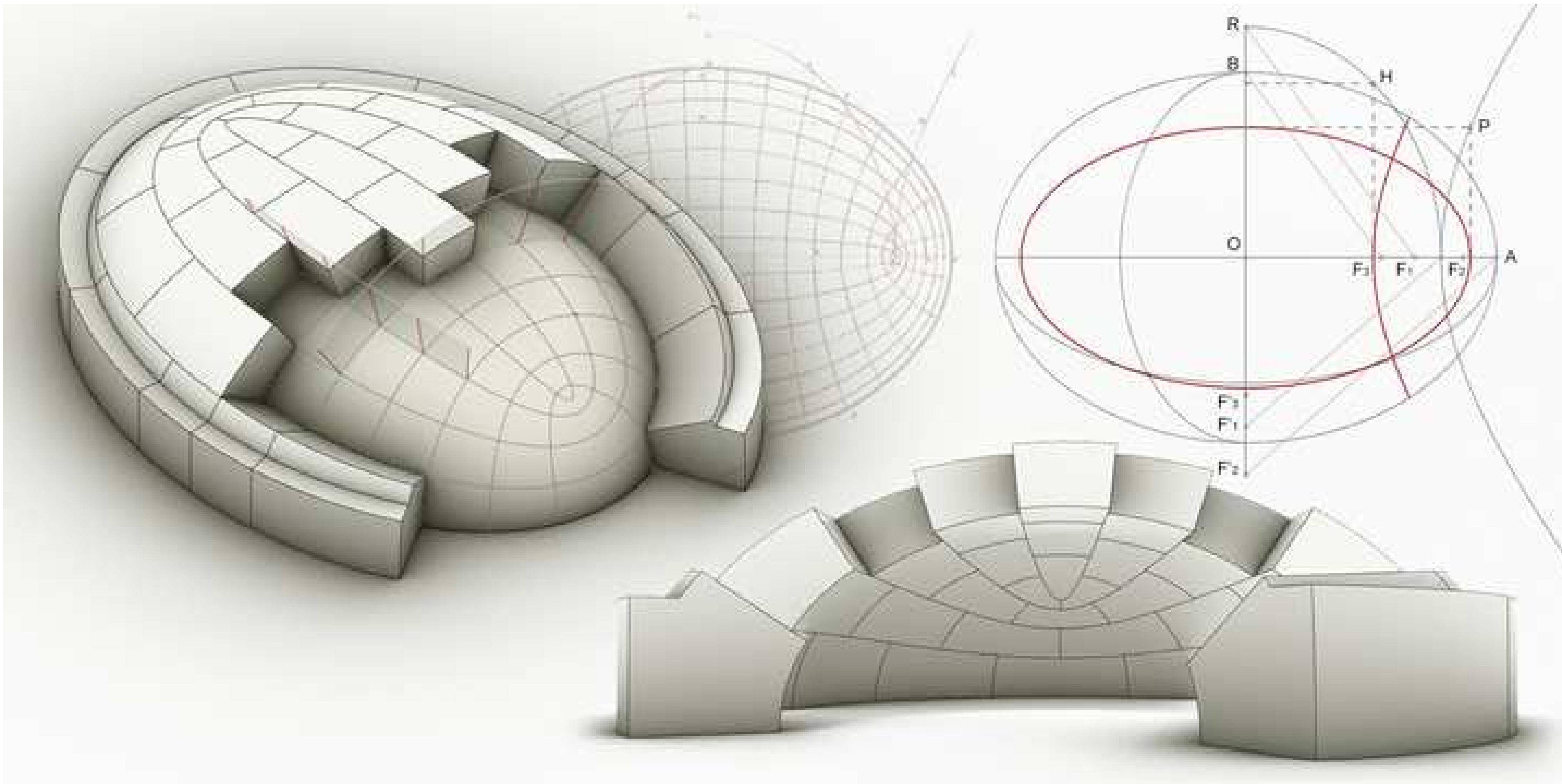

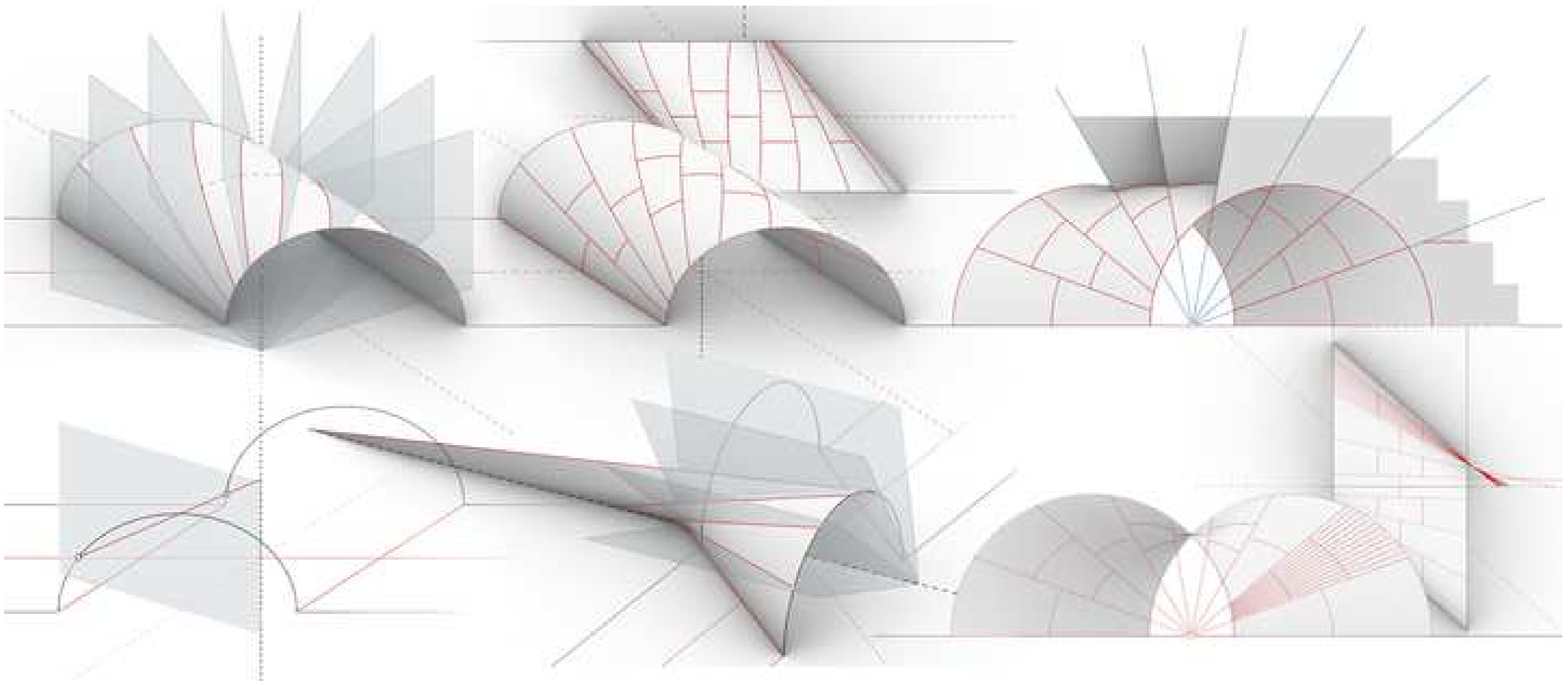

Captions

Fig 1) The principal lines of curvature of an ellipsoid

Fig 2) The developable surfaces normal to the ellipsoid surface

Fig 3) Monge's graphical construction of the ellipsoidal apparatus

Fig 4) The confocal quadric surfaces

Fig 5) Models of the apparatuses for small crossing from Adhemar's treatise

Fig 6) The helicoidal surfaces of junction

Fig 7) The helicoidal apparatus and its net

Fig 8) The three different ashlars

Fig 9) The construction of the *Coupe de Arriere Voussure de Marseille*

Fig 10) The model of the *Coupe de Arriere Voussure de Marseille* from the treatise of Hachette (1828: A.Pl.3)

Fig 11) Two ruled surfaces having a non-singular generatrix in common meet, in general, in no more than two points of this generatrix. If they touch in three points of the same generatrix, they will be tangent along the entire generatrix. The three tangent planes are identified by the straight line in common and by the three straight lines of the paraboloid