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Guido Gherardi
Eugenio Orlandelli

SUPER-STRICT IMPLICATIONS[1]

Abstract

This paper introduces the logics of super-strict implications, where a super-strict implication is a strengthening of C.I. Lewis' strict implication that avoids not only the paradoxes of material implication but also those of strict implication. The semantics of super-strict implications is obtained by strengthening the (normal) relational semantics for strict implication. We consider all logics of super-strict implications that are based on relational frames for modal logics in the modal cube. it is shown that all logics of super-strict implications are connexive logics in that they validate Aristotle's Theses and (weak) Boethius's Theses. A proof-theoretic characterisation of logics of super-strict implications is given by means of G3-style labelled calculi, and it is proved that the structural rules of inference are admissible in these calculi. It is also shown that validity in the $\mathsf{S5}$-based logic of super-strict implications is equivalent to validity in G. Priest's negation-as-cancellation-based logic. Hence, we also give a cut-free calculus for Priest's logic.

Keywords: Strict implication, paradoxes of implication, connexive implication, sequent calculi, structural rules.

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1. Introduction

Given its centrality in deductive reasoning, "implication seems to be the most important connective" [18, p. 167]. Nevertheless there are different competing formal explications of implication. Just to mention some cases, we have *material implication* ($\supset$); its modalised versions such as *strict implication* ($\multimap$) and *variably strict implication* ($\Box\!\!\rightarrow$); *relevant implications*; and *connexive implications*. This paper proposes a new pair of modalised versions of material implication which avoid the paradoxes of strict implication and generate a family of connexive logics with a simple relational semantics.

Strict implication has been introduced by C.I. Lewis [6] to overcome the paradoxes of material implication:

$$\neg A \supset (A \supset B) \tag{MI1}$$

$$B \supset (A \supset B) \tag{MI2}$$

But, as connexivists [13] and relevantists [1] argued, strict implication is plagued by its own paradoxes:

$$\bot \multimap B \qquad [\text{and} \qquad \Box\neg A \supset (A \multimap B)] \tag{SI1}$$

$$A \multimap \top \qquad [\text{and} \qquad \Box B \supset (A \multimap B)] \tag{SI2}$$

We agree with connexivists and relevantists in taking the paradoxes of strict implication to be as questionable as the paradoxes of material implication. If, following C.I. Lewis, it is not maintained that 'A implies B' follows from the hypothesis that A is false *simpliciter* (MI1) or from the hypothesis that B is true *simpliciter* (MI2), then it is unnatural to maintain that it follows from the hypothesis that A is inconsistent/impossible (SI1) or from the hypothesis that B is a logical/necessary truth (SI2).

This is not to say that we have quarrels with material implication, as is witnessed by the fact that we will consider logics containing material implication: we maintain that there are uses of 'A implies B' for which material implication is not appropriate, and that for these uses Lewis' strict implication fares no better than material implication. If 'A implies B' means that the truth of B *depends*, in one way or another, on that of

A, in all the aforementioned cases we should accept that 'A does not imply B' – i.e., among the validities governing implication we do not find (SI1) and (SI2) but, instead, their negations. We find no reasonable sense of 'depending' under which the truth of some B might depend on the truth of a sentence A that cannot be true (apart, possibly, the degenerate case in which B itself cannot be true that will be disregarded in this paper). Analogously, the truth of some B that cannot be false does not seem to depend on the truth of any other sentence.[2]

Another way of making the same point goes as follows. If 'A implies B' means that establishing the truth of A is a good way of establishing the truth of B – i.e., if the key role of an implication is its use in *Modus Ponens* – then we immediately see that '$\bot$ implies B' is useless for inquiring into the truth of B. Analogously, 'A implies $\top$' is a roundabout way for reaching the truth of $\top$ (from this perspective (SI1) is way more problematic than (SI2)).

This paper studies two strengthenings of $⥽$ – called *super-strict implications* (SSI) – that are designed to validate 'A does not imply B' when A is logically impossible or B is logically necessary. *Weak SSI* ($\rhd$) does not validate (SI1) nor (SI2), but it validates the negation of (SI1). *Strong SSI* ($\blacktriangleright$) does not validate (SI1) nor (SI2), but it validates their negations.

Semantically, SSI are obtained by modifying the truth condition for $⥽$ in such a way that (SI1) and (SI2) no longer hold. As is well known, C.I. Lewis' strict implication can be defined in terms of the modal operator $\Box$ as:

$$A ⥽ B \quad \equiv \quad \Box(A \supset B)$$

If we express this in terms of the relational semantics for normal modalities (for C.I. Lewis' stronger calculi $\mathsf{S4}$ and $\mathsf{S5}$), we obtain the following semantic clause:

> $A ⥽ B$ holds at a world w if and only if B holds at every w-accessible world satisfying A.

Needless to say, if A is semantically equivalent to $\bot$ then this semantic clause has an unsatisfiable antecedent and, hence, (SI1) is valid. Analo-

[2] Observe that intuitionistic implication ($\to$) with its BHK-interpretation – "a proof of $A \to B$ is a construction which permits us to transform any proof of A into a proof of B" [21, p. 9] – might seem to express an appropriate notion of dependency, but it doesn't fare better than material and strict implcations since it allows for the dummy transformation when $A \equiv \bot$ or $B \equiv \top$.

gously, if B is equivalent to $\top$ then the consequent of this semantic clause is always satisfied and (SI2) is valid.

Weak SSI avoids the first paradox since

> $A \rhd B$ holds at a world w whenever not only B holds at every w-accessible world satisfying A but also A holds at some w-accessible world.

Hence, $\rhd$ does not validate the paradoxes of strict implication: it validates the negation of (SI1). Strong SSI does better in this respect in that it validates also the negation of the second paradox (SI2):

> $A \blacktriangleright B$ holds at a world w iff the two clauses for $\rhd$ are satisfied and, moreover, B does not hold at some w-accessible world.

Having sketched the semantical interpretation of the notions of truth-value dependency and SSI that we subscribe, the next step will be that of introducing the class of logics that we are going to consider. Logics of SSI will be defined semantically as sets of formulas that are valid in some class of relational frames. In particular, we consider the logics determined by the classes of frames defined *via* the following well-known properties of the accessibility relation: seriality, reflexivity, transitivity, symmetry, and Euclideaness (see Tables 1 and 2). Proof-theoretically, we will characterise each logic by means of a G3-style labelled calculus [12, Chapter 11] where all structural rules will be shown to be admissible (both syntactically and semantically).

By means of these calculi it will be shown that all logics of SSI are connexive logics – see [14] for a recent introduction to connexive logics – in that they include both *Aristotle's Theses* ($\multimap$ stands for either $\rhd$ or $\blacktriangleright$):

$$\neg(A \multimap \neg A) \qquad\qquad \neg(\neg A \multimap A) \tag{AT1 \& AT2}$$

and *(weak) Boethius' Theses*:

$$(A \multimap B) \supset \neg(A \multimap \neg B) \qquad (A \multimap \neg B) \supset \neg(A \multimap B) \tag{BT1 \& BT2}$$

Observe that the validity of Aristotle's and Boethius' Theses is a natural outcome for an implication that expresses a notion of 'truth-value dependency'. Even if we have not tried to give a precise meaning to this notion of truth-value dependency, one minimal property that we can ascribe is the impossibility for both a proposition and its negation to depend on the same proposition. Accordingly, SSI validate Boethius' Theses. Another property that we can ascribe to this notion of truth-value dependency is that no proposition depends on its own negation. Whence, SSI validate Aristotle's Theses. SSI are connexive implications (though they do not validate the *strong Boethius' Theses* that are obtained from the weak ones by replacing $\supset$ with $\multimap$). To sum up, SSI are simple modality-based non-paradoxical and connexive implications where the connexion between antecedent and consequent of a true implication is some kind of 'truth-value dependency'.

Outline Section 2 discusses some related works and gives some motivations for SSI. Section 3 introduces the syntax and semantics of logics of SSI. Section 4 introduces labelled calculi for them and Section 5 proves the admissibility of the structural rules of inference. In Section 6 it is proved that these calculi are sound and complete with respect to their intended semantics. In Section 7 the semantics of the S5-based logic of SSI is shown to be equivalent to the semantics for a connexive logic considered by G. Priest [16]. Section 8 sketches some future works.

2. Super-strict implications and related works

SSI are interesting for at least two different but related reasons. First, they explicate the idea of implication as expressing some kind of truth-value dependency: 'A implies B' is taken to mean that establishing A is a way to establish B. Under this reading of implication the paradoxes of both material and strict implication do not hold. SSI validate instead the negation of the first paradox of strict implication (SI1); strong SSI validates also the negation of the second paradox (SI2). The second reason for studying SSI is that they generate a family of connexive logics having a very simple relational semantics. SSI show that we can achieve connexivity by adding to the truth conditions for strict implication the condition that the antecedent of a true implication has to be possible. Moreover, this additional condition is not just a formal machinery to achieve connexivity: it can be philosophically justified by taking implication to express some kind of truth-value dependency. In this section we compare SSI with some related approaches in order to highlight the main novelties of SSI.

The idea of strengthening C.I. Lewis' $⥽$ to obtain a more apt formal explication of implication is not new. The most famous example of this phenomenon is D. Lewis' and Stalnaker's logic of *variably strict implication* $\Box\!\!\rightarrow$ (a.k.a. the logic of counterfactual). We will briefly return to variably strict implication in Section 8; here we just highlight that the goal of proponents of variably strict implication is orthogonal to our own, since $\Box\!\!\rightarrow$ is designed to avoid the transitivity of $\prec$ and not its paradoxes (SI1) and (SI2).

A proposal that looks similar to SSI is made by E. J. Lowe in a series of papers [8, 9] where he proposes to capture indicative conditionals by explicating 'A implies B' in the modal language as $\Box(A \supset B) \wedge (\Diamond A \vee \Box B)$. Lowe explicitly rejects a modal analogous of weak SSI because of the following counterexample: "If n were the greatest natural number, then there would be a natural number greater than n" [9, p. 48]. We are not particularly moved by this counterexample since our intuitions about its truth value differ from Lowe's ones. While Lowe maintains that a conditional with a necessary consequent must be true even if its antecedent is impossible, we believe, to the opposite, that no formula is implied by an impossible formula. This is motivated by our understanding of implication as expressing some kind of truth-value dependency. 'A implies B' means that establishing A allows us to establish B. If A is impossible then it cannot be used to establish the truth of some B, not even when B is necessarily true as in Lowe's example: we might have many different ways to establish B, but A is not among them.[3] Hence, when A is impossible we take 'A implies B' to be (logically) false. Notice also that under Lowe's explication of implication the second paradox of strict implication (SI2) turns out to be valid. Even if Lowe's proposal were able to explicate indicative conditionals, it would not provide a non-paradoxical implication. SSI are instead meant to provide simple non-paradoxical modalised implication that need not be formal explications of indicative conditionals in natural language. For this reason we are not moved by the counterexamples given in [4] either.

Another proposal that is quite similar in spirit to the present one is Hitchcock's [5] *enthymematic consequence*: $A \Vdash B$ iff it is impossible that A is true and B is false, but both A and $\neg B$ are possibly true. Obviously, strong SSI can be seen as an object language representation of Hitchcock's consequence relation; and, analogously, weak SSI can be seen as an ob-

[3] If we consider strong SSI then Lowe's example turn out to be false for the very same reason used by Lowe to argue that it is true.

ject language representation of B. Bolzano's consequence relation since he required the premisses be consistent, see [2].

A connexive implication that is somehow similar to SSI is C. Pizzi's *consequential implication* [15]: A consequentially implies B if and only if $A \dashv\!\!3\, B$ and, moreover, A and B cannot have incompatible modal status. If the underlying modal logic is at least the serial logic KD, consequential implication validates Aristotle's Theses and weak Boethius' Theses. Moreover, connexive consequential implication does not validate strong Boethius' Theses and, as shown in [15], a consequential implication validating Strong Boethius' would be a commutative operator. One of the main differences between Pizzi's consequential implication and SSI is that the former is semantically analysed only via a translation in the modal language and its formal definition makes essential use of modal notions. The definition of SSI, instead, does not depend on modalities and, as will be shown in Section 3, the (normal) modal operators $\Box$ and $\Diamond$ can be defined in terms of weak SSI in a non-circular way. Another striking difference – which can be taken to show that SSI generate a better behaved family of connexive logics than consequential implication – is that SSI validate Aristotle's and Boethius' Theses even on structures falsifying the seriality axiom $D := \Box A \supset \Diamond A$ (see Example 4.1), whereas consequential implication is connexive only in the presence of the seriality axiom. All in all, SSI seem better behaved than consequential implication for introducing connexive implications with a simple relational semantics: modal notions should not be involved in the definition of a connexive implication (but they might be defined in terms of connexive implication) and, moreover, it might be interesting to consider modality-based connexive logics where the seriality axiom fails (even if seriality expresses the connexive idea that A and $\neg A$ express incompatible propositions).

Finally, another formal semantics for a connexive logic that is very similar to the one for SSI is the one adopted by G. Priest [16] to model the cancellation account of negation [19]. Section 7 will compare our proposal with Priest's one. Here we just anticipate that Priest's motivation is completely independent from ours. Priest's aim is that of making precise the cancellation account of negation and showing how this "gives rise to a semantics for a simple connexivist logic" [16, p. 141]. As we will see in Proposition 7.1, validity (but not consequence) in Priest's connexive logic can be seen as a particular case of our more general approach obtained by considering the S5-based logic of SSI. Hence, in providing proof systems for logics of SSI, we will provide also a proof system for validities in Priest's cancellation account of negation. As we will argue in Section 7, the cancellation account of negation cannot be formalised suitably by the given formal semantics, which is instead appropriate to capture the idea of implication as expressing some kind of truth-value dependency.

3. Syntax and semantics

The language ($\mathcal{L}$) of SSI is determined by the following grammar, where p ranges over a denumerable set of propositional variables $\mathcal{P}$:

$$A \ ::= \ p \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid A \rhd A \mid A \blacktriangleright A \qquad (\mathcal{L})$$

Parentheses follow the usual conventions (where SSI bind lighter than all other operators). We will use p, q, r as metavariables for propositional variables and A, B, C for arbitrary formulas. $\top$ is short for $\bot \supset \bot$ and $\neg A$ is short for $A \supset \bot$. Moreover, $A \multimap B$ will be short for both $A \rhd B$ and $A \blacktriangleright B$. Given a formula A, its *weight* $\mathtt{w}(A)$ is thus defined: $\mathtt{w}(p) = \mathtt{w}(\bot) = 0$ and $\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1$ for $\circ \in \{\wedge, \vee, \supset, \rhd, \blacktriangleright\}$.

As in relational semantics for modal logics, a *frame* $\mathcal{F}$ is a pair $\langle W, R\rangle$ composed of a non-empty set of worlds and an accessibility relation; a *model* $\mathcal{M}$ is a triple $\langle W, R, V\rangle$ where V is a valuation function mapping each propositional variable in $\mathcal{P}$ to a subset of W. If $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{F} = \langle W, R\rangle$, we say that the model $\mathcal{M}$ is *based* on the frame $\mathcal{F}$ (*$\mathcal{F}$-model*, for short).

The truth of a formula A at a world w of a model $\mathcal{M}$ (in symbols $\models^{\mathcal{M}}_{w} A$) is defined as usual for the classical operators and for the SSI is defined as follows:

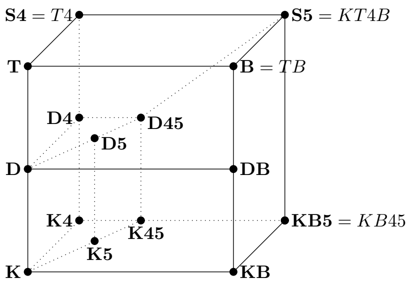
$$\begin{array}{lll} \models^{\mathcal{M}}_{w} A \rhd B & \text{iff} & \text{for all } v \in W\text{, if } wRv \text{ and } \models^{\mathcal{M}}_{v} A\text{, then } \models^{\mathcal{M}}_{v} B \quad \& \\ & & \text{some } v \in W \text{ is such that } wRv \text{ and } \models^{\mathcal{M}}_{v} A \\ \\ \models^{\mathcal{M}}_{w} A \blacktriangleright B & \text{iff} & \text{for all } v \in W\text{, if } wRv \text{ and } \models^{\mathcal{M}}_{v} A\text{, then } \models^{\mathcal{M}}_{v} B \quad \& \\ & & \text{some } v \in W \text{ is such that } wRv \text{ and } \models^{\mathcal{M}}_{v} A \quad \& \\ & & \text{some } v \in W \text{ is such that } wRv \text{ and } \not\models^{\mathcal{M}}_{v} B \end{array}$$

Truth in a model ($\models^{\mathcal{M}} A$) and *validity* in a frame ($\mathcal{F} \models A$) or in a class $\mathcal{C}$ of frames ($\mathcal{C} \models A$) are defined in the usual way. A formula is *valid* if it

Table 1. Modal correspondence results.

Name	Property	Modal axiom
(D)	*seriality* $\forall w \exists v(wRv)$	$\Box A \supset \Diamond A$
(T)	*reflexivity* $\forall w(wRw)$	$\Box A \supset A$
(4)	*transitivity* $\forall w, v, u(wRv \wedge vRu \supset wRu)$	$\Box A \supset \Box\Box A$
(B)	*symmetry* $\forall w, v(wRv \supset vRw)$	$A \supset \Box\Diamond A$
(5)	*Euclideaness* $\forall w, v, u(wRv \wedge wRu \supset vRu)$	$\Diamond A \supset \Box\Diamond A$

Table 2. Cube of normal modalities.



is valid in the class of all frames. Given a class $\mathcal{C}$ of frames (a $\mathcal{C}$-model is a model based on a frame in $\mathcal{C}$ and) its *(local) consequence relation* is thus defined:

$$\Gamma \models^{\mathcal{C}} A \quad \text{iff} \quad \text{for each world } w \text{ of a } \mathcal{C}\text{-model, if all formulas in } \Gamma \text{ are true at } w \text{ then } A \text{ is true at } w$$

A *logic* of SSI is defined as the set of $\mathcal{L}$-formulas that are valid in a class of frames. In particular, we consider the logics determined by the classes of frames defined *via* the well-known properties of the accessibility relation given in Table 1. We follow the standard naming conventions for $\mathcal{L}^{\Box}$-logics, see Table 2.

Now we present some important properties of logics of SSI.

PROPOSITION 3.1.

1. None of the paradoxes of material implication – i.e., (MI1) and (MI2) – is valid for $\multimap$.
2. None of the paradoxes of strict implication is valid for $\multimap$.
3. The negation of (SI1) is valid for $\multimap$; the negation of (SI2) is valid for $\blacktriangleright$.
4. The connexive principles (AT1 & AT2) and (BT1 & BT2) are valid for SSI.

5. Strong Boethius' thesis: $(A \multimap B) \multimap \neg(A \multimap \neg B)$ is not valid.

6. SSI are not reflexive: in general $A \multimap A$ is not valid.

7. Contraposition is valid for $\blacktriangleright$ ($A \blacktriangleright B \models \neg B \blacktriangleright \neg A$) but not for $\rhd$ ($A \rhd B \not\models \neg B \rhd \neg A$).

PROOF: The proof of items 1, 2, 5, 6, and 7 is left to the reader. Example 4.1 gives a syntactic proof of some of the claims made in items 3 and 4. $\square$

We finish this section by presenting the relationship between SSI and modal operators. It is immediate to notice that $\Box$ and $\Diamond$ can be defined in terms of weak SSI as follows:

$$\Diamond A \equiv A \rhd \top \qquad \Box A \equiv \neg(\neg A \rhd \top) \tag{Def. $\Box/\Diamond$}$$

Moreover, since $A \prec B \equiv \Box(A \supset B)$, we can express strict implication in terms of weak SSI as follows:

$$A \prec B \equiv \neg((A \wedge \neg B) \rhd \top) \tag{Def. $\prec$}$$

Finally, we can define strong SSI in terms of the weak one as follows:

$$A \blacktriangleright B \equiv ((A \rhd B) \wedge (\neg B \rhd \top)) \tag{Def. $\blacktriangleright$}$$

Neither the unary modalities, nor strict/weak SSI can be expressed in terms of strong SSI. This can be seen as a reason to prefer $\rhd$ over $\blacktriangleright$ despite the fact that $\rhd$ does not validate the negation of the paradox (SI2). Observe also that over serial frames the translation of $\Box A$ can be simplified into $\top \rhd A$.

It is also possible to express SSI in terms of the standard modal language $\mathcal{L}^{\Box}$: $A \rhd B$ is expressed by $\Diamond A \wedge \Box(A \supset B)$ and $A \blacktriangleright B$ by $\Diamond A \wedge \Box(A \supset B) \wedge \Diamond\neg B$. Thus, the logics of SSI are embeddable in the corresponding modal logics. Formally, we have the following translation $\mathtt{t}$ mapping $\mathcal{L}$-formulas into $\mathcal{L}^{\Box}$-formulas.

DEFINITION 3.2. Let A be an $\mathcal{L}$-formula, the $\mathcal{L}^{\Box}$-formula $\mathtt{t}(A)$ is thus defined:

$$\begin{array}{lll} \mathtt{t}(A) \equiv A & \text{iff} & A \text{ is a propositional variable or } \bot \\ \mathtt{t}(B \circ C) \equiv \mathtt{t}(B) \circ \mathtt{t}(C) & \text{iff} & \circ \in \{\wedge, \vee, \supset\} \\ \mathtt{t}(B \rhd C) \equiv \Diamond \mathtt{t}(B) \wedge \Box(\mathtt{t}(B) \supset \mathtt{t}(C)) & & \\ \mathtt{t}(B \blacktriangleright C) \equiv \Diamond \mathtt{t}(B) \wedge \Box(\mathtt{t}(B) \supset \mathtt{t}(C)) \wedge \Diamond(\neg \mathtt{t}(C)) & & \end{array}$$

It is immediate to see that the following theorem holds.

THEOREM 3.3. *Let $\mathcal{C}$ be a class of frames. The $\mathcal{L}$-formula A is valid in $\mathcal{C}$ if and only if the $\mathcal{L}^{\Box}$-formula $\mathtt{t}(A)$ is valid in $\mathcal{C}$ (w.r.t. $\mathcal{C}$-models for the modal language).*

This entails that all the logics of SSI we consider are decidable. In particular, [3] shows that the satisfiability problem is NP-complete for the modal logics $\mathsf{S5}$ and $\mathsf{KD45}$ and PSPACE-complete for the logics K, D, T, and $\mathsf{S4}$. In translating an $\mathcal{L}$-formula A into the corresponding $\mathcal{L}^{\Box}$-formula $\mathtt{t}(A)$ we have an exponential blowup in the weight of the formula since, e.g., $\mathtt{t}(B)$ occurs twice in $\mathtt{t}(B \rhd C)$. Nevertheless, if $|A|$ stands for the number of subformulas of A, the translation $\mathtt{t}$ has linear complexity. Given that the complexity algorithms in [3] depend on $|A|$ rather than on $\mathtt{w}(A)$, we conclude that the satisfiability problem is PSPACE-complete for the K-, D-, T-, and $\mathsf{S4}$-based logics of SSI and it is NP-complete for the $\mathsf{S5}$- and $\mathsf{KD45}$-based ones.[4]

4. Labelled calculi

We introduce G3-style labelled calculi for logics of SSI. We assume the reader is familiar with labelled calculi for modal logics, see [12, Chapter 11]. First of all, we introduce a set LAB of fresh variables, called *labels*. Labels will be denoted by $w, v, u, \ldots$ and will be used to represent worlds. Then, we extend the set of formulas by adding *relational atoms* of shape $w\mathcal{R}v$ – expressing that v is accessible from w. Moreover, we replace each $\mathcal{L}$-formula A with the *labelled formulas* $w : A$ – expressing that A holds at w. Finally, in analogy with [10, 11], we add *(existential* and *universal) forcing* formulas of shape $\Vdash^{\exists} A$ and $\Vdash^{\forall} A$ ($A \in \mathcal{L}$) – expressing that A holds at some/all worlds accessible from w.[5] The *weight* of a formula E

[4]Thanks are due to an anonymous reviewer for spotting a mistake in our original argument.

[5]As is explained in Fn.6, forcing formulas are needed to obtain complete rules for $\blacktriangleright$.

of the extended language, $\mathtt{w}(E)$, is a pair $\langle n, \ell \rangle$ (ordered lexicographically) where n is the weight of the $\mathcal{L}$-formula used to construct E or 0 if E is a relational atom and ℓ is 1 if E is a forcing formula, else it is 0. This definition is designed to have $\mathtt{w}(w : A_1 \blacktriangleright A_2) > \mathtt{w}(w \Vdash^{\exists/\forall} A_i) > \mathtt{w}(w : A_i)$, see the proof of Lemma 5.1. Given a formula E of this extended language, $E[w/v]$ is the formula obtained by substituting each occurrence of v in E with an occurrence of w. A *labelled sequent* is an expression:

$$\Gamma \Rightarrow \Delta$$

where Γ is a finite multiset composed of labelled formulas, forcing formulas, and relational atoms, and Δ is a finite multiset of labelled and forcing formulas only. Substitution of labels is extended to sequents by applying it componentwise.

The rules of the calculus $\mathsf{G3SS.L}$ for the logic of SSI over frames for the normal modal logic L are given in Table 3. The calculus $\mathsf{G3SS.K}$ contains all the rules in Table 3 but the non-logical ones. If L is an extension of K from Table 2, the calculus $\mathsf{G3SS.L}$ extends $\mathsf{G3SS.K}$ with the non-logical rules expressing the semantic properties of frames for L (a calculus contains the contracted rule instance *Euclid*^{c} iff it contains *Euclid*). To illustrate, the calculus $\mathsf{G3SS.S4}$ contains all rules of Table 3 but the non-logical rules *Ser*, *Sym*, *Euclid*, and *Euclid*^{c}. Observe that in a derivation there can be at most one instance of one of the rules $L \rhd'$ and $L \blacktriangleright'$ (some relational atom will occur in all nodes of the tree above this rule instance); moreover, as will be shown in Corollary 5.4, these rules are eliminable from calculi where rule *Ser* is admissible. We allow ourselves to use the following admissible rules:

$$\frac{\Gamma \Rightarrow \Delta, w : A}{w : \neg A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\neg} \quad \text{and} \quad \frac{w : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : \neg A}\ {\scriptstyle R\neg} \quad \text{and} \quad \frac{}{\Gamma \Rightarrow \Delta, w : \top}\ {\scriptstyle R\top}$$

A $\mathsf{G3SS.L}$-*derivation* of a sequent $\Gamma \Rightarrow \Delta$ is a tree of sequents, whose leaves are initial sequents, whose root is $\Gamma \Rightarrow \Delta$, and which grows according to the rules of $\mathsf{G3SS.L}$. The *height* of a $\mathsf{G3SS.L}$-derivation is the number of nodes of its longest branch. We say that $\Gamma \Rightarrow \Delta$ is $\mathsf{G3SS.L}$-derivable (with height n), and we write $\mathsf{G3SS.L} \vdash^{(n)} \Gamma \Rightarrow \Delta$, if there is a $\mathsf{G3SS.L}$-derivation (of height at most n) of $\Gamma \Rightarrow \Delta$. A rule is said to be *(height-preserving) admissible* in $\mathsf{G3SS.L}$, if, whenever its premisses are $\mathsf{G3SS.L}$-derivable (with

Table 3. Rules of the calculi for logics of SSI.

initial sequents: $w : p, \Gamma \Rightarrow \Delta, w : p$, with p atomic

logical rules:

$$\frac{}{w : \bot, \Gamma \Rightarrow \Delta}\; L\bot$$

$$\frac{w : A, w : B, \Gamma \Rightarrow \Delta}{w : A \wedge B, \Gamma \Rightarrow \Delta}\; L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, w : A \quad \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \wedge B}\; R\wedge$$

$$\frac{w : A, \Gamma \Rightarrow \Delta \quad w : B, \Gamma \Rightarrow \Delta}{w : A \vee B, \Gamma \Rightarrow \Delta}\; L\vee \qquad \frac{\Gamma \Rightarrow \Delta, w : A, w : B}{\Gamma \Rightarrow \Delta, w : A \vee B}\; R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, w : A \quad w : B, \Gamma \Rightarrow \Delta}{w : A \supset B, \Gamma \Rightarrow \Delta}\; L\supset \qquad \frac{w : A, \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \supset B}\; R\supset$$

$$\frac{w\mathcal{R}u, w\mathcal{R}v, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta, v : A \qquad w\mathcal{R}u, w\mathcal{R}v, v : B, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta}\; L\rhd, u \text{ fresh}$$

$$\frac{w\mathcal{R}u, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta}{w : A \rhd B, \Gamma \Rightarrow \Delta}\; L\rhd', u \text{ fresh, no relational atom in } \Gamma$$

$$\frac{w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma \Rightarrow \Delta, u : B \qquad w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, v : A}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B}\; R\rhd, u \text{ fresh}$$

$$\frac{w \Vdash^{\exists} A, w\mathcal{R}v, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta, v : A, w \Vdash^{\forall} B \qquad w\mathcal{R}v, v : B, w \Vdash^{\exists} A, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta, w \Vdash^{\forall} B}{w\mathcal{R}v, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta}\; L\blacktriangleright$$

$$\frac{w \Vdash^{\exists} A, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta, w \Vdash^{\forall} B}{w : A \blacktriangleright B, \Gamma \Rightarrow \Delta}\; L\blacktriangleright', \text{ no relational atom in } \Gamma$$

$$\frac{w\mathcal{R}u, u : A, \Gamma \Rightarrow \Delta, u : B \qquad \Gamma \Rightarrow \Delta, w \Vdash^{\exists} A \qquad w \Vdash^{\forall} B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : A \blacktriangleright B}\; R\blacktriangleright, u \text{ fresh}$$

forcing rules:

$$\frac{w\mathcal{R}u, u : A, \Gamma \Rightarrow \Delta}{w \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\; L{\Vdash^{\exists}}, u \text{ fresh} \qquad \frac{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w \Vdash^{\exists} A, v : A}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w \Vdash^{\exists} A}\; R{\Vdash^{\exists}}$$

$$\frac{v : A, w\mathcal{R}v, w \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\; L{\Vdash^{\forall}} \qquad \frac{w\mathcal{R}u, \Gamma \Rightarrow \Delta, u : A}{\Gamma \Rightarrow \Delta, w \Vdash^{\forall} A}\; L{\Vdash^{\forall}}, u \text{ fresh}$$

non-logical rules:

$$\frac{w\mathcal{R}u, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\; Ser, u \text{ fresh} \qquad \frac{w\mathcal{R}w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\; Ref \qquad \frac{w\mathcal{R}u, w\mathcal{R}v, v\mathcal{R}u, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, v\mathcal{R}u\Gamma \Rightarrow \Delta}\; Trans$$

$$\frac{v\mathcal{R}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, \Gamma \Rightarrow \Delta}\; Sym \qquad \frac{v\mathcal{R}u, w\mathcal{R}v, w\mathcal{R}u, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w\mathcal{R}u, \Gamma \Rightarrow \Delta}\; Euclid \qquad \frac{v\mathcal{R}v, w\mathcal{R}v, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, \Gamma \Rightarrow \Delta}\; Euclid^{c}$$

height at most n), also its conclusion is G3SS.L-derivable (with height at most n). In each rule depicted in Table 3, Γ and Δ are called *contexts*, the formulas occurring in the conclusion are called *principal*, and those occurring in the premisses only are called *active*.

Example 4.1. The following sequents are G3SS.L-derivable:

1. The negation of (SI1): $\Rightarrow w : \neg(\bot \multimap A)$
2. The negation of (SI2): $\Rightarrow w : \neg(A \blacktriangleright \top)$
3. First Aristotle's Thesis (AT1): $\Rightarrow w : \neg(A \multimap \neg A)$
4. Second Aristotle's Thesis (AT2): $\Rightarrow w : \neg(\neg A \multimap A)$
5. First Boethius' Thesis (BT1): $\Rightarrow w : (A \multimap B) \supset \neg(A \multimap \neg B)$
6. Second Boethius' Thesis (BT2): $\Rightarrow w : (A \multimap \neg B) \supset \neg(A \multimap B)$

Proof: For simplicity, we assume $\multimap$ is $\rhd$.

1.

$$\dfrac{\dfrac{\dfrac{}{w\mathcal{R}u, u : \bot, w : \bot \rhd A \Rightarrow}\,L\bot}{w : \bot \rhd A \Rightarrow}\,L\rhd'}{\Rightarrow w : \neg(\bot \rhd A)}\,R\neg$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{w\mathcal{R}u, w \Vdash^{\exists} A, w : A \blacktriangleright \top \Rightarrow u : \top}\,R\top}{w \Vdash^{\exists} A, w : A \blacktriangleright \top \Rightarrow w \Vdash^{\forall} \top}\,R{\Vdash^{\forall}}}{w : A \blacktriangleright \top \Rightarrow}\,L\blacktriangleright'}{\Rightarrow w : \neg(A \blacktriangleright \top)}\,R\neg$$

3.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{w\mathcal{R}v, v : A, w\mathcal{R}u, u : A, w : A \rhd \neg A \Rightarrow u : A}\,L.5.1 \qquad \dfrac{\dfrac{}{w\mathcal{R}v, v : A, w\mathcal{R}u, u : A, w : A \rhd \neg A \Rightarrow u : A}\,L.5.1}{w\mathcal{R}v, v : A, u : \neg A, w\mathcal{R}u, u : A, w : A \rhd \neg A \Rightarrow}\,L\neg}{w\mathcal{R}u, u : A, w : A \rhd \neg A \Rightarrow}\,L\rhd}{w : A \rhd \neg A \Rightarrow}\,L\rhd'}{\Rightarrow w : \neg(A \rhd \neg A)}\,R\neg$$

4. Analogous to the derivation of AT1.

5.

$$\dfrac{\dfrac{\dfrac{}{wRu, u:A, w:A \rhd \neg B, v:A, w:A \rhd B \Rightarrow v:A}\,{\scriptstyle L.5.1} \qquad \dfrac{\dfrac{}{u:A \Rightarrow u:A}\,{\scriptstyle L.5.1} \quad \dfrac{\dfrac{}{u:A \Rightarrow u:A}\,{\scriptstyle L.5.1} \quad \dfrac{\dfrac{}{u:B \Rightarrow u:B}\,{\scriptstyle L.5.1}}{u:B, u:\neg B \Rightarrow}\,{\scriptstyle L\neg}}{u:B, w\mathcal{R}u, u:A, w:A \rhd \neg B \Rightarrow}\,{\scriptstyle L\rhd}}{w\mathcal{R}u, u:A, v:B, w:A \rhd \neg B, v:A, w:A \rhd B \Rightarrow}\,{\scriptstyle L\rhd}}{\dfrac{\dfrac{\dfrac{w\mathcal{R}v, v:A, w:A \rhd \neg B, w:A \rhd B \Rightarrow}{w:A \rhd \neg B, w:A \rhd B \Rightarrow}\,{\scriptstyle L\rhd'}}{w:A \rhd B \Rightarrow w:\neg(A \rhd \neg B)}\,{\scriptstyle R\neg}}{\Rightarrow w:(A \rhd B) \supset \neg(A \rhd \neg B)}\,{\scriptstyle R\supset}}\,{\scriptstyle L\rhd}$$

6. Analogous to the derivation of BT1. □

5. Structural rules of inference

In this section we prove the admissibility of the structural rules of inference: weakening and contraction will be shown to be height-preserving admissible and cut will be shown to be admissible. Moreover, we'll show that all rules are height-preserving invertible. All proofs of this section will be based on (the non-modal fragment of) those in [10, 12] for the labelled calculi for normal and non-normal modal logics. In particular, in the proof of all lemmas/theorems we will not consider the propositional and non-logical cases since their proof can be found in [12] nor the forcing ones whose proof is in [10].

Lemma 5.1. *$w : A, \Gamma \Rightarrow \Delta, w : A$ and $w \Vdash^{\exists/\forall} A, \Gamma \Rightarrow \Delta, w \Vdash^{\exists/\forall} A$ – with A arbitrary $\mathcal{L}$-formula – are derivable in* G3SS.L*.*

Proof: By simultaneous induction on the weight of $w : A$ and $w \Vdash^{\exists/\forall} A$. We consider only the case when $A \equiv B \rhd C$ and, without loss of generality, we assume $\Gamma \equiv w\mathcal{R}v$ (if no relational atom is in Γ, we use $L \rhd'$ instead of $L \rhd$) and $\Delta \equiv \emptyset$. We have the following derivation (omitting all formulas that, bottom-up, become useless):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{u':B \Rightarrow u':B}\,{\scriptstyle IH} \quad \dfrac{}{u':C \Rightarrow u':C}\,{\scriptstyle IH}}{w\mathcal{R}u', u':B, w\mathcal{R}u, u:B, w:B \rhd C \Rightarrow u':C}\,{\scriptstyle L\rhd} \quad \dfrac{}{u:B \Rightarrow u:B}\,{\scriptstyle IH}}{w\mathcal{R}u, u:B, w:B \rhd C \Rightarrow w:B \rhd C, v:B}\,{\scriptstyle R\rhd} \qquad \dfrac{\dfrac{\dfrac{}{u':B \Rightarrow u':B}\,{\scriptstyle IH} \quad \dfrac{}{u':C \Rightarrow u':C}\,{\scriptstyle IH}}{w\mathcal{R}u', u':B, w:B \rhd C \Rightarrow u':C}\,{\scriptstyle L\rhd} \quad \dfrac{}{u:B \Rightarrow u:B}\,{\scriptstyle IH}}{w\mathcal{R}u, v:C, u:B, w:B \rhd C \Rightarrow w:B \rhd C}\,{\scriptstyle R\rhd}}{w\mathcal{R}v, w:B \rhd C \Rightarrow w:B \rhd C}\,{\scriptstyle L\rhd}$$

To prove the lemma for $w : A_1 \blacktriangleright A_2, \Gamma \Rightarrow \Delta, w : A_1 \blacktriangleright A_2$ it is essential that $\mathtt{w}(w : A_1 \blacktriangleright A_2) > \mathtt{w}(\Vdash^{\forall/\exists} A_i)$; to prove it for $w \Vdash^{\forall/\exists} A, \Gamma \Rightarrow \Delta, w \vdash^{\forall/\exists} A$ it is essential that $\mathtt{w}(\Vdash^{\forall/\exists} A) > \mathtt{w}(w : A)$. □

Lemma 5.2. *If* $\mathsf{G3SS.L} \vdash^h \Gamma \Rightarrow \Delta$ *then* $\mathsf{G3SS.L} \vdash^h \Gamma[u_2/u_1] \Rightarrow \Delta[u_2/u_1]$.

Proof: The proof is by induction on height h of the derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. Suppose the last step of $\mathcal{D}$ is by the following instance of $R\triangleright$:

$$\dfrac{w\mathcal{R}v_2, w\mathcal{R}v_1, v_2 : A, \Gamma \Rightarrow \Delta, v_2 : B \qquad w\mathcal{R}v_1, \Gamma \Rightarrow \Delta, w\ A \triangleright B, v_1 : A}{w\mathcal{R}v_1, \Gamma \Rightarrow \Delta, w : A \triangleright B}\ {\scriptstyle R\triangleright,\ v_2 \text{ fresh}}$$

Let us consider a label v_3 that is new to $\mathcal{D}$ and not in $\{u_1, u_2\}$. We transform $\mathcal{D}$ into the following derivation $\mathcal{D}[u_2/u_1]$ having same height as $\mathcal{D}$:

$$\dfrac{\dfrac{\dfrac{w\mathcal{R}v_2, w\mathcal{R}v_1, v_2 : A, \Gamma \Rightarrow \Delta, v_2 : B}{w\mathcal{R}v_3, w\mathcal{R}v_1, v_3 : A, \Gamma \Rightarrow \Delta, v_3 : B}\ {\scriptstyle IH[v_3/v_2]}}{w[u_2/u_1]\mathcal{R}v_3, v_3 : A, (w\mathcal{R}v_1, \Gamma \Rightarrow \Delta)[u_2/u_1], v_3 : B}\ {\scriptstyle IH[u_2/u_1]} \quad \dfrac{w\mathcal{R}v_1, \Gamma \Rightarrow \Delta, w\ A \triangleright B, v_1 : A}{(w\mathcal{R}v_1, \Gamma \Rightarrow \Delta, wA \triangleright B, v_1 : A)[u_2/u_1]}\ {\scriptstyle IH[u_2/u_1]}}{(w\mathcal{R}v_1, \Gamma \Rightarrow \Delta, w : A \triangleright B)[u_2/u_1]}\ {\scriptstyle R\triangleright}$$

The transformations for rules $L\triangleright^{(\prime)}$, $L\blacktriangleright^{(\prime)}$, and $R\blacktriangleright$ are similar and can thus be omitted. $\square$

Next is height-preserving admissibility of weakening.

Theorem 5.3 (Weakening). *If* $\mathsf{G3SS.L} \vdash^h \Gamma \Rightarrow \Delta$ *then* $\mathsf{G3SS.L} \vdash^h \Pi, \Gamma \Rightarrow \Delta, \Sigma$.

Proof: By induction on the height h of $\mathcal{D}$. If the last step of $\mathcal{D}$ is by a rule *Rule* for weak or strong SSI, then we start by applying Lemma 5.2 to the derivation of its premisses in order to replace its *eigenvariable*, if any, with variables new to Π, Σ and to $\mathcal{D}$. Next, we apply the inductive hypothesis, and an instance of *Rule*. $\square$

The following corollary of Theorem 5.3 shows that rules $L\triangleright'$ and $L\blacktriangleright'$ are needed only for logics of SSI defined by non-serial classes of frames.

Corollary 5.4. Rules $L\triangleright'$ and $L\blacktriangleright'$ are eliminable from $\mathsf{G3SS.L}$ when $\mathsf{L} \supseteq \mathsf{D}$.

Proof: Suppose the last step of $\mathcal{D}$ is by the following instance of $L\triangleright'$:

$$\dfrac{w\mathcal{R}u, u : A, w : A \triangleright B, \Gamma \Rightarrow \Delta}{w : A \triangleright B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\triangleright'}$$

We transform $\mathcal{D}$ into the following $L\triangleright'$-free derivation (v new to $\mathcal{D}$):

$$\dfrac{\dfrac{w\mathcal{R}u, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w\mathcal{R}u, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta, v : A} \, {\scriptstyle Thm.5.3} \quad \dfrac{w\mathcal{R}u, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, v : B, w\mathcal{R}u, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta} \, {\scriptstyle Thm.5.3}}{\dfrac{w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta}{w : A \rhd B, \Gamma \Rightarrow \Delta} \, Ser} \, {\scriptstyle L\rhd}$$

The transformation for rule $L \blacktriangleright'$ is analogous. □

To prove that contraction is height-preserving admissible, we need height-preserving invertibility of each rule – i.e., the derivability (with height n) of a sequent that can be the conclusion of a rule instance entails the derivability (with height n) of the premisses of that rule instance.

Lemma 5.5 (Inversion). *Each rule of* G3SS.L *is height-preserving invertible.*

Proof: The height-preserving invertibility of the rules for SSI with respect to premisses with repetition of all principal formulas follows from Theorem 5.3 (we have cases of 'Kleene-invertibility'). Hence, we have to consider only the invertibility of rule $R \rhd$ with respect to its leftmost premiss, and that of $R \blacktriangleright$ with respect to each one of its premisses. Let's consider inversion or $R \rhd$ w.r.t. its leftmost premiss (the proof is similar for the leftmost premiss of $R \blacktriangleright$). Suppose the conclusion of $\mathcal{D}$ is:

$$w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B$$

If the last step of $\mathcal{D}$ is by an instance of $R \rhd$ there is nothing to prove. Else, the last step of $\mathcal{D}$ is by a rule *Rule* with either one or two premisses $w\mathcal{R}v, \Gamma' \Rightarrow \Delta', w : A \rhd B$ and $w\mathcal{R}v, \Gamma'' \Rightarrow \Delta'', w : A \rhd B$. Let u be some fresh variable, we apply the inductive hypothesis to the derivations of the premisses in order to obtain derivations (having same height) of the sequents $w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma' \Rightarrow \Delta', u : B$ and $w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma'' \Rightarrow \Delta'', u : B$. By applying an instance of *Rule* we obtain a derivation (having same height of $\mathcal{D}$) of:

$$w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma \Rightarrow \Delta, u : B$$

Next, we prove hp-inversion of $R \blacktriangleright$ w.r.t. its second premiss (the proof is similar for the third one). Let's suppose $w : A \blacktriangleright B$ is not principal in the last step of $\mathcal{D}$, which is by *Rule* and has premiss(es) $\Gamma' \Rightarrow \Delta', w : A \blacktriangleright B$ (and $\Gamma'' \Rightarrow \Delta'', w : A \blacktriangleright B$). We apply the inductive hypothesis to the premiss(es) and then an instance of *Rule* to conclude $\Gamma \Rightarrow \Delta, w \Vdash^{\exists} A$. □

Now we can prove height-preserving admissibility of contraction.

Theorem 5.6 (Contraction). *If* $\mathsf{G3SS.L} \vdash^h \Gamma, \Gamma \Rightarrow \Delta, \Delta$ *then* $\mathsf{G3SS.L} \vdash^h \Gamma \Rightarrow \Delta$.

Proof: The proof is by induction on the height h of the derivation $\mathcal{D}$ of the premiss. We assume, w.l.o.g., we are contracting a single formula (occurring in the antecedent or in the consequent). We consider only the cases where the last step of $\mathcal{D}$ is by a rule *Rule* for $\multimap$. If the contraction formula is not principal in *Rule*, we apply the inductive hypothesis to the premiss(es) of *Rule* and then an instance of the same rule. If, instead, the contraction formula is principal in *Rule* then we apply Lemmas 5.5 and 5.2 to its premiss(es) without repetition of the principal formulas; next, we apply the inductive hypothesis to each (modified) premiss and we conclude by an instance of *Rule*. To illustrate, if the last step of $\mathcal{D}$ is:

$$\dfrac{w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma \Rightarrow \Delta, w : A \rhd B, u : B \qquad w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, w : A \rhd B, v : A}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, w : A \rhd B}\ {\scriptstyle R\rhd}$$

we transform $\mathcal{D}$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma \Rightarrow \Delta, w : A \rhd B, u : B}{w\mathcal{R}u_1, w\mathcal{R}u, w\mathcal{R}v, u_1 : A, u : A, \Gamma \Rightarrow \Delta, u_1 : B, u : B}\ {\scriptstyle Lem.5.5}}{w\mathcal{R}u, w\mathcal{R}u, w\mathcal{R}v, u : A, u : A, \Gamma \Rightarrow \Delta, u : B, u : B}\ {\scriptstyle Lem.5.2\,[u/u_1]}}{w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma \Rightarrow \Delta, u : B}\ {\scriptstyle IH} \qquad \dfrac{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, w : A \rhd B, v : A}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, v : A}\ {\scriptstyle IH}}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B}\ {\scriptstyle R\rhd}$$

□

We are now ready to prove cut elimination.

Theorem 5.7 (Cut). *Let* C *be either a labelled or forcing formula. If* $\mathsf{G3SS.L} \vdash \Gamma \Rightarrow \Delta, C$ *and* $\mathsf{G3SS.L} \vdash C, \Pi \Rightarrow \Sigma$ *then* $\mathsf{G3SS.L} \vdash \Gamma, \Pi \Rightarrow \Delta, \Sigma$.

Proof: As usual for G3-style calculi, see [12], the proof considers an uppermost instance of *Cut* and proceeds by lexicographical induction on the pair $\langle$weight of the cut formula, cut-height$\rangle$, where the *cut-height* is the sum of the height of the derivations of the two premisses of cut.

It is useful to organise the proof in three cases: (i) at least one premiss has a derivation of height 0; (ii) the cut formula is not principal in at least one of the premisses; (iii) the cut formula is principal in both premisses. We consider only cases (ii) and (iii) where the last step of the derivation of some premiss is by a rule for $\multimap$.

In case (ii) we can permute the cut upwards in the derivation of a premiss where the cut formula is not principal in the last step (if needed, we use Lemma 5.2 to replace *eigenvariables* with the appropriate labels). Suppose, e.g., the cut formula C is not principal in the last step of the left premiss, which is by an instance of $L\rhd$. We transform:

$$\dfrac{\dfrac{\begin{matrix}\vdots\mathcal{D}_{11}\\ w\mathcal{R}u, w\mathcal{R}v, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta, C, v : A\end{matrix} \quad \begin{matrix}\vdots\mathcal{D}_{12}\\ w\mathcal{R}u, w\mathcal{R}v, v : B, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta, C\end{matrix}}{w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta, C}\,{\scriptstyle L\rhd} \quad \begin{matrix}\vdots\mathcal{D}_{2}\\ C, \Pi \Rightarrow \Sigma\end{matrix}}{w\mathcal{R}v, w : A \rhd B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\,{\scriptstyle Cut}$$

into the following derivation (where u_1 is a fresh label):

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\mathcal{D}_{11}\\ w\mathcal{R}u, w\mathcal{R}v, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta, v : A, C\end{matrix}}{w\mathcal{R}u_1, w\mathcal{R}v, u_1 : A, w : A \rhd B, \Gamma \Rightarrow \Delta, v : A, C}\,{\scriptstyle L.5.2} \quad \begin{matrix}\vdots\mathcal{D}_{2}\\ C, \Pi \Rightarrow \Sigma\end{matrix}}{w\mathcal{R}u_1, w\mathcal{R}v, u_1 : A, w : A \rhd B, \Gamma, \Pi \Rightarrow \Delta, \Sigma, v : A}\,{\scriptstyle Cut} \quad \dfrac{\dfrac{\begin{matrix}\vdots\mathcal{D}_{12}\\ w\mathcal{R}u, w\mathcal{R}v, v : B, u : A, w : A \rhd B, \Gamma \Rightarrow \Delta, C\end{matrix}}{w\mathcal{R}u_1, w\mathcal{R}v, v : B, u_1 : A, w : A \rhd B, \Gamma \Rightarrow \Delta, C}\,{\scriptstyle L.5.2} \quad \begin{matrix}\vdots\mathcal{D}_{2}\\ C, \Pi \Rightarrow \Sigma\end{matrix}}{w\mathcal{R}u_1, w\mathcal{R}v, v : B, u_1 : A, w : A \rhd B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\,{\scriptstyle Cut}}{w\mathcal{R}v, w : A \rhd B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\,{\scriptstyle L\rhd}$$

where we have two instances of *Cut* that are admissible having lesser cut-height.

If we are in Case (iii) and the cut formula is not of shape $w : A \multimap B$, see [10, Theorem 4.9]. If, instead, the cut formula has shape $w : A \rhd B$ and the right premiss is by rule $L\rhd$ we have the following derivation:

$$\dfrac{\dfrac{\vdots\mathcal{D}_{11} \qquad \vdots\mathcal{D}_{12}}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B}\,{\scriptstyle R\rhd} \qquad \dfrac{\vdots\mathcal{D}_{21} \qquad \vdots\mathcal{D}_{22}}{w : A \rhd B, w\mathcal{R}u, \Pi \Rightarrow \Sigma}\,{\scriptstyle L\rhd}}{w\mathcal{R}v, \Gamma, w\mathcal{R}u, \Pi \Rightarrow \Delta, \Sigma}\,{\scriptstyle Cut}$$

Where:

- $\mathcal{D}_{11}$ is: $\begin{matrix}\vdots\\ w\mathcal{R}v_1, v_1 : A, w\mathcal{R}v, \Gamma \Rightarrow \Delta, v_1 : B\end{matrix}$
- $\mathcal{D}_{12}$ is: $\begin{matrix}\vdots\\ w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, v : A\end{matrix}$
- $\mathcal{D}_{21}$ is: $\begin{matrix}\vdots\\ w\mathcal{R}u_1, u_1 : A, w\mathcal{R}u, w : A \rhd B, \Pi \Rightarrow \Sigma, u : A\end{matrix}$
- $\mathcal{D}_{22}$ is: $\begin{matrix}\vdots\\ w\mathcal{R}u_1, u : B, u_1 : A, w\mathcal{R}u, w : A \rhd B, \Pi \Rightarrow \Sigma\end{matrix}$

We transform it into the following derivation containing some instances of *Cut* that are admissible by inductive hypothesis (here and in the following derivations, instances of cut marked with (†) have lesser cut-height, and those marked with (‡) have a cut formula of lower weight; moreover Γ^k stands for k copies of Γ):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_\alpha\\(w\mathcal{R}v)^3,(w\mathcal{R}u)^2,\Gamma^2,\Pi^2\Rightarrow\Delta^2,\Sigma^2,u:A\end{array}\qquad\begin{array}{c}\vdots\,\mathcal{D}_\beta\\u:A,w\mathcal{R}v,w\mathcal{R}u,\Gamma,\Pi\Rightarrow\Delta,\Sigma\end{array}}{(w\mathcal{R}v)^4,(w\mathcal{R}u)^3,\Gamma^3,\Pi^3\Rightarrow\Delta^3,\Sigma^3}\,{\scriptstyle Cut\ (\ddagger)}}{w\mathcal{R}v,w\mathcal{R}u,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\,{\scriptstyle Thm.5.6}$$

Where:

- $\mathcal{D}_\alpha$ is the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{12}\\w\mathcal{R}v,\Gamma\Rightarrow\Delta,v:A,w:A\rhd B\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_{2}\\w:A\rhd B,w\mathcal{R}u,\Pi\Rightarrow\Sigma\end{array}}{w\mathcal{R}u,w\mathcal{R}v,\Gamma,\Pi\Rightarrow\Delta,\Sigma,v:A}\,{\scriptstyle(\dagger)}\qquad\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{1}\\w\mathcal{R}v,\Gamma\Rightarrow\Delta,w:A\rhd B\end{array}\quad\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{21}\\w:A\rhd B,u_1:A,w\mathcal{R}u_1,w\mathcal{R}u,\Pi\Rightarrow\Sigma,u:A\end{array}}{w:A\rhd B,v:A,w\mathcal{R}v,w\mathcal{R}u,\Pi\Rightarrow\Sigma,u:A}\,{\scriptstyle L.5.2}}{v:A,w\mathcal{R}u,(w\mathcal{R}v)^2,\Gamma,\Pi\Rightarrow\Delta,\Sigma,u:A}\,{\scriptstyle(\dagger)}}{(w\mathcal{R}u)^2,(w\mathcal{R}v)^3,\Gamma^2,\Pi^2\Rightarrow\Delta^2,\Sigma^2,u:A}\,{\scriptstyle(\ddagger)}$$

- $\mathcal{D}_\beta$ is the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\w\mathcal{R}v_1,w\mathcal{R}v,v_1:A,\Gamma\Rightarrow\Delta,v_1:B\end{array}}{w\mathcal{R}u,w\mathcal{R}v,u:A,\Gamma\Rightarrow\Delta,u:B}\,{\scriptstyle L.5.2}\qquad\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{1}\\w\mathcal{R}v,\Gamma\Rightarrow\Delta,w:A\rhd B\end{array}\quad\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{22}\\w:A\rhd B,u:B,w\mathcal{R}u_1,w\mathcal{R}u,u_1:A,\Pi\Rightarrow\Sigma\end{array}}{w:A\rhd B,u:B,(w\mathcal{R}u)^2,u:A,\Pi\Rightarrow\Sigma}\,{\scriptstyle L.5.2}}{u:B,w\mathcal{R}v,(w\mathcal{R}u)^2,u:A,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\,{\scriptstyle(\dagger)}}{(u:A)^2,(w\mathcal{R}v)^2,(w\mathcal{R}u)^3,\Gamma^2,\Pi\Rightarrow\Delta^2,\Sigma}\,{\scriptstyle Cut\ (\ddagger)}}{u:A,w\mathcal{R}v,w\mathcal{R}u,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\,{\scriptstyle Thm.5.6}$$

If the cut formula has shape $w : A \rhd B$ and the right premiss is by rule $L\rhd'$, we transform the following derivation (the conclusion of $\mathcal{D}_{11}$ is $w\mathcal{R}v_1, v_1 : A, \Gamma \Rightarrow \Delta, v_1 : B$):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\\mathcal{S}\end{array}\qquad\begin{array}{c}\vdots\,\mathcal{D}_{12}\\w\mathcal{R}v,\Gamma\Rightarrow\Delta,w:A\rhd B,v:A\end{array}}{w\mathcal{R}v,\Gamma\Rightarrow\Delta,w:A\rhd B}\,{\scriptstyle R\rhd}\qquad\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{21}\\w:A\rhd B,u:A,w\mathcal{R}u,\Pi\Rightarrow\Sigma\end{array}}{w:A\rhd B,\Pi\Rightarrow\Sigma}\,{\scriptstyle L\rhd'}}{w\mathcal{R}v,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\,{\scriptstyle Cut}$$

into the following one where we have some admissible cuts:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{12}\\w\mathcal{R}v,\Gamma\Rightarrow\Delta,w:A\rhd B,v:A\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_{2}\\w:A\rhd B,\Pi\Rightarrow\Sigma\end{array}}{w\mathcal{R}v,\Gamma,\Pi\Rightarrow\Delta,\Sigma,v:A}\,{\scriptstyle(\dagger)}\qquad\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{1}\\w\mathcal{R}v,\Gamma\Rightarrow\Delta,w:A\rhd B\end{array}\quad\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{21}\\w:A\rhd B,u:A,w\mathcal{R}u,\Pi\Rightarrow\Sigma\end{array}}{w:A\rhd B,v:A,w\mathcal{R}v,\Pi\Rightarrow\Sigma}\,{\scriptstyle L.5.2}}{v:A,(w\mathcal{R}v)^2,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\,{\scriptstyle(\dagger)}}{(w\mathcal{R}v)^3,\Gamma^2,\Pi^2\Rightarrow\Delta^2,\Sigma^2}\,{\scriptstyle(\ddagger)}}{w\mathcal{R}v,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\,{\scriptstyle Thm.5.6}$$

If the cut formula has shape $w : A \blacktriangleright B$ and the right premiss is by rule $L \blacktriangleright$, we have the following derivation:

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_{11}}{u : A, w\mathcal{R}u, \Gamma \Rightarrow \Delta, u : B} \qquad \overset{\vdots\,\mathcal{D}_{12}}{\Gamma \Rightarrow \Delta, w \Vdash^{\exists} A} \qquad \overset{\vdots\,\mathcal{D}_{13}}{w \Vdash^{\forall} B, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta, w : A \blacktriangleright B}\;{\scriptstyle R\blacktriangleright} \qquad \vdots\,\mathcal{D}_2}{w\mathcal{R}v, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\;{\scriptstyle Cut}$$

Where $\mathcal{D}_2$ is the following derivation:

$$\dfrac{\overset{\vdots\,\mathcal{D}_{21}}{w \Vdash^{\exists} A, w : A \blacktriangleright B, w\mathcal{R}v, \Pi \Rightarrow \Sigma, w \Vdash^{\forall} B, v : A} \qquad \overset{\vdots\,\mathcal{D}_{22}}{v : B, w \Vdash^{\exists} A, w : A \blacktriangleright B, w\mathcal{R}v, \Pi \Rightarrow \Sigma, w \Vdash^{\forall} B}}{w : A \blacktriangleright B, w\mathcal{R}v, \Pi \Rightarrow \Sigma}\;{\scriptstyle L\blacktriangleright}$$

We transform it into the following derivation containing some admissible instances of cut:

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_{\alpha}}{(w\mathcal{R}v)^2, \Gamma^4, \Pi \Rightarrow \Delta^4, \Sigma, v : B} \qquad \overset{\vdots\,\mathcal{D}_{\beta}}{v : B, w\mathcal{R}v, \Gamma^3, \Pi \Rightarrow \Delta^3, \Sigma}}{(w\mathcal{R}v)^3, \Gamma^7, \Pi^2 \Rightarrow \Delta^7, \Sigma^2}\;(\ddagger)}{w\mathcal{R}v, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\;{\scriptstyle Thm.5.6}$$

Where:

- $\mathcal{D}_{\alpha}$ is the following derivation:

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_{12}}{\Gamma \Rightarrow \Delta, w \Vdash^{\exists} A} \qquad \dfrac{\overset{\vdots\,\mathcal{D}_{1}}{\Gamma \Rightarrow \Delta, w : A \blacktriangleright B} \quad \overset{\vdots\,\mathcal{D}_{21}}{w : A \blacktriangleright B, w \Vdash^{\exists} A, w\mathcal{R}v, \Pi \Rightarrow \Sigma, v : A, w \Vdash^{\forall} B}}{w \Vdash^{\exists} A, w\mathcal{R}v, \Gamma, \Pi \Rightarrow \Delta, \Sigma, v : A, w \Vdash^{\forall} B}\;(\dagger)}{w\mathcal{R}v, \Gamma^2, \Pi \Rightarrow \Delta^2, \Sigma, v : A, w \Vdash^{\forall} B}\;(\ddagger) \qquad \overset{\vdots\,\mathcal{D}_{13}}{w \Vdash^{\forall} B, \Gamma \Rightarrow \Delta}}{w\mathcal{R}v, \Gamma^3, \Pi \Rightarrow \Delta^3, \Sigma, v : A}\;(\ddagger) \qquad \dfrac{\overset{\vdots\,\mathcal{D}_{11}}{u : A, w\mathcal{R}u, \Gamma \Rightarrow \Delta, u : B}}{v : A, w\mathcal{R}v, \Gamma \Rightarrow \Delta, v : B}\;{\scriptstyle L.5.2}}{(w\mathcal{R}v)^2, \Gamma^4, \Pi \Rightarrow \Delta^4, \Sigma, v : B}\;(\ddagger)$$

- $\mathcal{D}_{\beta}$ is the following derivation:

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_{12}}{\Gamma \Rightarrow \Delta, w \Vdash^{\exists} A} \qquad \dfrac{\overset{\vdots\,\mathcal{D}_{1}}{\Gamma \Rightarrow \Delta, w : A \blacktriangleright B} \quad \overset{\vdots\,\mathcal{D}_{22}}{w : A \blacktriangleright B, w \Vdash^{\exists} A, v : B, w\mathcal{R}v, \Pi \Rightarrow \Sigma, w \Vdash^{\forall} B}}{w \Vdash^{\exists} A, v : B, w\mathcal{R}v, \Gamma, \Pi \Rightarrow \Delta, \Sigma, w \Vdash^{\forall} B}\;(\dagger)}{w\mathcal{R}v, v : B, \Gamma^2, \Pi \Rightarrow \Delta^2, \Sigma, w \Vdash^{\forall} B}\;(\ddagger) \qquad \overset{\vdots\,\mathcal{D}_{13}}{w \Vdash^{\forall} B, \Gamma \Rightarrow \Delta}}{v : B, w\mathcal{R}v, \Gamma^3, \Pi \Rightarrow \Delta^3, \Sigma}\;(\ddagger)$$

Finally, if the cut formula has shape $w : A \blacktriangleright B$ and the right premiss is by rule $L \blacktriangleright'$, we transform the following derivation (the conclusion of $\mathcal{D}_{11}$ is $w\mathcal{R}u, u : A, \Gamma \Rightarrow \Delta, u : B$):

$$\dfrac{\dfrac{\begin{matrix}\vdots\mathcal{D}_{11}\\ \mathcal{S}\end{matrix}\quad\begin{matrix}\vdots\mathcal{D}_{12}\\ \Gamma\Rightarrow\Delta, w\Vdash^{\exists} A\end{matrix}\quad\begin{matrix}\vdots\mathcal{D}_{13}\\ w\Vdash^{\forall} B,\Gamma\Rightarrow\Delta\end{matrix}}{\Gamma\Rightarrow\Delta, w:A\blacktriangleright B}{\scriptstyle R\blacktriangleright} \quad \dfrac{\begin{matrix}\vdots\mathcal{D}_{21}\\ w:A\blacktriangleright B, w\Vdash^{\exists} A,\Pi\Rightarrow\Sigma, w\Vdash^{\forall} B\end{matrix}}{w:A\blacktriangleright B,\Pi\Rightarrow\Sigma}{\scriptstyle L\blacktriangleright'}}{\Gamma,\Pi\Rightarrow\Delta,\Sigma}{\scriptstyle Cut}$$

into the following derivation containing some admissible instances of cut:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\mathcal{D}_{12}\\ \Gamma\Rightarrow\Delta, w\Vdash^{\exists} A\end{matrix}\quad\dfrac{\begin{matrix}\vdots\mathcal{D}_{1}\\ \Gamma\Rightarrow\Delta, w:A\blacktriangleright B\end{matrix}\quad\begin{matrix}\vdots\mathcal{D}_{21}\\ w:A\blacktriangleright B, w\Vdash^{\exists} A,\Pi\Rightarrow\Sigma, w\Vdash^{\forall} B\end{matrix}}{w\Vdash^{\exists} A,\Gamma,\Pi\Rightarrow\Delta,\Sigma,w\Vdash^{\forall} B}{\scriptstyle (\dagger)}}{\Gamma^2,\Pi\Rightarrow\Delta^2,\Sigma,w\Vdash^{\forall} B}{\scriptstyle (\ddagger)}\quad\begin{matrix}\vdots\mathcal{D}_{13}\\ w\Vdash^{\forall} B,\Gamma\Rightarrow\Delta\end{matrix}}{\Gamma^3,\Pi\Rightarrow\Delta^3,\Sigma}{\scriptstyle (\ddagger)}}{\Gamma,\Pi\Rightarrow\Delta,\Sigma}{\scriptstyle T.5.6}$$

$\square$

6. Soundness and completeness

In this section it is proved that the calculus G3SS.L is sound and complete with respect to L-frames. In particular, the proof of completeness will be a modular Tait-Shütte-Takeuti style proof: we define an exhaustive proof-search procedure that either outputs a G3SS.L-derivation or allows to build a countermodel based on an L-frame.

6.1. Soundness

In order to show that G3SS.L is sound with respect to L-frames we extend the notion of validity to sequents. We begin with some preliminary definitions.

Let $\mathcal{M} = \langle W, R, V\rangle$ be a model and let $\sigma : LAB \longrightarrow W$ be a function mapping labels to worlds of the model $\mathcal{M}$. We say that:

- $\mathcal{M}, \sigma$ *satisfies* the relational atom $w\mathcal{R}v$ iff $\sigma(w)R\sigma(v)$;
- $\mathcal{M}, \sigma$ *satisfies* the forcing formula $w \Vdash^{\forall/\exists} A$ iff each/some v such that $\sigma(w)R\sigma(v)$ is such that $\models^{\mathcal{M}}_{\sigma(v)} A$;

- $\mathcal{M}, \sigma$ *satisfies* the labelled formula $w : A$ iff $\models^{\mathcal{M}}_{\sigma(w)} A$.

A sequent $\Gamma \Rightarrow \Delta$ is $(\mathsf{L}\text{-})$*valid* iff each pair $\mathcal{M}, \sigma$ (with $\mathcal{M}$ based on an L-frame) satisfying all formulas in Γ satisfies some formula in Δ.

THEOREM 6.1 (Soundness). *If a sequent $\mathcal{S}$ is derivable in* $\mathsf{G3SS.L}$*, then it is* L*-valid.*

PROOF: The proof is by induction on the height of the derivation $\mathcal{D}$ of $\mathcal{S}$. The base cases hold trivially. For the inductive steps, we have to check that each rule of $\mathsf{G3SS.L}$ preserves validity over L-frames. First, we prove that the logical rules preserve validity. We consider only rules for $\triangleright$ and for $\blacktriangleright$ (the other cases are as in [10, Theorem 5.3]).

For rule $L\triangleright$, assume the last step of $\mathcal{D}$ has shape (u not in the conclusion):

$$\frac{w\mathcal{R}u, w\mathcal{R}v, u : A, w : A \triangleright B, \Gamma \Rightarrow \Delta, v : A \qquad w\mathcal{R}u, w\mathcal{R}v, v : B, u : A, w : A \triangleright B, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w : A \triangleright B, \Gamma \Rightarrow \Delta}\ L\triangleright$$

We need prove that each pair $\mathcal{M}, \sigma$ satisfying (all formulas in) $w\mathcal{R}v, w : A \triangleright B, \Gamma$ (Π, for short) satisfies some formula in Δ. Let $\mathcal{M}, \sigma'$ be a generic pair satisfying all of $w\mathcal{R}u, u : A, \Pi$. By induction on the left premiss, we know that $\mathcal{M}, \sigma'$ satisfies also some formula in Δ or $v : A$. In the first case we are done (u is not in Π, Δ and, hence, $\mathcal{M}, \sigma'$ is a generic pair satisfying Π). In the second case $\mathcal{M}, \sigma'$ satisfies $v : A \supset B$, and, hence, it satisfies $v : B$. By induction on the second premiss, we obtain that $\mathcal{M}, \sigma'$ satisfies some formula in Δ and we are done.

For rule $L\triangleright'$, assume the last step of $\mathcal{D}$ has shape (u not in the conclusion):

$$\frac{w\mathcal{R}u, u : A, w : A \triangleright B, \Gamma \Rightarrow \Delta}{w : A \triangleright B, \Gamma \Rightarrow \Delta}\ L\triangleright'$$

Let us consider a generic pair $\mathcal{M}, \sigma$ satisfying $w : A \triangleright B$ and all formulas in Γ. Since $w : A \triangleright B$ is satisfied, we know there is a world of the model $\mathcal{M}$ accessible from $\sigma(w)$ where A is true. Let σ' be like σ except for the label u that is mapped on that world. The pair $\mathcal{M}, \sigma'$ satisfies $w\mathcal{R}u, u : A, w : A \triangleright B, \Gamma$ and, by inductive hypothesis, it satisfies also some formula in Δ. We conclude that $\mathcal{M}, \sigma$ satisfies some formula in Δ (since u is not in $w : A \triangleright B, \Gamma \Rightarrow \Delta$).

For rule $R \rhd$, assume the last step of $\mathcal{D}$ has shape (u not in the conclusion):

$$\dfrac{w\mathcal{R}u, w\mathcal{R}v, u : A, \Gamma \Rightarrow \Delta, u : B \qquad w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, v : A}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B}\ {\scriptstyle R\rhd,\, u \text{ fresh}}$$

We have to prove that each pair $\mathcal{M}, \sigma$ satisfying all formulas in $w\mathcal{R}v, \Gamma$ (Π, for short) satisfies the formula $w : A \rhd B$ or some formula in Δ. By induction hypothesis applied to the second premiss, we know that $\mathcal{M}, \sigma$ satisfies some formula in Δ or $w : A \rhd B$ or $v : A$. The non trivial case happens when $\mathcal{M}, \sigma$ satisfies no formula in Δ but it satisfies $v : A$. In this case, we know that for the world $\sigma(v)$ both $\sigma(w)\mathcal{R}\sigma(v)$ and $\models^{\mathcal{M}}_{\sigma(v)} A$ hold, and hence there exists a pair $\mathcal{M}, \sigma'$ such that σ' differs from σ possibly only on u and such that it satisfies $w\mathcal{R}u, u : A, \Pi$. Let us consider then any generic pair of this kind. Observe that also $\mathcal{M}, \sigma'$ satisfies no formula in Δ, since u does not occur in Δ. Hence, by induction on the left premiss, $\mathcal{M}, \sigma'$ satisfies $u : B$. Therefore, we have seen that A is true in some world related to $\sigma(w)$ and, by genericity of σ', that any world related to $\sigma(w)$ satisfies $A \supset B$. We conclude that every pair $\mathcal{M}, \sigma$ satisfying all of Π but none of Δ must satisfy $w : A \rhd B$.

For rule $L \blacktriangleright$, assume the last step of $\mathcal{D}$ has shape:

$$\dfrac{w \Vdash^{\exists} A, w\mathcal{R}v, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta, v : A, w \Vdash^{\forall} B \qquad w\mathcal{R}v, v : B, w \Vdash^{\exists} A, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta, w \Vdash^{\forall} B}{w\mathcal{R}v, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\blacktriangleright}$$

We have to prove that each pair $\mathcal{M}, \sigma$ satisfying all formulas in $w\mathcal{R}v, w : A \blacktriangleright B$ satisfies some formula in Δ. The satisfaction of $w : A \blacktriangleright B$ guarantees the satisfaction of $w \Vdash^{\exists} A$ and the non satisfaction of $w \Vdash^{\forall} B$. Finally, it guarantees that $\not\models^{\mathcal{M}}_{\sigma(v)} A$ or $\models^{\mathcal{M}}_{\sigma(v)} B$. In the second case, the induction hypothesis applied to the second premiss will show that $\mathcal{M}, \sigma$ satisfies some formula in Δ. In the first case, the induction hypothesis applied to the first premiss will show again that $\mathcal{M}, \sigma$ satisfies some formula in Δ.

The proof for rule $L \blacktriangleright'$ is straightforward. Assume the last step of $\mathcal{D}$ has shape (no relational atom in Γ):

$$\dfrac{w \Vdash^{\exists} A, w : A \blacktriangleright B, \Gamma \Rightarrow \Delta, w \Vdash^{\forall} B}{w : A \blacktriangleright B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\blacktriangleright'}$$

Suppose that $\mathcal{M}, \sigma$ satisfies all of $w : A \blacktriangleright B, \Gamma$. The satisfiability of $w : A \blacktriangleright B$ entails that there is a world accessible from w where A is true. Thanks to the inductive hypothesis the pair $\mathcal{M}, \sigma$ satisfies some formula in Δ or $w \Vdash^{\forall} B$. The latter is impossible since $\mathcal{M}$ contains a world accessible from w where B is false (otherwise $\mathcal{M}, \sigma$ would not satisfy $w : A \blacktriangleright B$). We conclude that $\mathcal{M}, \sigma$ satisfies some formula in Δ.

For rule $R \blacktriangleright$, assume the last step of $\mathcal{D}$ has shape (u not in the conclusion):

$$\frac{w\mathcal{R}u, u : A, \Gamma \Rightarrow \Delta, u : B \qquad \Gamma \Rightarrow \Delta, w \Vdash^{\exists} A \qquad w \Vdash^{\forall} B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : A \blacktriangleright B}\ {\scriptstyle R\blacktriangleright}$$

We have to prove that each pair $\mathcal{M}, \sigma$ satisfying all formulas in Γ satisfies the formula $w : A \blacktriangleright B$ or some formula in Δ. By applying the induction hypothesis to the second premiss, we know that $\mathcal{M}, \sigma$ satisfies some formula in Δ or $w \Vdash^{\exists} A$. In the first case we are done. Otherwise, suppose that $\mathcal{M}, \sigma$ satisfies no formula in Δ but it satisfies $w \Vdash^{\exists} A$. There exists then a world related to $\sigma(w)$ in which A is true. Moreover, the induction hypothesis applied to the third premiss will show that $\mathcal{M}, \sigma$ cannot satisfy $w \Vdash^{\forall} B$, for otherwise we would infer the satisfaction of some formula in Δ. Therefore there must exist some world related to $\sigma(w)$ that falsifies B. Finally, an argument analogous to that used for $R \rhd$ will show that every world related to $\sigma(w)$ satisfies $A \supset B$. Overall, we conclude that $\mathcal{M}, \sigma$ satifies $w : A \blacktriangleright B$.

Each non-logical rule preserves validity over frames satisfying the corresponding semantic properties; cf. [12, Thm. 12.13]. □

6.2. Completeness

Definition 6.2 (Saturation). A branch $\mathcal{B}$ of a G3SS.L-proof-search tree for a sequent $\mathcal{S}$ (see procedure 6.5) is L-*saturated* if it satisfies the following conditions, where $\mathbf{\Gamma}$ ($\mathbf{\Delta}$) is the union of the antecedents (succedents) occurring in that branch,

1. no $w : p$ occurs in $\mathbf{\Gamma} \cap \mathbf{\Delta}$;
2. no $w : \bot$ occurs in $\mathbf{\Gamma}$;
3. if $w : A \wedge B$ is in $\mathbf{\Gamma}$ ($w : A \vee B \in \mathbf{\Delta} / w : A \supset B \in \mathbf{\Delta}$), then both $w : A$

and $w : B$ are in $\mathbf{\Gamma}$ ($w : A, w : B \in \mathbf{\Gamma}/w : A \in \mathbf{\Gamma}$ and $w : B \in \mathbf{\Delta}$, resp.);

4. if $w : A \wedge B$ is in $\mathbf{\Delta}$ ($w : A \vee B$ is in $\mathbf{\Gamma}/w : A \supset B$ is in $\mathbf{\Gamma}$), then at least one of $w : A$ and $w : B$ is in $\mathbf{\Delta}$ ($w : A \in \mathbf{\Gamma}$ or $w : B \in \mathbf{\Gamma}/w : A \in \mathbf{\Delta}$ or $w : B \in \mathbf{\Gamma}$, resp.);
5. if $w : A \rhd B$ is in $\mathbf{\Gamma}$, then
   (a) for some u, both $w\mathcal{R}u$ and $u : A$ are in $\mathbf{\Gamma}$; and
   (b) $v : A$ is in $\mathbf{\Delta}$ or $v : B$ is in $\mathbf{\Gamma}$, for any v such that $w\mathcal{R}v$ is in $\mathbf{\Gamma}$;
6. if $w : A \rhd B$ is in $\mathbf{\Delta}$, then,
   (a) for some u, both $w\mathcal{R}u$ and $u : A$ are in $\mathbf{\Gamma}$ and $u : B$ is in $\mathbf{\Delta}$; or
   (b) $v : A$ is in $\mathbf{\Delta}$ for any v such that $w\mathcal{R}v$ is in $\mathbf{\Gamma}$;
7. if $w : A \blacktriangleright B$ is in $\mathbf{\Gamma}$, then
   (a) for some u_1, u_2, all of $w\mathcal{R}u_1$ and $w\mathcal{R}u_2$ and $u_1 : A$ are in $\mathbf{\Gamma}$, moreover $u_2 : B$ is in $\mathbf{\Delta}$; and
   (b) $v : A$ is in $\mathbf{\Delta}$ or $v : B$ is in $\mathbf{\Gamma}$, for any v such that $w\mathcal{R}v$ is in $\mathbf{\Gamma}$;
8. if $w : A \blacktriangleright B$ is in $\mathbf{\Delta}$, then
   (a) for some u, $u : A$ in $\mathbf{\Gamma}$ and $u : B$ is in $\mathbf{\Delta}$; or
   (b) $v : A$ is in $\mathbf{\Delta}$ for any v such that $w\mathcal{R}v$ is in $\mathbf{\Gamma}$; or
   (c) $v : B$ is in $\mathbf{\Gamma}$, for any v such that $w\mathcal{R}v$ is in $\mathbf{\Gamma}$;

9_R. if *Rule* is a non-logical rule of G3SS.L, then for any set of principal formulas of *Rule* that are in $\mathbf{\Gamma}$ also the corresponding active formulas are in $\mathbf{\Gamma}$.

DEFINITION 6.3. Let $\mathcal{B}$ be L-saturated. The model $\mathcal{M}^{\mathcal{B}} = \langle W, R, V \rangle$ is thus defined:

1. W is the set of labels occurring in $\mathbf{\Gamma} \cup \mathbf{\Delta}$;
2. for each w, $v \in W$, wRv iff the formula $w\mathcal{R}v$ is in $\mathbf{\Gamma}$;
3. $V(p)$ is the set of all w such that $w : p$ is in $\mathbf{\Gamma}$.

Moreover, $\sigma_=$ denotes the identity function on LAB.

Observe that $\mathcal{M}^{\mathcal{B}}$ is well-defined thanks to clauses 1 and 2 of Def. 6.2.

LEMMA 6.4. *Let $\mathcal{B}$ be an L-saturated branch. Then*

1. *for any $\mathcal{L}$-formula A occurring in $\mathcal{B}$ we have that $\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_=(w)} A$ iff $w : A$ is in $\mathbf{\Gamma}$; and*
2. *$\mathcal{M}^{\mathcal{B}}$ is based on a frame for L.*

PROOF: Claim (2) follows by Definition 6.2.9_R and by construction of $\mathcal{M}^{\mathcal{B}}$ (Definition 6.3) and of $\sigma_=$.

The proof of claim (1) is by induction on the construction of A. The base case holds by construction of $\mathcal{M}^{\mathcal{B}}$ and of $\sigma_=$, and the inductive cases depend on Definition 6.2.3–8.

To illustrate, we consider the case when A has shape $w : B \rhd C$ and it occurs in $\mathcal{B}$.

If $w : B \rhd C$ is in $\mathbf{\Gamma}$, then we have to show that $\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_=(w)} B \rhd C$. By Definition 6.2.5(a), we know that, for some u, $w\mathcal{R}u$ and $u : B$ are in $\mathbf{\Gamma}$. Hence, $\sigma_=(w)R\sigma_=(u)$ and, by induction, $\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_=(u)} B$. Moreover, by Definition 6.2.5(b), we have that all v such that $w\mathcal{R}v$ is in $\mathbf{\Gamma}$ are such that either $v : B \in \mathbf{\Delta}$ or $v : C \in \mathbf{\Gamma}$. In both cases we have that if $\sigma_=(w)R\sigma_=(v)$ then $\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_=(v)} B \supset C$. We conclude that $\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_=(w)} B \rhd C$.

If, instead, $w : B \rhd C$ is in $\mathbf{\Delta}$, then we have to show that $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_=(w)} B \supset C$. By Definition 6.2.6, we know that, either for some u, $w\mathcal{R}u$ and $u : A$ are in $\mathbf{\Gamma}$ and $u : C \in \mathbf{\Delta}$, or $v : B \in \mathbf{\Delta}$ for all v such that $w\mathcal{R}v \in \mathbf{\Gamma}$. In the first case we have that $\sigma_=(w)R\sigma_=(u)$ and, by induction, $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_=(u)} B \supset C$. In the latter case we have that $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_=(v)} B$ for all v such that $\sigma_=(w)R\sigma_=(v)$. In both cases we conclude that $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_=(w)} B \rhd C$.

The case $w : B \blacktriangleright C \in \mathbf{\Gamma}$ is analogous to the corresponding one for $\rhd$ with the only further requirement of proving the existence of some world related to $\sigma_=(w)$ that falsifies B. But this is guaranteed by Definition 6.2.7(a), according to which there is some u_2 such that $w\mathcal{R}u_2$ is in $\mathbf{\Gamma}$ and $u_2 : B$ is in $\mathbf{\Delta}$. This means that $\sigma(w)R\sigma(u_2)$ and, by induction, that $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_=(u_2)} B$.

For $w : B \blacktriangleright C \in \mathbf{\Delta}$, to prove that $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_=(w)} B \blacktriangleright C$ one has to prove that at least one of the following conditions holds: (i) the existence of a world related to w that falsifies $B \supset C$, (ii) the falsity of B in every world related to $\sigma(w)$, (iii) the truth of C in every world related to $\sigma(w)$. For (i), we argue analogously to the case $w : B \rhd C \in \mathbf{\Delta}$. For (ii), one has to consider Definition 6.2.8(b), according to which for every v such that

$w\mathcal{R}v \in \mathbf{\Gamma}$, it holds $v : B \in \mathbf{\Delta}$. By induction, this means $\not\models^{\mathcal{M}^{\mathcal{B}}}_{\sigma_{=}(v)} B$. Since by construction of $\mathcal{M}^{\mathcal{B}}$, all worlds related to w are of type $\sigma(v)$ for $w\mathcal{R}v \in \mathbf{\Gamma}$, we are done. The proof for (iii) runs analogously to (ii) through Definition 6.2.8(c). □

Procedure 6.5. A G3SS.L-proof-search tree for a sequent $\mathcal{S}$ is a tree of sequents that has $\mathcal{S}$ as root and whose branches grow according to the following procedure: if the leaf is an initial sequent or an instance of rule $L\bot$ the branch stops growing, else either no instance of rules of G3SS.L is applicable root first to it, or k instances are (where rules *Ser* and *Ref* are applied once w.r.t. each label occurring in the branch). In the first case, the branch stops growing; in this case it is immediate to see that we have a finite L-saturated branch. In the second case, we apply the k rule instances that are applicable in some order (each one will be applied to all end-sequents that are generated at the previous step). If the tree never stops growing then, by König's Lemma, it has an infinite branch which, as the reader can easily check, is L-saturated.[6]

Theorem 6.6 (Completeness). *If a sequent $\mathcal{S}$ is* L*-valid then it is derivable in* G3SS.L*.*

Proof: The proof is in three steps. First, in Def. 6.2, we define a notion of L-saturated branch of a proof-search for a sequent $\mathcal{S}$, where, intuitively, a branch is L-saturated when all applicable instances of G3SS.L-rules have been applied. Then, with Def. 6.3 and Lemma 6.4, we show that an L-saturated branch allows us to define a countermodel for $\mathcal{S}$ that is based on a frame for L. Finally, we give a root first G3SS.L-proof-search procedure, Prop. 6.5, that either outputs a G3SS.L-derivation of $\mathcal{S}$ – and, by Thm. 6.1, $\mathcal{S}$ is L-valid – or it outputs an L-saturated branch – and, therefore, $\mathcal{S}$ has a countermodel based on an appropriate frame. □

Corollary 6.7. The structural rules of inference – i.e., (left) weakening, (left) contraction, and cut – are semantically admissible in G3SS.L.

[6] Forcing formulas are needed to ensure that, when applying root-first all possible instances of rule $R\blacktriangleright$ (see Table 3), we obtain a saturated branch (see Def. 6.2.8). If in the second and third premisses of $R\blacktriangleright$ we had $v : A$ and $v : B$ (as it happens for rule $R\triangleright$) in place of the forcing formulas, we would obtain a completed proof search with an open branch that is not saturated because for each v such that $w\mathcal{R}v \in \mathbf{\Gamma}$ we would have that either $v : A \in \mathbf{\Delta}$ or $v : B \in \mathbf{\Gamma}$.

7. Priest's cancellation account of negation and SSI

In [16] G. Priest has considered the cancellation account of negation – i.e. the idea that "$\neg A$ deletes, neutralizes, erases, cancels $A[\dots]$ so that $\neg A$ together with A leaves nothing" [19, p. 205] – and he modeled it by means of a relational semantics that is almost identical to the one we gave for the $\mathsf{S5}$-based logic of SSI. If we represent Priest's non-symmetric and symmetric implication by $\triangleright$ and $\blacktriangleright$, respectively, we have that a $\mathcal{L}$-formula is valid in Priest's semantics if and only if it is valid in the class of $\mathsf{S5}$-frames for logics of SSI.

Priest considers a language containing one of $\triangleright$ and $\blacktriangleright$. Priest's models are relational models without an accessibility relation – i.e., $\mathfrak{M} = \langle W, V\rangle$ – and truth of a formula in a world of a model is defined as in Section 3 save that in the truth-clauses for $\triangleright$ and $\blacktriangleright$ the quantifiers are not restricted to accessible worlds. To illustrate, $\triangleright$ is thus defined:

$$\models^{\mathfrak{M}}_{w} A \triangleright B \quad \text{iff} \quad \exists u \in W(\models^{\mathfrak{M}}_{u} A) \text{ and } \forall v \in W(\models^{\mathfrak{M}}_{v} A \text{ implies } \models^{\mathfrak{M}}_{v} B)$$

The notions of truth in a model and validity are defined as in Section 3, but the definition of logical consequence differs since in Priest's approach it is a metatheoretical version of $\triangleright$ or $\blacktriangleright$: for A to be a consequence of Γ, Γ must have a model and, possibly, $\neg A$ must have a model. This gives an highly non-Tarskian consequence relation in that, in both cases, it is neither reflexive, nor monotone, nor transitive.

This is not the place to discuss in full details the account of negation as cancellation (see [23] for some criticisms) nor the way Priest models it. We just note that Priest's semantics is not apt to model the cancellation account of negation since it adhere to a perfectly classical definition of negation. The non-classical elements are only the definition of implication and that of logical consequence. It follows that Priest's semantics has only a partial overlap with the cancellation account of negation: they share the claim that nothing follows from a contradiction (*alone*). But the cancellation account of negation cannot be reduced to this claim. If $\neg A$ deletes A so that Γ is extensionally identical to $\Gamma \cup \{A, \neg A\}$, then, if B is a consequence of some set of sentences Γ, it must be the case that B is a consequence of the set $\Gamma \cup \{A, \neg A\}$. Nevertheless in Priest's semantics nothing follows from a set containing contradictory formulas.

Priest's notion of validity is easily seen to be equivalent to that for the

S5-based logic of super strict implication (analogously to what happens in standard modal logics, an accessibility relation that is an equivalence relation is equivalent to a universal accessibility relation). Formally we have the following result:

PROPOSITION 7.1. Let A be any $\mathcal{L}$-formula containing only one of $\triangleright$ and $\blacktriangleright$, $\mathcal{C}^{\mathfrak{M}}$ the class of all Priest's models and $\mathcal{C}^{\mathsf{S5}}$ the class of all frames from Section 3 where R is an equivalence relation, then $\mathcal{C}^{\mathfrak{M}} \models A \quad \text{iff} \quad \mathcal{C}^{\mathsf{S5}} \models A$.

We find this result interesting since it entails that the logics of SSI are connexive logics having the same set of S5-validities of Priest's one, that are obtained without having to tamper with the Tarskian notion of consequence relation nor with the classical explosion model of negation [19]. Moreover, this entails that in Section 4 we have introduced a cut-free sequent calculus that characterizes validity in Priest's formal semantics.

All in all, the logics of SSI can be seen as modal extensions of classical logic that satisfies

> the central concern of connexive logic [which] consists of developing connexive systems that are naturally motivated conceptually or in terms of applications, that admit of a simple and plausible semantics, and that can be equipped with proof systems possessing nice proof-theoretical properties, such as the eliminability of the cut-rule. [14, p. 381]

8. Future works

For brevity, we have considered only the logics of SSI based on frames for normal modal logics. It should be possible to consider also the logics of SSI based on frames for C.I. Lewis's non-normal systems S1, S2 and S3. In particular, it would be interesting to present the S2-based logic of SSI since C.I. Lewis believed it the more likely correct logic of strict implication. It would also be interesting to see if for these weaker systems we can still use labelled calculi like the one we gave in Section 4, or if we have to introduce calculi more akin to those for non-normal modal logics presented in [10]. It might also be interesting to study some modifications of SSI, e.g., their reflexivizations (which should be very similar to Pizzi's consequential implication) and their constructive analogous. It would also be interesting to see if it is possible to have a SSI validating strong Boethius' Thesis without making the implication commutative as for consequential implications. Another problem that remains open is to give a complete axiomatisation of logics of SSI; we conjecture that this can be done by using the transfer methodology used in [17]. Finally, it would be extremely interesting to check if it is possible to give a proof-theoretical characterisation of the logics of SSI by means of some internal calculus such as hypersequents or nested sequents.

In this paper we have argued that the logics of SSI are connexive logics. In the future we plan to investigate whether SSI have some feature of the other main family of implications that avoid the paradoxes of strict implications, namely relevant implications. The philosophical motivation behind the introduction of SSI is quite similar to that behind relevant implications. Very roughly, we wanted 'A implies B' to be true just in case the truth of B *depends* on the truth of A and relevant logicians just in case the truth of A *is relevant* to the truth of B. It might well be that our notion of *dependance* is nothing but some notion of *relevance*. It is difficult to find a precise formal explication of what relevance is. Apart from the falsification of the paradoxes of material and strict implication, one typical condition for an implication being relevant is its having a *variable-sharing property*: if 'A implies B' is true then there must be some (all) propositional variables that are common to A and B [1, p. 33]. Hence, to see whether SSI are relevant implications we have to check whether they satisfy some variable-sharing property.

In the introduction we have claimed that the motivation for introducing SSI is orthogonal to D. Lewis' [7] and Stalnaker's [20] one for introducing variably strict implications. SSI are designed to avoid the paradoxes of strict implication (SI1) and (SI2). Variably strict implications, instead, are designed to avoid other properties of strict implications: monotonicity, contraposition and, last but not least, transitivity. Even if there is a partial overlap between them in that monotonicity fails also for SSI and contraposition fails for weak SSI, no one of the two approaches can be seen as a subsystem of the other in that SSI validate transitivity and variably strict implications validate the paradoxes of strict implication. Just as we tweaked strict implication to overcome its paradoxes, it should be possible to tweak variably strict implications in order to obtain variably SSI. Weak variably SSI has been considered and discharged by D. Lewis under the name of 'would counterfactual', cf [7, p. 25]; a connexive and variably strict implication has been considered in [22]. If variably strict implications approximate the logic of counterfactual conditionals, it might well be that variably SSI approximate the logic of indicative conditionals.

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Guido Gherardi
University of Bologna
Department of Philosophy and Communication Studies
I-40126, Via Zamboni 38
Bologna, Italy
e-mail: Guido.gherardi@unibo.it

Eugenio Orlandelli
University of Bologna
Department of Philosophy and Communication Studies
I-40126, Via Zamboni 38
Bologna, Italy
e-mail: eugenio.orlandelli@unibo.it