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# DIRICHLET PROBLEM FOR SECOND-ORDER ABSTRACT DIFFERENTIAL EQUATIONS

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Abstract. We study the well-posedness in the space of continuous functions of the Dirichlet boundary value problem for a homogeneous linear second-order differential equation $u'' + Au = 0$, where $A$ is a linear closed densely defined operator in a Banach space. We give necessary conditions for the well-posedness, in terms of the resolvent operator of $A$. In particular we obtain an estimate on the norm of the resolvent at the points $k^2$, where $k$ is a positive integer, and we show that this estimate is the best possible one, but it is not sufficient for the well-posedness of the problem. Moreover we characterize the bounded operators for which the problem is well-posed.

## 1. Introduction

We consider the Dirichlet boundary value problem

$$
\begin{gathered}
u''(t) + Au(t) = 0\,, \quad t \in [0, \pi]\,,\\
u(0) = x_0\,,\\
u(\pi) = x_\pi\,.
\end{gathered}
$$

where $A$ is a linear closed densely defined operator in a (real or complex) Banach space. We are interested in the uniform well-posedness of the problem in the sense of continuous functions, that is we ask that for every $x_0, x_\pi$ in the domain of $A$ there exists a unique solution $u$ such that $Au$ and $u''$ are continuous; moreover we require that the solution depends continuously on the boundary values.

Unlike most of the articles about abstract Dirichlet problems, we do not suppose that $-A$ is a positive operator. We give some necessary conditions for the uniform well-posedness in terms of the resolvent operator of $A$, in particular we prove that if the problem is uniformly well-posed then $k^2$ belongs to the resolvent set of $A$ for every positive integer $k$ and $(k^2 I - A)^{-1}$ has norm bounded by $C/k$ for a suitable $C \in \mathbb{R}^+$. We give examples showing that this is the best possible estimate of the resolvent, but it is not sufficient for the well-posedness.

Finally we show that if $A$ is bounded then there is uniform well-posedness if and only if $k^2$ belongs to the resolvent set of $A$ for every positive integer $k$.

The Dirichlet problem for an abstract second order equation (homogeneous or non-homogeneous) has been studied in various papers. Usually it is supposed that

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$-A$ is a positive operator, we refer to Section 4 of the review paper [2], and the articles quoted therein.

In [1] and [3] maximal regularity in the $L^p$ sense for the Dirichlet problem for the non-homogeneous equation $u'' + Au = f$ is characterized.

In [3] it is supposed that $A = -B^2$, where $B$ is the generator of an exponentially stable analytic semigroup. They prove (see Corollary 3.4) that, if there is maximal $L^p$ regularity for the Cauchy problem for the first order equation $u' - Bu = f$, then there is maximal $L^p$ regularity for the Dirichlet problem. In UMD spaces the converse implication holds.

In [1, Theorem 6.3] the authors prove that if $A$ is an arbitrary closed operator in a UMD space, then there is maximal $L^p$ regularity for the Dirichlet problem if and only if $k^2$ belongs to the resolvent set of $A$ for every positive integer $k$ and $\{k^2(k^2I - A)^{-1} \mid k \in \mathbb{N}\}$ is R-bounded.

We recall also [6] and [5] where the existence of solutions of the Dirichlet problem for particular boundary values is studied under the hypothesis that $A$ is a positive self-adjoint operator in a Hilbert space.

This paper is organized as follows. In Section 2 we show that the uniform well-posedness of the Dirichlet problem is equivalent to the existence of a suitable strongly continuous operator valued function. In Section 3 we give necessary conditions for the uniform well-posedness. Section 4 contains two examples showing that the estimate on the norm of the resolvent operator of $A$ obtained in Theorem 3.1 is the best possible one, but it is not sufficient to ensure the uniform well-posedness. In Section 5 we characterize uniform well-posedness in case $A$ is a bounded operator.

## 2. Well-posed problems

In what follows $X$ will be a Banach space over the field $\mathbb{K}$ (real or complex numbers) and $A$ a closed linear operator from $\mathcal{D}(A) \subseteq X$ to $X$ with dense domain; $\rho(A)$ will denote the resolvent set of $A$. We denote with $\mathcal{L}(X)$ the space of linear bounded operators in $X$. Finally $\mathbb{N}$ will denote the set of positive integers and $\mathbb{N}_0$ the set of non-negative integers.

We study the second-order abstract differential equation

$$u''(t) + Au(t) = 0\,, \quad t \in [0,\pi]\,, \tag{2.1}$$

and the Dirichlet boundary value problem for this equation

$$\begin{aligned} u''(t) + Au(t) = 0\,, \quad t \in [0,\pi]\,, \\ u(0) = x_0\,, \\ u(\pi) = x_\pi\,. \end{aligned} \tag{2.2}$$

We call *solution* of equation (2.1) a function $v : [0,\pi] \to X$ such that

(1) $v \in C^2\big([0,\pi], X\big) \cap C\big([0,\pi], \mathcal{D}(A)\big)$;
(2) for all $t \in [0,\pi]$, $v''(t) + Av(t) = 0$.

We call *solution* of problem (2.2) a solution $v$ of equation (2.1) such that $v(0) = x_0$ and $v(\pi) = x_\pi$.

Obviously a solution of this problem can exist only if $x_0, x_\pi \in \mathcal{D}(A)$.

We say that problem (2.2) is *uniformly well-posed* if

(1) for all $x_0, x_\pi \in \mathcal{D}(A)$, problem (2.2) has solution;

(2) there exists $C \in \mathbb{R}^+$ such that for any solution $v$ of equation (2.1) we have

$$\sup_{t\in[0,\pi]} \|v(t)\| \le C\big(\|v(0)\| + \|v(\pi)\|\big)\,.$$

Condition (2) implies the uniqueness of the solution of problem (2.2).

**Theorem 2.1.** *Let $X$ be a Banach space and $A$ be a linear closed densely defined operator in $X$. Problem* (2.2) *is uniformly well-posed if and only if there exists $S : [0,\pi] \to \mathcal{L}(X)$ such that:*

1. *for all $x \in X$, the function $S(\cdot)x$ is continuous;*
2. *for all $x \in X$, we have $S(0)x = 0$, $S(\pi)x = x$;*
3. *for all $x \in \mathcal{D}(A)$, the function $S(\cdot)x$ is solution of equation* (2.1)*;*
4. *if $v\colon [0,\pi] \to X$ is solution of equation* (2.1)*, then*

$$v(t) = S(t)v(\pi) + S(\pi - t)v(0)\,.$$

*Proof.* Suppose that problem (2.2) is uniformly well-posed. If $x \in \mathcal{D}(A)$, let $v$ be the unique solution of the problem

$$\begin{gathered} u''(t) + Au(t) = 0\,, \quad t \in [0,\pi]\,,\\ u(0) = 0\,,\\ u(\pi) = x\,; \end{gathered}$$

for $t \in [0,\pi]$ put $\widetilde{S}(t)x = v(t)$. Then $\widetilde{S}(t)$ is a linear operator from $\mathcal{D}(A)$ to $X$ and, because of the uniform well-posedness, there exists $C \in \mathbb{R}^+$ such that $\big\|\widetilde{S}(t)x\big\| \le C\|x\|$, for every $x \in \mathcal{D}(A)$. Since $\mathcal{D}(A)$ is dense in $X$, $\widetilde{S}(t)$ can be extended to a bounded linear operator $S(t)$ from $X$ to $X$ and $\big\|S(t)\big\| \le C$. Obviously $S$ satisfies conditions (2) and (3).

If $v$ is solution of equation (2.1) then it is solution of the Dirichlet problem

$$\begin{gathered} u''(t) + Au(t) = 0\,, \quad t \in [0,\pi]\,,\\ u(0) = v(0)\,,\\ u(\pi) = v(\pi)\,. \end{gathered}$$

The function $t \mapsto S(t)v(\pi) + S(\pi - t)v(0)$ is solution of the same problem, hence, by the uniqueness of the solution, we have

$$v(t) = S(t)v(\pi) + S(\pi - t)v(0)\,.$$

Therefore (4) is satisfied.

If $x \in \mathcal{D}(A)$ the function $S(\cdot)x$ is solution of equation (2.1), hence it is continuous. If $x \in X$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(A)$ converging to $x$. The functions $S(\cdot)x_n$ are continuous and

$$\sup_{t\in[0,\pi]} \big\|S(t)x_n - S(t)x\big\| \le C\|x_n - x\| \xrightarrow[n\to\infty]{} 0\,.$$

Since it is the uniform limit of continuous functions $S(\cdot)x$ is continuous. Therefore condition (1) is satisfied.

Conversely suppose that there exists $S$ satisfying conditions (1)–(4). Since, for all $x \in X$, the function $S(\cdot)x$ is continuous, it is bounded, hence by the uniform boundedness principle $\big\{S(t) \bigm| t \in [0,\pi]\big\}$ is bounded, therefore there exists $C \in \mathbb{R}^+$ such that $\big\|S(t)\big\| \le C$, for all $t \in [0,\pi]$.

If $x_0, x_\pi \in \mathcal{D}(A)$, let

$$v\colon [0,\pi] \to X\,, \quad v(t) = S(t)x_\pi + S(\pi - t)x_0\,.$$

By (3) $v \in C^2\big([0,\pi], X\big) \cap C\big([0,\pi], \mathcal{D}(A)\big)$, and for all $t \in [0,\pi]$ we have

$$v''(t) + Av(t) = S''(t)x_\pi + S''(\pi - t)x_0 + AS(t)x_\pi + AS(\pi - t)x_0 = 0\,;$$

moreover, by (2)

$$v(0) = S(0)x_\pi + S(\pi)x_0 = x_0\,, \quad v(\pi) = S(\pi)x_\pi + S(0)x_0 = x_\pi\,.$$

Hence problem (2.2) has solution.

From (4) and the estimate $\big\|S(t)\big\| \le C$ it follows that if $v$ is solution of equation (2.1) then

$$\begin{aligned} \|v(t)\| &= \big\|S(t)v(0) + S(\pi - t)v(0)\big\| \\ &\le \big\|S(t)\big\|\,\|v(0)\| + \big\|S(\pi - t)\big\|\,\|v(\pi)\| \\ &\le C\big(\|v(0)\| + \|v(\pi)\|\big)\,. \end{aligned}$$

Hence problem (2.2) is uniformly well-posed. □

**Theorem 2.2.** *Let $X$ be a Banach space and $A$ be a linear closed densely defined operator in $X$. If problem* (2.2) *is uniformly well-posed, then, for all $m \in \mathbb{N} \setminus \{1\}$, the same problem is uniformly well-posed for the operator $m^{-2}A$.*

*Proof.* Let $S$ be the strongly continuous operator valued function whose existence is guaranteed by Theorem 2.1. Let $T\colon [0,\pi] \to \mathcal{L}(X)$ defined as follows. If $m$ is even

$$T(t) = \sum_{j=1}^{m/2} \Big( S\Big(\frac{(2j-1)\pi + t}{m}\Big) - S\Big(\frac{(2j-1)\pi - t}{m}\Big)\Big), \tag{2.3}$$

if $m$ is odd

$$T(t) = \sum_{j=1}^{(m-1)/2} \Big( S\Big(\frac{2j\pi + t}{m}\Big) - S\Big(\frac{2j\pi - t}{m}\Big)\Big) + S\Big(\frac{t}{m}\Big). \tag{2.4}$$

It is easy to verify that $T$ satisfies conditions (1)–(3) of Theorem 2.1 with respect to the operator $m^{-2}A$.

To prove that also condition (4) is satisfied it is sufficient to show that the unique solution of the problem

$$\begin{gathered} u''(t) + \frac{1}{m^2}\,Au(t) = 0\,, \quad t \in [0,\pi]\,, \\ u(0) = 0\,, \\ u(\pi) = 0\,, \end{gathered}$$

is the identically zero function.

Suppose that $v \in C^2\big([0,\pi], X\big) \cap C\big([0,\pi], \mathcal{D}(A)\big)$ is solution of this problem. Let $w\colon \mathbb{R} \to X$ be the $2\pi$-periodic repetition of the odd extension of $v$. Since $v(0) = v(\pi) = 0$, $w \in C^1(\mathbb{R}, X) \cap C(\mathbb{R}, \mathcal{D}(A))$. Moreover $w$ is twice continuously differentiable in $\mathbb{R} \setminus \{j\pi \,|\, j \in \mathbb{Z}\}$, with second derivative equal to $-m^{-2}Aw$. In the points $j\pi$ the second derivative from the left and the second derivative from the right are both equal to $-m^{-2}Aw(j\pi) = 0$, hence $w$ is twice differentiable also at these points and $w \in C^2(\mathbb{R}, X) \cap C(\mathbb{R}, \mathcal{D}(A))$. Let $z\colon [0,\pi] \to X$ be defined by $z(t) = w(mt)$. Then $z$ is solution of the problem

$$\begin{gathered} u''(t) + Au(t) = 0\,, \quad t \in [0,\pi]\,, \\ u(0) = 0\,, \\ u(\pi) = 0\,; \end{gathered}$$

hence it is identically zero. Therefore $v = 0$. □

## 3. Necessary conditions

In this section we give necessary conditions for the uniform well-posedness of problem (2.2) in terms of the resolvent operator of $A$. First of all we have a condition on the resolvent set and an estimate of the resolvent operator of $A$.

**Theorem 3.1.** *Let $X$ be a Banach space and $A$ be a linear closed densely defined operator in $X$ such that problem (2.2) is uniformly well-posed. Then for all $k \in \mathbb{N}$, $k^2 \in \rho(A)$ and there exists $C \in \mathbb{R}^+$ such that, for all $k \in \mathbb{N}$,*

$$\big\|(k^2 I - A)^{-1}\big\| \le \frac{C}{k}\,. \tag{3.1}$$

The fact that $k^2$ does not belongs to the point spectrum of $A$ is a particular case of [4, Theorem 1].

*Proof.* Let $S$ be the operator valued function whose existence is guaranteed by Theorem 2.1 and $C = \sup_{t\in[0,\pi]}\big\|S(t)\big\|$.

For $k \in \mathbb{N}$ the operator $k^2 I - A$ is injective. Indeed if $x \in \mathcal{D}(A)$ is such that $k^2 x = Ax$, it is easy to check that the function $t \mapsto \sin(kt)x$ is solution of the problem

$$\begin{gathered} u''(t) + Au(t) = 0\,, \quad t \in [0,\pi]\,, \\ u(0) = 0\,, \\ u(\pi) = 0\,. \end{gathered}$$

The unique solution of this problem is the identically zero function, hence $x = 0$.

For $k \in \mathbb{N}$ and $x \in X$ let

$$R_k x = \frac{(-1)^{k+1}}{k} \int_0^\pi \sin(ks) S(s) x\, ds\,.$$

In this way an operator $R_k \in \mathcal{L}(X)$ is defined; moreover

$$\|R_k x\| \le \frac{1}{k} \int_0^\pi |\sin(ks)|\, \big\|S(s)x\big\|\, ds \le \frac{1}{k} \int_0^\pi |\sin(ks)|\, C\|x\|\, ds = \frac{2C}{k}\, \|x\|\,,$$

hence $\|R_k\| \le 2C/k$. For all $x \in \mathcal{D}(A)$ we have

$$\begin{aligned} AR_k x &= \frac{(-1)^{k+1}}{k} \int_0^\pi \sin(ks) AS(s) x\, ds = \frac{(-1)^k}{k} \int_0^\pi \sin(ks) S''(s) x\, ds \\ &= \frac{(-1)^k}{k} \big[\sin(ks) S'(s) x\big]_0^\pi - (-1)^k \int_0^\pi \cos(ks) S'(s) x\, ds \\ &= -(-1)^k \big[\cos(ks) S(s) x\big]_0^\pi - (-1)^k k \int_0^\pi \sin(ks) S(s) x\, ds \\ &= -x + k^2 R_k x\,. \end{aligned}$$

If $x \in X$ let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(A)$ converging to $x$. Then $R_k x_n \to R_k x$ and $AR_k x_n \to -x + k^2 R_k x$. Since $A$ is closed this proves that $R_k x \in \mathcal{D}(A)$ and $AR_k x = -x + k^2 R_k x$. Hence, for all $x \in X$, we have $(k^2 I - A)R_k x = x$. Therefore $R_k$ is a right inverse of $k^2 I - A$.

To prove that it is also a left inverse, observe that if $x \in \mathcal{D}(A)$ then

$$(k^2 I - A)R_k(k^2 - A)x = (k^2 I - A)x\,,$$

hence $R_k(k^2 - A)x = x$ since $k^2 I - A$ is injective.

Therefore $k^2 I - A$ is invertible and $(k^2 I - A)^{-1} = R_k$. This proves the theorem. ☐

To prove another necessary condition, we need a result about Fourier series.

**Lemma 3.2.** *The series of functions $\sum_{k=1}^{\infty}\big((-1)^{k+1}/k\big)\sin(kt)$ converges pointwise for $t \in [0,\pi]$, the sequence of partial sums is uniformly bounded and*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\,\sin(kt) = \begin{cases} \frac{1}{2}\,t\,, & \textit{if } 0 \le t < \pi\,,\\ 0 & \textit{if } t = \pi\,. \end{cases}$$

*Proof.* We have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\,\sin(kt) = \sum_{k=1}^{\infty}\frac{1}{k}\,\sin\big(k(\pi - t)\big)$$

and the convergence of this series is proved in [9, Chapter I, (2.8)]; the uniform boundedness of the partial sums of this series is proved in [9, Chapter II, Section 9]. ☐

**Theorem 3.3.** *Let $X$ be a Banach space and $A$ be a linear closed densely defined operator in $X$ such that problem (2.2) is uniformly well-posed. Then for all $m \in \mathbb{N}$ the series $\sum_{k=1}^{\infty}\big((mk)^2 I - A\big)^{-1}$ converges in operator norm.*

*Moreover, if $S$ is the operator valued function introduced in Theorem 2.1, then*

$$\sum_{k=1}^{\infty}\big(k^2 I - A\big)^{-1}x = \frac{1}{2}\int_0^{\pi} tS(t)x\,dt$$

*and, for $m \in \mathbb{N}\setminus\{1\}$,*

$$\sum_{k=1}^{\infty}\big((mk)^2 I - A\big)^{-1}x = \frac{1}{2}\int_0^{\pi} tS(t)x\,dt - \frac{(m-2j+1)\pi}{2m}\sum_{j=1}^{[m/2]}\int_{(m-2j)\pi/m}^{(m-2j+2)\pi/m} S(t)x\,dt$$

*Proof.* From the proof of Theorem 3.1 we know that for all $k \in \mathbb{N}$ and for all $x \in X$

$$(k^2 I - A)^{-1}x = \frac{(-1)^{k+1}}{k}\int_0^{\pi}\sin(kt)S(t)x\,dt\,,$$

Then

$$\begin{aligned} &\Big\|\sum_{j=1}^{k}\big(j^2 I - A\big)^{-1}x - \frac{1}{2}\int_0^{\pi} tS(t)x\,dt\Big\| \\ &= \Big\|\int_0^{\pi}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}\sin(jt)S(t)x\,dt - \frac{1}{2}\int_0^{\pi} tS(t)x\,dt\Big\| \end{aligned}$$

$$\begin{aligned}&\le \int_0^\pi \Big\| \Big( \sum_{j=1}^k \frac{(-1)^{j+1}}{j} \sin(jt) - \frac{1}{2}\, t \Big) S(t)x \Big\| \, dt \\ &\le \int_0^\pi \Big| \sum_{j=1}^k \frac{(-1)^{j+1}}{j} \sin(jt) - \frac{1}{2}\, t \Big| \, dt \; C\|x\|.\end{aligned}$$

By Lemma 3.2 the series $\sum_{j=1}^\infty \big((-1)^{j+1}/j\big)\sin(jt)$ converges pointwise to $t/2$ and the partial sums are uniformly bounded. Therefore, by the dominated convergence theorem, the integral tends to 0 as $k$ tends to $\infty$. This proves the theorem if $m = 1$.

By Theorem 2.2 the same is true for the operator $m^{-2}A$. Since

$$\big(k^2 I - m^{-2}A\big)^{-1} = m^2\big((mk)^2 I - A\big)^{-1},$$

the series $\sum_{k=1}^\infty \big((mk)^2 I - A\big)^{-1}$ converges in operator norm. The expression of the sum of the series can easily be obtained from what we have just proved and equalities (2.3) and (2.4). □

## 4. Examples

In this section we see two examples showing that the estimate of the resolvent stated in Theorem 3.1 cannot be improved but it is not sufficient to guarantee the uniform well-posedness. For the first example we need two lemmas.

**Lemma 4.1.** *Let $K\colon [0,\pi]^2 \to \mathbb{R}$ be such that*

$$K(t,s) = \begin{cases} \frac{(t-\pi)s}{\pi}, & \text{if } s \le t, \\ \frac{t(s-\pi)}{\pi}, & \text{if } s > t. \end{cases}$$

(1) *If $f \in C\big([0,\pi], X\big)$ then the function $z\colon [0,\pi] \to X$ such that*

$$z(t) = \int_0^\pi K(t,s) f(s)\, ds$$

*belongs to $C^2\big([0,\pi], X\big)$, with $z'' = f$ and $z(0) = z(\pi) = 0$.*

(2) *If $z \in C^2\big([0,\pi], X\big)$, with $z(0) = z(\pi) = 0$ then, for all $t \in [0,\pi]$, we have*

$$z(t) = \int_0^\pi K(t,s) z''(s)\, ds\,.$$

The proof of the above lemma is an easy consequence of the fundamental theorem of calculus.

**Lemma 4.2.** *Let $(v_k)_{k\in\mathbb{N}}$ be a sequence in $C^2\big([0,\pi], X\big)$ such that*

(1) *$(v_k)_{k\in\mathbb{N}}$ converges pointwise to $v \in C\big([0,\pi], X\big)$;*
(2) *$(v_k'')_{k\in\mathbb{N}}$ converges uniformly to $w \in C\big([0,\pi], X\big)$.*

*Then $v \in C^2\big([0,\pi], X\big)$ and $v'' = w$.*

*Proof.* Let

$$z_k\colon [0,\pi] \to X\,, \quad z_k(t) = v_k(t) - \frac{\pi - t}{\pi}\, v_k(0) - \frac{t}{\pi}\, v_k(\pi)\,.$$

We have $z_k(0) = z_k(\pi) = 0$ and $z_k'' = v_k''$. Hence, by Lemma 4.1, for all $t \in [0,\pi]$, we have

$$v_k(t) = z_k(t) + \frac{\pi - t}{\pi}\, v_k(0) + \frac{t}{\pi}\, v_k(\pi)$$

$$= \int_0^\pi K(t,s) v_k''(s)\, ds + \frac{\pi - t}{\pi}\, v_k(0) + \frac{t}{\pi}\, v_k(\pi)\,.$$

By (2) we can pass to the limit under the integral sign, hence

$$v(t) = \int_0^\pi K(t,s) w(s)\, ds + \frac{\pi - t}{\pi}\, v(0) + \frac{t}{\pi}\, v(\pi)\,.$$

Therefore, by Lemma 4.1, $v$ is twice differentiable and $v'' = w$. □

**Example 4.3.** Let $X = \ell^p(\mathbb{N}, \mathbb{K})$, with $1 \le p < \infty$. Choose $\alpha$ such that $0 < \alpha < 1$ and let $A$ be the operator in $X$ defined by

$$\mathcal{D}(A) = \big\{ x \in X \;\big|\; (n^2 x_n)_{n\in\mathbb{N}} \in X \big\}\,, \quad (Ax)_n = (n-\alpha)^2 x_n\,.$$

The operator $A$ is linear, it is easy to prove that it is closed and has dense domain, since the domain contains the eventually zero sequences.

For $t \in [0, \pi]$ let

$$S(t) \colon X \to X\,, \quad \big(S(t)x\big)_n = \frac{\sin\big((n-\alpha)t\big)}{\sin\big((n-\alpha)\pi\big)}\, x_n\,.$$

We prove that $S$ satisfies conditions (1)–(4) of Theorem 2.1, hence the Dirichlet problem for the operator $A$ is uniformly well-posed. Since

$$\sum_{n=1}^{\infty} \Big| \frac{\sin\big((n-\alpha)t\big)}{\sin\big((n-\alpha)\pi\big)}\, x_n \Big|^p \le \sum_{n=1}^{\infty} \frac{1}{\big(\sin(\alpha\pi)\big)^p}\, |x_n|^p\,,$$

$S(t) \in \mathcal{L}(X)$ with

$$\big\| S(t) \big\| \le \big(\sin(\alpha\pi)\big)^{-1}\,. \tag{4.1}$$

A similar argument shows that if $x \in \mathcal{D}(A)$ then $S(t)x \in \mathcal{D}(A)$. Moreover we have $AS(t)x = S(t)Ax$.

The function $\big(S(\cdot)x\big)_n$ is continuous, for every $x \in X$ and $n \in \mathbb{N}$; therefore if $x$ is an eventually zero sequence then $S(\cdot)x$ is continuous. If $x \in X$ then it is the limit of a sequence of eventually zero elements of $X$, hence, by estimate (4.1), $S(\cdot)x$ is the uniform limit of a sequence of continuous functions, therefore it is continuous. This proves (1).

Property (2) is obvious.

The function $\big(S(\cdot)x\big)_n$ is of class $C^2$, for every $x \in X$ and $n \in \mathbb{N}$, and

$$\frac{d^2}{dt^2}\big(S(t)x\big)_n = \frac{d^2}{dt^2}\Big( \frac{\sin\big((n-\alpha)t\big)}{\sin\big((n-\alpha)\pi\big)}\, x_n \Big) = -(n-\alpha)^2\, \frac{\sin\big((n-\alpha)t\big)}{\sin\big((n-\alpha)\pi\big)}\, x_n\,.$$

Therefore if $x \in \mathcal{D}(A)$ then

$$\frac{d^2}{dt^2}\big(S(t)x\big)_n = -\big(AS(t)x\big)_n\,.$$

Hence if $x$ is eventually zero then $S(\cdot)x$ is solution of equation (2.1). Now let $x \in \mathcal{D}(A)$ and, for $m \in \mathbb{N}$, let $x^{(m)} \in X$ such that

$$\big(x^{(m)}\big)_n = \begin{cases} x_n\,, & \text{for } n \le m\,,\\ 0\,, & \text{for } n > m\,. \end{cases}$$

We have

$$\sup_{t\in[0,\pi]} \big\| S(t)x^{(m)} - S(t)x \big\| \le \sup_{t\in[0,\pi]} \big\| S(t) \big\|\, \big\| x^{(m)} - x \big\|$$

$$\le \frac{1}{\sin(\alpha\pi)} \Big( \sum_{n=m+1}^{\infty} |x_n|^p \Big)^{1/p} \xrightarrow[m\to\infty]{} 0\,,$$

and

$$\begin{aligned} \sup_{t\in[0,\pi]} \big\| S(t)Ax^{(m)} - S(t)Ax \big\| &\le \sup_{t\in[0,\pi]} \big\| S(t) \big\| \, \big\| Ax^{(m)} - Ax \big\| \\ &\le \frac{1}{\sin(\alpha\pi)} \Big( \sum_{n=m+1}^{\infty} (n-\alpha)^{2p} |x_n|^p \Big)^{1/p} \xrightarrow[m\to\infty]{} 0\,. \end{aligned}$$

Hence the sequence of functions $S(\cdot)x^{(m)}$ converges uniformly to $S(\cdot)x$ and the sequence $S''(\cdot)x^{(m)} = -AS(\cdot)x^{(m)} = -S(\cdot)Ax^{(m)}$ converges uniformly to the function $-S(\cdot)Ax = -AS(\cdot)x$. By Lemma 4.2 $S(\cdot)x \in C^2\big([0,\pi],X\big)$ and $S''(\cdot)x = -AS(\cdot)x$. Therefore (3) is satisfied.

If $v\colon [0,\pi] \to X$ is solution of equation (2.1), then for all $n \in \mathbb{N}$ we have

$$v_n''(t) + (n-\alpha)^2 v_n(t) = 0\,.$$

The functions $t \mapsto \sin\big((n-\alpha)t\big)$ and $t \mapsto \sin\big((n-\alpha)(\pi - t)\big)$ are two linearly independent solutions of this equation, therefore there exist $c_1, c_2 \in \mathbb{K}$ such that

$$v_n(t) = c_1 \sin\big((n-\alpha)t\big) + c_2 \sin\big((n-\alpha)(\pi-t)\big)\,.$$

By putting $t = \pi$ or $t = 0$ in this equality we get

$$c_1 = \frac{v_n(\pi)}{\sin\big((n-\alpha)\pi\big)}\,, \quad c_2 = \frac{v_n(0)}{\sin((n-\alpha)\pi\big)}\,,$$

hence

$$v_n(t) = \frac{\sin\big((n-\alpha)(\pi-t)\big)}{\sin\big((n-\alpha)\pi\big)}\, v_n(0) + \frac{\sin\big((n-\alpha)t\big)}{\sin\big((n-\alpha)\pi\big)}\, v_n(\pi)\,,$$

that is

$$v(t) = S(\pi - t)v(0) + S(t)v(\pi)\,.$$

Therefore (4) is satisfied.

If $k \in \mathbb{N}$ it is easy to check that $k^2 I - A$ is invertible and, for all $x \in X$, for all $n \in \mathbb{N}$,

$$\big((k^2 - A)^{-1}x\big)_n = \frac{1}{k^2 - (n-\alpha)^2}\, x_n\,.$$

If we denote with $e_k$ the $k$-th element of the canonical basis of $X$, we have

$$\big\|(k^2 - A)^{-1}\big\| \ge \big\|(k^2 - A)^{-1} e_k\big\| = \frac{1}{k^2 - (k-\alpha)^2} = \frac{1}{2k\alpha - \alpha^2} \ge \frac{1}{2k\alpha}\,.$$

Therefore estimate (3.1) is the best possible one with respect to the dependence on $k$. Moreover the best constant $C$ can be arbitrarily large.

The next example is based on the results of [8]. We recall some facts from that article.

Let $X$ be a Banach space and $(y_n)_{n\in\mathbb{N}}$ be a basis of $X$. We denote with $P_n$ the $n$-th projection operator associated with this basis, i.e. $P_n \in \mathcal{L}(X)$ is such that, for all $x \in X$, we have $x = \sum_{n=1}^{\infty} P_n(x) y_n$. By the uniform boundedness principle $M_0 = \sup_{n\in\mathbb{N}} \big\|\sum_{k=1}^{n} P_n\big\| < \infty$.

We associate with every scalar valued sequence $a = (a_n)_{n\in\mathbb{N}}$ a linear operator $A$ in $X$ defined by

$$\mathcal{D}(A) = \Big\{x \in X \mid \sum_{n=1}^{\infty} a_n P_n(x) y_n \text{ is convergent}\Big\}\,,$$
$$Ax = \sum_{n=1}^{\infty} a_n P_n(x) y_n\,.$$

Let $a$ be a scalar valued bounded sequence; we set $\|a\|_\infty = \sup_{n\in\mathbb{N}} |a_n|$ and $V(a) = \sum_{n=1}^{\infty} |a_{n+1} - a_n|$. We denote with $BV$ the space of the sequences of bounded variation, i.e. such that $V(a) < \infty$.

**Lemma 4.4** ([8, Lemma 2.4])**.** *Let $a$ be a scalar valued sequence. Then the operator $A$ associated with $a$ is densely defined and closed. Moreover if $a \in BV$, then $A \in \mathcal{L}(X)$, with $\|A\| \le M_0\big(\|a\|_\infty + V(a)\big)$.*

**Lemma 4.5** ([8, Lemma 2.5])**.** *Let $a$ be a scalar valued sequence, $A$ be the operator associated with $a$ and $\lambda \in \mathbb{K} \setminus \{a_n \,|\, n \in \mathbb{N}\}$. Then $\lambda I - A$ is one-to-one and $(\lambda I - A)^{-1}$ is the operator associated with the sequence $\big((\lambda - a_n)^{-1}\big)_{n\in\mathbb{N}}$. In particular $\lambda \in \rho(A)$ if and only if for all $x \in X$ the series $\sum_{n=1}^{\infty}(\lambda - a_n)^{-1} P_n(x) y_n$ is convergent.*

We say that the basis $(y_n)_{n\in\mathbb{N}}$ of the Banach space $X$ is *unconditional* if, for all $x \in X$, the series $\sum_{n=1}^{\infty} P_n(x) y_n$ is unconditionally convergent. Otherwise we say that the basis is *conditional* (see [7, Chapter II, Definition 14.1]).

Every Banach space with a basis has an unconditional basis (see [7, Chapter II, Theorem 23.2]).

**Lemma 4.6** ([7, Chapter II, Theorem 16.1, 1 $\Leftrightarrow$ 8])**.** *Let $(y_n)_{n\in\mathbb{N}}$ be a conditional basis of the space $X$. Then there exists $\overline{x} \in X$ and a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\{-1, 1\}$ such that*

$$\sup_{n\in\mathbb{N}} \Big\| \sum_{j=1}^{n} \varepsilon_j P_j(\overline{x}) \Big\| = \infty\,.$$

**Example 4.7.** Let $X$ be a Banach space with a conditional basis $(y_n)_{n\in\mathbb{N}}$ and let $P_n$ be the $n$-th projection operator associated with this basis.

Let $(\ell_n)_{n\in\mathbb{N}}$ be a sequence of non-negative integers, unbounded and such that $\ell_1 = 0$ and $\ell_{n+1} - \ell_n \in \{0, 1\}$. Set

$$a_n = \Big(2\ell_n + \frac{1}{2}\Big)^2$$

and let $A$ be the operator associated with the sequence $(a_n)_{n\in\mathbb{N}}$. By Lemma 4.4, $A$ is closed and densely defined.

We prove that $A$ satisfies the conditions of Theorem 3.1. Let $k \in \mathbb{N}$. For every $n \in \mathbb{N}$ we have $a_n \ne k^2$, hence, by Lemma 4.5, $(k^2 I - A)$ is one-to-one and, if we set $b_n = 1/(k^2 - a_n)$, then $(k^2 I - A)^{-1}$ is the operator associated with the sequence $(b_n)_{n\in\mathbb{N}}$. For every $\ell \in \mathbb{N}$ we have

$$\Big| k^2 - \Big(2\ell + \frac{1}{2}\Big)^2 \Big| \ge \Big| k^2 - \Big(k - \frac{1}{2}\Big)^2 \Big| = k - \frac{1}{4}\,,$$

hence

$$\|b\|_\infty = \frac{1}{\inf_{n\in\mathbb{N}} |k^2 - a_n|} \le \frac{1}{k - (1/4)}\,.$$

Let $\overline{n} = \max\{n \in \mathbb{N} \,|\, 2\ell_n < k\}$; the sequence $b_n$ is positive and non-decreasing for $1 \le n \le \overline{n}$ while is negative and non-decreasing for $n \ge \overline{n}+1$. Hence

$$\begin{aligned} V(b) &= \sum_{n=1}^{\infty} |b_{n+1} - b_n| \\ &= \sum_{n=1}^{\overline{n}-1} (b_{n+1} - b_n) + b_{\overline{n}} - b_{\overline{n}+1} + \sum_{n=\overline{n}+1}^{\infty} (b_{n+1} - b_n) \\ &= b_{\overline{n}} - b_1 + b_{\overline{n}} - b_{\overline{n}+1} + \lim_{n\to\infty} b_n - b_{\overline{n}+1} \\ &\le 2b_{\overline{n}} - 2b_{\overline{n}+1} \le \frac{4}{k-(1/4)}\,. \end{aligned}$$

Therefore, by Lemmas 4.4 and 4.5, $k \in \rho(A)$ and there exists $C \in \mathbb{R}^+$ such that $\|(k^2 I - A)^{-1}\| \le C/k$.

Now we prove that $A$ satisfies the condition of Theorem 3.3 with $m = 1$, that is the series $\sum_{k=1}^{\infty} (k^2 I - A)^{-1}$ converges in operator norm.

First of all for all $k \in \mathbb{N}$ and all $n \in \mathbb{N}_0$ we have

$$\begin{aligned} \sum_{j=k}^{\infty} \frac{1}{j^2 - \big(n + (1/2)\big)^2} &= \lim_{p\to\infty} \sum_{j=k}^{p} \frac{1}{j^2 - \big(n+(1/2)\big)^2} \\ &= \frac{1}{2n+1} \lim_{p\to\infty} \sum_{j=k}^{p} \Big( \frac{1}{j-n-(1/2)} - \frac{1}{j+n+(1/2)} \Big) \\ &= \frac{1}{2n+1} \lim_{p\to\infty} \Big( \sum_{\ell=k-n}^{p-n} \frac{1}{\ell-(1/2)} - \sum_{\ell=k+n+1}^{p+n+1} \frac{1}{\ell-(1/2)} \Big) \\ &= \frac{1}{2n+1} \lim_{p\to\infty} \Big( \sum_{\ell=k-n}^{k+n} \frac{1}{\ell-(1/2)} - \sum_{\ell=p-n+1}^{p+n+1} \frac{1}{\ell-(1/2)} \Big) \\ &= \frac{1}{2n+1} \sum_{\ell=k-n}^{k+n} \frac{1}{\ell-(1/2)}\,. \end{aligned}$$

Hence

$$\sum_{j=k}^{\infty} \frac{1}{j^2 - \big(n+(1/2)\big)^2} = \frac{1}{2n+1} \sum_{\ell=k-n}^{k+n} \frac{1}{\ell-(1/2)}\,; \tag{4.2}$$

in particular

$$\sum_{j=1}^{\infty} \frac{1}{j^2 - \big(n+(1/2)\big)^2} = \frac{1}{2n+1} \sum_{\ell=1-n}^{1+n} \frac{1}{\ell-(1/2)} = \frac{2}{(2n+1)^2}\,,$$

where the last equality can be easily proved by induction on $n$. Therefore, for all $n \in \mathbb{N}$, we have

$$\sum_{j=1}^{\infty} \frac{1}{j^2 - \big(2\ell_n + (1/2)\big)^2} = \frac{2}{(4\ell_n+1)^2}\,.$$

Let $B$ be the operator in $X$ associated with the sequence $\big(2/\big(4\ell_n+1\big)^2\big)_{n\in\mathbb{N}}$. This sequence is non-negative and decreasing, hence it is bounded and has bounded variation, therefore, by Lemma 4.4, $B$ is bounded. We prove that

$$\sum_{k=1}^{\infty}(k^2 I - A)^{-1} = B\,.$$

The operator $B - \sum_{j=1}^{k-1}(j^2 I - A)^{-1}$ is associated with the sequence whose $n$-th term, taking into account (4.2), is

$$\begin{aligned}\frac{2}{\big(4\ell_n + 1\big)^2} - \sum_{j=1}^{k-1}\frac{1}{j^2 - \big(2\ell_n + (1/2)\big)^2} &= \sum_{j=k}^{\infty}\frac{1}{j^2 - \big(2\ell_n + (1/2)\big)^2}\\ &= \frac{1}{4\ell_n + 1}\sum_{j=k-2\ell_n}^{k+2\ell_n}\frac{1}{j-(1/2)}\,,\end{aligned}$$

If we put

$$C_{k,\ell} = \frac{1}{2\ell+1}\sum_{j=k-\ell}^{k+\ell}\frac{1}{j-(1/2)}\,,$$

we have to prove that the sequence $(C_{k,2\ell_n})_{n\in\mathbb{N}}$ has least upper bound and variation converging to 0 as $k \to 0$. Obviously

$$\sup_{n\in\mathbb{N}}|C_{k,2\ell_n}| \le \sup_{\ell\in\mathbb{N}_0}|C_{k,\ell}|\,.$$

Moreover, since $\ell_{n+1} - \ell_n \in \{0,1\}$ each difference $C_{k,2\ell_{n+1}} - C_{k,2\ell_n}$ either is null or is equal to $C_{k,2\ell+2} - C_{k,2\ell}$ for a suitable $\ell \in \mathbb{N}_0$. Hence

$$\sum_{n=1}^{\infty}|C_{k,2\ell_{n+1}} - C_{k,2\ell_n}| \le \sum_{\ell=0}^{\infty}|C_{k,2\ell+2} - C_{k,2\ell}| \le \sum_{\ell=0}^{\infty}|C_{k,\ell+1} - C_{k,\ell}|\,.$$

For all $k \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$ we have $C_{k,\ell} > 0$. Indeed, if $\ell < k$ then $C_{k,\ell} > 0$, since each term in the sum is positive. If $\ell \ge k$ then

$$\sum_{j=k-\ell}^{k+\ell}\frac{1}{j-(1/2)} = \sum_{j=k-\ell}^{\ell-k+1}\frac{1}{j-(1/2)} + \sum_{j=\ell-k+2}^{k+\ell}\frac{1}{j-(1/2)} = \sum_{j=\ell-k+2}^{k+\ell}\frac{1}{j-(1/2)} > 0\,.$$

If we put $c_{k,j} = 1/\big(k+j-(1/2)\big)$, we have

$$c_{k,-j} + c_{k,j} = \frac{2k-1}{\big(k-(1/2)\big)^2 - j^2}$$

Hence it is easy to prove that

$$2c_{k,0} < c_{k,-1} + c_{k,1} < c_{k,-2} + c_{k,2} < \dots < c_{k,-k+1} + c_{k,k-1}$$

and, if $k \le j$, $c_{k,-j} + c_{k,j} < 0$.

We observe that $C_{k,\ell}$ is the mean of $c_{k,-\ell}, c_{k,-\ell+1}, \dots, c_{k,\ell--}, c_{k,\ell}$ and $C_{k,\ell+1}$ is the mean of the same numbers plus $c_{k,-\ell-1}$ and $c_{k,\ell+1}$. If $\ell+1 < k$ the mean of $c_{k,-\ell-1}$ and $c_{k,\ell+1}$ is greater then the mean of $c_{k,-\ell}, \dots, c_{k,\ell}$, hence $C_{k,\ell+1} > C_{k,\ell}$. If $\ell+1 \ge k$ the mean of $c_{k,-\ell-1}$ and $c_{k,\ell+1}$ is negative, while the mean of $c_{k,-\ell}, \dots, c_{k,\ell}$ is positive, hence $C_{k,\ell+1} < C_{k,\ell}$. Therefore $\sup\{C_{k,\ell} \mid \ell \in \mathbb{N}_0\} = C_{k,k-1}$.

Moreover

$$\sum_{\ell=0}^{\infty}|C_{k,\ell+1} - C_{k,\ell}| = \sum_{\ell=0}^{k-2}(C_{k,\ell+1} - C_{k,\ell}) + \sum_{\ell=k-1}^{\infty}(C_{k,\ell} - C_{k,\ell+1})$$

$$
\begin{aligned}
&= C_{k,k-1} - C_{k,0} + C_{k,k-1} - \lim_{\ell\to\infty} C_{k,\ell+1} \\
&< 2C_{k,k-1}\,.
\end{aligned}
$$

Since

$$
\begin{aligned}
C_{k,k-1} &= \frac{1}{2k-1}\sum_{j=1}^{2k-1}\frac{1}{j-(1/2)} \\
&\le \frac{1}{2k-1}\Big(2+\int_1^{2k-1}\frac{1}{x-(1/2)}\,dx\Big) \\
&= \frac{1}{2k-1}\big(2+\log(4k-3)\big) \xrightarrow[k\to\infty]{} 0\,,
\end{aligned}
$$

the statement about the sequence $(C_{k,\ell_n})_{n\in\mathbb{N}}$ is proved.

Let $v\colon[0,\pi]\to X$ be a solution of equation (2.1). Then for all $n\in\mathbb{N}$ we have

$$
\frac{d^2}{dt^2}P_n\big(v(t)\big) = P_n\big(v''(t)\big) = -P_n\big(Av(t)\big) = -a_nP_n\big(v(t)\big)\,.
$$

Hence the function $P_n\circ v$ is solution of the equation $u''(t)+a_nu(t)=0$, that is $u''(t)+\big(2\ell_n+(1/2)\big)^2u(t)=0$. Hence there exist $c_1,c_2\in\mathbb{K}$ such that

$$
P_n\big(v(t)\big) = c_1\sin\Big(\Big(2\ell_n+\frac12\Big)(\pi-t)\Big) + c_2\sin\Big(\Big(2\ell_n+\frac12\Big)t\Big).
$$

Obviously we have $c_1 = P_n\big(v(0)\big)$, $c_2 = P_n\big(v(\pi)\big)$.

Now we choose a particular sequence $(\ell_n)_{n\in\mathbb{N}}$. By Lemma 4.6, there exist $\overline{x}\in X$ and a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, in $\{-1,1\}$ such that

$$
\sup_{n\in\mathbb{N}}\Big\|\sum_{k=1}^n\varepsilon_kP_k(\overline{x})\Big\| = \infty\,.
$$

The sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ can be chosen such that $\varepsilon_1=1$. We put $\varepsilon_0=1$ and, for $n\in\mathbb{N}$, $\ell_n=(1/2)\sum_{k=1}^n|\varepsilon_k-\varepsilon_{k-1}|$. In this way we have defined a non-decreasing sequence of integer numbers, with $\ell_1=0$ and $\ell_{n+1}-\ell_n\in\{0,1\}$. Since the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ must have an infinite number of changes of sign, the sequence $\ell_n$ is unbounded. Moreover it is easy to show, by induction, that, for all $n\in\mathbb{N}$, we have $(-1)^{\ell_n}=\varepsilon_n$.

If $v$ is solution of the problem

$$
\begin{aligned}
u''(t)+Au(t)&=0\,,\\
u(0)&=0\,,\\
u(\pi)&=\overline{x}\,,
\end{aligned}
\tag{4.3}
$$

then

$$
P_n\big(v(t)\big) = \sin\Big(\Big(2\ell_n+\frac12\Big)t\Big)P_n(\overline{x})\,,
$$

hence

$$
P_n\Big(v\Big(\frac{\pi}{2}\Big)\Big) = \sin\Big(\ell_n\pi+\frac{\pi}{4}\Big)P_n(\overline{x}) = \frac{(-1)^{\ell_n}}{\sqrt2}\,P_n(\overline{x}) = \frac{\varepsilon_n}{\sqrt2}\,P_n(\overline{x})\,.
$$

Since the series $\sum_{k=1}^\infty\varepsilon_kP_k(\overline{x})$ doesn't converge, problem (4.3) doesn't have solution.

## 5. Bounded operators

If the operator $A$ is bounded there is a simple characterization of uniform well-posedness of the Dirichlet problem. For the proof of the theorem we need a result about Fourier series.

**Lemma 5.1.** *The series of functions* $\sum_{k=1}^{\infty}\big((-1)^{k+1}/k^3\big)\sin(kt)$ *converges uniformly for* $t\in[0,\pi]$ *and*

$$\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}\,\sin(kt)=\frac{\pi}{6}\,t-\frac{1}{6\pi}\,t^3\,.$$

*Proof.* The uniform convergence of the series is obvious, since it is uniformly estimated by the harmonic series of exponent 3. Let

$$f\colon[-\pi,\pi]\to\mathbb{R}\,,\quad f(t)=\frac{\pi}{6}\,t-\frac{1}{6\pi}\,t^3\,.$$

It is easy to show that, for all $k\in\mathbb{N}$, we have

$$\int_0^{\pi}f(t)\sin(kt)\,dt=\frac{(-1)^{k+1}}{k^3}\,.$$

Hence the Fourier coefficients of the odd function $f$ are equal to $2(-1)^{k+1}/(\pi k^3)$ (see [9] Chapter I, (4.8)). Since $f\in C^1\big([-\pi,\pi],\mathbb{R}\big)$ and its $2\pi$-periodic repetition has the same regularity, by [9, Chapter II,Theorem (8.1)] its Fourier series converges to $f$. $\square$

**Theorem 5.2.** *Let $X$ be a Banach space and $A\in\mathcal{L}(X)$. Problem* (2.2) *is uniformly well-posed if and only if* $\{k^2\mid k\in\mathbb{N}\}\subseteq\rho(A)$.

*Proof.* The necessity of the condition is a consequence of Theorem 3.1.

We prove that, if $\{k^2\mid k\in\mathbb{N}\}\subseteq\rho(A)$, there exists $S\colon[0,\pi]\to X$ that satisfies conditions (1)–(4) of Theorem 2.1; hence problem (2.2) is uniformity well-posed.

It is well known that if $\lambda\in\mathbb{K}$ is such that $|\lambda|>\|A\|$ then $\lambda\in\rho(A)$ and $(\lambda I-A)^{-1}=\sum_{n=0}^{\infty}\lambda^{-n-1}A^n$, hence

$$\big\|(\lambda I-A)^{-1}\big\|\le\sum_{n=0}^{\infty}|\lambda|^{-n-1}\|A\|^n=\frac{1}{|\lambda|-\|A\|}\,.$$

Therefore there exists $M\in\mathbb{R}^+$ such that, for all $k\in\mathbb{N}$, we have

$$\big\|(k^2I-A)^{-1}\big\|\le\frac{M}{k^2}\,.$$

For $t\in[0,\pi]$ and $x\in X$ put

$$S(t)x=\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}\,\sin(kt)\,(k^2I-A)^{-1}A^2x+\Big(\frac{\pi}{6}\,t-\frac{1}{6\pi}\,t^3\Big)Ax+\frac{t}{\pi}\,x\,.$$

We have

$$\begin{aligned}&\big\|S(t)x\big\|\\&\le\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^3}\,|\sin(kt)|\,\big\|(k^2I-A)^{-1}\big\|\,\|A\|^2\|x\|+\Big(\frac{\pi}{6}\,t-\frac{1}{6\pi}\,t^3\Big)\|A\|\,\|x\|+\frac{t}{\pi}\,\|x\|\\&\le\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{M}{k^5}\|A\|\,\|x\|+\frac{\pi^2}{6}\|A\|\,\|x\|+\|x\|=C\|x\|\,.\end{aligned}$$

Hence $S(t) \in \mathcal{L}(X)$. This estimate shows also that the series is uniformly convergent, therefore the function $S(\cdot)x$ is continuous. Hence (1) is satisfied.

Obviously (2) is satisfied.

It is easy to check that the series of first and second derivatives are uniformly convergent, hence $S(\cdot)x \in C^2\big([0,\pi], X\big)$ and, for all $t \in [0,\pi]$,

$$\begin{aligned}
&S''(t) + AS(t)\\
&= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\,\sin(kt)k^2(k^2I-A)^{-1}A^2x - \frac{t}{\pi}\,Ax\\
&\quad + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}\,\sin(kt)A(k^2I-A)^{-1}A^2x + \Big(\frac{\pi}{6}\,t - \frac{1}{6\pi}\,t^3\Big)A^2x + \frac{t}{\pi}\,Ax\\
&= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\,\sin(kt)A^2x + \Big(\frac{\pi}{6}\,t - \frac{1}{6\pi}\,t^3\Big)A^2x = 0\,,
\end{aligned}$$

the last equality follows from Lemma 5.1. Therefore condition (3) is satisfied.

To prove (4) we first show the uniqueness of the solution of problem (2.2). Let $v \in C^2\big([0,\pi], X\big)$ be a solution of

$$\begin{gathered}
u''(t) + Au(t) = 0\,, \quad t \in [0,\pi]\,,\\
u(0) = 0\,,\\
u(\pi) = 0\,.
\end{gathered}$$

For $k \in \mathbb{N}$ let

$$a_k = \int_0^{\pi} \sin(kt)v(t)\,dt\,.$$

Then

$$\begin{aligned}
Aa_k &= \int_0^{\pi} \sin(kt)Av(t)\,dt\\
&= -\int_0^{\pi} \sin(kt)v''(t)\,dt\\
&= -\big[\sin(kt)v'(t)\big]_0^{\pi} + \int_0^{\pi} k\cos(kt)v'(t)\,dt\\
&= \big[k\cos(kt)v(t)\big]_0^{\pi} + \int_0^{\pi} k^2\sin(kt)v(t)\,dt = k^2a_k\,.
\end{aligned}$$

Hence $(k^2I - A)a_k = 0$. Since $k^2I - A$ is injective, this proves that $a_k = 0$. Therefore all the Fourier coefficients of the $2\pi$-periodic repetition of the odd extension of $v$ are null, hence $v(t) = 0$ for a.e. $t$. Since $v$ is continuous $v(t) = 0$ for all $t$.

If $x_0, x_\pi \in X$, then the function $t \mapsto S(\pi - t)x_0 + S(t)x_\pi$ is solution of problem (2.2), but the solution is unique, hence every solution of problem (2.2) coincides with that function. Therefore (4) is satisfied. □

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