Finite groups with small centralizers of word-values

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Abstract. Given a positive integer m and a group-word w, we consider a finite group G such that $w(G) \neq 1$ and all centralizers of non-trivial w-values have order at most m. We prove that if $w = v(x_1^{q_1}, \ldots, x_k^{q_k})$, where v is a multilinear commutator word and $q_1, \ldots, q_k$ are p-powers for some prime p, then the order of G is bounded in terms of w and m only. Similar results hold when w is the nth Engel word or the word $w = [x^n, y_1, \ldots, y_k]$ with $k \geq 1$.

1. Introduction

The present paper grew out of the observation made in [**3**] that if w is a multilinear commutator word and G is a profinite group in which all w-values have finite centralizers, then either $w(G) = 1$ or G is finite. Here, as usual, $w(G)$ denotes the subgroup generated by the w-values in G. By a multilinear commutator word we mean a word which is obtained by nesting commutators, but using always different variables. Such words are also known under the name of outer commutator words and are precisely the words that can be written in the form of multilinear Lie monomials.

Naturally, a corresponding result for finite groups must be of quantitative nature, so in this paper we deal with the following family of questions.

Let m be a positive integer, w a group-word, and G a finite group such that $w(G) \neq 1$ and $C_G(x)$ has order at most m for each nontrivial w-value x of G. Does it follows that the order of G is bounded in terms of m and w only?

It is not difficult to see that for some words w the answer is negative. In particular, this happens when $w = x^n$, with $n > 1$ (see Section 3).

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On the other hand, it is well known that if all nontrivial elements of a finite group G have centralizers of order at most m, then the order of G is m-bounded. An easy way to see this is via observation that every Sylow p-subgroup P of G, having nontrivial centre, has order at most m. Therefore $|P| \leq m$ and so $|G| \leq m!$. In the published literature one can find results containing better bounds for $|G|$ (in particular, a result by Isaacs [**7**] says that $|G| \leq m^2$ whenever G is soluble).

In this paper we show that there are other words for which the question has an affirmative answer.

Theorem 1.1. *Let p be a prime, let $q_1, \dots, q_k$ be p-powers and $w = w(x_1, \dots, x_k)$ a multilinear commutator word of weight at least 2. Set $w_0 = w(x_1^{q_1}, \dots, x_k^{q_k})$. Assume that G is a finite group such that $w_0(G) \neq 1$ and $|C_G(x)| \leq m$ for every nontrivial w_0-value x of G. Then the order of G is (w_0, m)-bounded.*

Recall that the nth Engel word is defined inductively by $[x,_1 y] = [x, y]$, and $[x,_n y] = [[x_{n-1} y], y]$.

Theorem 1.2. *Let $w = [x,_n y]$ be the nth Engel word or the word $w = [x^n, y_1, \dots, y_k]$ with $k \geq 1$. Let m be an integer and G a finite group with the properties that $w(G) \neq 1$ and $|C_G(x)| \leq m$ for every nontrivial w-value x of G. Then the order of G is (w, m)-bounded.*

Throughout the paper we use the expression "$(a, b, \dots)$-bounded" to mean that a quantity is finite and bounded by a certain number depending only on the parameters $a, b, \dots$. Given a word w and a group G, we will denote by G_w the set of all values of w in G.

2. Preliminaries

Most of results given in this section are well known and are included simply for the reader's convenience.

Lemma 2.1. *Let φ be an automorphism of a finite group G. Then*

i) *$|C_{G/N}(\varphi)| \leq |C_G(\varphi)|$ for every φ-invariant normal subgroup N of G.*

ii) *If $|\varphi|$ and $|G|$ are coprime and $[G, \varphi, \varphi] = 1$, then $\varphi = 1$.*

Proof. The first statement is Lemma 2.12 in [**8**]. The second one is a well-known result about coprime action (see for instance Theorem 3.6 in [**5**]) □ □

Lemma 2.2. [**8**, Corollary 2.7] *Let p be a prime and let φ be an automorphism of order p^s of a finite abelian p-group A. If $|C_A(\varphi)| = p^m$, then the rank of A is at most mp^s.*

From Lemmas 2.1 and 2.2, using Burnside's Basis Theorem, we deduce the following result.

Corollary 2.3. *If a finite p-group P admits an automorphism φ of order p^s with exactly p^m fixed points, then P is generated by at most mp^s elements.*

Lemma 2.4. [**8**, Theorem 12.15] *If a finite p-group P admits an automorphism φ of order p^s with exactly p^m fixed points, then P contains a characteristic subgroup of (p, m, s)-bounded index which is soluble of (p, s)-bounded derived length.*

The following theorem, due to Hartley, depends on the classification of finite simple groups.

Theorem 2.5. [**6**, Theorem A] *There exists an integer-valued function f such that if G is a finite group containing an element x with $|C_G(x)| \leq m$, then G has a soluble normal subgroup of index at most $f(m)$.*

Let $Z_i(G)$ denote the ith term of the upper central series of a group G.

Lemma 2.6. *Let G be a finite group having an element g such that $|C_G(g)| \leq m$. Then $|Z_i(G)| \leq m^i$ for each i.*

Proof. Indeed, by Lemma 2.1 (i) we have

$$|Z_j(G)/Z_{j-1}(G)| \leq |C_{G/Z_{j-1}(G)}(gZ_{j-1}(G))| \leq |C_G(g)| \leq m.$$

In other words, all factors of the upper central series of G have orders at most m. Hence, the lemma follows. □ □

It will be convenient to fix the following hypothesis:

Hypothesis 1. *Let w be a group-word, m an integer and G a finite group such that $w(G) \neq 1$ and $|C_G(x)| \leq m$ for every nontrivial w-value x of G.*

Note that under Hypothesis 1, every w-value in G has order at most m.

3. Proof of Theorem 1.1

We recall that multilinear commutator words and their subcommutators are recursively defined as follows: the word $w = x$ in one variable x is a multilinear commutator word of weight 1 with x as its only subcommutator. If u and v are multilinear commutator words of weight r and s involving disjoint sets of variables, the word $w = [u, v]$

is a multilinear commutator word of weight $r+s$, and its subcommutators are w itself and all subcommutators of u and v. All multilinear commutator words are obtained in this way.

An important family of multilinear commutator words is formed by the derived words δ_k, on 2^k variables, which are defined recursively by

$$\delta_0 = x_1, \qquad \delta_k = [\delta_{k-1}(x_1, \ldots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})].$$

Of course $\delta_k(G) = G^{(k)}$ is the kth derived group of G. We will need the following well-known result (see for example [**9**, Lemma 4.1]).

Lemma 3.1. *Let G be a group and let w be a multilinear commutator word on k variables. Then each δ_k-value is a w-value.*

Let G be a group and let $w = w(x_1, \ldots, x_k)$ be a word. The marginal subgroup $w^*(G)$ of G corresponding to the word w is defined as the set of all $x \in G$ such that

$$w(g_1, \ldots, xg_i, \ldots, g_k) = w(g_1, \ldots, g_i x, \ldots, g_k) = w(g_1, \ldots, g_i, \ldots, g_k)$$

for all $g_1, \ldots, g_k \in G$ and $1 \leq i \leq k$. It is well known that $w^*(G)$ is a characteristic subgroup of G and that $[w^*(G), w(G)] = 1$. If w is a multilinear commutator word and N is a normal subgroup of a group G containing no nontrivial w-values, then N is contained $w^*(G)$ and, in particular, it centralizes $w(G)$ (see e.g. [**10**, Theorem 2.3] or the comment after Lemma 4.1 in [**3**]).

Proposition 3.2. *Assume Hypothesis 1 with $w = w(x_1, \ldots, x_k)$ being a multilinear commutator word. Then the order of G is m-bounded.*

Proof. By Theorem 2.5, G has a soluble normal subgroup H of m-bounded index. Let

$$1 = A_0 \leq A_1 \leq \cdots \leq A_t = H$$

be a characteristic series of H with the property that each section A_{i+1}/A_i is abelian. Let i be the greatest integer such that $A_i \cap G_w = 1$. As mentioned above, A_i is contained in the marginal subgroup of G, and hence it centralizes $w(G)$. In particular $|A_i| \leq m$.

If $i = t$, then the order of G is m-bounded, since $A_t = H$ has m-bounded index in G. Otherwise, $i < t$ and, by the maximality of i, the subgroup A_{i+1} contains a nontrivial w-value. So A_{i+1}/A_i contains a nontrivial w-value of G/A_i and, by Lemma 2.1 (i), G/A_i satisfies Hypothesis 1. It follows from Lemma **??** that $|G/A_i|$ is m-bounded. As A_i has m-bounded order, the order of G is m-bounded as well. □ □

In what follows we require an additional hypothesis.

Hypothesis 2. *Let p be a prime and let $q_1, \dots, q_k$ be p-powers. For a multilinear commutator word $w = w(x_1, \dots, x_k)$ of weight at least 2, set $w_0 = w(x_1^{q_1}, \dots, x_k^{q_k})$. Let G be a finite group such that $w_0(G) \neq 1$ and $|C_G(x)| \leq m$ for every nontrivial w_0-value x of G.*

If $u = u(x_r, \dots, x_s)$ is a subcommutator of w, we denote by u_0 the word $u_0 = u(x_r^{q_r}, \dots, x_s^{q_s})$.

Lemma 3.3. *Assume Hypothesis 2 and let T be a normal subgroup of G containing no nontrivial w_0-values of G. Let R be the subgroup generated by the u_0-values of G lying in T, where u ranges over all subcommutators of w. Then R has (w,m)-bounded order.*

Proof. The proof is by induction on the number of subcommutators v of w such that T $\cap G_{v_0} = 1$. Let u be a subcommutator of w of maximal weight such that $T \cap G_{u_0} \neq 1$. By maximality, there exists a subcommutator v of w such that $v = [u, u']$ and $T \cap G_{v_0} = 1$. Let $1 \neq a \in T \cap G_{u_0}$. For every $b \in T \cap G_{u_0'}$ observe that

$$[a, b] \in T \cap G_{v_0} = 1.$$

Hence $[a, u_0'(G)] = 1$. Set $R_1 = \langle T \cap G_{u_0} \rangle$. Since $w_0(G) \leq u_0'(G)$, it follows that $R_1 \leq C_G(w_0(G))$. In particular R_1 has order at most m. By Lemma 2.1 (i), G/R_1 satisfies Hypothesis 2. Moreover $(T/R_1) \cap (G/R_1)_{u_0} = 1$. Note that R/R_1 is generated by all v_0-values of G/R_1 lying in T/R_1, where v ranges over all subcommutators of w. Hence, by induction, R/R_1 has (w,m)-bounded order. As the order of R_1 is (w,m)-bounded, the lemma follows. □ □

Remark 3.4. Note that any multilinear commutator word of weight at least 2 has the words x and $[x, y]$ as subcommutators. Let $q = \max\{q_1, \dots, q_k\}$. Taking into account that $G_{x^q} \leq G_{x^{q_i}}$ and $G_{[x^q, y^q]} \leq G_{[x^{q_i}, y^{q_j}]}$, we observe that, under the assumptions of Lemma 3.3, the quotient group T/R has exponent dividing q and it contains no nontrivial commutators of the form $[x^q, y^q]$ for $x, y \in G$.

Lemma 3.5. *Assume Hypothesis 2 and let T be a normal p'-subgroup of G. Then $|T|$ is (w,m)-bounded.*

Proof. Since T is a p'-group, it follows that $T_{w_0} = T_w$. If $w(T) \neq 1$, by Proposition 3.2, we have that $|T|$ is m-bounded. So assume $w(T) = 1$. In particular, by Lemma 3.1, T is soluble with derived length at most k. We use induction on the derived length of T. Let A be the last term of the derived series of T. Note that A is normal in G. If there exists a nontrivial element in $A \cap G_{w_0}$, then $|G|$ is m-bounded by Lemma **??** and the same holds for $|T|$. So, let $A \cap G_{w_0} = 1$

and let R be the subgroup defined in Lemma 3.3. Then $|R|$ is (w, m)-bounded and, by Remark 3.4, the exponent of A/R divides q. Since A is a p'-subgroup, $A/R = 1$, hence $A \leq R$ has (w, m)-bounded order. Moreover, G/A satisfies Hypothesis 2. By induction, $|T/A|$ is (w, m)-bounded. Hence $|T|$ is (w, m)-bounded. □ □

Lemma 3.6. *Assume Hypothesis 2 and let T be a normal p-subgroup of G. Then $|T|$ is (w_0, m)-bounded.*

Proof. Assume that $T \cap G_{w_0} = 1$. Then the subgroup R of T defined in Lemma 3.3 has (w, m)-bounded order. By Remark 3.4, T/R has exponent dividing q and it contains no nontrivial commutators of the form $[x^q, y^q]$ for $x, y \in G$. As $R \cap G_{w_0} \leq T \cap G_{w_0} = 1$, by Lemma 2.1 (i), we can pass to the quotient G/R and assume without loss of generality that $R = 1$. Let g be a nontrivial element of G and $a \in T$. Since $[g^{-qa}, g^q]T = [g^{-q}, g^q]T$ is trivial in G/T, we see that $[g^{-qa}, g^q] \in T$. Thus

$$[g^{-qa}, g^q] \in T \cap G_{[x^q, y^q]} = 1.$$

As $[a, g^q, g^q] = [g^{-qa}, g^q]$, it follows that $[a, g^q, g^q] = 1$ for every $a \in T$. In particular, if $g \in G_{w_0}$ and $g = g^r g^s$, where g^r is a p-element and g^s is a p'-element, then

$$[T, g^{sq}, g^{sq}] = 1.$$

It follows from Lemma 2.1 (ii) that $[T, g^{sq}] = 1$. As g^s is a p'-element, we obtain that $[T, g^s] = 1$, so $C_P(g^r) = C_P(g)$. Therefore g induces a p-automorphism on T, which has order at most m and at most m fixed points.

Now assume that $T \cap G_{w_0} \neq 1$. Then any nontrival element in $T \cap G_{w_0}$ induces a p-automorphism on T, which has order at most m and at most m fixed points.

In both cases - when $T \cap G_{w_0} = 1$ and when $T \cap G_{w_0} \neq 1$ - we deduce from Lemma 2.4 that T has a soluble characteristic subgroup T_0 of (w_0, m)-bounded index and (w_0, m)-bounded derived length.

We now argue by induction on the derived length of T_0. Let A be the last term of the derived series of T_0. Note that A is normal in G. If there exists a nontrivial element in $A \cap G_{w_0}$, then $|G|$ is m-bounded by Lemma **??** and the same holds for $|T|$. So asssume that $A \cap G_{w_0} = 1$. As in Lemma 3.3, let R_A be the subgroup generated by the u_0-values of G lying in A, where u ranges over all subcommutators of w. Then $|R_A|$ is (w, m)-bounded and, by Remark 3.4, the exponent of A/R_A divides q. Note that, by Lemmas 2.1 and 2.2, A/R_A has (w_0, m)-bounded rank. It follows that $|A/R_A|$ is (w_0, m)-bounded, hence $|A|$ is (w_0, m)-bounded as well. Since G/A satisfies Hypothesis 2, by induction on the derived length of T_0, we get that $|T/A|$ is (w_0, m)-bounded. Hence $|T|$ is (w_0, m)-bounded. □ □

Let π be a set of primes dividing the order of a finite soluble group G. Recall that the π-length $l_\pi(G)$ of G is the number of π-factors in the shortest (π, π')-series of G. It is a well-known corollary of the Hall-Higman theory that $l_\pi(G)$ is bounded in terms of the derived length of a Hall π-subgroup of G (see for example [**1**] for an explicit bound).

Proof of Theorem 1.1. Assume Hypothesis 2. It follows from Theorem 2.5 that G has a normal soluble subgroup N of m-bounded index. Let H be a Hall p'-subgroup of N. Note that $H_{w_0} = H_w$. If $w(H) = 1$, then it follows from Lemma 3.1 that H has w-bounded derived length. If $w(H) \neq 1$, then H satisfies Hypothesis 1 for the multilinear word w, hence it has m-bounded order by Proposition 3.2. In both cases, the derived length of H is (w, m)-bounded. The Hall-Higman theory shows that the p-length of N is (w, m)-bounded. Thus N has a normal series of (w, m)-bounded length in which every factor is either a p-group or a p'-group and every subgroup in the series is normal in G.

Let T be the last term of the series. Either by Lemma 3.5 or by Lemma 3.6 we get that $|T|$ is (w_0, m)-bounded. If $w_0(G)$ is contained in T, then by Lemma **??** we conclude that the order of G is (w_0, m)-bounded. Otherwise, G/T contains a nontrivial w_0-value, hence, by Lemma 2.1 (i), it satisfies Hypothesis 2. By induction on the length of the above series, the order of G/T is (w_0, m)-bounded. Hence the order of G is (w_0, m)-bounded. □ □

Remark 3.7. The conclusion of Theorem 1.1 does not hold in general when $w = x$ is the multilinear word of length 1.

Indeed, consider the word x^n with $n \geq 2$. Let K a finite abelian group of exponent n and H a group of prime order p such that $p \nmid n$. Let B be the base group of the wreath product $K \wr H$ and let G be the semidirect product of $[B, H]$ by H. We see that G is a Frobenius group in which x^n-values are conjugate to elements of H and hence their centralizers have order p. On the other hand, K can be chosen of arbitrary big order so there are no bounds on the order of G here.

4. Proof of Theorem 1.2

A subgroup H of a group G is said to be a right n-Engel subgroup if all elements of H are right n-Engel, that is

$$[h,_n g] = 1$$

for every $h \in H$ and $g \in G$.

It is well known that right n-Engel elements belong to the hypercentre of G. The following theorem is a special case of Theorem 1 in [**2**].

Theorem 4.1. *Let G be a finite d-generator group and let H be a normal right n-Engel subgroup of G. Then there exists an integer $r = r(d, n)$, depending only on d and n, such that $H \leq Z_r(G)$.*

Let w be a word and m an integer. Recall that a finite group G satisfies Hypothesis 1 if $w(G) \neq 1$ and $|C_G(x)| \leq m$ for every nontrivial w-value x.

Lemma 4.2. *Assume Hypothesis 1 with w being the nth Engel word, and let F be a normal subgroup of G such that $F \cap G_w = 1$. Then F has (w, m)-bounded order.*

Proof. From the fact that $F \cap G_w = 1$ we deduce that F is a right n-Engel subgroup. Hence F is nilpotent. Let p be a prime divisor of $|F|$ and let P be the Sylow p-subgroup of F. As P is normal in G, we have $[P,_n g] = 1$ for every $g \in G$. When g is a p'-element, we deduce from Lemma 2.1 (ii) that g centralizes P. Thus $G/C_G(P)$ is a p-group.

Let x be a nontrivial w-value of G. Then x induces a p-automorphism of P of order at most m with at most m fixed points. Thus by Corollary 2.3, P is generated by at most mp^m elements. Moreover, since $1 \neq |C_P(x)| \leq m$, we have $p \leq m$. Thus every Sylow p-subgroup P of F is generated by at most m^{m+1} elements. Being nilpotent, F is generated by at most m^{m+1} elements, as well. So $G_1 = F\langle x\rangle$ is generated by at most $m^{m+1} + 1$ elements.

Since $F \cap G_w = 1$, it follows that $[f,_n g] = 1$ for every $f \in F$ and $g \in G_1$. We deduce from Theorem 4.1 that there exists a (w, m)-bounded integer i such that $F \leq Z_i(G_1)$. Since, by Lemma 2.6, $|Z_i(G_1)| \leq m^i$, we conclude that F has (w, m)-bounded order. □ □

The next lemma is the analogue of the previous one for the word $w = [x^n, y_1, \ldots, y_k]$.

Lemma 4.3. *Assume Hypothesis 1 with $w = [x^n, y_1, \ldots, y_k]$ with $k \geq 1$, and let F be a normal subgroup of G such that $F \cap G_w = 1$. Then F has (w, m)-bounded order.*

Proof. From the fact that $F \cap G_w = 1$ we deduce that for every $g \in G$,

$$[[g^n, F],_{k-1} G] = 1,$$

hence $[g^n, F] \leq Z_{k-1}(G)$. So, F centralizes g^n in the quotient group $G/Z_{k-1}(G)$ and therefore $[G^n, F] \leq Z_{k-1}(G)$. Note that $Z_{k-1}(G)$ has

(w,m)-bounded order by Lemma 2.6. If $w(G) \le Z_{k-1}(G)$, then the order of G is (w,m)-bounded by Lemma **??**, and the lemma follows. Otherwise $G/Z_{k-1}(G)$ contains a nontrivial w-value $\bar{g}$. For short we will use the bar notation to denote the images of subgroups in the quotient group $\bar{G} = G/Z_{k-1}(G)$. Since $\bar{g} \in \bar{G}^n$, we deduce that $\bar{F}$ centralizes $\bar{g}$, hence $|\bar{F}| \le m$. Therefore F has (w,m)-bounded order, as desired. □ □

Lemma 4.4. *Assume Hypothesis 1 with w being either the nth Engel word or the word $w = [x^n, y_1, \dots, y_k]$ with $k \ge 1$, and let F be a normal nilpotent subgroup of G cointaing a nontrivial w-value of G. Then F has (w,m)-bounded order.*

Proof. Let j be the greatest integer such that $Z_j(F) \cap G_w = 1$. Then by Lemmas 4.2 and 4.3 we deduce that $Z_j(F)$ has (w,m)-bounded order. Note that our assumptions imply that $Z_j(F) \ne F$.

For short we will use the bar notation to denote the images of subgroups and elements in the quotient group $\bar{G} = G/Z_j(F)$. Since $Z_{j+1}(F) \cap G_w \ne 1$ and $Z_j(F) \cap G_w = 1$, the centre $Z(\bar{F}) = \overline{Z_{j+1}(F)}$ contains a nontrivial w-value $\bar{g}$ of $\bar{G}$. Thus $\bar{F}$ is a subgroup of $C_{\bar{G}}(\bar{g})$, whose order is at most m by Lemma 2.1 (i). We conclude that F has (w,m)-bounded order. □ □

Recall that the Fitting height of a soluble group G is the length of a shortest series of G with nilpotent factors.

Proof of Theorem 1.2. Assume Hypothesis 1 with w being either the nth Engel word or the word $w = [x^n, y_1, \dots, y_k]$ with $k \ge 1$. By Theorem 2.5, G has a soluble normal subgroup H of m-bounded index. We claim that there exists an (w,m)-bounded integer e such that H belongs to the class $Y(e,w)$ of finite groups G in which every w-value has order dividing e and $w(P)$ has exponent dividing e for every p-Sylow subgroup of G. Indeed, if P is a p-Sylow subgroup of H, then either $w(P) = 1$ or, by Lemma 4.4, the order of $w(P)$ is (w,m)-bounded.

Lemma 9 in [**4**] states that the Fitting height of any soluble group in $Y(e,w)$ is (e,w)-bounded. Therefore the Fitting height h of H is (w,m)-bounded. We argue by induction on h.

Let T be the Fitting subgroup of H. Either by Lemma 4.4 or by Lemmas 4.2 and 4.3 we get that $|T|$ is (w,m)-bounded. If $w(G)$ is contained in T, then by Lemma **??** we conclude that the order of G is (w,m)-bounded. Otherwise, G/T contains a nontrivial w-value, hence, by Lemma 2.1 (i), it satisfies Hypothesis 2. By induction, the order of G/T is (w,m)-bounded, hence G has (w,m)-bounded order. □ □

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